

Perturbative approach to Quantum BGK Modes

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Abstract

First-order quantum corrections to the classical Bernstein-Greene-Kruskal equilibria with zero phase velocity are considered. The Wigner-Poisson system with periodic boundary conditions is solved by using a perturbative approach about the classical solution. The numerical results indicate that the most important quantum corrections to the classical BGK solution occur near the phase-space separatrices between classically trapped and untrapped particles.

1 Classical and quantum BGK modes

In this contribution, we study the quantum analogue of the classical steady-state solutions of the Vlasov-Poisson system (BGK modes), introduced by Bernstein, Greene and Kruskal in 1957 [1] and describing the spatially inhomogeneous equilibria of a collisionless unmagnetized plasma with immobile ions. The classical system is characterized by the electron distribution function $f(x, p)$ and by the self-consistent electric field $E(x)$, which are the unknown quantities and obey the nonlinear, stationary, one-dimensional Vlasov-Poisson system [2]

$$p \frac{\partial f}{\partial x} - E \frac{\partial f}{\partial p} = 0 \quad (1)$$

$$\frac{\partial E}{\partial x} = 1 - \int f dp, \quad (2)$$

with x and p the space and momentum variables. In equations (1)-(2) above, we have used the momentum variable p instead of the usual velocity variable v , since the quantum formulation uses p as a phase-space variable together with x . In the classical case, we simply have $p = mv$. Here, all quantities are dimensionless; x is measured in units of the Debye length $\lambda_D = \sqrt{T/(4\pi ne^2)}$, p in units of mv_{th} , with $v_{th} = \sqrt{T/m}$ the thermal speed, f in units of $n/(mv_{th})$ and E in units of $4\pi ne\lambda_D$ (m is the electron mass, e the electron charge, T the plasma temperature and n the plasma density). Furthermore, we impose periodic boundary conditions in space, of the type $f(0, p) = f(L, p)$. However, instead of prescribing the period L at the outset, the distribution function $f(x, p)$ is assigned at $x = 0$ and the period L is then determined from the equations. We recall that, due to the Galilean invariance of the equations, any solution $(f(x, v), E(x))$ of (1)-(2) can be transformed into a solution of the time-dependent Vlasov-Poisson system with an arbitrary phase velocity V . These are BGK modes with non-zero phase velocity. Here, however, we consider only steady-state solutions, that is BGK modes with zero phase speed. The solutions of (1)-(2) can be characterized in the following way.

First, note that the single-particle energy $\mathcal{E} \equiv p^2/2 - \phi(x)$ is a constant of motion, therefore $f(x, p) = F(\mathcal{E})$ satisfies the Vlasov equation identically. The electrostatic potential $\phi(x)$ partitions the electron population into three groups of particles: trapped particles, untrapped particles with positive momentum and untrapped particles with negative momentum. The curves that divide the trapped and untrapped populations of the phase space are separatrices of a given energy ϵ_s . Then, one can think of a problem with four components: the three groups of particles and the electrostatic potential. In general, three of these components can be fixed with some arbitrariness and the fourth one is then found by solving the equations. In our approach, we assign the distribution function over the entire phase space; Poisson's equation then becomes a differential equation for the electrostatic potential $\phi(x)$. The distribution function is assigned at one boundary, $x = 0$, by $f(0, p) = m(p)(1 + \alpha)$, for a given function $m(p)$ and α arbitrary. In general, ϵ_s and α are small positive numbers. As they vary, a family of BGK equilibria is obtained, which we identify as BGK equilibria generated by the function $m(p)$. The electrostatic potential is subject to the boundary conditions $\phi(0) = -\epsilon_s$ and $\phi'(0) = 0$. These boundary conditions and the numerical generation of BGK modes are discussed in more detail in [3]. If $\phi(x)$ is the unknown electrostatic potential, the equations of the separatrices are given by

$$p(x) = \pm\sqrt{2(\phi(x) + \epsilon_s)} = \pm p_0(x).$$

Then for $-p_0(x) < p < p_0(x)$ (trapped particles) the distribution is left

arbitrary, $f(x, p) = F_t(\mathcal{E})$ and for $p < -p_0(x)$ or $p > p_0(x)$ (untrapped particles) we take $f(x, p) = f(0, \bar{p})$, where \bar{p} is the momentum obtained by considering the constant energy curve passing through the point (x, p) of the phase space and tracing it back to $x = 0$. The value of \bar{p} is given by $p^2/2 - \phi(x) = \bar{p}^2/2 + \epsilon_s$. Summarizing, $f(x, p) = F(\mathcal{E})$ where

$$F(\mathcal{E}) = \begin{cases} F_-(\mathcal{E}) = f(0, -\sqrt{2(\mathcal{E} - \epsilon_s)}), & \mathcal{E} > \epsilon_s \text{ and } p < -p_0(x) \\ F_t(\mathcal{E}), & -\phi_{MAX} < \mathcal{E} < \epsilon_s \\ F_+(\mathcal{E}) = f(x, \sqrt{2(\mathcal{E} - \epsilon_s)}), & \mathcal{E} > \epsilon_s \text{ and } p > p_0(x) \end{cases} \quad (3)$$

where ϕ_{MAX} is the maximum value of the potential. The distribution function thus constructed is a functional of the electrostatic potential $\phi(x)$. Therefore, Poisson's equation can be written as:

$$-\frac{d^2\phi}{dx^2} = 1 - \mathcal{R}(\phi) \quad (4)$$

with $\mathcal{R}(\phi) = \int f(x, p)dp = \int ||J||_\phi F(\mathcal{E})d\mathcal{E}$, where $||J||_\phi$ is the Jacobian of the transformation from the momentum variable to the energy variable for fixed x . For an arbitrary function $w(x, p) = W(\mathcal{E})$, we have

$$\begin{aligned} \int_{-\infty}^{+\infty} w(x, p)dp &= \int ||J||_\phi W(\mathcal{E})d\mathcal{E} = \int_{\epsilon_s}^{\infty} \frac{(W_-(\mathcal{E}) + W_+(\mathcal{E}))d\mathcal{E}}{\sqrt{2(\mathcal{E} + \phi(x))}} + \\ &+ 2 \int_{-\phi(x)}^{\epsilon_s} \frac{W_t(\mathcal{E})d\mathcal{E}}{\sqrt{2(\mathcal{E} + \phi(x))}} \end{aligned} \quad (5)$$

where W_+ , W_- and W_t correspond to the untrapped (with $p > 0$ and $p < 0$) and trapped particles, respectively. Then,

$$\mathcal{R}(\phi) = \int_{\epsilon_s}^{\infty} \frac{(F_-(\mathcal{E}) + F_+(\mathcal{E}))d\mathcal{E}}{\sqrt{2(\mathcal{E} - \phi)}} + 2 \int_{-\phi(x)}^{\epsilon_s} \frac{F_t(\mathcal{E})d\mathcal{E}}{\sqrt{2(\mathcal{E} - \phi)}}. \quad (6)$$

With this expression, eq. (4) becomes a second order differential equation for the potential $\phi(x)$, equipped with the boundary conditions $\phi(0) = -\epsilon_s$ and $\phi'(0) = 0$. In our numerical approach, the integrals appearing in (6) are calculated by standard Laguerre and Chebychev collocation rules and the differential equation for the potential is solved by using a standard marching technique from the boundary at $x = 0$. The periodicity length L is then extracted from the solution of the differential equation by using the condition $\phi(L) = \phi(0)$. In our former calculations of classical BGK modes [3], the computed period was consistent with the periodicity length determined by using

the potential theory for BGK modes (see [1] and [4]). The amplitude of the potential is governed by the parameter α . For $\alpha = 0$ we have $f(0, p) = m(p)$ and the only solution is the one with zero field and a uniform distribution. As α becomes positive, a BGK mode with nonzero field amplitude is generated.

The quantum version of the Vlasov-Poisson system is the Wigner-Poisson system [5, 6], in which the Wigner distribution f_W takes the place of the classical distribution function and obeys an evolution equation where the acceleration term is given by a pseudo-differential operator, in place of the usual differential operator of the Boltzmann equation. The steady-state Wigner-Poisson system is:

$$p \frac{\partial f_W}{\partial x} - \frac{i}{\tilde{\epsilon}} \Theta(f_W) = 0 \quad (7)$$

$$\frac{\partial E}{\partial x} = 1 - \int f_W dp, \quad (8)$$

where $f_W(x, p)$ is the Wigner function, $\tilde{\epsilon} = \hbar / (m v_{th} \lambda_D)$ and the pseudodifferential operator Θ is defined by

$$\Theta(f_W)(x, p) = \int d\eta \delta\phi(x, \eta) \hat{f}_W(x, \eta) e^{ip\eta/\tilde{\epsilon}}, \quad (9)$$

with

$$\hat{f}_W(x, \eta) = \frac{1}{2\pi\tilde{\epsilon}} \int dp f_W(x, p) e^{ip\eta/\tilde{\epsilon}} \quad (10)$$

the Fourier transform of the Wigner function and $\delta\phi(x, \eta) = \phi(x + \eta/2) - \phi(x - \eta/2)$ the symbol of the pseudodifferential operator. On the system (7)-(8) we impose the same boundary conditions used for the classical Vlasov-Poisson system (1)-(2), that is $f_0(0, p) = m(p)(1 + \alpha)$, $\phi_0(0) = -\epsilon_s$ and $\phi'(0) = 0$. The steady-state Wigner-Poisson system (7)-(8) admits spatially periodic solutions, as was proved by Lange, Toomire and Zweifel [7]. Furthermore, it is well known that the Wigner-Poisson system reduces to the classical Vlasov-Poisson system as $\tilde{\epsilon} \rightarrow 0$ [6, 8]. It is therefore natural to look for small quantum corrections to the classical BGK equilibria by using perturbation methods. In this work, we shall develop only a first-order theory and produce numerical results showing the quantum corrections to the classical BGK equilibria generated by a two-stream distribution.

2 First-Order perturbation theory and quantum correction

As a first step, we consider the asymptotic expansion of the pseudodifferential operator in powers of $\tilde{\epsilon}$. It can be shown that

$$-\frac{i}{\tilde{\epsilon}}\Theta(f_W)(x, p) = -E(x)\frac{\partial f_W}{\partial p} + \frac{\tilde{\epsilon}^2}{24}\frac{\partial^2 E}{\partial x^2}\frac{\partial^3 f_W}{\partial p^3} + \mathcal{O}(\tilde{\epsilon}^4). \quad (11)$$

By setting $\epsilon = \tilde{\epsilon}^2/24$, the steady-state Wigner-Poisson system (7)-(8) becomes:

$$p\frac{\partial f_W}{\partial x} - E(x)\frac{\partial f_W}{\partial p} + \epsilon\frac{\partial^2 E}{\partial x^2}\frac{\partial^3 f_W}{\partial p^3} + \mathcal{O}(\epsilon^2) = 0 \quad (12)$$

$$\frac{\partial E}{\partial x} = 1 - \int f_W dp. \quad (13)$$

We now introduce a perturbative expansion of f_W and ϕ in powers of ϵ :

$$f_W(x, p) = f_0(x, p) + \epsilon f_1(x, p) + \mathcal{O}(\epsilon^2) \quad (14)$$

$$\phi(x) = \phi_0(x) + \epsilon \phi_1(x) + \mathcal{O}(\epsilon^2). \quad (15)$$

The expansion of the potential (15) induces an analogous expansion of $E(x) = -\phi'(x)$: $E(x) = E_0(x) + \epsilon E_1(x) + \mathcal{O}(\epsilon^2)$, where $E_n(x) = -\phi'_n(x)$, $n = 0, 1, \dots$. To order ϵ^0 we have:

$$p\frac{\partial f_0}{\partial x} - E_0\frac{\partial f_0}{\partial p} = 0 \quad (16)$$

$$\frac{\partial E_0}{\partial x} = 1 - \int f_0 dp \quad (17)$$

which is the classical steady-state Vlasov-Poisson system. If we impose the boundary conditions $f_0(0, p) = m(p)(1 + \alpha)$, $\phi_0(0) = -\epsilon_s$ and $\phi'_0(0) = 0$, the solution to order ϵ^0 is simply the family of classical BGK equilibria generated by $m(p)$, that is $f_0(x, p) = F(\mathcal{E})$, with $\mathcal{E} \equiv p^2/2 - \phi_0(x)$ and $F(\mathcal{E})$ given by (3). To order ϵ we have:

$$p\frac{\partial f_1}{\partial x} - E_0\frac{\partial f_1}{\partial p} - E_1\frac{\partial f_0}{\partial p} + \frac{\partial^2 E_0}{\partial x^2}\frac{\partial^3 f_0}{\partial p^3} = 0 \quad (18)$$

$$\frac{\partial E_1}{\partial x} = - \int f_1 dp, \quad (19)$$

subject to the boundary conditions $f_1(0, p) = 0$, $\phi_1(0) = 0$ and $\phi'_1(0) = 0$. In order to solve equations (18)-(19), it is convenient to adopt the transformation from the variables x and p to the variables x and \mathcal{E} . While the lowest

order Wigner distribution is a function of \mathcal{E} alone (that is $f_0(x, p) = F(\mathcal{E})$), the first order correction gives the variation of the Wigner function along the curves of constant \mathcal{E} , and so $f_1(x, p) = F_1(x, \mathcal{E})$. The partial derivatives of f_0 and f_1 in the new variables x and \mathcal{E} then become:

$$\begin{aligned}\frac{\partial f_0}{\partial p} &= p(x)F'(\mathcal{E}) \\ \frac{\partial^3 f_0}{\partial p^3} &= p(x) \left[p(x)^2 F'''(\mathcal{E}) + 3F''(\mathcal{E}) \right] \\ \frac{\partial f_1}{\partial x} &= \frac{\partial F_1}{\partial x} - \phi_0(x) \frac{\partial F_1}{\partial \mathcal{E}} \\ \frac{\partial f_1}{\partial p} &= p(x) \frac{\partial F_1}{\partial \mathcal{E}},\end{aligned}$$

where $p(x) = \pm \sqrt{2[\mathcal{E} + \phi_0(x)]}$. The differential equation (18) for f_1 then becomes

$$p(x) \left[\frac{\partial F_1}{\partial x} - E_1 F'(\mathcal{E}) + \frac{\partial^2 E_0}{\partial x^2} \left(p(x)^2 F'''(\mathcal{E}) + 3F''(\mathcal{E}) \right) \right] = 0, \quad (20)$$

whose solution is

$$\begin{aligned}F_1(x, \mathcal{E}) &= C(\mathcal{E}) - F'(\mathcal{E})\phi_1(x) + 3\phi_0''(x)F''(\mathcal{E}) + \\ &- \left[\phi_0'(x)^2 - 2\phi_0''(x)(\mathcal{E} + \phi_0(x)) \right] F'''(\mathcal{E}).\end{aligned}$$

The integration constant $C(\mathcal{E})$ can be determined from the boundary conditions. Since $f_1(0, p) = 0$ we have $F_1(0, \mathcal{E}) = 0$, and by using that $\phi_0''(0) = -\alpha$ and $\phi_1(0) = 0$, we obtain

$$C(\mathcal{E}) = -\alpha [3F''(\mathcal{E}) + 2(\mathcal{E} - \epsilon_s)F'''(\mathcal{E})]$$

and thus

$$\begin{aligned}F_1(x, \mathcal{E}) &= -F'(\mathcal{E})\phi_1(x) + [3(\phi_0''(x) - \alpha)] F''(\mathcal{E}) + \\ &+ 2 \left[\mathcal{E}(\phi_0''(x) - \alpha) + \phi_0''(x)\phi_0(x) + \alpha\epsilon_s - \frac{\phi_0'(x)^2}{2} \right] F'''(\mathcal{E}).\end{aligned} \quad (21)$$

The function ϕ_1 is still unknown and must be obtained from equation (19), which involves an integration of f_1 over the momentum space. By using (5) to perform the integration in the variable \mathcal{E} , equation (19) becomes:

$$\frac{d^2 \phi_1}{dx^2}(x) + \kappa^2(x)\phi_1(x) = g(x), \quad (22)$$

where $\kappa(x)$ and $g(x)$ are periodic functions of period L (the period of the zero order solution) and are given by

$$\begin{aligned}\kappa^2(x) &= \int \|J\|_{\phi_0} F'(\mathcal{E}) d\mathcal{E} \\ g(x) &= [\phi_0''(x) - \alpha][3\delta(x) + 2\beta(x)] + 2 \left[\phi_0''(x)\phi_0(x) + \alpha\epsilon_s - \frac{\phi_0'(x)^2}{2} \right] \gamma(x),\end{aligned}$$

where

$$\begin{aligned}\delta(x) &= \int \|J\|_{\phi_0} F''(\mathcal{E}) d\mathcal{E} \\ \beta(x) &= \int \|J\|_{\phi_0} \mathcal{E} F'''(\mathcal{E}) d\mathcal{E} \\ \gamma(x) &= \int \|J\|_{\phi_0} F''''(\mathcal{E}) d\mathcal{E}\end{aligned}$$

are periodic functions of period L . We shall choose the boundary conditions, and thus the zero order BGK equilibrium, so that $\kappa^2(x) > 0$ and therefore $\kappa(x)$ is real (and chosen positive). Equation (22) for ϕ_1 has to be solved subject to the boundary conditions $\phi_1(0) = \phi_1'(0) = 0$. To obtain the numerical solution of equation (22), we use the same techniques used for the classical BGK modes. Substituting for ϕ_1 in (21) gives the first-order correction f_1 to the Wigner function.

3 Numerical results

In this Section, we consider BGK equilibria generated by a two-stream distribution,

$$m(p) = \frac{1}{2\sqrt{2\pi}} \left[e^{-(p-p_s)^2/2} + e^{-(p+p_s)^2/2} \right] \quad (23)$$

with $p_s = 2$. The trapped particle distribution must be chosen so that the existence conditions for the classical BGK equilibrium that gives the zero-order solution are satisfied [4]. Following our previous calculations of BGK modes [3], the trapped particle distribution of equation (3) is taken constant, that is $F_i(\mathcal{E}) = f(0,0)$. Of course, other choices are possible, but then the existence of the zero-order solution would also have to be addressed.

In Figure 1 we show the classical BGK equilibrium, generated by the function (23) with $\epsilon_s = 0.5$ and $\alpha = 0.05$, in the phase-space region around the separatrices. The period of this solution is $L \approx 17$. Figures 2, 3 and 4 show the first-order quantum solution for $\epsilon = 0.004$. Figure 2 shows the BGK quantum equilibrium Wigner distribution in the phase-space region around

the separatrices, while Figure 3 shows the quantum correction $\phi_1(x)$ to the self-consistent potential over the interval $0 \leq x \leq 2L$. Figure 4 shows the self-consistent Quantum BGK potential $\phi_0(x) + \epsilon\phi_1(x)$ (dashed line) together with the classical BGK potential (solid line), over the interval $0 \leq x \leq 2L$. We first note that ϕ_1 is not periodic; if we look at ϕ_1 over several periods of length L , it shows secular behaviour. Therefore, the straightforward expansion introduced in (14)-(15) is nonuniform, and it is meaningful only over a spatial scale of one period. This suggests that the problem should be studied by other perturbative techniques, such as the multiple-scale analysis, which are better designed to approach the secular behaviour. From these results, we see that the quantum correction to the BGK potential amounts to a decrease of the amplitude and a shortening of the period. We also see that, for small ϵ , the differences between the classical distribution function and the Wigner function as determined by our straightforward perturbative expansion, are most evident near the separatrices.

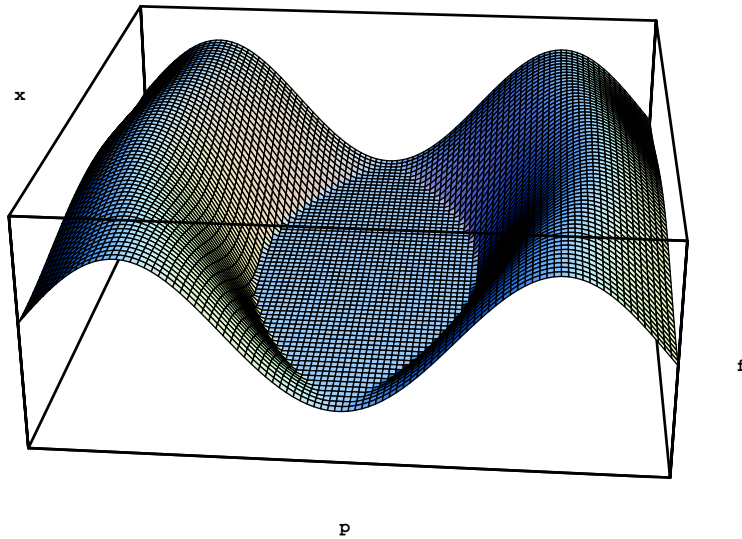


Figure 1: Classical BGK equilibrium distribution $f(x, p)$ as a function of x and p around the trapped particle region. Here, $0 \leq x \leq L$ and $-3 \leq p \leq 3$.

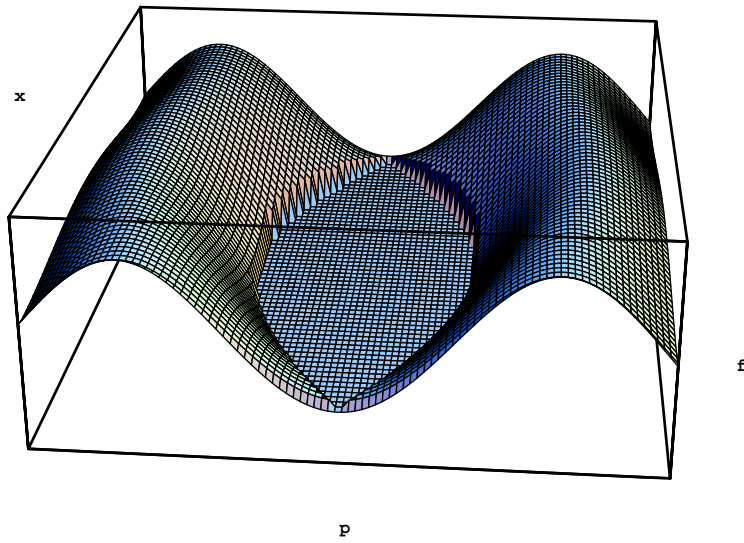


Figure 2: Quantum BGK equilibrium Wigner distribution $f_W(x, p) \approx f_0(x, p) + \epsilon f_1(x, p)$ as a function of x and p around the trapped particle region. Here, $0 \leq x \leq L$ and $-3 \leq p \leq 3$.

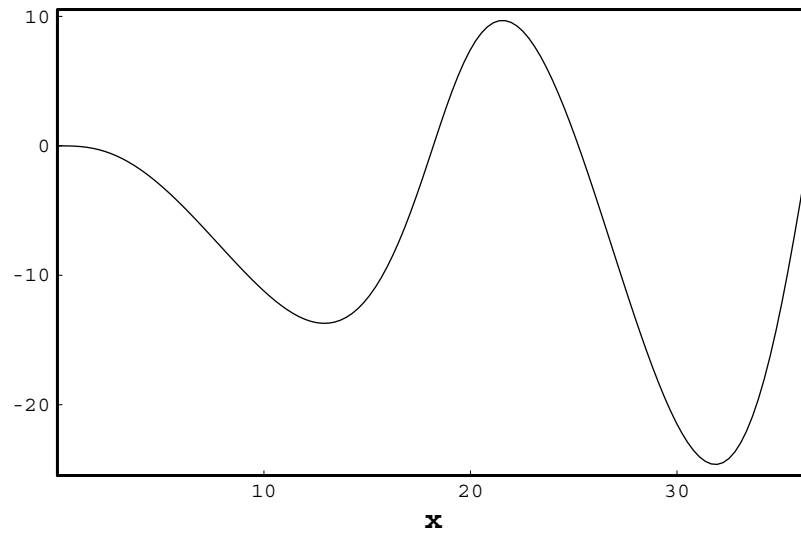


Figure 3: Quantum correction, $\phi_1(x)$, to the classical BGK potential as a function of x , $0 \leq x \leq 2L$.

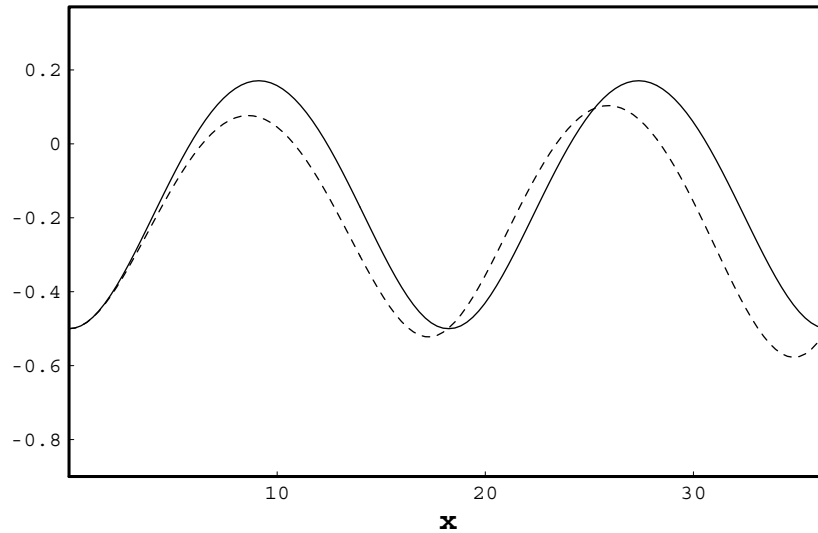


Figure 4: Classical BGK potential $\phi(x)$ (solid line) and quantum BGK potential $\phi_0(x) + \epsilon\phi_1(x)$ (dashed line) as a function of x , $0 \leq x \leq 2L$.

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