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PERIODIC TRAVELING WAVES IN A TAUT CABLE ON A BILINEAR ELASTIC SUBSTRATE

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- some numerical simulations, based on a finite difference method, are performed to confirm the analytical findings;
- the stability of the proposed waves is discussed theoretically and numerically, also by using return maps in phase space.

Literature

Transport equations

- M. A. Johnson, Nonlinear stability of periodic traveling wave solutions of the generalized Korteweg-de Vries equation, SIAM J. Math. Anal., 41 (2009) 1921-1947.
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Model

Governing equation (Klein-Gordon equation)

$$\frac{\partial^2 w}{\partial t^2} - v^2 \frac{\partial^2 w}{\partial x^2} + \gamma(w)\, w = 0$$

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Traveling wave solutions

With $w(x,t) = W(s) = W(x - \hat{c}\, t)$ we have (with $\xi = s/L$ and $c = \hat{c}/v$)

$$(c^2 - 1)\, W''(\xi) + k(W)\, W(\xi) = 0 \quad (c \text{ propagation speed})$$

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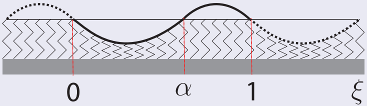
Piecewise constant stiffness

$k(W) = k_1, \quad W \le 0$ (compression)

$k(W) = k_2, \quad W > 0$ (tension)



$$\frac{d^2 W_1}{d\xi^2}(\xi) + a^2 W_1(\xi) = 0, \quad W_1(\xi) \le 0 \qquad a^2 = \frac{k_1}{c^2 - 1}$$

$$\frac{d^2 W_2}{d\xi^2}(\xi) + b^2 W_2(\xi) = 0, \quad W_2(\xi) > 0. \qquad b^2 = \frac{k_2}{c^2 - 1}$$

We are interested in "simple waves", i.e. waves which cross the zero baseline within one wavelength. The problem is to find α and c, and the explicit wave form. Note that if (W_1, W_2) is a solution, so is $A\,(W_1, W_2)$, $\forall A > 0$.

Solution procedure

Matching at endpoints

The solution is obtained by imposing continuity of $W(\xi)$ and $W'(\xi)$ at the boundaries and at the internal point $\xi = \alpha$ (i.e., W is a $C^1([0,1])$ function):

$$W_1(0) = W_1(\alpha) = W_2(\alpha) = W_2(1) = 0$$
$$W_1'(0) = W_2'(1)\,; \quad W_1'(\alpha) = W_2'(\alpha)$$

implying $\alpha\, a = \pi$ and $(1-\alpha)\, b = \pi$.

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Solution

$$\alpha = \frac{\sqrt{k_2}}{\sqrt{k_1}+\sqrt{k_2}} = \frac{1}{1+\sqrt{k_1/k_2}}$$

$$c^2 = 1 + \frac{1}{\pi^2}\,\frac{k_1\,k_2}{(\sqrt{k_1}+\sqrt{k_2})^2}$$

$$W_1(\xi) = -A\sqrt{\frac{k_2}{k_1}}\sin\left(\frac{\xi\pi}{\alpha}\right), \quad 0 \le \xi \le \alpha$$

$$W_2(\xi) = A\,\sin\left(\frac{(\xi-\alpha)\,\pi}{1-\alpha}\right), \quad \alpha \le \xi \le 1$$

For given k_1 and k_2, c and α are uniquely determined by the equations; the amplitude remains undetermined.

Numerical example

Numerical scheme

Numerical simulations were performed by an FFD algorithm, with the boundary condition at $x = 0$ coinciding with the analytical solution and zero initial condition on $\mathbb{R}^+$:

$$\frac{\partial^2 u}{\partial t^2} - v^2 \frac{\partial^2 u}{\partial x^2} + \gamma(u)\, u = 0, \quad x \geq 0,$$

$$u(0,t) = \varphi(t) = w(0,t), \quad w(x,0) = 0, \qquad \frac{\partial w}{\partial t}(x,0) = 0$$

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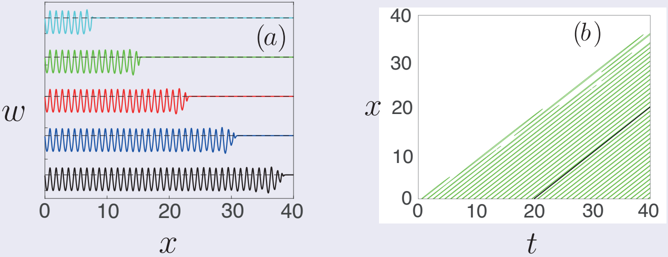
Progression of the wave train in time and as countour plot ($k_1 = 1$, $k_2 = 5$)



Particular case

Unilateral substrate

When $k_2 \to 0$, the substrate becomes unilateral. In this limit $\alpha \to 0$, $c \to 1$ and $W_1(\xi) \to 0$, which means that the wave propagates with the same speed as in the absence of the substrate and the compression region reduces to one point ($\xi = 0$), that is the solution remains in the tension part and the derivative has a jump.

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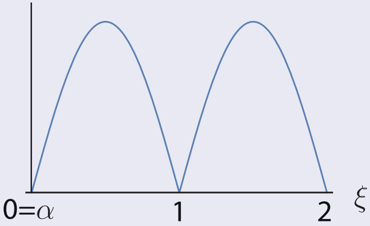

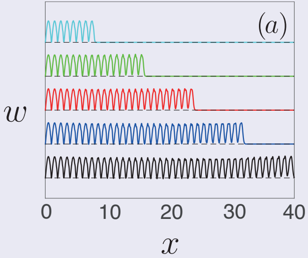

- The conclusion is that periodic waves with regular profile on a perfectly unilateral substrate, crossing the region $w < 0$ (where $k_1 > 0$) do not exist.
- The same conclusion holds when $k_2 \to \infty$ (with the sign reversed - unilaterally rigid substrate)

Stability of traveling waves

General considerations

- The system of equations can be interpreted as a 2D map from $(W(0), W'(0)) = (W_0, W'_0)$ at $\xi = 0$ to $(W(1), W'(1)) = (W_1, W'_1)$ at $\xi = 1$;

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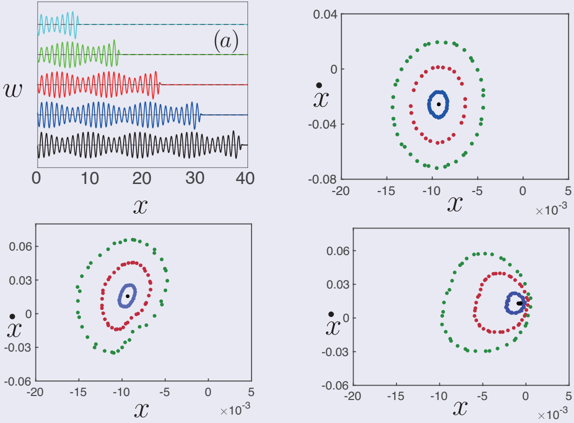
Return map

$(f(t), \dot{f}(t)) \to (f(t+\tau), \dot{f}(t+\tau))$ where $f(t) = w(x_0, t)$ with $x_0 > 0$ a point on the simulation domain and τ the period. The return maps are shown in the next figures for $k_1 = 1, k_2 = 1$, $k_1 = 1, k_2 = 5$ and $k_1 = 0.001, k_2 = 1$.

Stability of traveling waves

Simulation results



Here $k_1 = k_2 = 1$, $k_1 = 1$, $k_2 = 5$, $k_1 = 0.001$, $k_2 = 1$,
$\varepsilon = 0$ (black dot), 0.001 (blue dots), 0.003 (red dots) and 0.005 (green dots).

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- the numerical simulations, based on a finite difference method, confirm the analytical findings;
- the waves' stability is discussed theoretically and numerically, also by using return maps in phase space; we found that the fixed point of the return map is a center, implying stability of the periodic traveling waves.

Farewell

THANK YOU