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Bifurcation diagrams for singularly perturbed system: the multi-dimensional case.

Matteo Franca*

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Abstract

We consider a singularly perturbed system where the fast dynamics of the unperturbed problem exhibits a trajectory homoclinic to a critical point. We assume that the slow time system admits a unique critical point, which undergoes a bifurcation as a second parameter varies: transcritical, saddle-node, or pitchfork. We generalize to the multi-dimensional case the results obtained in a previous paper where the slow-time system is 1-dimensional. We prove the existence of a unique trajectory $(\breve{x}(t,\varepsilon,\lambda),\breve{y}(t,\varepsilon,\lambda))$ homoclinic to a centre manifold of the slow manifold. Then we construct curves in the 2-dimensional parameters space, dividing it in different areas where $(\breve{x}(t,\varepsilon,\lambda),\breve{y}(t,\varepsilon,\lambda))$ is either homoclinic, heteroclinic, or unbounded. We derive explicit formulas for the tangents of these curves. The results are illustrated by some examples.

Keywords. Singular perturbation, homoclinic trajectory, transcritical bifurcation central manifold.
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1 Introduction

In this paper we consider the following singularly perturbed system:

$$\begin{cases} \dot{x} = \varepsilon f(x,y,\varepsilon,\lambda) \\ \dot{y} = g(x,y,\varepsilon,\lambda) \end{cases} \tag{1.1}$$

*Dipartimento di Scienze Matematiche, Università Politecnica delle Marche, Via Brecce Bianche 1, 60131 Ancona - Italy. Partially supported by G.N.A.M.P.A. - INdAM (Italy) and MURST (Italy)

where $x \in \mathbb{R}^{m+1}$, $y \in \mathbb{R}^n$, ε and λ are small real parameters and $f(x,y,\varepsilon,\lambda)$, $g(x,y,\varepsilon,\lambda)$ are C^r–functions in their arguments bounded with their derivatives, $r \geq 3$. We assume that for $\varepsilon = \lambda = 0$ (1.1) admits a homoclinic trajectory $(0,h(t))$. Our purpose is to look for conditions ensuring the persistence of a bounded trajectory close to $(0,h(t))$ for ε and λ small, assuming that the slow time system undergoes a bifurcation as λ varies. This paper generalizes previous results obtained in [8] assuming that $x \in \mathbb{R}$, i.e. $m = 0$.

We suppose that the following conditions hold:

(i) for any $x \in \mathbb{R}^{m+1}$, we have

$$g(x,0,0,0) = 0,$$

(ii) the infimum over $x \in \mathbb{R}^{m+1}$ of the moduli of the real parts of the eigenvalues of the jacobian matrix $\frac{\partial g}{\partial y}(x,0,0,0)$ is greater than a positive number Λ^g.

(iii) the equation

$$\dot{y} = g(0,y,0,0)$$

has a solution $h(t)$ homoclinic to the origin $0 \in \mathbb{R}^n$

(iv) $\dot{h}(t)$ is the unique bounded solution of the linear variational system:

$$\dot{y} = \frac{\partial g}{\partial y}(0,h(t),0,0)y \tag{1.2}$$

up to a scalar multiple.

According to condition (ii), for any $x = (x_0, x_1, \ldots, x_m) \in \mathbb{R}^{m+1}$, the linear system $\dot{y} = \frac{\partial g}{\partial y}(x,0,0,0)y$ has exponential dichotomy on $\mathbb{R}$ with projections, say, $P^0(x)$. Let $\mathrm{rank}[P^0(x)] = p$, p being the number of eigenvalues of $\frac{\partial g}{\partial y}(x,0,0,0)$ with positive real parts: we stress that p is constant. From assumptions (ii) and (iii) it follows that the linear system (1.2) and its adjoint

$$\dot{y} = -\Big[\frac{\partial g}{\partial y}(0,h(t),0,0)\Big]^* y \tag{1.3}$$

have exponential dichotomies on both $\mathbb{R}_+$ and $\mathbb{R}_-$, see [6, 8] for more details. Here and later we use the shorthand notation $\pm$ to represent both the $+$ and $-$ equations and functions. Observe that $\mathrm{rank}(P^+) = \mathrm{rank}(P^-) = p$ and the projections of the dichotomy of (1.3) on $\mathbb{R}_\pm$ are $\mathbf{I} - [P^\pm]^*$. From (iv) it follows that (1.3) has a unique bounded solution on $\mathbb{R}$, up to a multiplicative constant. We denote one of these solutions by $\psi(t)$. Note that

$\psi := \psi(0)$ satisfies $\mathcal{N}[P^+]^* \cap \mathcal{R}[P^-]^* = \text{span}(\psi) = [\mathcal{R}P^+ \cap \mathcal{N}P^-]^\perp$; we assume w.l.o.g. that $|\psi(0)| = 1$. Condition (i) implies the existence of $\varepsilon_0 > 0$, $\lambda_0 > 0$ and a function $v(x, \varepsilon, \lambda)$ which is defined for $x \in \mathbb{R}^{m+1}$ small enough, $|\lambda| \leq \lambda_0$ and $|\varepsilon| \leq \varepsilon_0$, such that $v(x, 0, 0) \equiv 0$ and the manifold $\mathcal{M}^c(\varepsilon, \lambda) := \{(x, y) \mid y = v(x, \varepsilon, \lambda)\}$ is an invariant centre manifold for the flow of (1.1) (see for example [2, 12]). We will refer to $\mathcal{M}^c(\varepsilon, \lambda)$ as the "slow" manifold, since we have the following: if $\bar{x} = O(|\varepsilon| + |\lambda|)$ and $(x(t, \varepsilon, \lambda), y(t, \varepsilon, \lambda))$ is the solution of (1.1) such that $(x(0, \varepsilon, \lambda), y(0, \varepsilon, \lambda)) = (\bar{x}, v(\bar{x}, \varepsilon, \lambda))$, then $\|\dot{y}(0, \varepsilon, \lambda)\| = O[(|\lambda| + |\varepsilon|)]$. Moreover $v(x, \varepsilon, \lambda)$ is C^{r-1} and bounded with its derivatives. Using the flow of (1.1) we can extend the local manifold $y = v(x, \varepsilon, \lambda)$ outside a neighborhood of the origin: in such a case the manifold is not anymore a graph on the x coordinates.
In fact when $g(x, 0, \lambda, \varepsilon) \equiv 0$, then $v(x, \varepsilon, \lambda) \equiv 0$; anyhow, for $|x|$ small enough, passing to the new variable $\tilde{y} = y - v(x, \varepsilon, \lambda)$ and replacing f by $\tilde{f}(x, \tilde{y}, \varepsilon, \lambda) = f(x, \tilde{y} - v(x, \varepsilon, \lambda), \varepsilon, \lambda)$, and g by $\tilde{g}(x, \tilde{y}, \varepsilon, \lambda) = g(x, \tilde{y} - v(x, \varepsilon, \lambda), \varepsilon, \lambda)$ we can assume that the slow manifold is defined by $\tilde{y} = 0$. We also wish to emphasize that even if $v(x, \varepsilon, \lambda)$ is unknown we can get some information on its derivatives using the fact that $y = v(x, \varepsilon, \lambda)$ is invariant. E.g. if $\frac{\partial g}{\partial \lambda}(0, 0, 0, 0) = 0$ then $\frac{\partial v}{\partial \lambda}(0, 0, 0) = 0$.

1.1 Remark. All our arguments are local, i.e. we just consider what happens in a small (ε and λ independent) neighborhood $\Omega_h \subset \mathbb{R}^{m+n+1}$ of the graph of the unperturbed homoclinic $(0, h(t))$, obtained for $\varepsilon = \lambda = 0$. We stress that a priori the slow manifold $\mathcal{M}^c(\varepsilon, \lambda)$ may be not unique: this fact follows from centre manifold theory see [6, 13, 14]. However bounded trajectories, if any, belong to all the slow manifolds. Moreover there is a smooth conjugation between the dynamics of all the slow manifolds. This lack of uniqueness, together with an analogous uniqueness problem concerning centre manifold theory which is explained few lines below, will be discussed in more details in the Appendix in section 5. From now on we choose a slow manifold $\mathcal{M}^c$ which is globally defined, and we work on this, unless specified.

Let $x_c(t, \xi, \varepsilon, \lambda)$ be the solution of the initial value problem:

$$\dot{x} = f(x, v(x, \varepsilon, \lambda), \varepsilon, \lambda) \qquad x(0) = \xi \tag{1.4}$$

So $(x_c(t, \xi, \varepsilon, \lambda), v(x_c(t, \xi, \varepsilon, \lambda), \varepsilon, \lambda))$ describes the flow on the slow manifold $\mathcal{M}^c$, and (1.4) is the so called "slow time" system.

The behavior of homoclinic and heteroclinic trajectories subject to singular perturbation has been studied in several papers, see e.g. [1, 2, 4, 5, 6, 8, 12, 15]. In particular in [6] the authors built up a theory to prove the existence of solutions homoclinic to $\mathcal{M}^c$, for the perturbed problem (1.1) assuming conditions (i)–(iv) and giving transversality conditions of several different types. They refine previous results obtained in [4].

This paper along with [8] are thought as a sequel of [6]. Here and in [8] we assume that the "slow time" system (1.4) undergoes a bifurcation as λ changes sign for $\varepsilon = 0$. In [8] we assumed that $x \in \mathbb{R}$ is a scalar so the solution $(\tilde{x}(t, \varepsilon, \lambda), \tilde{y}(t, \varepsilon, \lambda))$ homoclinic to the slow manifold is unique. Then we derived further Melnikov conditions which enable us to divide the ε, λ space in different sets in which $(\tilde{x}(t, \varepsilon, \lambda), \tilde{y}(t, \varepsilon, \lambda))$ has different behavior: it is homoclinic, heteroclinic or it does not converge to critical points either in the past or in the future. The purpose of this paper is to extend the results of [8] to the multidimensional case $x \in \mathbb{R}^{m+1}$ for $m > 0$. Let Λ_i for $i = 0, 1, \dots, m$ be the eigenvalues of $\frac{\partial f}{\partial x}(0, 0, 0, 0)$; we assume

h) $\Lambda_0 = 0, \quad$ and $\Lambda^f := 1/2 \min\{|\mathrm{Re}[\Lambda_i]| \mid i = 1, \dots, m\} > 0$

From **h)** it follows that for $\varepsilon = \lambda = 0$ the origin of (1.4) admits a, possibly non-unique, one dimensional centre manifold $C = C(0, 0)$, which for ε and λ small persists and will be denoted by $C(\varepsilon, \lambda)$. We assume further that (1.4) undergoes a bifurcation as the parameters vary: we develop in details the case where (1.4) is subject to either a transcritical or a saddle-node bifurcation (both in the non-degenerate case). Following [13] section 5, for centre-manifold we mean a manifold $C(\varepsilon, \lambda)$ which is invariant for (1.4) and which has the following property:

$$\xi \in C(\varepsilon, \lambda) \quad \text{implies} \quad \lim_{|t| \to \infty} ||x_c(t, \xi, \varepsilon, \lambda)|| \exp(-|t\Lambda^f|) = 0 \tag{1.5}$$

Then it follows that C in the origin is tangent to the eigenvector corresponding to the eigenvalues of (1.4) with null real part (in our case it is 1-dimensional). For ε and λ small an invariant manifold, denoted by $C(\varepsilon, \lambda)$, with the property (1.5) persist, its dimension is preserved and its tangent in the origin varies smoothly, see again [13]. We emphasize that C and $C(\varepsilon, \lambda)$ may be not unique; however if we have two different centre manifolds C_1 and C_2 (and consequently two manifolds $C_1(\varepsilon, \lambda)$ and $C_2(\varepsilon, \lambda)$, as ε, λ vary) their dynamics is conjugated.

The main new aspect with respect to the $m = 0$ case is the following. In the $m > 0$ case we need Proposition 3.1, which selects via implicit function theorem $(\breve{x}(t, \varepsilon, \lambda), \breve{y}(t, \varepsilon, \lambda))$ between the trajectories homoclinic to $\mathcal{M}^c$. Such a trajectory is asymptotic in the past to a centre-unstable manifold of (1.4) and in the future to a centre-stable manifold of (1.4) whose intersection is $C(\varepsilon, \lambda)$ (see section 5 in [13] and the appendix for a rigorous definition and a discussion on uniqueness).

Then, generalizing the ideas of [8], we find Melnikov conditions sufficient to divide the ε, λ space in different sets, say α, β, γ, in which we can specify if $(\breve{x}(t,\varepsilon,\lambda), \breve{y}(t,\varepsilon,\lambda))$ is homoclinic ($(\varepsilon,\lambda) \in \alpha$), heteroclinic ($(\varepsilon,\lambda) \in \beta$), or leaves Ω_h for some $t \in \mathbb{R}$ ($(\varepsilon,\lambda) \in \gamma$). This is the content of Theorems 3.4, 3.7 which regard respectively the case where (1.4) undergoes a transcritical or a saddle-node bifurcation. We emphasize that, while in the $m = 0$ case $(\breve{x}(t,\varepsilon,\lambda), \breve{y}(t,\varepsilon,\lambda))$ is always unique, when $m > 0$ we may lose uniqueness as a consequence of the lack of uniqueness of centre manifolds. However even when centre manifold is not unique, the trajectory $(\breve{x}(t,\varepsilon,\lambda), \breve{y}(t,\varepsilon,\lambda))$ is unique if $(\varepsilon,\lambda) \in \alpha, \beta$ (i.e. when it is bounded), while uniqueness is lost if $(\varepsilon,\lambda) \in \gamma$, but we have the same behavior for all the trajectories $(\breve{x}(t,\varepsilon,\lambda), \breve{y}(t,\varepsilon,\lambda))$.
We stress that we have explicit formulas for the derivatives of the curves defining the border of the sets,α, β, γ. These formulas generalize the ones found in [8], and they are a bit cumbersome due also to the "new" contribution given by the strongly stable and unstable directions of (1.4) (whose not trivial computation is the second new aspect with respect to [8]).

For sake of clarity from now on we choose one centre manifold, denoted by $C(\varepsilon,\lambda)$, postponing to the appendix further discussions on this lack of uniqueness problem. We denote by $\mathcal{M}^c(C(\varepsilon,\lambda))$ the centre manifold of (1.1) within the slow manifold, i.e.

$$\mathcal{M}^c(C(\varepsilon,\lambda)) := \{(\xi, v(\xi,\varepsilon,\lambda)) \mid \xi \in C(\varepsilon,\lambda)\}$$

After a C^{r-2} smooth transformation, we may straighten $C(\varepsilon,\lambda)$ and the corresponding centre-unstable and centre-stable manifolds. So if **h)** holds we can assume w.l.o.g. that (1.4) has the following form, see Theorem 5.8 in [13]:

$$\begin{aligned}
\dot{x}_0 = &\ f_0(x, v(x,\varepsilon,\lambda)\varepsilon,\lambda) := C_0(x_0,\varepsilon,\lambda) + C_a(x,\varepsilon,\lambda)x_a + C_b(x,\varepsilon,\lambda)x_b \\
\dot{x}_a = &\ [A(\varepsilon,\lambda) + F_a(x,\varepsilon,\lambda)]x_a \\
\dot{x}_b = &\ [B(\varepsilon,\lambda) + F_b(x,\varepsilon,\lambda))]x_b
\end{aligned} \tag{1.6}$$

where $x_0 \in \mathbb{R}$, $x_a \in \mathbb{R}^l$, $x_b \in \mathbb{R}^{m-l}$, A and B are matrices with respectively l positive and $m-l$ negative eigenvalues, $C_0 \in C^{r-1}$, $C_a, C_b, F_a, F_b \in C^{r-2}$, C_0 vanishes along with its first derivative in x for $\varepsilon = \lambda = 0$, $F_a(0)$, $F_b(0)$ are null, $F_a(x_0, x_a, 0, 0, 0) \equiv 0$, $F_b(x_0, 0, x_b, 0, 0) \equiv 0$. This way $x_b = 0$, $x_a = 0$ and $(x_a, x_b) = 0$ define respectively centre-unstable, centre-stable, and centre manifolds. We mainly focus on the transcritical and saddle-node case (non-degenerate), so, following subsection 11.2 in [14] (see also the introduction of [8]), up to a further change of variables we can assume w.l.o.g. that f_0 has one of the following form:

$$f_0(x,v(x,\varepsilon,\lambda),\varepsilon,\lambda) = (x_0)^2 - b(\varepsilon)\lambda^2 + o(x^2) \tag{1.7}$$
$$f_0(x,v(x,\varepsilon,\lambda),\varepsilon,\lambda) = (x_0)^2 - a(\varepsilon)\lambda + o(x^2) \tag{1.8}$$

where $a(\varepsilon)$ and $b(\varepsilon)$ are positive C^{r-1} functions and the terms contained in $o(x^2)$ are C^{r-1} in x and ε and C^{r-2} in λ, see subsection 11.1 in [14] or the introduction in [8] for details.

1.2 Remark. We need f and g to be at least C^r with $r \geq 3$, because we lose one order of regularity to define the slow manifold $\mathcal{M}^c$, and a further order to pass from (1.6) to the normal form (1.7) or (1.8). So if $v \equiv 0$ (e.g. when $g(x,0,\varepsilon,\lambda) \equiv 0$) and if (1.4) is in normal form, then we do not lose any regularity and we may start from f and g just C^1.

Our purpose is to find trajectories of (1.1) which are close for any $t \in \mathbb{R}$ to the homoclinic trajectory $(0, h(t))$ of the unperturbed system, and to understand when they are homoclinic, heteroclinic or they leave Ω_h for some $t \in \mathbb{R}$ (and so they are close to $(0, h(t))$ at most for t in a half-line). The techniques can be applied also to bifurcations of higher order, i.e. when the first nonzero term of the expansion of f in x has degree 3 or more (in this case we need to assume f at least C^4 or more in the x and λ variable). However in such a case to obtain a complete unfolding of the singularity more parameters are needed. In fact we just sketch the case of pitchfork bifurcation. Again, following subsection 11.2 of [14], we see that, up to changes in variables and parameters, we may reduce to f of the form

$$f_0(x,v(x,\varepsilon,\lambda),\varepsilon,\lambda) = [x_0^2 - a(\varepsilon)\lambda][x_0 - b(\varepsilon)\lambda] + o(x^3) \tag{1.9}$$

where $a(\varepsilon)$ and $b(\varepsilon)$ are C^{r-1} positive functions and the $o(x^3)$ is C^{r-1} in ε and C^{r-2} in λ.

The paper is divided as follows. In section 2 we briefly review some facts, proved in [6]: we construct the solutions asymptotic to the slow manifold $\mathcal{M}^c$ either in the past or in the future, then we match them via implicit function theorem, to construct a solution homoclinic to $\mathcal{M}^c$. In section 3 we prove our main results: in Theorem 3.1 we show that for any ε, λ small enough and any centre manifold in $\mathcal{M}^c(C(\varepsilon,\lambda))$ there is a unique solution $(\breve{x}(t,\varepsilon,\lambda), \breve{y}(t,\varepsilon,\lambda))$ homoclinic to it; then through Theorems 3.4 and 3.7 (in subsections 3.1 and 3.2 respectively) we show which is the behavior of the solution $(\breve{x}(t,\varepsilon,\lambda), \breve{y}(t,\varepsilon,\lambda))$ as ε and λ vary, respectively in the transcritical and in the saddle-node case. So we give sufficient conditions in order to have homoclinic, heteroclinic or no solutions lying in Ω_h for any $t \in \mathbb{R}$, as the parameters vary. Finally we explain how the same methods can be extended to describe pitchfork and higher degree bifurcations in subsection 3.3. We illustrate our results drawing some bifurcation diagrams. In section 4 we construct examples for which we can explicitly compute the derivatives of the bifurcation curves appearing in the diagrams. Section 5 is an Appendix in which we discuss the lack of uniqueness problems deriving from centre manifold theory, and we explain how some unicity may be recovered even if $\mathcal{M}^c(C(\varepsilon,\lambda))$ is not unique.

We collect here some notation which will be in force in the whole paper: **Notation.** Let $x = (x_0, x_1, \dots, x_m) \in \mathbb{R}^{m+1}$: we denote by $x_a = (x_1, \dots, x_l)$, by $x_b = (x_{l+1}, \dots, x_m)$, by $x_{\hat{0}} = (x_a, x_b) = (x_1, \dots, x_m)$; therefore $x = (x_0, x_{\hat{0}}) = (x_0, x_a, x_b)$. Moreover in the whole paper Ω_x, Ω denote small neighborhood of the origin respectively in $\mathbb{R}^{m+1}$ and in $\mathbb{R}^{n+m+1}$, while Ω_0 and Ω_h denote neighborhoods of $(0, h(0))$ and of $\{(0, h(t)) \mid t \in \mathbb{R}\}$ (in $\mathbb{R}^{n+m+1}$) respectively. All these neighborhoods are independent of ε and λ. If f_0 satisfies (1.7) (respectively (1.8)) the origin of (1.4) undergoes a transcritical bifurcation (respectively a saddle node bifurcation). Let $u(\varepsilon,\lambda) = (u_0(\varepsilon,\lambda), u_1(\varepsilon,\lambda), \dots, u_m(\varepsilon,\lambda))$, $s(\varepsilon,\lambda) = (s_0(\varepsilon,\lambda), s_1(\varepsilon,\lambda), \dots, s_m(\varepsilon,\lambda))$ be the zeroes of $f(x, v(x,\varepsilon,\lambda), \varepsilon, \lambda) = 0$. We denote by $U(\varepsilon,\lambda) = (u(\varepsilon,\lambda), v(u(\varepsilon,\lambda),\varepsilon,\lambda))$, $S(\varepsilon,\lambda) = (s(\varepsilon,\lambda), v(s(\varepsilon,\lambda),\varepsilon,\lambda))$ the critical points of (1.1) when they exist. When f_0 satisfies either (1.7) or (1.8), (1.4) admits two critical points for $\lambda > 0$, i.e. $u(\varepsilon,\lambda), s(\varepsilon,\lambda) \in \mathbb{R}^{m+1}$. Note that $u(\varepsilon,\lambda)$ and $s(\varepsilon,\lambda)$ (as well as their heteroclinic connection) are contained in each centre manifold of $C(\varepsilon,\lambda)$, see section 5 in [13] or section 3 in this article for details. Moreover u is unstable, while s is stable with respect to the flow of (1.4) restricted to $C(\varepsilon,\lambda)$.

From the implicit function theorem we easily find that u_i and s_i are C^{r-2} functions of ε and λ, whose derivatives

$$\frac{\partial u_i}{\partial \varepsilon}(0,0) = \frac{\partial s_i}{\partial \varepsilon}(0,0)\,, \quad \frac{\partial u_i}{\partial \lambda}(0,0) = \frac{\partial s_i}{\partial \lambda}(0,0)\,, \qquad \text{for } i = 1, \dots, m \tag{1.10}$$

can be explicitly computed. However u_0 and s_0 are C^{r-2} functions of λ if (1.7) holds, and of $\nu = \sqrt{\lambda}$ if (1.8) holds, and we have

$$\frac{\partial u_0}{\partial \lambda}(0,0) = -\frac{\partial s_0}{\partial \lambda}(0,0)\,, \qquad \frac{\partial u_0}{\partial \varepsilon}(0,0) = 0 = \frac{\partial s_0}{\partial \varepsilon}(0,0) \tag{1.11}$$

in the former case, while $\frac{\partial u_0}{\partial \nu}(0,0) = -\frac{\partial s_0}{\partial \nu}(0,0)$ in the latter. The assumptions used in the main Theorems are the following:

(v)

$$\int_{-\infty}^{\infty} \psi^*(t) \frac{\partial g}{\partial x_0}(0, h(t), 0, 0) dt \neq 0$$

(vi)

$$B_0 + B_m \neq \pm \frac{\partial u_0}{\partial \lambda}(0,0)\,,$$

where the computable constants B_0 and B_m are given in (2.20) and (3.17).

2 Solutions homoclinic to $\mathcal{M}^c$.

In this section we construct $\mathcal{M}^{cu}$ and $\mathcal{M}^{cs}$ which are (locally) invariant manifolds of solutions that approach the slow manifold $y = v(x, \varepsilon, \lambda)$ at an exponential rate. In [5, 6] the following result has been proved.

2.1 Theorem. *[6] Let f and g be bounded C^r functions, $r \geq 2$, with bounded derivatives, satisfying conditions (i)-(iv) of the Introduction and let the numbers β and σ satisfy $0 < r\sigma < \beta < \Lambda^g$. Then, given suitably small positive numbers μ_1 and μ_2, there exist positive numbers ρ_0, λ_0, $\varepsilon_0(< 2\sigma/N$, where N is a bound for the derivatives of $f(x,0,0,0))$, such that for $|\varepsilon| \leq \varepsilon_0$, $|\lambda| \leq \lambda_0$, $|\xi^\pm| \leq \rho_0$, $\zeta^+ \in \mathcal{R}(P^+)$, $|\zeta^+| \leq \mu_1$, $\zeta^- \in \mathcal{N}(P^-)$, $|\zeta^-| \leq \mu_2$, there exists a unique solution*

$$(x^\pm(t), y^\pm(t)) = (x^\pm(t, \xi^\pm, \zeta^\pm, \lambda), y^\pm(t, \xi^\pm, \zeta^\pm, \varepsilon, \lambda))$$

of (1.1) defined respectively for $t \geq 0$ and for $t \leq 0$ such that

$$e^{|\beta t|}|x^+(t) - x_c(\varepsilon t, \xi^+, \varepsilon, \lambda)| \leq \mu_1, \quad e^{|\beta t|}|y^+(t) - v(x^+(t), \varepsilon, \lambda)| \leq \mu_1 \quad (2.1)$$

for $t \geq 0$, and

$$e^{|\beta t|}|x^-(t) - x_c(\varepsilon t, \xi^-, \varepsilon, \lambda)| \leq \mu_2, \quad e^{|\beta t|}|y^-(t) - v(x^-(t), \varepsilon, \lambda)| \leq \mu_2 \quad (2.2)$$

for $t \leq 0$, and

$$P^+[y^+(0) - v(x^+(0), \varepsilon, \lambda)] = \zeta^+\,, \quad (\mathbf{I} - P^-)[y^-(0) - v(x^-(0), \varepsilon, \lambda)] = \zeta^- \quad (2.3)$$

Moreover $y^\pm(t, \xi^\pm, \zeta^\pm, \varepsilon, \lambda) - v(x^\pm(t, \xi^\pm, \zeta^\pm, \varepsilon, \lambda), \varepsilon, \lambda)$ and $x^\pm(t, \xi^\pm, \zeta^\pm, \varepsilon, \lambda) - x_c(\varepsilon t, \xi^\pm, \varepsilon, \lambda)$ are C^{r-1} in the parameters $(\xi^\pm, \zeta^\pm, \varepsilon, \lambda)$ and for $k = 1, \ldots, r-1$, their k^{th} derivatives also satisfy the estimate (2.2) with β replaced by $\beta - k\sigma$ and μ_1 and μ_2 replaced by possibly larger constants. Also there is a constant N_1 such that for $t \leq 0$

$$\begin{aligned} &e^{|\beta t|}|x^-(t, \xi^-, \zeta^-, \varepsilon, \lambda) - x_c(\varepsilon t, \xi^-, \varepsilon, \lambda)| \leq N_1|\varepsilon||\zeta_-|, \\ &e^{|\beta t|}|y^-(t, \xi^-, \zeta^-, \varepsilon, \lambda) - v(x^-(t, \xi^-, \zeta^-, \varepsilon, \lambda), \varepsilon, \lambda))| \leq N_1|\zeta_-|. \end{aligned} \quad (2.4)$$

and for $t \geq 0$

$$\begin{aligned} &e^{|\beta t|}|x^+(t, \xi^+, \zeta^+, \varepsilon, \lambda) - x_c(\varepsilon t, \xi^+, \varepsilon, \lambda)| \leq N_1|\varepsilon||\zeta_+|, \\ &e^{|\beta t|}|y^+(t, \xi^+, \zeta^+, \varepsilon, \lambda) - v(x^+(t, \xi^+, \zeta^+, \varepsilon, \lambda), \varepsilon, \lambda))| \leq N_1|\zeta_+|. \end{aligned} \quad (2.5)$$

Following section 2.1 in [6], using Theorem 2.1 we define the local centre-unstable and centre-stable manifolds near the origin in $\mathbb{R}^{m+n+1}$ as follows

$$\begin{aligned}\mathcal{M}^{cu}_{loc} &:= \{(x^-(0,\xi^-,\zeta^-,\varepsilon,\lambda), y^-(0,\xi^-,\zeta^-,\varepsilon,\lambda)) \;:\; |\zeta^-|<\mu_0,\, |\xi^-|<\rho_0\},\\ \mathcal{M}^{cs}_{loc} &:= \{(x^+(0,\xi^+,\zeta^+,\varepsilon,\lambda), y^+(0,\xi^+,\zeta^+,\varepsilon,\lambda)) \;:\; |\zeta^+|<\mu_0,\, |\xi^+|<\rho_0\}.\end{aligned}$$

In [6] it has been proved that $\mathcal{M}^{cu}_{loc}$ and $\mathcal{M}^{cs}_{loc}$ are respectively negatively and positively invariant for (1.1). Thus, going respectively forward and backward in t, we can construct from $\mathcal{M}^{cu}_{loc}$ and $\mathcal{M}^{cs}_{loc}$ the global manifold $\mathcal{M}^{cu}$ and $\mathcal{M}^{cs}$, see Lemma 2.3 in section 2.2 in [6]. Therefore $\mathcal{M}^{cu}$ and $\mathcal{M}^{cs}$ are respectively $p+m+1$ and $n-p+m+1$ dimensional immersed manifolds of $\mathbb{R}^{n+m+1}$, made up by the trajectories asymptotic to $\mathcal{M}^c$ resp. in the past and in the future.

Following the discussion after Theorems 2.1 and 2.2 in [6], we see that the k^{th} derivatives of $x^+(t,\xi,\zeta^+,\varepsilon,\lambda)$ and of $x^-(t,\xi,\zeta^-,\varepsilon,\lambda)$ with respect to $(\xi,\zeta^\pm,\varepsilon,\lambda)$ are bounded above in absolute value by $C_k e^{(k+1)\sigma|t|}$ for $t\in\mathbb{R}$, where C_k is a constant and $\sigma > N\varepsilon_0$ is a positive number that satisfies $0 < r\sigma < \beta < \Lambda^g$. Finally, because of uniqueness of $(x^\pm(t,\xi^\pm,\zeta^\pm,\varepsilon,\lambda)$, $y^\pm(t,\xi,\zeta^\pm,\varepsilon,\lambda))$, we see that the following properties hold:

$$\begin{aligned}x^\pm(t,\xi^\pm,v(\xi^\pm,\varepsilon,\lambda),\varepsilon,\lambda) &= x_c(\varepsilon t,\xi^\pm,\varepsilon,\lambda),\\ y^\pm(t,\xi^\pm,v(\xi^\pm,\varepsilon,\lambda),\varepsilon,\lambda) &= v(x_c(\varepsilon t,\xi^\pm,\varepsilon,\lambda),\varepsilon,\lambda)\end{aligned}\tag{2.6}$$

and

$$x^\pm(t,\xi^\pm,\zeta^\pm,0,\lambda) = \xi^\pm \tag{2.7}$$

see [6]. Since $x_c(0,\xi,\varepsilon,\lambda)=\xi$, we see that the slow manifold $\mathcal{M}^c$ defined by $y=v(\xi,\varepsilon,\lambda)$ is contained in the intersection between $\mathcal{M}^{cu}$ and $\mathcal{M}^{cs}$.

Using section 2.3 in [6] we can define a foliation of $\mathcal{M}^{cu}_{loc}$ and $\mathcal{M}^{cs}_{loc}$ as follows. Let $\xi\in\mathbb{R}^{m+1}$, $|\xi|$ sufficiently small, we set

$$\begin{aligned}\mathcal{M}^{cu}(\xi) &:= \{(x^-(t,\xi,\zeta^-,\varepsilon,\lambda), y^-(t,\xi,\zeta^-,\varepsilon,\lambda)) \mid |\zeta^-|<\mu_0,\, \zeta^-\in\mathcal{N}P^-, t\in\mathbb{R}\}\\ \mathcal{M}^{cs}(\xi) &:= \{(x^+(t,\xi,\zeta^+,\varepsilon,\lambda), y^+(t,\xi,\zeta^+,\varepsilon,\lambda)) \mid |\zeta^+|<\mu_0,\, \zeta^+\in\mathcal{R}P^+, t\in\mathbb{R}\}.\end{aligned}$$

From section 2.3 in [6] we see that that $\mathcal{M}^{cu}(\xi)$ and $\mathcal{M}^{cs}(\xi)$ are p and $n-p$ manifolds for any $\xi\in\mathbb{R}^{m+1}$, and that $\mathcal{M}^{cu}=\cup_{\xi\in\mathbb{R}^{m+1}}\mathcal{M}^{cu}(\xi)$, $\mathcal{M}^{cs}=\cup_{\xi\in\mathbb{R}^{m+1}}\mathcal{M}^{cs}(\xi)$, are the global centre-unstable and centre-stable manifolds defined above. Moreover given $\bar{\xi},\tilde{\xi}\in\mathbb{R}^{m+1}$ then either $\mathcal{M}^{cu}(\bar{\xi})$ and $\mathcal{M}^{cu}(\tilde{\xi})$ coincide or they do not intersect; similarly either $\mathcal{M}^{cs}(\bar{\xi})$ and $\mathcal{M}^{cs}(\tilde{\xi})$ coincide or they do not intersect: thus $\mathcal{M}^{cu}(\xi)$ and $\mathcal{M}^{cs}(\xi)$ define indeed foliations for $\mathcal{M}^{cu}$ and $\mathcal{M}^{cs}$. In section 4 we use the following sets:

$$\begin{aligned}\mathcal{M}^{cu}(C(\varepsilon,\lambda)) &:= \{\mathcal{M}^{cu}(\xi),\ \mid \xi\in C(\varepsilon,\lambda)\}\\ \mathcal{M}^{cs}(C(\varepsilon,\lambda)) &:= \{\mathcal{M}^{cs}(\xi),\ \mid \xi\in C(\varepsilon,\lambda)\},\end{aligned}\tag{2.8}$$

where $C(\varepsilon,\lambda)$ is a centre manifold of (1.4). Observe that $\mathcal{M}^{cu}(C(\varepsilon,\lambda))$ and $\mathcal{M}^{cs}(C(\varepsilon,\lambda))$ are resp. $p+1$ and $n-p+1$ dimensional immersed manifolds which are invariant for the flow of (1.1). Moreover

$$(x(0),y(0)) \in \mathcal{M}^{cu}(C(\varepsilon,\lambda)) \implies \lim_{t\to-\infty}(x(t),y(t)-v(x(t),\varepsilon,\lambda))\mathrm{e}^{-|\Lambda^f \varepsilon t|}=0$$
$$(x(0),y(0)) \in \mathcal{M}^{cs}(C(\varepsilon,\lambda)) \implies \lim_{t\to\infty}(x(t),y(t)-v(x(t),\varepsilon,\lambda))\mathrm{e}^{-|\Lambda^f \varepsilon t|}=0\,.$$

Let $A \subset \mathbb{R}^{m+1}$ be a set, we define $\mathrm{dist}(\xi,A)=\inf\{|\xi-\eta| \mid \eta\in A\}$. We borrow from [6] a theorem which ensures the existence of solutions of (1.1) homoclinic to $\mathcal{M}^c$.

2.2 Theorem. *[6] Let f and g be bounded C^r functions, $r\geq 2$, with bounded derivatives, satisfying conditions (i)–(v) of the Introduction. Then there exist positive numbers ρ_0, λ_0, ε_0 such that for any $|\varepsilon|<\varepsilon_0$, $|\lambda|<\lambda_0$ there is a family of solutions $(\tilde{x}(t,\xi_{\hat{0}},\varepsilon,\lambda),\tilde{y}(t,\xi_{\hat{0}},\varepsilon,\lambda))$ depending on $\xi_{\hat{0}}\in\mathbb{R}^m$, $|\xi_{\hat{0}}|<\rho_0$, such that $(\tilde{x}(t),\tilde{y}(t))\in(\mathcal{M}^{cs}\cap\mathcal{M}^{cu})\setminus\mathcal{M}^c$ and*

$$\lim_{|t|\to\infty} \mathit{dist}\big((\tilde{x}(\xi_{\hat{0}},\varepsilon,\lambda,t),\tilde{y}(\xi_{\hat{0}},\varepsilon,\lambda,t)),\mathcal{M}^c\big)=0$$

In fact a local type of uniqueness in ensured: there is a neighborhood Ω^0 of $(0,h(0))$ such that, if $(x(t),y(t))\in(\mathcal{M}^{cs}\cap\mathcal{M}^{cu})$ and $(x(0),y(0))\in\Omega^0$, then $(x(t),y(t))$ coincides with one of the solutions $(\tilde{x}(t,\xi_{\hat{0}},\varepsilon,\lambda),\tilde{y}(t,\xi_{\hat{0}},\varepsilon,\lambda))$ constructed through theorem 2.2, for a certain $\xi_{\hat{0}}\in\mathbb{R}^m$.

We sketch the proof since some details will be useful later on. To prove theorem 2.2 Battelli and Palmer in [6] look for a bifurcation function whose zeroes correspond to solutions of the system

$$\begin{cases} x^+(-T,\xi^+,\zeta^+,\varepsilon,\lambda)=x^-(T,\xi^-,\zeta^-,\varepsilon,\lambda)=\xi \\ y^+(-T,\xi^+,\zeta^+,\varepsilon,\lambda)=y^-(T,\xi^-,\zeta^-,\varepsilon,\lambda) \end{cases} \tag{2.9}$$

where $T>0$, and $|\xi^\pm|<\rho_0$. Set

$$K(\xi^+,\xi^-,\zeta^+,\zeta^-,\varepsilon,\lambda):=y^+(-T,\xi^+,\zeta^+,\varepsilon,\lambda)-y^-(T,\xi^-,\zeta^-,\varepsilon,\lambda)$$

They apply Liapunov-Schmidt reduction to system (2.9) and rewrite it as follows

$$\begin{cases} x^+(-T,\xi^+,\zeta^+,\varepsilon,\lambda)=x^-(T,\xi^-,\zeta^-,\varepsilon,\lambda)=\xi \\ K(\xi^+,\xi^-,\zeta^+,\zeta^-,\varepsilon,\lambda)-[\psi^*K(\xi^+,\xi^-,\zeta^+,\zeta^-,\varepsilon,\lambda)]\psi=0 \\ \psi^*K(\xi^+,\xi^-,\zeta^+,\zeta^-,\varepsilon,\lambda)=0 \end{cases} \tag{2.10}$$

Using several times the implicit function theorem and exponential dichotomy estimates, they express $\xi^\pm$ as functions of the variables $(\xi,\zeta^\pm,\varepsilon,\lambda)$, then

$\zeta^{\pm}$ as functions of remaining, and they end up with unique C^{r-1} functions $\bar{\zeta}^{\pm}(\xi,\varepsilon,\lambda)$, and $\bar{\xi}^{\pm}(\xi,\varepsilon,\lambda)$ which are the unique solutions of the first two equations in (2.10), see pages 448-453 in [6] for more details. Set

$$\begin{array}{c}\bar{x}^{\pm}(t,\xi,\varepsilon,\lambda) := x^{\pm}(t,\bar{\xi}^{\pm}(\xi,\varepsilon,\lambda),\bar{\zeta}^{\pm}(\xi,\varepsilon,\lambda),\varepsilon,\lambda) \\ \bar{y}^{\pm}(t,\xi,\varepsilon,\lambda) := y^{\pm}(t,\bar{\xi}^{\pm}(\xi,\varepsilon,\lambda),\bar{\zeta}^{\pm}(\xi,\varepsilon,\lambda),\varepsilon,\lambda)\end{array},$$

Since for $\varepsilon = 0$, $\dot{x} = 0$ (see (1.1)), using (2.9) it follows that

$$x^{\pm}(t,\bar{\xi}^{\pm}(\xi,0,\lambda),\bar{\zeta}^{\pm}(\xi,0,\lambda),0,\lambda) \equiv \bar{\xi}^{\pm}(\xi,0,\lambda) \equiv \xi \tag{2.11}$$

for any ξ and λ. Hence

$$\begin{array}{l}\frac{\partial}{\partial\xi_i}\bar{\xi}^{\pm}(\xi,\varepsilon,\lambda)\lfloor_{(\xi,\varepsilon,\lambda)=(0,0,0)} = \begin{cases} 0 & \text{for } i = 1,\ldots,m \\ 1 & \text{for } i = 0 \end{cases} \\ \frac{\partial}{\partial\lambda}\bar{\xi}^{\pm}(\xi,\varepsilon,\lambda)\lfloor_{(\xi,\varepsilon,\lambda)=(0,0,0)} = 0\end{array} \tag{2.12}$$

Moreover, following [6], we see that

$$\frac{\partial\bar{\xi}^{\pm}}{\partial\varepsilon}(\xi,\varepsilon,\lambda)\lfloor_{(0,0,0)} = \int_0^{\pm\infty} f(0,h(s),0,0)ds \tag{2.13}$$

Hence we are left with solving the bifurcation equation:

$$G(\xi,\varepsilon,\lambda) = \psi^*[\bar{y}^+(-T,\xi,\varepsilon,\lambda) - \bar{y}^-(T,\xi,\varepsilon,\lambda)] = 0 \tag{2.14}$$

Following [6] we see that

$$\frac{\partial G}{\partial\xi_i}(0,0,0) = -\int_{-\infty}^{+\infty}\psi^*(t)\frac{\partial g}{\partial x_i}(0,h(t),0,0)dt\,, \quad \text{i=0,\ldots,m} \tag{2.15}$$

$$\frac{\partial G}{\partial\lambda}(0,0,0) = -\int_{-\infty}^{+\infty}\psi^*(t)\frac{\partial g}{\partial\lambda}(0,h(t),0,0)\mathrm{d}t \tag{2.16}$$

$$\frac{\partial G}{\partial\varepsilon}(0,0,0) = -\int_{-\infty}^{\infty}\psi^*(s)\left[\frac{\partial g}{\partial\varepsilon}(s) + \frac{\partial g}{\partial x}(s)\left(\int_0^s f(t)dt\right)\right]ds \tag{2.17}$$

where $g(s)$ stands for $g(0,h(s),0,0)$, $f(s)$ for $f(0,h(s),0,0)$. Therefore, if (v) holds $\frac{\partial}{\partial\xi_0}G(0,0,0) \neq 0$; so via Implicit Function Theorem we obtain a C^{r-1} function $\tilde{\xi}_0(\xi_{\hat{0}},\varepsilon,\lambda)$ such that $\tilde{\xi}_0(0,0,0) = 0$ and $G(\tilde{\xi}_0(\xi_{\hat{0}},\varepsilon,\lambda),\xi_{\hat{0}},\varepsilon,\lambda) = 0$. Hence, for any $(\xi_{\hat{0}},\varepsilon,\lambda) \in \mathbb{R}^{m+2}$ small enough, there is a unique solution of (1.1) which is homoclinic to the slow manifold $\mathcal{M}^c$, i.e.:

$$\begin{array}{l}\tilde{x}(t,\xi_{\hat{0}},\varepsilon,\lambda) = \begin{cases}\bar{x}^+(t-T,\tilde{\xi}_0(\xi_{\hat{0}},\varepsilon,\lambda),\xi_{\hat{0}},\varepsilon,\lambda) & t \geq 0, \\ \bar{x}^-(t+T,\tilde{\xi}_0(\xi_{\hat{0}},\varepsilon,\lambda),\xi_{\hat{0}},\varepsilon,\lambda) & t \leq 0.\end{cases} \\ \tilde{y}(t,\xi_{\hat{0}},\varepsilon,\lambda) = \begin{cases}\bar{y}^+(t-T,\tilde{\xi}_0(\xi_{\hat{0}},\varepsilon,\lambda),\xi_{\hat{0}},\varepsilon,\lambda) & t \geq 0, \\ \bar{y}^-(t+T,\tilde{\xi}_0(\xi_{\hat{0}},\varepsilon,\lambda),\xi_{\hat{0}},\varepsilon,\lambda) & t \leq 0.\end{cases}\end{array} \tag{2.18}$$

We evaluate all the derivatives, which will be useful in next section

$$\frac{\partial\tilde{\xi}_0}{\partial\xi_j}(\xi_{\hat{0}},\varepsilon,\lambda)\lfloor_{(0,0,0)} = -\frac{\int_{-\infty}^{\infty}\psi^*(t)\frac{\partial g}{\partial x_j}(0,h(t),0,0)dt}{\int_{-\infty}^{\infty}\psi^*(t)\frac{\partial g}{\partial x_0}(0,h(t),0,0)dt} \tag{2.19}$$

$$B_0 := \frac{\partial\tilde{\xi}_0}{\partial\lambda}(\xi_{\hat{0}},\varepsilon,\lambda)\lfloor_{(0,0,0)} = -\frac{\int_{-\infty}^{\infty}\psi^*(t)\frac{\partial g}{\partial\lambda}(0,h(t),0,0)dt}{\int_{-\infty}^{\infty}\psi^*(t)\frac{\partial g}{\partial x_0}(0,h(t),0,0)dt} \tag{2.20}$$

$$\frac{\partial\tilde{\xi}_0}{\partial\varepsilon}(\xi_{\hat{0}},\varepsilon,\lambda)\lfloor_{(0,0,0)} = -\frac{\frac{\partial G}{\partial\varepsilon}}{\frac{\partial G}{\partial\xi_0}} = \mathfrak{A} + A_m , \qquad \text{where} \tag{2.21}$$

$$\begin{aligned}\mathfrak{A} &:= -\frac{\int_{-\infty}^{\infty}\psi^*(s)\frac{\partial g}{\partial\varepsilon}(s)ds + \int_{-\infty}^{\infty}(\psi^*(s)\frac{\partial g}{\partial x_0}(s)\int_0^s f_0(t)dt)ds}{\int_{-\infty}^{\infty}\psi^*(s)\frac{\partial g}{\partial x_0}(s)ds} ,\\ A_m &:= -\frac{\sum_{i=1}^m[\int_{-\infty}^{\infty}(\psi^*(s)\frac{\partial g}{\partial x_i}(s)\int_0^s f_i(t)dt)ds]}{\int_{-\infty}^{\infty}\psi^*(s)\frac{\partial g}{\partial x_0}(s)ds} ,\end{aligned} \tag{2.22}$$

and $g(s)$ stands for $g(0,h(s),0,0)$, $f_i(s)$ for $f_i(0,h(s),0,0)$ and similarly for their derivatives. This concludes the proof of Theorem 2.2. We stress that in [6] the authors just require $\frac{\partial G}{\partial\xi}(0,0,0) \neq 0$ (i.e. $\frac{\partial G}{\partial\xi_j}(0,0,0) \neq 0$ for a certain $j \in \{0,1,\dots,m\}$) and use such a condition and the implicit function theorem to construct the solution defined in (2.18). Our request is slightly more restrictive: we need (v), i.e. $\frac{\partial G}{\partial\xi_0}(0,0,0) \neq 0$ (so we ask the j-coordinate to be the 0 one).

2.3 Remark. We emphasize that Theorem 2.2 allows to specify the trajectory of the slow manifold which is approached by the solution (2.18) of (1.1): this fact will be used in the next section. More precisely the orbit (2.18) approaches the trajectory $(x_c(\varepsilon t,\xi,\varepsilon,\lambda), v(x_c(\varepsilon t,\xi,\varepsilon,\lambda),\varepsilon,\lambda))$ of the slow manifold such that $x_c(\mp\varepsilon T,\xi,\varepsilon,\lambda) = \tilde{\xi}^{\pm}$, where $\xi = (\tilde{\xi}_0(\xi_{\hat{0}},\varepsilon,\lambda),\xi_{\hat{0}})$ and $\tilde{\xi}^{\pm} = \tilde{\xi}^{\pm}(\xi_{\hat{0}},\varepsilon,\lambda)$.

3 Existence of Homoclinic and Heteroclinic solutions.

In this section we state and prove our main results. In Theorem 3.1, we select a trajectory $(\breve{x}(t,\varepsilon,\lambda),\breve{y}(t,\varepsilon,\lambda))$ defined in (3.14), homoclinic to the centre manifold within the slow manifold, denoted by $\mathcal{M}^c(C(\varepsilon,\lambda))$. The lack of uniqueness problems are discussed in the appendix.

We recall that, when **h)** holds, (1.4) admits at least a centre-manifold which continue to exist if ε and λ are small, as long as the critical points persist. We choose one of them and denote it by $C(\varepsilon, \lambda)$. Moreover, with a C^{r-2} change of coordinates (we recall that (1.4) is just C^{r-1}) and losing some regularity, we can flatten $C(\varepsilon, \lambda)$ and make it coincide with the x_0 axis, for $\varepsilon = \lambda = 0$: i.e. we pass from (1.4) to (1.6), see Theorem 5.8 in [13] (in fact something more can be said when either the unstable or the stable directions do not exist, i.e. $l = 0$ or $m - l = 0$ respectively, see section 5 in [13]).

Assume to fix the ideas that f_0 is either as in (1.7) or as in (1.8) and $\varepsilon, \lambda > 0$, so that we have two critical points $s(\varepsilon, \lambda)$ and $u(\varepsilon, \lambda)$ which are respectively stable and unstable for the restriction of (1.6) to $C(\varepsilon, \lambda)$. Then $u(\varepsilon, \lambda)$ and $s(\varepsilon, \lambda)$ admit respectively a $l + 1$ and a l dimensional unstable manifolds, as critical points of (1.6), denoted by $W^u(u(\varepsilon, \lambda))$ and $W^u(s(\varepsilon, \lambda))$; similarly they admit the $m - l$ and $m - l + 1$ dimensional stable manifolds $W^s(u(\varepsilon, \lambda))$ and $W^s(s(\varepsilon, \lambda))$: all these manifolds are uniquely defined. Let Ω_x be a neighborhood of the origin in $\mathbb{R}^{m+1}$: we can define local centre-unstable $W^u(C(\varepsilon, \lambda))$ and centre-stable manifolds $W^s(C(\varepsilon, \lambda))$ (not unique) having the following properties

$$\begin{array}{ll} \text{if } \ \xi \in (W^u(C(\varepsilon, \lambda)) \cap \Omega_x)\,, & \text{then } \ \lim_{t \to -\infty} |x_c(t, \xi, \varepsilon, \lambda)| \mathrm{e}^{\Lambda^f t} = 0 \\ \text{if } \ \xi \in (W^s(C(\varepsilon, \lambda)) \cap \Omega_x)\,, & \text{then } \ \lim_{t \to \infty} |x_c(t, \xi, \varepsilon, \lambda)| \mathrm{e}^{-\Lambda^f t} = 0 \end{array} \tag{3.1}$$

where Λ^f is defined in **h)**.

Note that $C(\varepsilon, \lambda)$ is obtained as intersection between $W^u(C(\varepsilon, \lambda))$ and $W^s(C(\varepsilon, \lambda))$. Observe that $C(\varepsilon, \lambda)$ is divided by $u(\varepsilon, \lambda)$ into two open components: one, say $C^-(\varepsilon, \lambda)$, is the graph of a trajectory which becomes unbounded as $t \to +\infty$, the other is made up by trajectories converging to $s(\varepsilon, \lambda)$ as $t \to +\infty$. Similarly $s(\varepsilon, \lambda)$ divides $C(\varepsilon, \lambda)$ into two open components: $C^+(\varepsilon, \lambda)$ is the graph of a trajectory which becomes unbounded as $t \to -\infty$, and the other is made up by trajectories converging to $u(\varepsilon, \lambda)$ as $t \to -\infty$. Analogously the $m-l$ dimensional manifold $W^s(u(\varepsilon, \lambda)) \cap \Omega_x$ splits $W^s(C(\varepsilon, \lambda))$ into two relatively open components: $W^s(s(\varepsilon, \lambda)) \cap \Omega_x$ and a further component, say $W^{s,n}(C(\varepsilon, \lambda))$ which is made up by trajectories which leave Ω_x for $t > 0$. The fact that we have just two components easily follows from an analysis of the tangent spaces. Moreover the l dimensional manifold $W^u(s(\varepsilon, \lambda)) \cap \Omega_x$ splits $W^u(C(\varepsilon, \lambda))$ into $W^u(u(\varepsilon, \lambda)) \cap \Omega_x$ and a further component, say $W^{u,n}(C(\varepsilon, \lambda))$ which is made up by trajectories which leave Ω_x for $t > 0$, Note that all the manifolds $W^u(C(\varepsilon, \lambda))$, $W^u(s(\varepsilon, \lambda))$, $W^s(C(\varepsilon, \lambda))$, $W^s(u(\varepsilon, \lambda))$ have their tangent planes coinciding with the coordinate axes.

Now we represent all these manifolds, $W^u(C(\varepsilon, \lambda))$, $W^s(C(\varepsilon, \lambda))$, $W^u(s(\varepsilon, \lambda))$, $W^s(u(\varepsilon, \lambda))$ of $\Omega_x \subset \mathbb{R}^{m+1}$ as graphs, introducing some functions h_j. Consider $W^u(C(\varepsilon, \lambda))$ and $W^s(C(\varepsilon, \lambda))$; there are open neighborhoods of the origin $A_{0,a} \subset \mathbb{R}^{l+1}$, $A_{0,b} \subset \mathbb{R}^{m-l+1}$, and smooth functions $h_{(0,a)} : A_{0,a} \times$

$[-\varepsilon_0,\varepsilon_0]\times[-\lambda_0,\lambda_0]\to\mathbb{R}^{m-l}$, $h_{(0,b)}: A_{0,b}\times[-\varepsilon_0,\varepsilon_0]\times[-\lambda_0,\lambda_0]\to\mathbb{R}^l$ such that $\xi=(\xi_0,\xi_a,\xi_b)\in W^u(C(\varepsilon,\lambda))\cap\Omega_x$ if and only if $\xi_b=h_{(0,a)}(\xi_0,\xi_a,\varepsilon,\lambda)$, while $\xi\in W^s(C(\varepsilon,\lambda))\cap\Omega_x$ if and only if $\xi_a=h_{(0,b)}(\xi_0,\xi_b,\varepsilon,\lambda)$.
Moreover there are smooth functions $h_a: A_a\times[-\varepsilon_0,\varepsilon_0]\times[-\lambda_0,\lambda_0]\to\mathbb{R}$, $h_b: A_b\times[-\varepsilon_0,\varepsilon_0]\times[-\lambda_0,\lambda_0]\to\mathbb{R}$ such that $\xi=(\xi_0,\xi_a,\xi_b)\in W^u(s(\varepsilon,\lambda))\cap\Omega_x$ if and only if $\xi_0=h_a(\xi_a,\varepsilon,\lambda)$ and $\xi_b=h_{(0,a)}(\xi_0,\xi_a,\varepsilon,\lambda)$, while $\xi=(\xi_0,\xi_a,\xi_b)\in W^s(u(\varepsilon,\lambda))\cap\Omega_x$ if and only if $\xi_0=h_b(\xi_b,\varepsilon,\lambda)$ and $\xi_a=h_{(0,b)}(\xi_0,\xi_b,\varepsilon,\lambda)$. Moreover

$$\frac{\partial h_{(0,a)}}{\partial(\xi_0,\xi_a)}=0\,,\quad \frac{\partial h_{(0,b)}}{\partial(\xi_0,\xi_b)}=0\,,\quad \frac{\partial h_a}{\partial\xi_a}=0\,,\quad \frac{\partial h_b}{\partial\xi_b}=0 \tag{3.2}$$

where the derivatives are evaluated for $\varepsilon=\lambda=\xi_0=0$, $\xi_a=0$, $\xi_b=0$. Such an orthogonality condition is a consequence of the particular form of system (1.6), which is obtained precisely flattening stable, unstable and centre invariant manifolds.

In order to divide the parameter space (ε,λ) in subsets in which $(\breve{x}(t,\varepsilon,\lambda)$, $\breve{y}(t,\varepsilon,\lambda))$ has different behavior, we need to compute the derivatives of the functions $h_{(0,a)}$, $h_{(0,b)}$, h_a, h_b with respect to all the variables. Since

$$\begin{array}{ll} s_b(\varepsilon,\lambda)=h_{(0,a)}(s_0(\varepsilon,\lambda),s_a(\varepsilon,\lambda),\varepsilon,\lambda)\,, & s_0(\varepsilon,\lambda)=h_a(s_a(\varepsilon,\lambda),\varepsilon,\lambda)\\ u_a(\varepsilon,\lambda)=h_{(0,b)}(u_0(\varepsilon,\lambda),u_b(\varepsilon,\lambda),\varepsilon,\lambda)\,, & u_0(\varepsilon,\lambda)=h_b(u_b(\varepsilon,\lambda),\varepsilon,\lambda)\end{array} \tag{3.3}$$

from (3.2) and (3.3) we get

$$\begin{array}{ll} \frac{\partial}{\partial\varepsilon}h_{(0,a)}((0,0),0,0)=\frac{\partial u_b}{\partial\varepsilon}(0,0)\,, & \frac{\partial}{\partial\varepsilon}h_a(0,0,0)=-\frac{\partial u_0}{\partial\varepsilon}(0,0)=0\\ \frac{\partial}{\partial\varepsilon}h_{(0,b)}((0,0),0,0)=\frac{\partial u_a}{\partial\varepsilon}(0,0)\,, & \frac{\partial}{\partial\varepsilon}h_b(0,0,0)=\frac{\partial u_0}{\partial\varepsilon}(0,0)=0\end{array} \tag{3.4}$$

where we used (1.10) and (1.11). When f_0 satisfies (1.7) we have

$$\begin{array}{ll} \frac{\partial}{\partial\lambda}h_{(0,a)}((0,0),0,0)=\frac{\partial u_b}{\partial\lambda}(0,0)\,, & \frac{\partial}{\partial\lambda}h_a(0,0,0)=-\frac{\partial u_0}{\partial\lambda}(0,0)\\ \frac{\partial}{\partial\lambda}h_{(0,b)}((0,0),0,0)=\frac{\partial u_a}{\partial\lambda}(0,0)\,, & \frac{\partial}{\partial\lambda}h_b(0,0,0)=\frac{\partial u_0}{\partial\lambda}(0,0)\end{array} \tag{3.5}$$

When f_0 satisfies (1.8) we need to introduce the variable $\mu:=\sqrt{\lambda}$ so that the critical points are smooth functions of ε and μ. Then reasoning as above and recalling that $\frac{\partial s_i}{\partial\mu}(0,0)=0$ for $i=1,\dots,m$ we find the following:

$$\begin{array}{ll} \frac{\partial}{\partial\mu}h_{(0,a)}((0,0),0,0)=0\,, & \frac{\partial}{\partial\mu}h_{(0,b)}((0,0),0,0)=0\\ \frac{\partial}{\partial\mu}h_a(0,0,0)=-\frac{\partial u_0}{\partial\mu}(0,0)\,, & \frac{\partial}{\partial\mu}h_b(0,0,0)=\frac{\partial u_0}{\partial\mu}(0,0)\end{array} \tag{3.6}$$

We recall that $\mathcal{M}^c(C(\varepsilon,\lambda))$ denotes the centre manifold within the slow manifold and that it is contained in $\mathcal{M}^{cu}(C(\varepsilon,\lambda))\cap\mathcal{M}^{cs}(C(\varepsilon,\lambda))$. We are going to prove the following.

3.1 Theorem. *Let f and g be $\mathcal{C}^r$ functions, $r \geq 2$, bounded with their derivatives, satisfying conditions (i)–(v) of the Introduction. Then there are $\varepsilon_0 > 0$ and $\lambda_0 > 0$ such that for any $0 < |\varepsilon| < \varepsilon_0$ and $|\lambda| < \lambda_0$ there exists a trajectory $(\breve{x}(t,\varepsilon,\lambda), \breve{y}(t,\varepsilon,\lambda))$ homoclinic to $\mathcal{M}^c(C(\varepsilon,\lambda))$ and such that $(\breve{x}(0,\varepsilon,\lambda), \breve{y}(0,\varepsilon,\lambda))$ lies in the neighborhood Ω_0 of $(0,h(0))$.*

We look for a trajectory homoclinic to $\mathcal{M}^c(C(\varepsilon,\lambda))$; we set

$$\tilde{\xi}^{\pm}(\xi_{\hat{0}},\varepsilon,\lambda) := (\tilde{\xi}_0^{\pm}(\xi_{\hat{0}},\varepsilon,\lambda), \tilde{\xi}_{\hat{0}}^{\pm}(\xi_{\hat{0}},\varepsilon,\lambda)) := \bar{\xi}^{\pm}\big(\tilde{\xi}_0(\xi_{\hat{0}},\varepsilon,\lambda), \xi_{\hat{0}}, \varepsilon, \lambda\big)\,, \tag{3.7}$$

In the next subsection we divide the parameters space in different subsets in which the solution $(\breve{x}(t,\varepsilon,\lambda), \breve{y}(t,\varepsilon,\lambda))$ constructed via Theorem 3.1 has a different asymptotic behavior: we need to evaluate all the derivatives of $\tilde{\xi}_0^{\pm}$.

Notation. *In the whole section we proceed to evaluate via implicit function theorem several derivatives, having quite long formulas. To deal with less cumbersome notation in the whole section we set, with a little abuse, $f_i(s)$ for $f_i(0,h(s),0,0)$, $g_i(s)$ for $g_i(0,h(s),0,0)$, $\frac{\partial f_i}{\partial x}(s)$ for $\frac{\partial f_i}{\partial x}(0,h(s),0,0)$ and similarly for all the possible derivatives of both f and g, for $i = 0,\dots,m,a,b,\hat{0}$.*

Using (2.11), for $\varepsilon = 0$ we find:

$$\tilde{\xi}^{\pm}(\xi_{\hat{0}},0,\lambda) = (\tilde{\xi}_0(\xi_{\hat{0}},0,\lambda), \xi_{\hat{0}})\,. \tag{3.8}$$

It follows that

$$\frac{\partial \tilde{\xi}_{\hat{0}}^{\pm}}{\partial \xi_{\hat{0}}}(\xi_{\hat{0}},0,\lambda) = \mathbf{I}\,, \qquad \frac{\partial \tilde{\xi}_{\hat{0}}^{\pm}}{\partial \lambda}(\xi_{\hat{0}},0,\lambda) = 0\,. \tag{3.9}$$

Using (2.13) and the first equality in (3.9) we find

$$\frac{\partial \tilde{\xi}_{\hat{0}}^{\pm}}{\partial \varepsilon}(\xi_{\hat{0}},0,0) = \Big(\int_0^{\pm\infty} f_1(s)ds, \dots, \int_0^{\pm\infty} f_m(s)ds\Big) = \int_0^{\pm\infty} f_{\hat{0}}(s)ds\,, \tag{3.10}$$

From (2.13) and (2.21), setting $A_0^{\pm} := \mathfrak{A} + \frac{\partial \bar{\xi}_0^{\pm}(0,0,0)}{\partial \varepsilon}$ we find

$$\begin{gathered}\frac{\partial \tilde{\xi}_0^{\pm}(0,0,0)}{\partial \varepsilon} = \frac{\partial \bar{\xi}_0^{\pm}(0,0,0)}{\partial \xi_0}\frac{\partial \tilde{\xi}_0(0,0,0)}{\partial \varepsilon} + \frac{\partial \bar{\xi}_0^{\pm}(0,0,0)}{\partial \varepsilon} = A_0^{\pm} + A_m\,, \\ A_0^{\pm} := -\frac{\int_{-\infty}^{\infty} \psi^*(s)\frac{\partial g}{\partial \varepsilon}(s)ds + \int_{-\infty}^{\infty}(\psi^*(s)\frac{\partial g}{\partial x_0}(s)\int_{\pm\infty}^{s} f_0(t)dt)ds}{\int_{-\infty}^{\infty}\psi^*(s)\frac{\partial g}{\partial x_0}(s)ds}\end{gathered} \tag{3.11}$$

We consider the solution $(\tilde{x}(t,\xi_{\hat{0}},\varepsilon,\lambda), \tilde{y}(t,\xi_{\hat{0}},\varepsilon,\lambda))$, and the corresponding functions $\tilde{\xi}^{\pm}(\xi_{\hat{0}},\varepsilon,\lambda) = (\tilde{\xi}_0^{\pm}(\xi_{\hat{0}},\varepsilon,\lambda), \tilde{\xi}_a^{\pm}(\xi_{\hat{0}},\varepsilon,\lambda), \tilde{\xi}_b^{\pm}(\xi_{\hat{0}},\varepsilon,\lambda))$ constructed through Theorem 2.2. We define the functions $H^+ : A_a \times A_b \times [-\varepsilon_0,\varepsilon_0] \times [-\lambda_0,\lambda_0] \to$

$\mathbb{R}^l$, $H^- : A_a \times A_b \times [-\varepsilon_0, \varepsilon_0] \times [-\lambda_0, \lambda_0] \to \mathbb{R}^{m-l}$, $H : A_a \times A_b \times [-\varepsilon_0, \varepsilon_0] \times [-\lambda_0, \lambda_0] \to \mathbb{R}^m$ as follows:

$$\begin{aligned} H^+(\xi_{\hat{0}}, \varepsilon, \lambda) &= \tilde{\xi}_a^+(\xi_{\hat{0}}, \varepsilon, \lambda) - h_{(0,b)}(\tilde{\xi}_0^+(\xi_{\hat{0}}, \varepsilon, \lambda), \tilde{\xi}_b^+(\xi_{\hat{0}}, \varepsilon, \lambda), \varepsilon, \lambda) \\ H^-(\xi_{\hat{0}}, \varepsilon, \lambda) &= \tilde{\xi}_b^-(\xi_{\hat{0}}, \varepsilon, \lambda) - h_{(0,a)}(\tilde{\xi}_0^-(\xi_{\hat{0}}, \varepsilon, \lambda), \tilde{\xi}_a^-(\xi_{\hat{0}}, \varepsilon, \lambda), \varepsilon, \lambda) \\ H(\xi_{\hat{0}}, \varepsilon, \lambda) &= (H^+(\xi_{\hat{0}}, \varepsilon, \lambda), H^-(\xi_{\hat{0}}, \varepsilon, \lambda)) \end{aligned} \tag{3.12}$$

We stress that $\tilde{\xi}^+(\xi_{\hat{0}}, \varepsilon, \lambda) \in W^s(C(\varepsilon, \lambda))$ and $\tilde{\xi}^-(\xi_{\hat{0}}, \varepsilon, \lambda) \in W^u(C(\varepsilon, \lambda))$ whenever $H(\xi_{\hat{0}}, \varepsilon, \lambda) = 0$. Observe that

$$\frac{\partial H^+}{\partial \xi_{\hat{0}}} = \frac{\partial \tilde{\xi}_a^+}{\partial \xi_{\hat{0}}} - \frac{\partial h_{(0,b)}}{\partial(x_0, x_b)} \left(\frac{\partial \tilde{\xi}_0^+}{\partial \xi_{\hat{0}}}, \frac{\partial \tilde{\xi}_b^+}{\partial \xi_{\hat{0}}} \right).$$

Using (3.9), (3.2) and repeating the argument for H^-, we find

$$\frac{\partial H}{\partial \xi_{\hat{0}}}(\xi_{\hat{0}}, \varepsilon, \lambda) \lfloor_{(0,0,0)} = \mathbf{I} \tag{3.13}$$

Moreover by construction we have $H(0,0,0) = 0$; so we can apply the implicit function theorem to find a smooth function $\breve{\xi}_{\hat{0}}(\varepsilon, \lambda)$ such that $H(\breve{\xi}_{\hat{0}}(\varepsilon, \lambda), \varepsilon, \lambda) \equiv 0$. As usual we set $\breve{\xi}^\pm(\varepsilon, \lambda) := \tilde{\xi}^\pm(\breve{\xi}_{\hat{0}}(\varepsilon, \lambda), \varepsilon, \lambda)$, and

$$(\breve{x}(t, \varepsilon, \lambda), \breve{y}(t, \varepsilon, \lambda)) := (\tilde{x}(t, \breve{\xi}_{\hat{0}}(\varepsilon, \lambda), \varepsilon, \lambda), \tilde{y}(t, \breve{\xi}_{\hat{0}}(\varepsilon, \lambda), \varepsilon, \lambda)) \tag{3.14}$$

The solution of (1.1) defined by (3.14) is homoclinic to $\mathcal{M}(C(\varepsilon, \lambda))$ and proves Theorem 3.1. We evaluate the derivatives of $\breve{\xi}_{\hat{0}}(\varepsilon, \lambda)$, which will be useful for the next step. From (3.13) we get $\frac{\partial \breve{\xi}_{\hat{0}}}{\partial \lambda}(0,0) = -\frac{\partial H}{\partial \lambda}(0,0,0)$, so using (3.9), (3.2), (3.5) and the implicit function theorem we find

$$\frac{\partial \breve{\xi}_{\hat{0}}}{\partial \lambda}(0,0) = -\left(\frac{\partial \tilde{\xi}_a^+}{\partial \lambda} - \frac{\partial h_{(0,b)}}{\partial \lambda}, \frac{\partial \tilde{\xi}_b^-}{\partial \lambda} - \frac{\partial h_{(0,a)}}{\partial \lambda} \right) = \frac{\partial u_{\hat{0}}}{\partial \lambda}(0,0) \tag{3.15}$$

Since $\frac{\partial \breve{\xi}_{\hat{0}}}{\partial \varepsilon}(0,0) = -\frac{\partial H}{\partial \varepsilon}(0,0,0)$, using (3.10) and (3.4) we find

$$\frac{\partial \breve{\xi}_{\hat{0}}}{\partial \varepsilon}(0,0) = \left(\int_{+\infty}^0 f_a(s) ds + \frac{\partial u_a}{\partial \varepsilon}(0,0), \int_{-\infty}^0 f_b(s) + \frac{\partial u_b}{\partial \varepsilon}(0,0) ds \right) \tag{3.16}$$

From

$$\frac{\partial \breve{\xi}_0^\pm}{\partial \lambda}(0,0) = \frac{\partial \tilde{\xi}_0^\pm}{\partial \xi_{\hat{0}}}(0,0,0) \frac{\partial \breve{\xi}_{\hat{0}}}{\partial \lambda}(0,0) + \frac{\partial \tilde{\xi}_0^\pm}{\partial \lambda}(0,0,0),$$

using (2.19), (2.20) and (3.15) we find the following

$$\begin{aligned} &\frac{\partial \breve{\xi}_0^{\pm}}{\partial \lambda}(0,0) = B_0 + B_m \,, \quad \text{where } B_0 \text{ is as in (2.20) and} \\ &B_m := -\frac{\sum_{j=1}^{m} \left[\frac{\partial u_j}{\partial \lambda}(0,0) \int_{-\infty}^{\infty} \psi^*(t) \frac{\partial}{\partial x_j} g(t) dt \right]}{\int_{-\infty}^{\infty} \psi^*(t) \frac{\partial}{\partial x_0} g(t) dt} \,. \end{aligned} \tag{3.17}$$

Hence in particular $\frac{\partial \breve{\xi}_0^-}{\partial \lambda}(0,0) = \frac{\partial \breve{\xi}_0^+}{\partial \lambda}(0,0)$. Similarly we find

$$\frac{\partial \breve{\xi}_0^{\pm}}{\partial \varepsilon}(0,0) = \frac{\partial \tilde{\xi}_0^{\pm}}{\partial \xi_{\hat{0}}}(0,0,0) \frac{\partial \breve{\xi}_{\hat{0}}}{\partial \varepsilon}(0,0) + \frac{\partial \tilde{\xi}_0^{\pm}}{\partial \varepsilon}(0,0,0) \,.$$

Hence using (2.19), (3.16), (3.11) we find

$$\begin{aligned} &\frac{\partial \breve{\xi}_0^{\pm}}{\partial \varepsilon}(0,0) = C_m + A_0^{\pm} + A_m \,, \quad \text{where } C_m := \frac{\sum_{i=1}^{m} \left[F_i \int_{-\infty}^{\infty} \psi^*(t) \frac{\partial g}{\partial x_i}(t) dt \right]}{\int_{-\infty}^{\infty} \psi^*(t) \frac{\partial g}{\partial x_0}(t) dt} \\ &F_j = \int_0^{+\infty} f_j(t) dt - \frac{\partial u_j}{\partial \varepsilon}(0,0) \,, \qquad F_k = \int_0^{-\infty} f_k(t) dt - \frac{\partial u_k}{\partial \varepsilon}(0,0) \end{aligned} \tag{3.18}$$

for $1 \le j \le l < k \le m$. We stress that the terms $A_0^{\pm}$ and B_0 are the same as in the one dimensional case $m = 0$, while A_m, B_m and C_m are new terms depending on the strongly stable and unstable directions of the slow manifold and they become trivially null when $m = 0$ (compare with [8]).
In the next subsections we see for which values of the parameters the solution defined by (3.14) is heteroclinic, homoclinic or leaves Ω_h. Now we distinguish between f satisfying (1.7) and (1.8).

3.1 Transcritical bifurcation.

We argue separately in each quadrant: we start from $\varepsilon > 0$ and $\lambda > 0$. The key point to understand the behavior in the future is to establish the mutual positions of $\breve{\xi}^+(\varepsilon, \lambda)$ and $W^s(u(\varepsilon, \lambda))$, while to understand the behavior in the past we need to know the positions of $\breve{\xi}^-(\varepsilon, \lambda)$ with respect to $W^u(s(\varepsilon, \lambda))$. So we define $J_1^{\pm} : [-\varepsilon_0, \varepsilon_0] \times [-\lambda_0, \lambda_0] \to \mathbb{R}$ as follows

$$\begin{aligned} J_1^+(\varepsilon, \lambda) &= \breve{\xi}_0^+(\varepsilon, \lambda) - h_b(\breve{\xi}_b^+(\varepsilon, \lambda), \varepsilon, \lambda) \,, \\ J_1^-(\varepsilon, \lambda) &= \breve{\xi}_0^-(\varepsilon, \lambda) - h_a(\breve{\xi}_a^-(\varepsilon, \lambda), \varepsilon, \lambda) \end{aligned} \tag{3.19}$$

We want to construct via implicit function theorem two curves, $\lambda_1^+(\varepsilon)$ and $\lambda_1^-(\varepsilon)$, satisfying $\lambda_1^\pm(0)=0$, and such that $J_1^\pm(\varepsilon,\lambda_1^\pm(\varepsilon))=0$. Then $(\breve{x}(t,\varepsilon,\lambda_1^+(\varepsilon)), \breve{y}(t,\varepsilon,\lambda_1^+(\varepsilon)))$ converges to $U(\varepsilon,\lambda_1^+(\varepsilon))$ as $t\to+\infty$, while $(\breve{x}(t,\varepsilon,\lambda_1^-(\varepsilon)), \breve{y}(t,\varepsilon,\lambda_1^-(\varepsilon)))$ converges to $S(\varepsilon,\lambda_1^-(\varepsilon))$ as $t\to-\infty$.

Using (3.17) and (3.5) we find

$$\begin{array}{l} \frac{\partial J_1^+}{\partial\lambda}(0,0)=\frac{\partial\breve{\xi}_0^+}{\partial\lambda}(0,0)-\frac{\partial h_b}{\partial\lambda}(0,0,0)=B_0+B_m-\frac{\partial u_0}{\partial\lambda}(0,0) \\ \frac{\partial J_1^-}{\partial\lambda}(0,0)=\frac{\partial\breve{\xi}_0^-}{\partial\lambda}(0,0)-\frac{\partial h_a}{\partial\lambda}(0,0,0)=B_0+B_m+\frac{\partial u_0}{\partial\lambda}(0,0) \end{array}\;; \tag{3.20}$$

so, if (vi) holds we can apply the implicit function theorem and construct the curves $\lambda_1^\pm(\varepsilon)$ (defined for $0\le\varepsilon\le\varepsilon_0$) such that $J_1^\pm(\varepsilon,\lambda_1^\pm(\varepsilon))=0$. Moreover

$$\begin{aligned} \frac{d}{d\varepsilon}\lambda_1^+(0) &= -\frac{\frac{\partial}{\partial\varepsilon}\breve{\xi}_0^+(0,0)-\frac{\partial u_0}{\partial\varepsilon}(0,0)}{\frac{\partial}{\partial\lambda}\breve{\xi}_0^+(0,0)-\frac{\partial u_0}{\partial\lambda}(0,0)} = -\frac{A_0^++A_m+C_m}{B_0+B_m-\frac{\partial u_0}{\partial\lambda}(0,0)} \\ \frac{d}{d\varepsilon}\lambda_1^-(0) &= -\frac{A_0^-+A_m+C_m}{B_0+B_m+\frac{\partial u_0}{\partial\lambda}(0,0)} \end{aligned} \tag{3.21}$$

3.2 Remark. The curves $\lambda_1^+(\varepsilon)$ and $\lambda_1^-(\varepsilon)$ may not intersect the open set $Q_1=\{(\varepsilon,\lambda)\mid\varepsilon>0\,,\;\lambda\ge 0\}$. If this is the case for any $(\varepsilon,\lambda)\in Q_1$ the trajectory $(\breve{x}(t,\varepsilon,\lambda),\breve{y}(t,\varepsilon,\lambda))$ does not converge respectively to U in the future neither to S in the past.

By construction $\breve{\xi}^+(\varepsilon,\lambda)\in W^s(u(\varepsilon,\lambda))$ if and only if $J_1^+(\varepsilon,\lambda)=0$. We recall that, if we restrict to a small neighborhood Ω_x of the origin, then $W^s(C(\varepsilon,\lambda))$ is divided by $W^s(u(\varepsilon,\lambda))$ in two relatively open components, $W^s(s(\varepsilon,\lambda))$ and $W^{s,n}(C(\varepsilon,\lambda))$. The following result is crucial in what follows (see also section 5 for a discussion concerning uniqueness problem related to centre manifold theory).

3.3 Remark. Assume $\varepsilon,\lambda>0$, then

$$\begin{array}{ll} \xi_0-h_b(\xi_b,\varepsilon,\lambda)<0\,, & \text{if and only if }\;\xi\in W^s(s(\varepsilon,\lambda))\,; \\ \xi_0-h_b(\xi_b,\varepsilon,\lambda)>0\,, & \text{if and only if }\;\xi\in W^{s,n}(C(\varepsilon,\lambda)) \end{array}$$

Proof. In Ω_x the manifold $W^s(C(\varepsilon,\lambda))$ is characterized by the property that $\xi\in W^s(C(\varepsilon,\lambda))$ if and only if $\xi_a=h_{0,b}(\xi_0,\xi_b,\varepsilon,\lambda)$; moreover $\xi\in W^s(u(\varepsilon,\lambda))$ if and only if we also have $\xi_0=h_b(\xi_b,\varepsilon,\lambda)$. Moreover $\xi_0-h_b(\xi_b,\varepsilon,\lambda)$ changes sign in $W^s(C(\varepsilon,\lambda))\cap\Omega_x$. It follows that we have the two alternatives: the one described in Remark 3.3 and its opposite (the one obtained changing sign in both the inequality).

From (3.2) and the fact that $u_0(\varepsilon,\lambda)-s_0(\varepsilon,\lambda)>0$ we see that

$$h_b(u_b(\varepsilon,\lambda),\varepsilon,\lambda)-h_b(s_b(\varepsilon,\lambda),\varepsilon,\lambda)<u_0(\varepsilon,\lambda)-s_0(\varepsilon,\lambda)\,.$$

Since $u(\varepsilon,\lambda) \in W^s(u(\varepsilon,\lambda))$ then $u_0(\varepsilon,\lambda) = h_b(u_b(\varepsilon,\lambda),\varepsilon,\lambda)$; using this fact in the previous inequality we get $s_0(\varepsilon,\lambda) - h_b(s_b(\varepsilon,\lambda),\varepsilon,\lambda) < 0$. Obviously $s(\varepsilon,\lambda) \in W^s(s(\varepsilon,\lambda))$, hence $\xi_0 - h_b(\xi_b,\varepsilon,\lambda) < 0$ in the whole connected component containing $s(\varepsilon,\lambda)$ and Remark 3.3 follows. $\square$

From Remark 3.3 we see that if $J_1^+(\varepsilon,\lambda) < 0$, then $\breve{\xi}^+(\varepsilon,\lambda) \in W^s(s(\varepsilon,\lambda))$ and the trajectory $x_c(t,\breve{\xi}^+(\varepsilon,\lambda),\varepsilon,\lambda)$ of (1.4) converges to $s(\varepsilon,\lambda)$; if $J_1^+(\varepsilon,\lambda) > 0$, then $\breve{\xi}^+(\varepsilon,\lambda) \in W^{s,n}(C(\varepsilon,\lambda))$, so there is $T > 0$ such that $x_c(T,\breve{\xi}^+(\varepsilon,\lambda),\varepsilon,\lambda) \notin \Omega_x$. In the former case $(\breve{x}(t,\varepsilon,\lambda),\breve{y}(t,\varepsilon,\lambda)) \to S(\varepsilon,\lambda)$ as $t \to +\infty$, while in the latter there is $T > 0$ such that $\breve{x}(T,\varepsilon,\lambda) \notin \Omega_x$ (obviously $\breve{\xi}^+(\varepsilon,\lambda_1^+(\varepsilon)) \in W^s(u(\varepsilon,\lambda_1^+(\varepsilon)))$ so $\breve{x}(t,\varepsilon,\lambda) \to U(\varepsilon,\lambda)$ as $t \to +\infty$). The analogous argument holds also for $W^u(s(\varepsilon,\lambda))$, $W^u(u(\varepsilon,\lambda))$ and in $W^{u,n}(C(\varepsilon,\lambda))$. Furthermore

$$
\begin{aligned}
J_1^+(\varepsilon,\lambda) &= J_1^+(\varepsilon,\lambda_1^+(\varepsilon)) + \tfrac{\partial J_1^+}{\partial\lambda}(\varepsilon,\lambda_1^+(\varepsilon))(\lambda-\lambda_1^+(\varepsilon)) + O((\lambda-\lambda_1^+(\varepsilon))^2)\\
J_1^-(\varepsilon,\lambda) &= J_1^-(\varepsilon,\lambda_1^-(\varepsilon)) + \tfrac{\partial J_1^-}{\partial\lambda}(\varepsilon,\lambda_1^-(\varepsilon))(\lambda-\lambda_1^-(\varepsilon)) + O((\lambda-\lambda_1^-(\varepsilon))^2)
\end{aligned}
\tag{3.22}
$$

From (3.20) we know the signs of $\frac{\partial}{\partial\lambda}J_1^\pm(\varepsilon,\lambda_1^\pm(\varepsilon))$; thus, using these two elementary observations we deduce for which values of ε,λ the point $\breve{\xi}^+(\varepsilon,\lambda)$ belongs to $W^u(u(\varepsilon,\lambda))$ or $W^{u,n}(C(\varepsilon,\lambda))$ (and to $W^s(u(\varepsilon,\lambda))$), and we obtain a detailed bifurcation diagram (we give some examples in figures 1, 2).

Now we assume $\lambda \le 0 < \varepsilon$, the critical points $u(\varepsilon,\lambda)$ and $s(\varepsilon,\lambda)$ are C^1 so u is stable and s is unstable for the flow of (1.4) restricted to $C(\varepsilon,\lambda)$. We look for the values of the parameters for which $\breve{\xi}^+(\varepsilon,\lambda) \in W^s(s(\varepsilon,\lambda))$ and $\breve{\xi}^-(\varepsilon,\lambda) \in W^u(u(\varepsilon,\lambda))$. So we define the smooth functions $\tilde{h}_b : A_b \times [-\varepsilon_0,\varepsilon_0] \times [-\lambda_0,\lambda_0] \to \mathbb{R}$, $\tilde{h}_a : A_a \times [-\varepsilon_0,\varepsilon_0] \times [-\lambda_0,\lambda_0] \to \mathbb{R}$ such that $\xi = (\xi_0,\xi_a,\xi_b) \in W^u(u(\varepsilon,\lambda)) \cap \Omega_x$ if and only if $\xi_0 = \tilde{h}_a(\xi_a,\varepsilon,\lambda)$ and $\xi_b = h_{(0,a)}(\xi_0,\xi_a,\varepsilon,\lambda)$, while $\xi \in W^s(s(\varepsilon,\lambda)) \cap \Omega_x$ if and only if $\xi_0 = \tilde{h}_b(\xi_b,\varepsilon,\lambda)$ and $\xi_a = h_{(0,b)}(\xi_0,\xi_b,\varepsilon,\lambda)$. Repeating for $\tilde{h}$ the argument developed for h, we see that the derivatives of $\tilde{h}_a$ and $\tilde{h}_b$ in $(0,0,0)$ with respect to ξ_i are null, but

$$
\begin{aligned}
\frac{\partial\tilde{h}_a}{\partial\lambda}(0,0,0) &= \frac{\partial u_0}{\partial\lambda}(0,0) \;= -\frac{\partial\tilde{h}_b}{\partial\lambda}(0,0,0)\\
\frac{\partial\tilde{h}_a}{\partial\varepsilon}(0,0,0) &= \frac{\partial u_0}{\partial\varepsilon}(0,0) = 0 = \frac{\partial\tilde{h}_b}{\partial\varepsilon}(0,0,0)
\end{aligned}
\tag{3.23}
$$

Then we define

$$
\begin{aligned}
J_4^+(\varepsilon,\lambda) &= \breve{\xi}_0^+(\varepsilon,\lambda) - \tilde{h}_b(\breve{\xi}_b^+(\varepsilon,\lambda),\varepsilon,\lambda)\,,\\
J_4^-(\varepsilon,\lambda) &= \breve{\xi}_0^-(\varepsilon,\lambda) - \tilde{h}_a(\breve{\xi}_a^-(\varepsilon,\lambda),\varepsilon,\lambda)
\end{aligned}
\tag{3.24}
$$

Once again $\breve{\xi}^+(\varepsilon,\lambda) \in W^s(s(\varepsilon,\lambda))$ if and only if $J_4^+(\varepsilon,\lambda)=0$, while $\breve{\xi}^-(\varepsilon,\lambda) \in W^u(u(\varepsilon,\lambda))$ if and only if $J_4^-(\varepsilon,\lambda)=0$ (whenever $\lambda \le 0 < \varepsilon$). Moreover arguing as above we see that $J_4^\pm(0,0)=0$ and

$$\begin{array}{l} \frac{\partial J_4^+}{\partial \lambda}(0,0) = B_0 + B_m + \frac{\partial u_0}{\partial \lambda}(0,0) = \frac{\partial J_1^-}{\partial \lambda}(0,0) \\ \frac{\partial J_4^-}{\partial \lambda}(0,0) = B_0 + B_m - \frac{\partial u_0}{\partial \lambda}(0,0) = \frac{\partial J_1^+}{\partial \lambda}(0,0) \end{array} ; \tag{3.25}$$

so, if (vi) holds we can apply the implicit function theorem and construct the curves $\lambda_4^\pm(\varepsilon)$ such that $J_4^\pm(\varepsilon,\lambda_4^\pm(\varepsilon)) \equiv 0$, and

$$\begin{array}{l} \frac{d}{d\varepsilon}\lambda_4^+(0) = -\frac{\frac{\partial}{\partial\varepsilon}J_4^+(0,0)}{\frac{\partial}{\partial\lambda}J_4^+(0,0)} = -\frac{A_0^+ + A_m + C_m}{B_0 + B_m + \frac{\partial u_0}{\partial\lambda}(0,0)} \\ \frac{d}{d\varepsilon}\lambda_4^-(0) = -\frac{\frac{\partial}{\partial\varepsilon}J_4^-(0,0)}{\frac{\partial}{\partial\lambda}J_4^-(0,0)} = -\frac{A_0^- + A_m + C_m}{B_0 + B_m - \frac{\partial u_0}{\partial\lambda}(0,0)} \end{array} \tag{3.26}$$

Obviously a fact analogous to Remark 3.2 holds also in this setting (and when $\varepsilon < 0$ as well, see below). So $W^s(s(\varepsilon,\lambda))$ divides $W^s(C(\varepsilon,\lambda))$ into two relatively open sets: $W^s(u(\varepsilon,\lambda))$, and say $W^{s,n}(C(\varepsilon,\lambda))$. If $\breve{\xi}^+(\varepsilon,\lambda) \in W^s(u(\varepsilon,\lambda))$, the solution $(\breve{x}(t,\varepsilon,\lambda),\breve{y}(t,\varepsilon,\lambda))$ defined in (3.14) converges to $S(\varepsilon,\lambda)$ as $t \to +\infty$, while if $\breve{\xi}^+(\varepsilon,\lambda) \in W^{s,n}(C(\varepsilon,\lambda))$, there is $T>0$ such that $(\breve{x}(T,\varepsilon,\lambda),\breve{y}(T,\varepsilon,\lambda)) \notin \Omega$. The analogous argument holds for $W^u(u(\varepsilon,\lambda))$ and $W^u(C(\varepsilon,\lambda))$. So, using a Taylor expansion analogous to (3.22), we can draw a detailed bifurcation diagram (we give some examples in figures 1, 2).

When $\varepsilon < 0$ we have an inversion in the stability properties of the critical points of (1.1) with respect to the stability properties of (1.4). Therefore if $\breve{\xi}^+ = \breve{\xi}^+(\varepsilon,\lambda) = (\breve{\xi}_0^+, \breve{\xi}_a^+, h_{(0,a)}(\breve{\xi}_0^+,\breve{\xi}_a^+,\varepsilon,\lambda)) \in W^u(C(\varepsilon,\lambda))$ then $\breve{x}(t,\varepsilon,\lambda)$ converges to $C(\varepsilon,\lambda)$ as $t\to+\infty$, while if $\breve{\xi}^- = \breve{\xi}^-(\varepsilon,\lambda) = (\breve{\xi}_0^-, h_{(0,b)}(\breve{\xi}_0^-,\breve{\xi}_b^-,\varepsilon,\lambda), \breve{\xi}_b^-) \in W^s(C(\varepsilon,\lambda))$ then $\breve{x}(t,\varepsilon,\lambda)$ converges to $C(\varepsilon,\lambda)$ as $t\to-\infty$. So we have to reverse the role of $\breve{\xi}^+$ and $\breve{\xi}^-$. Namely we set

$$\begin{array}{c} \tilde{H}^+(\xi_{\hat{0}},\varepsilon,\lambda) = \tilde{\xi}_b^+(\xi_{\hat{0}},\varepsilon,\lambda) - h_{(0,a)}(\tilde{\xi}_0^+(\xi_{\hat{0}},\varepsilon,\lambda),\varepsilon,\lambda),\tilde{\xi}_a^+(\xi_{\hat{0}},\varepsilon,\lambda),\varepsilon,\lambda) \\ \tilde{H}^-(\xi_{\hat{0}},\varepsilon,\lambda) = \tilde{\xi}_a^-(\xi_{\hat{0}},\varepsilon,\lambda) - h_{(0,b)}(\tilde{\xi}_0^-(\xi_{\hat{0}},\varepsilon,\lambda),\varepsilon,\lambda),\tilde{\xi}_b^-(\xi_{\hat{0}},\varepsilon,\lambda),\varepsilon,\lambda) \\ \tilde{H}(\xi_{\hat{0}},\varepsilon,\lambda) = (\tilde{H}^-(\xi_{\hat{0}},\varepsilon,\lambda),\tilde{H}^+(\xi_{\hat{0}},\varepsilon,\lambda)) \end{array} \tag{3.27}$$

Reasoning as in (3.13) we find again $\tilde{H}(0,0,0)=0$ and $\frac{\partial \tilde{H}}{\partial \xi_{\hat{0}}} = \mathbf{I}$, so we can apply the implicit function theorem to find $\breve{\xi}_{\hat{0}}(\varepsilon,\lambda)$ such that $\tilde{H}(\breve{\xi}_{\hat{0}}(\varepsilon,\lambda),\varepsilon,\lambda) \equiv 0$, and the solution defined by (3.14) is homoclinic to $\mathcal{M}(C(\varepsilon,\lambda))$.

Assume first $\lambda \geq 0$; arguing as in (3.15), (3.16) we find

$$\begin{aligned} \frac{\partial \breve{\xi}_{\hat{0}}}{\partial \lambda}(0,0) &= \frac{\partial u_{\hat{0}}}{\partial \lambda}(0,0) \\ \frac{\partial \breve{\xi}_{\hat{0}}}{\partial \varepsilon}(0,0) &= \left(\int_{-\infty}^{0} f_a(s)ds + \frac{\partial u_a}{\partial \varepsilon}(0,0), \int_{+\infty}^{0} f_b(s)ds + \frac{\partial u_b}{\partial \varepsilon}(0,0) \right) \end{aligned} \tag{3.28}$$

We stress that the formula for $\frac{\partial \xi_{\hat{0}}}{\partial \varepsilon}(0,0)$ has changed with respect to the $\varepsilon > 0$ case. When $\varepsilon \leq 0 \leq \lambda$ we define the following functions:

$$\begin{aligned} J_2^+(\varepsilon,\lambda) &= \breve{\xi}_0^+(\varepsilon,\lambda) - h_a(\breve{\xi}_a^+(\varepsilon,\lambda),\varepsilon,\lambda)\,, \\ J_2^-(\varepsilon,\lambda) &= \breve{\xi}_0^-(\varepsilon,\lambda) - h_b(\breve{\xi}_b^-(\varepsilon,\lambda),\varepsilon,\lambda) \end{aligned} \tag{3.29}$$

and we look for curves $\lambda_2^{\pm}(\varepsilon)$ such that $J_2^{\pm}(\varepsilon, \lambda_2^{\pm}(\varepsilon)) = 0$. Then the solution defined by (3.14) converges to $S(\varepsilon,\lambda)$ as $t \to +\infty$ when $(\varepsilon,\lambda) = (\varepsilon, \lambda_2^+(\varepsilon))$ and to $U(\varepsilon,\lambda)$ as $t \to -\infty$ when $(\varepsilon,\lambda) = (\varepsilon,\lambda_2^-(\varepsilon))$. Repeating the argument of (3.17) and (3.18) we find again

$$\frac{\partial \breve{\xi}_0^{\pm}}{\partial \lambda}(0,0) = B_0 + B_m \tag{3.30}$$

as in the $\varepsilon > 0$ case, but the formula for $\frac{\partial \breve{\xi}_0^{\pm}}{\partial \varepsilon}$ differs from the $\varepsilon > 0$ case:

$$\begin{aligned} &\frac{\partial \breve{\xi}_0^{\pm}}{\partial \varepsilon}(0,0) = A_0^{\pm} + A_m + \tilde{C}_m\,, \quad \text{where } \tilde{C}_m := \frac{\sum_{i=1}^{m} \left[\tilde{F}_i \int_{-\infty}^{\infty} \psi^*(t) \frac{\partial g}{\partial x_i}(t)dt \right]}{\int_{-\infty}^{\infty} \psi^*(t) \frac{\partial g}{\partial x_0}(t)dt} \\ &\tilde{F}_j = \int_0^{-\infty} f_j(t)dt - \frac{\partial u_j}{\partial \varepsilon}(0,0)\,, \quad \tilde{F}_k = \int_0^{+\infty} f_k(t)dt - \frac{\partial u_k}{\partial \varepsilon}(0,0) \end{aligned} \tag{3.31}$$

for $1 \leq j \leq l < l+1 \leq k \leq m$. So if (vi) holds we can apply the implicit function theorem to construct the curves $\lambda_2^{\pm}(\varepsilon)$ and we have the following formulas for the derivatives:

$$\begin{aligned} \frac{d}{d\varepsilon}\lambda_2^+(0) &= -\frac{\frac{\partial}{\partial \varepsilon} J_2^+(0,0)}{\frac{\partial}{\partial \lambda} J_2^+(0,0)} = -\frac{A_0^+ + A_m + \tilde{C}_m}{B_0 + B_m + \frac{\partial u_0}{\partial \lambda}(0,0)} \\ \frac{d}{d\varepsilon}\lambda_2^-(0) &= -\frac{\frac{\partial}{\partial \varepsilon} J_2^-(0,0)}{\frac{\partial}{\partial \lambda} J_2^-(0,0)} = -\frac{A_0^- + A_m + \tilde{C}_m}{B_0 + B_m - \frac{\partial u_0}{\partial \lambda}(0,0)} \end{aligned} \tag{3.32}$$

Then, using a Taylor expansion as in the $\varepsilon > 0$ case, we get a picture of the whole bifurcation diagram.

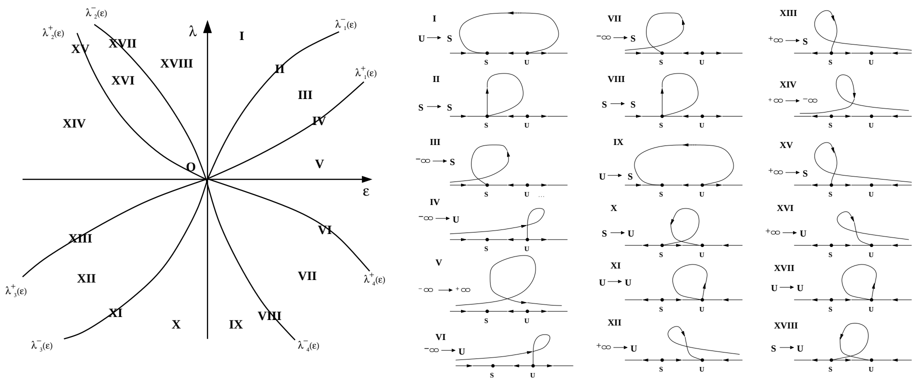

Figure 1: Example of bifurcation diagram in the transcritical case i.e. f_0 as in (1.7). Here we assume $\frac{\partial J^+}{\partial \lambda} < 0 < \frac{\partial J^-}{\partial \lambda}$, and $\frac{\partial \lambda_j^-}{\partial \varepsilon} < \frac{\partial \lambda_j^+}{\partial \varepsilon} < 0 < \frac{\partial \lambda_i^+}{\partial \varepsilon} < \frac{\partial \lambda_i^-}{\partial \varepsilon}$, for $i = 1, 3,\ j = 2, 4$.

When ε and λ are both negative we have a further change in the stability properties. So we define the functions

$$\begin{array}{c} J_3^+(\varepsilon,\lambda) = \breve{\xi}_0^+(\varepsilon,\lambda) - \tilde{h}_a(\breve{\xi}_a^+(\varepsilon,\lambda),\varepsilon,\lambda)\,, \\ J_3^-(\varepsilon,\lambda) = \breve{\xi}_0^-(\varepsilon,\lambda) - \tilde{h}_b(\breve{\xi}_b^-(\varepsilon,\lambda),\varepsilon,\lambda) \end{array} . \tag{3.33}$$

and we look for the curves $\lambda_3^\pm(\varepsilon)$ such that $J_3^\pm(\varepsilon, \lambda_3^\pm(\varepsilon)) \equiv 0$, so that the solution defined by (3.14) converges to $S(\varepsilon, \lambda)$ as $t \to -\infty$ and to $U(\varepsilon, \lambda)$ as $t \to +\infty$. Once again such curves can be constructed via implicit function theorem if (vi) holds, and we find:

$$\begin{aligned} \frac{d}{d\varepsilon}\lambda_3^+(0) &= -\frac{\frac{\partial}{\partial \varepsilon} J_3^+(0,0)}{\frac{\partial}{\partial \lambda} J_3^+(0,0)} = -\frac{A_0^+ + A_m + \tilde{C}_m}{B_0 + B_m - \frac{\partial u_0}{\partial \lambda}(0,0)} \\ \frac{d}{d\varepsilon}\lambda_3^-(0) &= -\frac{\frac{\partial}{\partial \varepsilon} J_3^-(0,0)}{\frac{\partial}{\partial \lambda} J_3^-(0,0)} = -\frac{A_0^- + A_m + \tilde{C}_m}{B_0 + B_m + \frac{\partial u_0}{\partial \lambda}(0,0)} \end{aligned} \tag{3.34}$$

We stress that a priori the curves $\lambda_i^\pm$ for $i = 1, 2, 3, 4$ may have all different tangent in the origin. This is not the case in the $m = 0$ case, see [8].

The bifurcation diagram changes according to the signs of the nonzero computable constants $\frac{\partial J_1^\pm}{\partial \lambda}(0,0)$ and of the following computable constants which may be zero

$$\frac{d}{d\varepsilon}\lambda_i^+(0), \qquad \frac{d}{d\varepsilon}\lambda_i^-(0), \qquad \frac{d}{d\varepsilon}\lambda_i^+(0) - \frac{d}{d\varepsilon}\lambda_i^-(0) \tag{3.35}$$

for $i = 1, 2, 3, 4$. To illustrate the meaning of Theorem 3.4 we draw some pictures for specific nonzero values of the constants given in (3.35), the other

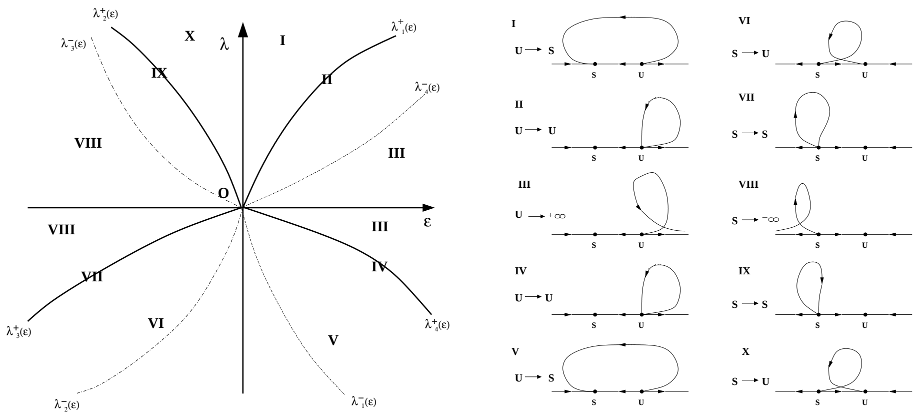

Figure 2: Example of bifurcation diagram in the transcritical case i.e. f_0 as in (1.7). Here we assume $\frac{\partial J_1^+}{\partial \lambda} < 0 < \frac{\partial J_1^-}{\partial \lambda}$, $\frac{d}{d\varepsilon}\lambda_i^- < 0 < \frac{d}{d\varepsilon}\lambda_i^+$ and $\frac{d}{d\varepsilon}\lambda_j^+ < 0 < \frac{d}{d\varepsilon}\lambda_j^-$ for $i = 1, 3$ and $j = 1, 4$.

possibilities can be obtained similarly (not all the combinations are effectively possible). In section 4 we construct a differential equation for which the values of these constants are explicitly computed.

3.4 Theorem. *Assume that Hypotheses (i)–(vi) of the Introduction hold and that f satisfies **h)** and (1.7). Then we can draw the bifurcation diagram for system (1.1), see figures 1, 2)*

3.5 Remark. Assume that ε and λ are both positive. When $(\breve{x}(t,\varepsilon,\lambda), \breve{y}(t,\varepsilon,\lambda))$ tends to $S(\varepsilon,\lambda)$ as $t \to +\infty$ (so that in particular $\lambda \neq \lambda_+^1(\varepsilon)$) it has a slow rate of convergence, i.e. $\|(\breve{x}(t,\varepsilon,\lambda), \breve{y}(t,\varepsilon,\lambda)) - S(\varepsilon,\lambda)\| \sim \exp(-K_1|\varepsilon\lambda|t)$ for some $K_1 > 0$ (independent of ε and λ). However when $(\breve{x}(t,\varepsilon,\lambda), \breve{y}(t,\varepsilon,\lambda))$ tends to $S(\varepsilon,\lambda)$ as $t \to +\infty$, that is $\lambda = \lambda_1^+(\varepsilon)$, we have faster convergence i. e. $\|(\breve{x}(t,\varepsilon,\lambda), \breve{y}(t,\varepsilon,\lambda)) - U(\varepsilon, \lambda_1^+(\varepsilon))\| \sim \exp(-\varepsilon K_2 t)$ for a certain $K_2 > 0$.

3.6 Remark. In the proof of Theorem 3.1 we have shown that, for each centre manifold $\mathcal{M}^c(C(\varepsilon,\lambda))$, there is exactly one trajectory $(\breve{x}(t,\varepsilon,\lambda), \breve{y}(t,\varepsilon,\lambda))$ homoclinic to it (unicity follows from the use of Implicit Function Theorem). We emphasize that when $(\breve{x}(t,\varepsilon,\lambda), \breve{y}(t,\varepsilon,\lambda))$ is bounded, i.e. when it is either a homoclinic or a heteroclinic trajectory, then it satisfies

$$\sup_{t\in\mathbb{R}} |(\breve{x}(t,\varepsilon,\lambda), \breve{y}(t,\varepsilon,\lambda)) - (0, h(t))| \leq K\varepsilon \tag{3.36}$$

for a certain $K > 0$.

If we consider the example in figure 1, such a fact happens if (ε,λ) are in the subsets **I**, **II**, **VIII**, **IX**, **XI**, **XVII**, **XVIII**. In the remaining cases

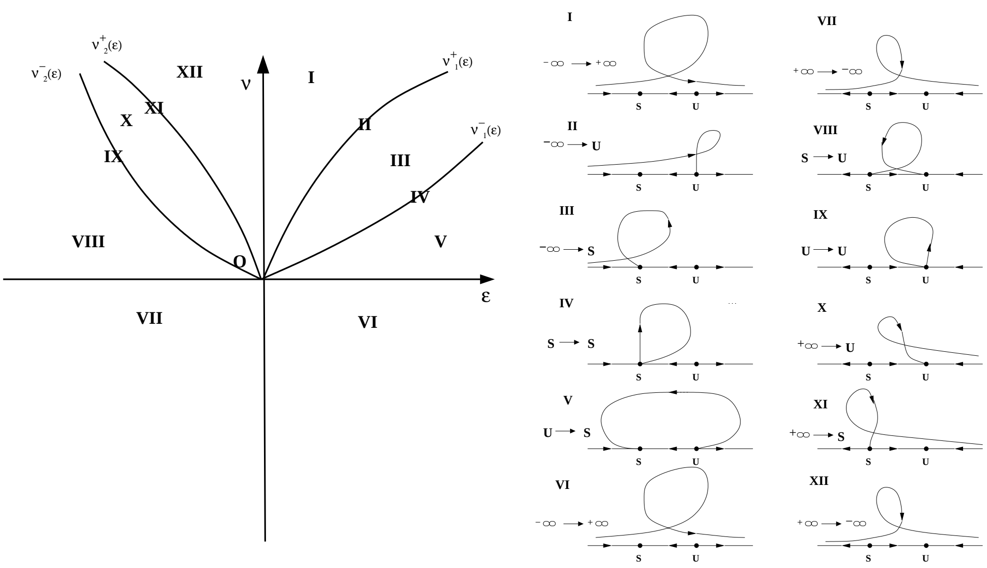

Figure 3: Example of bifurcation diagram in the saddle node case, i.e. f_0 as in (1.8). We assume $\frac{\partial \breve{\xi}_0^+}{\partial \varepsilon}(0,0) < 0 < \frac{\partial \breve{\xi}_0^-}{\partial \varepsilon}(0,0)$, $\frac{\partial \breve{\xi}_0^+}{\partial \varepsilon}(0,0) < 0 < \frac{\partial \breve{\xi}_0^-}{\partial \varepsilon}(0,0)$.

there are no trajectories satisfying (3.36). See Remark 5.3 for a discussion of uniqueness in relation with the multiplicity of centre manifolds.

When the computable constants given in (3.35) are null we cannot draw the bifurcation diagram in all details; see the end of section 3 in [8] for a more detailed discussion of this case.

3.2 Saddle-node bifurcation.

We briefly consider the case where f_0 satisfies (1.8) so that the origin undergoes a saddle-node bifurcation. We need to introduce the auxiliary variable $\nu = \sqrt{|\lambda|}$ and we observe that $u(\varepsilon, \nu^2)$ and $s(\varepsilon, \nu^2)$ are smooth functions (while they are just Holder functions of λ). Theorem 3.1 holds also in this setting, so there is a unique solution of (1.1) which is homoclinic to $\mathcal{M}^c(C(\varepsilon, \lambda))$. In fact Theorem 3.4 works too, with some minor changes, but condition (vi) is not needed anymore. Once again we have to argue separately in each quadrant of the parameters plane; we start from ε and λ positive, and we define

$$\tilde{J}_1^+(\varepsilon, \nu) = \breve{\xi}_0^+(\varepsilon, \nu^2) - h_b(\breve{\xi}_b^+(\varepsilon, \nu^2), \varepsilon, \nu^2)\,,$$
$$\tilde{J}_1^-(\varepsilon, \nu) = \breve{\xi}_0^-(\varepsilon, \nu^2) - h_a(\breve{\xi}_a^-(\varepsilon, \nu^2), \varepsilon, \nu^2)$$

and we repeat the analysis made in the previous subsection. The solution defined by (3.14) converges to U as $t \to +\infty$ if $\tilde{J}_1^+(\varepsilon, \nu) = 0$ and to S as

$t \to -\infty$ if $\tilde{J}_1^-(\varepsilon,\nu) = 0$. We stress that $\frac{\partial \breve{\xi}_0^{\pm}}{\partial \nu}(0,0) = 0$ since $\frac{\partial \lambda}{\partial \nu}(0) = 0$, therefore

$$\frac{\partial \tilde{J}_1^-}{\partial \nu}(0,0) = \frac{\partial u_0}{\partial \nu}(0,0) = -\frac{\partial \tilde{J}_1^+}{\partial \nu}(0,0)\,.$$

So we can apply the implicit function Theorem and construct smooth curves $\nu_1^{\pm}(\varepsilon)$ such that $\nu_1^{\pm}(0) = 0$, $\tilde{J}_1^{\pm}(\varepsilon, \nu_1^{\pm}(\varepsilon)) = 0$; note that (vi) is not necessary. Furthermore

$$\frac{d}{d\varepsilon}\nu_1^+(0) = \frac{A_0^+ + A_m + C_m}{\frac{\partial}{\partial \nu}u_0(0,0)}\,, \quad \frac{d}{d\varepsilon}\nu_1^-(0,0) = -\frac{A_0^- + A_m + C_m}{\frac{\partial}{\partial \nu}u_0(0,0)} \tag{3.37}$$

When $\varepsilon < 0 \le \lambda$ we define

$$\begin{aligned}\tilde{J}_2^+(\varepsilon,\nu) &= \breve{\xi}_0^+(\varepsilon,\nu^2) - h_a(\breve{\xi}_a^+(\varepsilon,\nu^2),\varepsilon,\nu^2)\,,\\ \tilde{J}_2^-(\varepsilon,\nu) &= \breve{\xi}_0^-(\varepsilon,\nu^2) - h_b(\breve{\xi}_b^-(\varepsilon,\nu^2),\varepsilon,\nu^2)\end{aligned}$$

and we find again curves $\nu_2^{\pm}(\varepsilon)$ such that $\nu_2^{\pm}(0) = 0$, $\tilde{J}_2^{\pm}(\varepsilon, \nu_2^{\pm}(\varepsilon)) = 0$, and

$$\frac{d}{d\varepsilon}\nu_2^+(0) = \frac{A_0^+ + A_m + \tilde{C}_m}{\frac{\partial}{\partial \nu}u_0(0,0)}\,, \quad \frac{d}{d\varepsilon}\nu_2^-(0,0) = -\frac{A_0^- + A_m + \tilde{C}_m}{\frac{\partial}{\partial \nu}u_0(0,0)} \tag{3.38}$$

The solution defined by (3.14) converges to S as $t \to +\infty$ if $\tilde{J}_2^+(\varepsilon,\nu) = 0$ and to U as $t \to -\infty$ if $\tilde{J}_2^-(\varepsilon,\nu) = 0$.

Obviously in both the cases for $\lambda < 0$ there are no critical points and hence no bounded trajectories. Arguing as in the previous subsection we obtain a result analogous to Theorem 3.4.

3.7 Theorem. *Assume that Hypotheses (i)–(v) of the Introduction hold and that f satisfies **h)** and (1.8). Then we can draw the bifurcation diagram for system (1.1).*

The bifurcation diagram of (1.1) described in Theorem 3.7 depends on the signs of the following computable constants:

$$\frac{d\nu_i^+}{d\varepsilon}(0), \qquad \frac{d\nu_i^-}{d\varepsilon}(0), \qquad \frac{d}{d\varepsilon}\nu_i^+(0) - \frac{d}{d\varepsilon}\nu_i^-(0). \tag{3.39}$$

We give again one example for illustrative purposes, see figure 3.

3.8 Remark. When ε is the only parameter involved in the bifurcation, so that f does not depend on λ, we can still perform our analysis, with some trivial (and simplifying) changes. When both f and g do not depend on λ, we cannot unfold completely the singularity. However the behavior of the

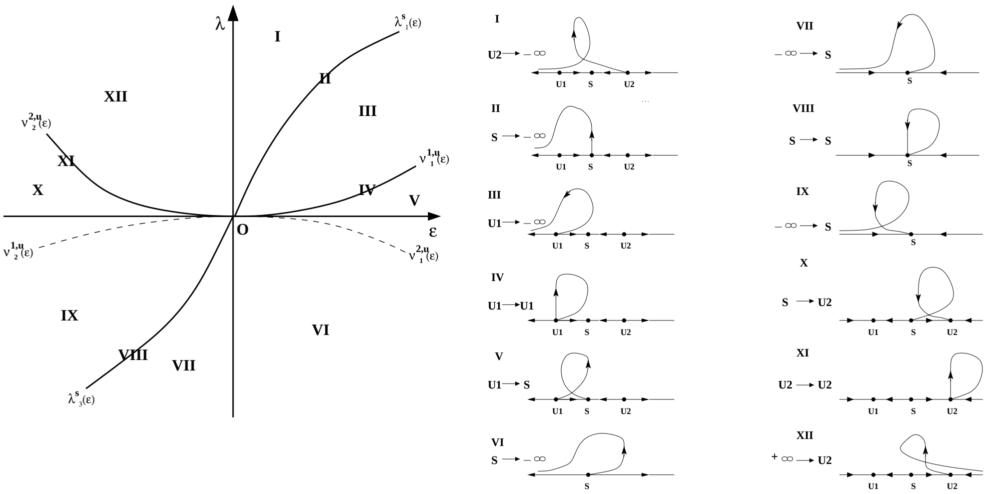

Figure 4: Bifurcation diagram in the pitchfork case, i.e. f_0 as in (1.9). We assume $A_0^{\pm} + A_m + C_m > 0$, $A_0^{\pm} + A_m + \tilde{C}_m > 0$, and $B_0 + B_m - \frac{\partial s_0}{\partial \lambda} > 0$.

solution $(\breve{x}(t,\varepsilon), \breve{y}(t,\varepsilon))$ defined by (3.14) is determined in the transcritical case by the signs of the following constants:

$$\begin{aligned}
K^{+} &= \frac{\partial \breve{\xi}_0^{+}}{\partial \varepsilon}(0) - \frac{\partial u_0}{\partial \varepsilon}(0) = A_0^{+} + A_m + C_m - \frac{\partial u_0}{\partial \varepsilon}(0)\,, \\
K^{-} &= \frac{\partial \breve{\xi}_0^{-}}{\partial \varepsilon}(0) + \frac{\partial u_0}{\partial \varepsilon}(0) = A_0^{-} + A_m + C_m + \frac{\partial u_0}{\partial \varepsilon}(0)\,, \\
\tilde{K}^{+} &= \frac{\partial \breve{\xi}_0^{+}}{\partial \varepsilon}(0) - \frac{\partial u_0}{\partial \varepsilon}(0) = A_0^{+} + A_m + \tilde{C}_m - \frac{\partial u_0}{\partial \varepsilon}(0)\,, \\
\tilde{K}^{-} &= \frac{\partial \breve{\xi}_0^{-}}{\partial \varepsilon}(0) + \frac{\partial u_0}{\partial \varepsilon}(0) = A_0^{-} + A_m + \tilde{C}_m + \frac{\partial u_0}{\partial \varepsilon}(0)\,,
\end{aligned} \tag{3.40}$$

see (3.18), (3.31). E.g. if $K^{\pm}$ are positive, we find that $(\breve{x}(t,\varepsilon), \breve{y}(t,\varepsilon))$ converges to $U(\varepsilon)$ as $t \to -\infty$ and leaves a neighborhood of the origin for t large, and the same happens for $\varepsilon < 0$, see Remark 3.6 in [8] for more details.

Reasoning in the same way it is easy to see that when f and g are independent from λ and (1.4) exhibits a saddle-node bifurcation, then $(\breve{x}(t,\varepsilon), \breve{y}(t,\varepsilon))$ is always a heteroclinic connection between U and S, and converges to the former in the past and to the latter in the future, since $s_0(\varepsilon) < \breve{\xi}_0^{\pm}(\varepsilon) < u_0(\varepsilon)$ for $\varepsilon > 0$; in fact $\frac{\partial s_0}{\partial \varepsilon}(0) = -\infty$ and $\frac{\partial u_0}{\partial \varepsilon}(0) = +\infty$.

3.3 Degree 3 or more.

In this subsection we show briefly how our methods can be applied to unfold singularities more degenerate than (1.7) and (1.8). We just sketch the case where (1.4) undergoes a pitchfork bifurcation, i.e. f_0 has the form (1.9) stressing that the construction can be easily generalized to describe singularities of higher order. We denote by u_0^1, s_0 and u_0^2 the x_0 coordinates of the critical points, and we set $U^1(\varepsilon,\lambda) = (u^1(\varepsilon,\lambda), v(u^1(\varepsilon,\lambda),\varepsilon,\lambda))$, $U^2(\varepsilon,\lambda) = (u^2(\varepsilon,\lambda), v(u^2(\varepsilon,\lambda),\varepsilon,\lambda))$, $S(\varepsilon,\lambda) = (s(\varepsilon,\lambda), v(s(\varepsilon,\lambda),\varepsilon,\lambda)))$. However to achieve a complete unfolding of the singularity one more parameter is needed.

Theorem 3.1 holds in this case too, so using the function H defined in (3.12) for $\varepsilon > 0$, and the function $\tilde{H}$ defined in (3.27) for $\varepsilon < 0$, via implicit function theorem we construct the smooth function $\breve{\xi}_{\hat{0}}(\varepsilon,\lambda)$ such that the solution defined by (3.14) is homoclinic to $\mathcal{M}^c(C(\varepsilon,\lambda))$.

Similarly to the saddle-node case the functions $u_0^1(\varepsilon,\lambda)$ and $u_0^2(\varepsilon,\lambda)$ are not smooth in the origin, so we need to introduce the parameter $\nu = \sqrt{\lambda}$. On the other hand the function $s_0(\varepsilon,\lambda)$ is smooth and its derivative with respect to ν is null; so, in order to apply the implicit function theorem, we have to work with $u_0^1(\varepsilon,\nu^2)$, $u_0^2(\nu^2)$ and $s_0(\varepsilon,\lambda)$.

Let us start assuming $\lambda \geq 0$ and $\varepsilon > 0$, in analogy to the previous subsection we define the functions $\bar{h}_b^1, \bar{h}_b^2 : A_b \times [-\varepsilon_0,\varepsilon_0] \times [-\lambda_0,\lambda_0] \to \mathbb{R}$, $\bar{h}_a : A_a \times [-\varepsilon_0,\varepsilon_0] \times [-\lambda_0,\lambda_0] \to \mathbb{R}$ such that $\breve{\xi} \in W^s(u^i(\varepsilon,\lambda)) \cap \Omega_x$ if and only if $\breve{\xi}_0 = \bar{h}_b^i(\breve{\xi}_b,\varepsilon,\nu)$ and $\breve{\xi}_a = h_{(0,b)}(\breve{\xi}_0,\breve{\xi}_b,\varepsilon,\nu^2)$ for $i = 1,2$, while $\breve{\xi} \in W^u(s(\varepsilon,\lambda)) \cap \Omega_x$ if and only if $\breve{\xi}_0 = \bar{h}_a(\breve{\xi}_a,\varepsilon,\lambda)$ and $\breve{\xi}_b = h_{(0,a)}(\breve{\xi}_0,\breve{\xi}_a,\varepsilon,\lambda)$. Again the derivatives of $\bar{h}_a$ and $\bar{h}_b^j$ in $(0,0,0)$ in ε and x_i are null, and

$$\frac{\partial \bar{h}_b^1}{\partial \nu}(0,0,0) = \frac{\partial u_0^1}{\partial \nu}(0,0) = -\frac{\partial \bar{h}_b^2}{\partial \nu}(0,0,0)\,, \qquad \frac{\partial \bar{h}_a}{\partial \lambda}(0,0) = \frac{\partial s_0}{\partial \lambda}(0,0)\,.$$

Then we define the functions

$$J_1^{i,u}(\varepsilon,\nu) = \breve{\xi}_0^+(\varepsilon,\nu^2) - \bar{h}_b^i(\breve{\xi}_b,\varepsilon,\nu)\,, \qquad J_1^s(\varepsilon,\lambda) = \breve{\xi}_0^-(\varepsilon,\lambda) - \bar{h}_a(\breve{\xi}_a,\varepsilon,\lambda)$$

for $i = 1,2$; obviously $J_1^{i,u}(0,0) = 0$ for $i = 1,2$ and $J_1^s(0,0) = 0$. We stress that $\frac{\partial}{\partial \nu}\breve{\xi}_0^+(\varepsilon,\nu^2) = 0$ for $(\varepsilon,\nu) = (0,0)$. To apply the implicit function theorem we just need to assume

(vi') $\frac{\partial u_0^1}{\partial \nu}(0,0) \neq 0\,,$ and $B_0 + B_m - \frac{\partial}{\partial \lambda} s_0(0,0) \neq 0$

So we prove the existence of curves $\nu_1^{i,u}(\varepsilon)$, $\lambda_1^s(\varepsilon)$ such that $(\breve{x}(t,\varepsilon,\lambda), \breve{y}(t,\varepsilon,\lambda))$ converges to U^i as $t \to +\infty$ when $\lambda = [\nu_1^{i,u}(\varepsilon)]^2$ for $i = 1,2$, and to S as

$t \to -\infty$ when $\lambda = \lambda_1^s(\varepsilon)$. Moreover

$$\begin{aligned}
\frac{d}{d\varepsilon}\nu_1^{1,u}(0) &= \frac{\frac{\partial}{\partial\varepsilon}\breve{\xi}_0^+(0,0) - \frac{\partial u_0^1}{\partial\varepsilon}(0,0)}{\frac{\partial u_0^1}{\partial\nu}(0,0)} = \frac{A_m + A_0^+ + C_m}{\frac{\partial u_0^1}{\partial\nu}(0,0)} \\
\frac{d}{d\varepsilon}\nu_1^{2,u}(0) &= \frac{\frac{\partial}{\partial\varepsilon}\breve{\xi}_0^+(0,0) - \frac{\partial u_0^2}{\partial\varepsilon}(0,0)}{\frac{\partial u_0^2}{\partial\nu}(0,0)} = -\frac{A_m + A_0^+ + C_m}{\frac{\partial u_0^1}{\partial\nu}(0,0)} \qquad (3.41) \\
\frac{d}{d\varepsilon}\lambda_1^s(0) &= -\frac{\frac{\partial}{\partial\varepsilon}\breve{\xi}_0^-(0,0) - \frac{\partial s_0}{\partial\varepsilon}(0,0)}{\frac{\partial}{\partial\lambda}\breve{\xi}_0(0,0) - \frac{\partial s_0}{\partial\lambda}(0,0)} = -\frac{A_m + A_0^- + C_m}{B_0 + B_m - \frac{\partial s_0}{\partial\lambda}(0,0)}\,,
\end{aligned}$$

When $\lambda \le 0$ the only critical point of (1.4) in a neighborhood of the origin is $s(\varepsilon,\lambda)$, which is unstable in the direction of $C(\varepsilon,\lambda)$. So we define the function $\bar{h}_b$ such that $\xi \in W^s(s) \cap \Omega_x$ if and only if $\xi_0 = \bar{h}_b(\xi_b,\varepsilon,\lambda)$ and $\xi_a = h_{(0,b)}(\xi_0,\xi_b,\varepsilon,\lambda)$ and

$$J_4^s(\varepsilon,\lambda) = \breve{\xi}_0^+(\varepsilon,\lambda) - \bar{h}_b(\breve{\xi}_b,\varepsilon,\lambda)\,.$$

Then via implicit function theorem we construct the curve $\lambda_4^s(\varepsilon)$ such that $(\breve{x}(t,\varepsilon,\lambda_4^s(\varepsilon)), \breve{y}(t,\varepsilon,\lambda_4^s(\varepsilon)))$ converges to S as $t \to +\infty$; moreover

$$\frac{d}{d\varepsilon}\lambda_4^s(0) = -\frac{\frac{\partial}{\partial\varepsilon}\breve{\xi}_0^+(0,0) - \frac{\partial s_0}{\partial\varepsilon}(0,0)}{\frac{\partial}{\partial\lambda}\breve{\xi}_0(0,0) - \frac{\partial s_0}{\partial\lambda}(0,0)} = -\frac{A_m + A_0^+ + C_m}{B_0 + B_m - \frac{\partial s_0}{\partial\lambda}(0,0)}\,. \qquad (3.42)$$

When $\lambda < 0 < \varepsilon$ the trajectory $(\breve{x}(t,\varepsilon,\lambda), \breve{y}(t,\varepsilon,\lambda))$ homoclinic to $\mathcal{M}(C(\varepsilon,\lambda))$ converges to S as $t \to -\infty$.

When $\varepsilon < 0$ as usual the critical points of (1.1) reverse their stability properties, so we have to redefine the auxiliary functions as we did in the previous section. When $\varepsilon < 0 \le \lambda$ we construct via implicit function theorem the curves $\nu_2^{1,u}(\varepsilon)$, $\nu_2^{2,u}(\varepsilon)$ and $\lambda_2^s(\varepsilon)$ with the following properties: the trajectory defined by (3.14) converges to U^i as $t \to -\infty$ when $\sqrt{\lambda}$ equals $\nu_2^{i,u}(\varepsilon)$ for $i = 1,2$, and to S as $t \to +\infty$ when $\lambda = \lambda_2^s(\varepsilon)$. Moreover

$$\begin{aligned}
&\frac{d}{d\varepsilon}\nu_2^{1,u}(0) = \frac{A_m + A_0^- + \tilde{C}_m}{\frac{\partial u_0^1}{\partial\nu}(0,0)}\,, \quad \frac{d}{d\varepsilon}\nu_2^{2,u}(0) = -\frac{A_m + A_0^- + \tilde{C}_m}{\frac{\partial u_0^1}{\partial\nu}(0,0)} \\
&\frac{d}{d\varepsilon}\lambda_2^s(0) = -\frac{A_m + A_0^+ + \tilde{C}_m}{B_0 + B_m - \frac{\partial s_0}{\partial\lambda}(0,0)}\,,
\end{aligned} \qquad (3.43)$$

Similarly when ε and λ are negative, we construct the curve $\lambda_3^s(\varepsilon)$, such that the trajectory defined by (3.14) converges to S as $t \to -\infty$. Moreover

$$\frac{d}{d\varepsilon}\lambda_3^s(0) = -\frac{A_m + A_0^- + \tilde{C}_m}{B_0 + B_m - \frac{\partial s_0}{\partial\lambda}(0,0)}\,.$$

Furthermore the trajectory $(\breve{x}(t,\varepsilon,\lambda),\breve{y}(t,\varepsilon,\lambda))$ homoclinic to $\mathcal{M}^c(C(\varepsilon,\lambda))$ converges to $S(\varepsilon,\lambda)$ as $t\to+\infty$. Now, similarly to the previous subsections, using a Taylor expansion analogous to (3.22), we can draw the bifurcation diagram for (1.1). Once again the bifurcation diagram depends on the sign of some computable constants, i. e. $B_0+B_m-\frac{\partial s_0}{\partial\lambda}(0,0)$, $\frac{\partial\nu_i^{1,u}}{\partial\varepsilon}$, for $i=1,2$, $\frac{\partial\lambda_i^s}{\partial\varepsilon}$ for $i=1,2,3,4$, see figure 4.

4 Examples.

In this section we construct examples for which the conditions of Theorems 3.1, 3.4, 3.7 are fulfilled and the derivatives of the bifurcation curves can be explicitly computed. Let us consider the following system:

$$\begin{cases} \dot{x}_0 &= \varepsilon\big[x_0^2-(\sigma_0\lambda)^2+\alpha y_1y_2+\omega_0(x,y,\varepsilon,\lambda)\big] := \varepsilon f_0(x,y,\varepsilon,\lambda) \\ \dot{x}_1 &= \varepsilon\big[x_1-\sigma_1\lambda+\beta y_1^2y_2+\omega_1(x,y,\varepsilon,\lambda)\big] \\ \dot{x}_2 &= \varepsilon\big[-x_2+\sigma_2\lambda+\gamma y_2+\omega_2(x,y,\varepsilon,\lambda)\big] \\ \dot{y}_1 &= y_2+x_0(a'y_1+a''y_2)+a'''x_1y_2+a^{iv}x_2y_1+\lambda y_2k(y_1)+O((|\lambda|+|\varepsilon|)|x||y|) \\ \dot{y}_2 &= y_1-(y_1)^3+x_0(b'y_1+b''y_2)+x_1y_1+\lambda h(y_1)+O((|\lambda|+|\varepsilon|)|x||y|) \end{cases} \tag{4.1}$$

where $\alpha,\beta,\gamma,\sigma_i,a',a'',a''',a^{iv},b',b''\in\mathbb{R}$, h and k are smooth functions satisfying $h(0)=0=k(0)$, $\omega_i(x,y,\varepsilon,\lambda)=O(|y||x|)+o(\varepsilon^2+\lambda^2+|x|^2)$ for $i=0,1,2$. The y component of (4.1) is constructed on the unperturbed problem

$$\begin{cases} \dot{y}_1 &= g_1(0,y,0,0) := & y_2 \\ \dot{y}_2 &= g_2(0,y,0,0) := & y_1-(y_1)^3 \end{cases} \tag{4.2}$$

which admits two homoclinic trajectories $\pm(\chi_1(t),\chi_2(t))$ where

$$\chi_1(t)=\frac{2\sqrt{2}}{e^t+e^{-t}}\,,\quad \chi_2(t)=-2\sqrt{2}\frac{e^t-e^{-t}}{(e^t+e^{-t})^2}$$

and $\chi_1^4/2-\chi_1^2+\chi_2^2=0$. So $\chi(t)=(0,0,0,\chi_1(t),\chi_2(t))$ and $-\chi(t)$ are homoclinic trajectory for (4.1) for $\varepsilon=\lambda=0$. Note that the adjoint variational systems $\dot{y}=-[\partial g/\partial y]^*(\pm\chi(t),0,0)y$ admits the unique (up to multiplicative constant) solutions $\pm\psi(t)=\pm(\{\chi_1(t)-[\chi_1(t)]^3\},-\chi_2(t))$.

We stress that for each slow manifold $\mathcal{M}^c$ there is a possibly different function v which satisfies $y=v(x,\varepsilon,\lambda)$ close to the origin in $\mathbb{R}^6$; however from $\frac{\partial g}{\partial\lambda}(0,0,0,0)=0=\frac{\partial g}{\partial\varepsilon}(0,0,0,0)$ we find $v(x,\varepsilon,\lambda)=O(\varepsilon^2+\lambda^2)$. Hence all the slow manifolds $\mathcal{M}^c$ satisfy

$$f_0(x,v(x,\varepsilon,\lambda),\varepsilon,\lambda)=x_0^2-(\sigma_0\lambda)^2+o(\varepsilon^2+\lambda^2+|x|^2)\,.$$

Moreover it is easy to check that the centre manifold within the slow manifold (which possibly is not unique) is tangent to the x_0 axis for ε and λ small. From a straightforward computation we find $\frac{\partial u}{\partial\lambda}(0,0) = (\sigma_0, \sigma_1, \sigma_2)$, $\frac{\partial s}{\partial\lambda}(0,0) = (-\sigma_0, \sigma_1, \sigma_2)$, and $\frac{\partial u_0}{\partial\varepsilon}(0,0) = \frac{\partial s_0}{\partial\varepsilon}(0,0) = 0$.

From further computations we get $\chi_1(0) = \sqrt{2}$, $\chi_2(0) = 0$, $\int_{\mathbb{R}} \chi_1^4 = \int_{\mathbb{R}} \chi_2^2 = \frac{16}{3}$, $\int_{\mathbb{R}} \chi_1^2 = 4$, $\int_{\mathbb{R}} \chi_1^6 = \frac{128}{15}$, $\int_{\mathbb{R}} \chi_1^2\chi_2^2 = \frac{16}{15}$, $\int_{\mathbb{R}} \chi_1^3 = \pi\sqrt{2}$, $\int_{\mathbb{R}} \chi_1^5 = \frac{3\pi}{\sqrt{2}}$.

$$X_0 = \int_{-\infty}^{\infty} \pm\psi^*(t)\frac{\partial g}{\partial x_0}(\pm\chi(t),0,0)dt = \int_{-\infty}^{\infty} \left[a'(\chi_1^2 - \chi_1^4(t)) - b''\chi_2^2(t)\right]dt =$$
$$= -\frac{4}{3}a' - \frac{16}{3}b'', \qquad X_1 = \int_{-\infty}^{\infty} \pm\psi^*(t)\frac{\partial g}{\partial x_1}(\pm\chi(t),0,0)dt = 0,$$
$$X_2 = \int_{-\infty}^{\infty} \pm\psi^*(t)\frac{\partial g}{\partial x_2}(\pm\chi(t),0,0)dt = -\frac{4}{3}a^{iv}$$

Moreover

$$\int_0^t f_0(\pm\chi(s),0,0)ds = \frac{\alpha}{2}[\chi_1^2(t) - \chi_1^2(0)]\,, \int_0^t f_1(\pm\chi(s),0,0)ds =$$
$$\pm\frac{\beta}{3}[\chi_1^3(t) - \chi_1^3(0)]\,, \int_0^t f_2(\pm\chi(s),0,0)ds = \pm\gamma[\chi_1(t) - \chi_1(0)]$$
$$F_1^\pm = \int_0^{-\infty} f_1(\pm\chi(t),0,0)dt - \frac{\partial s_1}{\partial\varepsilon}(0,0) = \mp\frac{2\sqrt{2}\beta}{3} = \tilde{F}_1^\pm$$
$$F_2^\pm = \int_0^{+\infty} f_2(\pm\chi(t),0,0)dt - \frac{\partial s_2}{\partial\varepsilon}(0,0) = \mp\gamma\sqrt{2} = \tilde{F}_2^\pm$$

and

$$K_0 = \int_{-\infty}^{\infty} \pm\psi^*(t)\frac{\partial g}{\partial x_0}(\pm\chi(t),0,0)\Big[\int_{\pm\infty}^t f_0(\pm\chi(s),0,0)ds\Big]dt = -\frac{8\alpha}{15}\left(3a' + b''\right)\,,$$
$$\int_{-\infty}^{\infty} \pm\psi^*(t)\frac{\partial g}{\partial\lambda}(\pm\chi(t),0,0)dt = \int_{-\infty}^{\infty} \pm\psi^*(t)\frac{\partial g}{\partial\varepsilon}(\pm\chi(t),0,0)dt = 0$$
$$K_1 = \int_{-\infty}^{\infty} \pm\psi^*(t)\frac{\partial g}{\partial x_1}(\pm\chi(t),0,0)\Big[\int_0^t f_1(\pm\chi(s),0,0)ds\Big]dt = 0$$
$$K_2^\pm = \int_{-\infty}^{\infty} \pm\psi^*(t)\frac{\partial g}{\partial x_2}(\pm\chi(t),0,0)\Big[\int_0^t f_2(\pm\chi(s),0,0)ds\Big]dt = \frac{\pm a^{iv}\gamma}{\sqrt{2}}(\frac{8}{3} - \pi)\,.$$

We stress that condition (v) is satisfied whenever $X_0 \neq 0$, so it is satisfied for both $\pm\chi$ when $a' \neq -4b''$. Condition (vi) is satisfied whenever $\sigma_1 X_1 + \sigma_2 X_2 \neq \pm\sigma_0 X_0$.

The values of F_i^+, $\tilde{F}_i^+$ and K_2^+ change to F_i^-, $\tilde{F}_i^-$ and K_2^- passing from χ

to $-\chi$, while the other values remain the same. For simplicity from now on we restrict our attention to $+\chi(t)$. So when (v) and (vi) hold, using (3.18), (3.31) and (3.17) we find:

$$
\begin{aligned}
&A_0^{\pm} + A_m = -\frac{K_0 + K_1 + K_2^{+}}{X_0}, \quad B_0 + B_m = -\frac{\sigma_1 X_1 + \sigma_2 X_2}{X_0}, \\
&C_m = \tilde{C}_m = \frac{F_1 X_1 + F_2 X_2}{X_0}.
\end{aligned}
$$

Thus, from (3.21), (3.26), (3.32), (3.34) we get the following:

$$
\frac{\partial \lambda_1^{\pm}}{\partial \varepsilon}(0) = \frac{\partial \lambda_4^{\mp}}{\partial \varepsilon}(0) = \frac{\partial \lambda_2^{\mp}}{\partial \varepsilon}(0) = \frac{\partial \lambda_3^{\pm}}{\partial \varepsilon}(0) = \frac{F_1 X_1 + F_2 X_2 - K_0 - K_2^{+}}{\sigma_1 X_1 + \sigma_2 X_2 \pm \sigma_0 X_0}. \tag{4.3}
$$

So we can draw explicitly the bifurcation diagram of (1.1), and our description is accurate at least at the first order. Furthermore from Remark 5.3, we can say if $(\breve{x}(t,\varepsilon,\lambda), \breve{y}(t,\varepsilon,\lambda))$ satisfies (3.36) (and in such a case it is uniquely defined and it is the unique trajectory satisfying (3.36)) and we can specify if it is a heteroclinic or a homoclinic, or if $(\breve{x}(t,\varepsilon,\lambda), \breve{y}(t,\varepsilon,\lambda))$ leaves Ω_h and so it does not satisfy (3.36). In this latter case we may have many trajectories $(\breve{x}(t,\varepsilon,\lambda), \breve{y}(t,\varepsilon,\lambda))$ with the same behavior.

If we replace f_0 in (4.1) by

$$
f_0(x,y,\varepsilon,\lambda) := x_0^2 - (\sigma_0 \lambda) + \alpha y_1 y_2 + \omega_{sn}(x,y,\varepsilon,\lambda)
$$

where $\omega_{sn}(x,y,\varepsilon,\lambda) = O(|y||x|) + o(\varepsilon^2 + \lambda + |x_0|^2 + |x_{\hat{0}}|)$ and $\sigma_0 > 0$, we get

$$
f_0(x, v(x,\varepsilon,\lambda), \varepsilon, \lambda) = x_0^2 - (\sigma_0 \lambda) + o(\varepsilon^2 + \lambda + |x_0|^2 + |x_{\hat{0}}|),
$$

so we have a saddle-node bifurcation. Once again condition (v) is satisfied whenever $X_0 \neq 0$, and using (3.37) we find

$$
\frac{\partial \nu_1^{\pm}}{\partial \varepsilon}(0) = \mp \frac{F_1 X_1 + F_2 X_2 - K_0 - K_2^{+}}{\sqrt{\sigma_0} X_0} = \frac{\partial \nu_2^{\mp}}{\partial \varepsilon}(0) \tag{4.4}
$$

So we can draw the bifurcation diagram of (1.1), also in this case.

If we replace f_0 in (4.1) by

$$
f_0(x,y,\varepsilon,\lambda) := (x_0 - \tilde{\sigma}_0 \lambda)(x_0^2 - \sigma_0 \lambda) + \alpha y_1 y_2 + \omega_p(x,y,\varepsilon,\lambda)
$$

where $\omega_p(x,y,\varepsilon,\lambda) = o(|y||x| + \varepsilon^2 + \lambda^2 + |x_0|^3 + |x_{\hat{0}}|^2)$ and $\sigma_0 > 0$, we get

$$
f_0(x, v(x,\varepsilon,\lambda), \varepsilon, \lambda) = (x_0 - \tilde{\sigma}_0 \lambda)(x_0^2 - \sigma_0 \lambda) + o(\varepsilon^2 + \lambda^2 + |x_0|^3 + |x_{\hat{0}}|^2),
$$

so we have a pitchfork bifurcation. We find $\frac{\partial \nu_1^{1,u}}{\partial \varepsilon}(0) = \frac{\partial \nu_2^{2,u}}{\partial \varepsilon}(0) = -\frac{\partial \nu_1^{2,u}}{\partial \varepsilon}(0) = -\frac{\partial \nu_2^{1,u}}{\partial \varepsilon}(0)$

$$\frac{\partial \nu_1^{1,u}}{\partial \varepsilon}(0) = -\frac{K_0 + K_2^+ + \alpha X_0 - F_1 X_1 - F_2 X_2}{\sqrt{\sigma_0} X_0}$$

$$\frac{\partial \lambda_i^s}{\partial \varepsilon}(0) = -\frac{K_0 + K_2^+ - F_1 X_1 - F_2 X_2}{\sigma_1 X_1 + \sigma_2 X_2 + \tilde{\sigma}_0 X_0} \qquad \text{for } i = 1, 2, 3, 4$$

Thus we obtain the bifurcation diagram of (1.1) in this case, too.

5 Appendix: lack of uniqueness problem.

In this appendix we want to discuss lack of uniqueness problems, depending on the use of centre manifold theory; in particular we want to show how we can recover some uniqueness result even if we have many centre manifolds. In Theorem 3.1, we have selected a trajectory $(\breve{x}(t, \varepsilon, \lambda), \breve{y}(t, \varepsilon, \lambda))$ defined in (3.14), homoclinic to the centre manifold within the slow manifold, denoted by $\mathcal{M}^c(C(\varepsilon, \lambda))$. Our construction faces two lack of uniqueness problems: the slow manifold $\mathcal{M}^c$ of (1.1) may be not unique, and, even when $\mathcal{M}^c$ is unique, the centre manifold within the slow manifold, i.e. $\mathcal{M}^c(C(\varepsilon, \lambda))$ may be not unique. However in any case bounded trajectories of (1.6) are contained in all the possible choices of the centre manifold $C(\varepsilon, \lambda)$, see section 5 in [13]. From this fact we will get that, if $(\breve{x}(t, \varepsilon, \lambda), \breve{y}(t, \varepsilon, \lambda))$ is bounded (a homoclinic or a heteroclinic trajectory), then it is unique even if the centre manifold $C(\varepsilon, \lambda)$ and the slow manifold $\mathcal{M}^c$ are not. While if $(\breve{x}(t, \varepsilon, \lambda), \breve{y}(t, \varepsilon, \lambda))$ leaves Ω_h, such a trajectory will depend both in the choice of the of the slow manifold $\mathcal{M}^c$ and on the choice of the centre manifold $C(\varepsilon, \lambda)$ within $\mathcal{M}^c$. However in this latter case all the trajectories $(\breve{x}(t, \varepsilon, \lambda), \breve{y}(t, \varepsilon, \lambda))$ will have the same behavior due to smooth conjugation between centre manifolds, see Theorem 5.4 in [13].

Let us assume that $\mathcal{M}^c$ is fixed: we recall some known facts related to the centre manifold $C(\varepsilon, \lambda)$, see section 5 in [13] for more details. If **h)** holds, then (1.4) admits a one-parameter family of centre-manifolds $C_i(\varepsilon, \lambda)$, $i \in I$, which may coincide (recovering uniqueness). We can flatten one of them, say $C_1(\varepsilon, \lambda)$, and pass to a system of the form (1.6). These centre manifolds continue to exist as long as (at least) one critical point persists. Assume this is the case and that there are two of such manifolds $C_1(\varepsilon, \lambda) \neq C_2(\varepsilon, \lambda)$ and $C_1(0, 0)$ is the x_0 axis: they both contain all the bounded trajectories of (1.6), in particular the critical points and their heteroclinic connections, if any. Moreover the dynamics in $C_1(\varepsilon, \lambda)$ and $C_2(\varepsilon, \lambda)$ are C^{r-2} conjugate, see

Theorem 5.4 in [13]. Assume to fix the ideas that f_0 is either as in (1.7) or as in (1.8) and $\varepsilon, \lambda > 0$: $s(\varepsilon, \lambda)$ and $u(\varepsilon, \lambda)$ exist, belong to $C_i(\varepsilon, \lambda)$ for any i and are respectively stable and unstable for the restriction of (1.6) to $C_i(\varepsilon, \lambda)$ for any i. The $l+1$ and a l dimensional unstable manifolds, $W^u(u(\varepsilon, \lambda))$ and $W^u(s(\varepsilon, \lambda))$ are uniquely defined, as well as the $m-l$ and $m-l+1$ dimensional stable manifolds $W^s(u(\varepsilon, \lambda))$ and $W^s(s(\varepsilon, \lambda))$. Moreover we can define a not unique $l+1$ dimensional centre-unstable manifold $W^u(C_i(\varepsilon, \lambda))$ of $s(\varepsilon, \lambda)$, and a not unique $m-l+1$ dimensional centre-stable manifold $W^s(C_i(\varepsilon, \lambda))$ of $u(\varepsilon, \lambda)$ with the following properties: they are respectively negatively and positively invariant for the flow of (1.6) and

$$\begin{array}{ll} \text{if } \xi \in \Omega_x \cap W^u(C_i(\varepsilon, \lambda))\,, & \text{then } \lim_{t\to-\infty} |x_c(t, \xi, \varepsilon, \lambda) - s(\varepsilon, \lambda)| \mathrm{e}^{\Lambda^f t} = 0 \\ \text{if } \xi \in \Omega_x \cap W^s(C_i(\varepsilon, \lambda))\,, & \text{then } \lim_{t\to\infty} |x_c(t, \xi, \varepsilon, \lambda) - u(\varepsilon, \lambda)| \mathrm{e}^{-\Lambda^f t} = 0 \end{array}$$

for any $i \in I$, where Λ^f is defined in **h)**; see section 5 of [13] for the construction. In fact we may replace $\lim_{t\to-\infty} |x_c(t, \xi, \varepsilon, \lambda) - s(\varepsilon, \lambda)| \mathrm{e}^{\Lambda^f t}$ simply by $\lim_{t\to-\infty} |x_c(t, \xi, \varepsilon, \lambda)| \mathrm{e}^{\Lambda^f t}$ in the definition of $W^u(C_i(\varepsilon, \lambda))$ and similarly for $W^s(C_i(\varepsilon, \lambda))$. By definition each centre manifold $C_i(\varepsilon, \lambda)$ is obtained as intersection between centre-unstable manifolds $W^u(C_i(\varepsilon, \lambda))$ and centre-stable manifolds $W^s(C_i(\varepsilon, \lambda))$ (in [13] they are actually constructed as such intersections).

The (uniquely defined) l dimensional manifold $W^u(s(\varepsilon, \lambda)) \cap \Omega_x$ is contained in each $l+1$ dimensional manifold $W^u_i(C(\varepsilon, \lambda))$ for any $i \in I$, and divides it into two open components. One component is $W^u(u(\varepsilon, \lambda)) \cap \Omega_x$ and it is in common to any $W^u_i(C(\varepsilon, \lambda))$ for $i \in I$, so it is uniquely defined; then we have a further component, $W^{u,n}_i(C(\varepsilon, \lambda))$ which is made up by trajectories which leaves Ω_0 for $t < 0$, and changes as i takes values in I (however it is always $l+1$ dimensional). Similarly any manifold $W^s_i(C(\varepsilon, \lambda))$ in $W^s(C(\varepsilon, \lambda))$ is made up by a common part uniquely defined, i.e. $[W^s(s(\varepsilon, \lambda)) \cup W^s(u(\varepsilon, \lambda))] \cap \Omega_x$, and a further not uniquely defined part, say $W^{s,n}_i(C(\varepsilon, \lambda))$ (see the beginning of section 3). Hence each $C_i(\varepsilon, \lambda)$ has a common part uniquely defined, say $C^0(\varepsilon, \lambda)$, and two not uniquely defined parts, say $C^-_i(\varepsilon, \lambda)$ and $C^+_i(\varepsilon, \lambda)$. $C^0(\varepsilon, \lambda)$ is $\{u(\varepsilon, \lambda), s(\varepsilon, \lambda)\} \cup [W^u(u(\varepsilon, \lambda)) \cap W^s(s(\varepsilon, \lambda))]$, which are respectively two critical points and their heteroclinic connection; $C^-_i(\varepsilon, \lambda)$ (respectively $C^+_i(\varepsilon, \lambda)$) is the graph of a trajectory that leaves Ω_x for $t = -T < 0$, (respectively for $t = T > 0$), which may be different for $i \in I$ (recall that bounded trajectories belong to all the centre manifolds, see [13]).

Now we proceed to examine when we have effectively non-uniqueness and when uniqueness is recovered in the arguments of section 3 of this paper.

5.1 Remark. The functions $h_{0,a}$ (and $h_{0,b}$) defined in section 3, depend on the choice of $W^u(C_i(\varepsilon, \lambda))$ (respectively the choice of $W^s(C(\varepsilon, \lambda))$), while h_a and

h_b are uniquely defined. However all the centre-unstable and centre-stable manifolds have the same Taylor expansions, so the derivatives evaluated in (3.4), (3.5), (3.6) are uniquely defined.

5.2 Remark. The function $J_1^+(\varepsilon, \lambda)$ in (3.19) is uniquely defined. In fact, even if $h_{(0,b)}$ is not unique $W^s(u(\varepsilon, \lambda))$ and consequently h_b are uniquely defined. Similarly $J_1^-(\varepsilon, \lambda)$ is uniquely defined. Analogously we see that the functions $J_i^{\pm}(\varepsilon, \lambda)$ defined in (3.24), (3.29), (3.33), and their analogous of section 3.2 and 3.3 do not depend on the choice of the centre manifold. It follows that, even if $\breve{\xi}(\varepsilon, \lambda)$ may depend on the choice of the centre manifold, the functions $\lambda_i^{\pm}(\varepsilon)$ defined for $i = 1, 2, 3, 4$, are independent from such a choice in the (1.7) case; the analogous result applies in the (1.8) and (1.9) case.

5.3 Remark. From Theorem 3.1 we know that, for each centre manifold $\mathcal{M}^c(C(\varepsilon, \lambda))$, there is exactly one trajectory $(\breve{x}(t, \varepsilon, \lambda), \breve{y}(t, \varepsilon, \lambda))$ homoclinic to it. However each centre manifold $C_i(\varepsilon, \lambda)$ has a common part, $C^0(\varepsilon, \lambda)$: if for $i = 1$, $\breve{\xi}(\varepsilon, \lambda) \in C^0(\varepsilon, \lambda)$ then we have the same trajectory $(\breve{x}(t, \varepsilon, \lambda), \breve{y}(t, \varepsilon, \lambda))$ for all the centre manifolds $C_i(\varepsilon, \lambda)$, and for all the slow manifolds $\mathcal{M}^c$ (again since bounded trajectories belong to all the centre manifolds and to all the slow manifolds); so in this case $(\breve{x}(t, \varepsilon, \lambda), \breve{y}(t, \varepsilon, \lambda))$ is really unique. If we consider the example in figure 1, such a fact happens if (ε, λ) are in the subsets **I**, **II**, **VIII**, **IX**, **XI**, **XVII**, **XVIII**. In this cases $(\breve{x}(t, \varepsilon, \lambda), \breve{y}(t, \varepsilon, \lambda))$ is bounded, and it is the unique trajectory of (1.1) which satisfies (3.36)

If for $i = 1$, $\breve{\xi}(\varepsilon, \lambda) \in C_1^{\pm}(\varepsilon, \lambda)$, then $(\breve{x}(t, \varepsilon, \lambda), \breve{y}(t, \varepsilon, \lambda))$ leaves Ω_h and there are no trajectories satisfying (3.36). Moreover we have as many $(\breve{x}(t, \varepsilon, \lambda), \breve{y}(t, \varepsilon, \lambda))$ as the number of centre-manifolds within the slow manifold $\mathcal{M}^c(C(\varepsilon, \lambda))$, so we may lose uniqueness (in the example of figure 1 this happens when (ε, λ) is in the subsets **III**, **IV**, **V**, **VI**, **VII**, **X**, **XII**, **XIII**, **XIV**, **XV**, **XVI**). However all these trajectories have the same behavior (the one sketched in figure 1).

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