Linear stability analysis for traveling waves of second order in time PDE'S

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LINEAR STABILITY ANALYSIS FOR TRAVELING WAVES OF SECOND ORDER IN TIME PDE'S

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ABSTRACT. We study traveling waves φ_c of second order in time PDE's $u_{tt} + \mathcal{L}u + N(u) = 0$. The linear stability analysis for these models is reduced to the question for stability of quadratic pencils in the form $\lambda^2 Id + 2c\lambda\partial_x + \mathcal{H}_c$, where $\mathcal{H}_c = c^2\partial_{xx} + \mathcal{L} + N'(\varphi_c)$.

If $\mathcal{H}_c$ is a self-adjoint operator, with a simple negative eigenvalue and a simple eigenvalue at zero, then we completely characterize the linear stability of φ_c. More precisely, we introduce an explicitly computable index $\omega^*(\mathcal{H}_c) \in (0, \infty]$, so that the wave φ_c is stable if and only if $|c| \geq \omega^*(\mathcal{H}_c)$. The results are applicable both in the periodic case and in the whole line case.

The method of proof involves a delicate analysis of a function $\mathcal{G}$, associated with $\mathcal{H}$, whose positive zeros are exactly the positive (unstable) eigenvalues of the pencil $\lambda^2 Id + 2c\lambda\partial_x + \mathcal{H}$. We would like to emphasize that the function $\mathcal{G}$ is not the Evans function for the problem, but rather a new object that we define herein, which fits the situation rather well.

As an application, we consider three classical models - the "good" Boussinesq equation, the Klein-Gordon-Zakharov (KGZ) system and the fourth order beam equation. In the whole line case, for the Boussinesq case and the KGZ system (and as a direct application of the main results), we compute explicitly the set of speeds which give rise to linearly stable traveling waves (and for all powers of p in the case of Boussinesq). This result is new for the KGZ system, while it generalizes the results of [2] and [1], which apply to the case $p = 2$. For the beam equation, we provide an explicit formula (depending of the function $\|\varphi_c'\|_{L^2}$), which works for all p and for both the periodic and the whole line cases.

Our results complement (and exactly match, whenever they exist) the results of a long line of investigation regarding the related notion of orbital stability of the same waves. Informally, we have found that in all the examples that we have looked at, our theory goes through, whenever the Grillakis-Shatah-Strauss theory applies. We believe that the results in this paper (or a variation thereof) will enable the linear stability analysis as well as asymptotic stability analysis for most models in the form $u_{tt} + \mathcal{L}u + N(u) = 0$.

1. INTRODUCTION AND MOTIVATION

The main motivation of our study is the following abstract second order in time nonlinear PDE

$$u_{tt} + \mathcal{L}_x u + N(u) = 0, \quad (t,x) \in \mathbf{R}^1_+ \times \mathbf{R}^d \quad \text{or} \quad (t,x) \in \mathbf{R}^1 \times [-L, L]^d, \tag{1}$$

where $\mathcal{L}_x$ is a given linear operator, acting on the x variable and $N(u)$ is the nonlinear term. We outline some relevant examples of interest, see Section 1.1 below, but notice that we do consider both periodic boundary conditions as well as vanishing at infinity solutions. These two scenarios will be considered simultaneously, since our method works equally well in both cases.

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Our interest is in the study of the stability properties of traveling waves, that is, solutions in the form $\varphi(x + \vec{c}t)$. Clearly, these satisfy the stationary PDE

$$\mathcal{L}_x\varphi + \sum_{i,j=1}^{d} c_i c_j \varphi_{x_i}\varphi_{x_j} + N(\varphi) = 0 \tag{2}$$

We take the ansatz $u = \varphi(x + \vec{c}t) + v(t, x + \vec{c}t)$ and plug it in (1). Taking into account (2) and dropping all quadratic and higher order terms in v, we arrive at the following

$$v_{tt} + 2\langle \vec{c}, \nabla_x v_t\rangle + \sum_{j,k=1}^{d} c_j c_k \frac{\partial^2 v}{\partial x_j \partial x_k} + \mathcal{L}v + N'(\varphi)v = 0 \tag{3}$$

Therefore, introducing the operator $H_{\vec{c}} = \mathcal{L}_x + \sum_{j,k=1}^{d} c_j c_k \partial_{x_j}\partial_{x_k} + N'(\varphi)$, we are lead to study the following problem

$$v_{tt} + 2\langle \vec{c}, \nabla_x v_t\rangle + Hu = 0. \tag{4}$$

This is the linearized stability problem for the nonlinear equation (1) in a vicinity of the special solution $\varphi(x + \vec{c}t)$. Before we move on with our exposition, we provide some informal definitions and discussions to motivate our interest in the problem. There are several notions of stability that are of interest. Generally, linear stability is easier to check than (and is often a prerequisite to) nonlinear stability. Even within the linear stability, we distinguish between spectral and linear stability. Namely, for an evolutionary problem in the form $z_t = \mathcal{M}z$, where $\mathcal{M}$ is a closed operator generating C_0 semigroup, we say that we have *spectral stability*, if $\sigma(\mathcal{M}) \subset \mathcal{Z}_- = \{\lambda : \Re\lambda \leq 0\}$. We say that the same problem has *linear stability*, if the solutions grow at infinity slower than any exponential[1], i.e. for every $\delta > 0$, $\lim_{t\to\infty} e^{-\delta t}\|z(t)\| = 0$. The relationship between spectral and linear stability has been well-explored and documented in the literature, and we will not dwell on it, except to point out that in principle (and in the presence of the so-called spectral mapping theorem for the generator $\mathcal{M}$), these are equivalent and amount to lack of exponentially growing modes, that is solutions in the form $\mathcal{M}\psi = \lambda\psi$ for $\lambda : \Re\lambda > 0$.

On the other hand, we have two distinct notions of nonlinear stability - orbital and asymptotic stability. Assuming for simplicity that the only invariance of the system is translation, orbital stability requires that a solution for the full nonlinear equation that starts close to a traveling wave stay close for all times to a (time-dependent) translate of the starting wave. Asymptotic stability requires a bit more, namely that the perturbed profile will actually converge to a (time-dependent) translate of the wave.

There is a large body of literature that deals with this problem in various models. We would like to point out that the powerful methods of Grillakis-Shatah-Strauss, [12], [13] reduce (in most cases) the problem of orbital stability to checking certain conditions on the linearized functionals. We mention in this regard the papers [4], [7], [23], [24], which treat models of interest to us in this paper. We also note that establishing orbital instability for a given problem seems to be harder and requires more problem specific efforts, [23].

Our interest here is the question of linear stability of such traveling waves φ_c, that is, whether there are exponentially growing solutions of (4) in the form $e^{\lambda t}\psi(x)$. We pose the following

Question: For which values of $\vec{c}$, the corresponding traveling wave φ_c determined by (2) is linearly/spectrally stable? More precisely, for which $\vec{c}$, the equation (4) has a solution in the form $e^{\lambda t}\psi$?

[1] Here, note that one cannot require bounded orbits. Indeed, even in the ODE case, if one takes a Jordan block A of size d, e^{tA} has polynomial growth t^d

Note that this question has been studied thoroughly in the last twenty years. Here, the methods are completely different, although the results in the end must be related with those obtained by orbital stability methods. After all one expects orbitaly stable waves to be linearly stable and vice versa (although this sometimes fails at points where the stability character changes).

The Evans function method has been used in connection to the linear stability of such waves, [2], [28], [29], [21]. However, the method has well-known limitations, in particular dealing with systems, which is the issue at hand here, since the equations are second order in time. In fact, the papers quoted here are among the few that deal with second order in time equations.

Another method that has been developed is the method of "indices counting", which basically relates the number of unstable eigenvalues of self-adjoint entries of $\mathcal{M}$ to the number of unstable eigenvalues of $\mathcal{M}$ itself. This has been mostly useful in the KdV and Schrödinger type systems, but it has also played important role in general Hamiltonian and dissipative systems, [15], [16], [17], [8], [9]. We would like to point out that some of these results have enabled the consideration of spatially periodic waves, [9], which is one of the goals of this project as well.

A third method for establishing mostly sufficient conditions for instability is through a direct construction of unstable modes, [23], [22]. In fact, there are many other impressive results in the literature, mostly for standing waves, which provide "instability by blow up" for such waves. These are done typically by constructing clever Lyapunov functionals, associated with the unstable modes.

Our goal in this paper is to develop a fairly general and systematic theory, which treats the question for linear stability (i.e. the existence/nonexistence of such ψ) of traveling waves of second order in time systems. *Indeed, we present a complete answer to this question, in the case of* (4)*, when H is self-adjoint, with at most one negative eigenvalue and $d = 1$.*

1.1. **Examples.** We consider three basic examples that fit this category, although there are numerous others to which our results will be applicable. The reader should also bear in mind that one might consider both periodic and whole line boundary conditions, for the models that we list below. To keep the discussion simple, we will mostly stick to the whole line case, the periodic cases will be addressed in a subsequent publication, [14].

1.1.1. *The "good" Boussinesq models.* Our first example is the "good' Boussinesq-type model,

$$u_{tt} + u_{xxxx} - u_{xx} + (u^p)_{xx} = 0, \quad (t, x) \in \mathbf{R}^1_+ \times \mathbf{R}^1 \tag{5}$$

This model was considered by Bona-Sachs, [4] as a model for propagation of small amplitude, long waves on the surface of water. This is indeed an equation that belongs to a family of Boussinesq models, which all have the same level of formal validity, however (5) exhibits some desirable features, like local well-posedness in various function spaces, [4]. Interestingly, global well-posedness for (10) does not hold, even if one requires smooth initial data with compact support. In fact, there are "instability by blow-up" results for such unstable traveling waves for this equation.

It is easy to see that there exists one-parameter family of traveling waves of the form $\varphi(x - ct), |c| \in (-1, 1)$, which obey the equation

$$c^2\varphi + \varphi'' - \varphi + \varphi^p = 0 \tag{6}$$

and which have the explicit form

$$\varphi_c(\xi) = \left[\left(\frac{p+1}{2}\right)(1 - c^2)\right]^{\frac{1}{p-1}} sech^{\frac{2}{p-1}}\left(\frac{\sqrt{1-c^2}(p-1)}{2}\xi\right).$$

One should of course recognize the standard *sech* solitons of the generalized KdV hierarchy. This is not surprising, since the governing ODE for the traveling waves of (10) is related to the corresponding ODE for gKdV.

In [4], the authors proved orbital stability for these solutions, provided $p < 5, \frac{\sqrt{p-1}}{2} < |c| < 1$, while nonlinear instability was established in [23]. Both of these results were achieved via the methods of the Grillakis-Shatah-Strauss theory (developed in [12], [13]) for orbital stability/instability.

1.1.2. *The Klein-Gordon-Zakharov system.* Consider the following system of coupled wave equations[2]

$$\left| \begin{array}{l} u_{tt} - u_{xx} + u + nu = 0 \quad (t,x) \in \mathbf{R}^1_+ \times \mathbf{R}^1 \\ n_{tt} - n_{xx} - \frac{1}{2}(|u|^2)_{xx} = 0, \end{array} \right. \tag{7}$$

which describes the interaction of a Langmuir wave and an ion acoustic wave in a plasma[3], [11]. Taking the traveling wave ansatz $u(t,x) = \varphi(x-ct), n(t,x) = \psi(x-ct)$, we derive from the second equation that $(c^2-1)\psi'' = \frac{1}{2}(\varphi^2)''$, whence $(c^2-1)\psi = \frac{\varphi^2}{2}$, since the functions φ, ψ decay at infinity. Using this relation in the first equation yields the following ODE for φ

$$-(1-c^2)\varphi'' + \varphi - \frac{\varphi^3}{2(1-c^2)} = 0 \tag{8}$$

It is well-known (and in fact can be checked easily, based on simple rescaling arguments in the ODE's governing traveling wave solutions for the KdV equation) that (7) admits an one parameter family of traveling wave solutions in the form $u(t,x) = \varphi(x-ct), n(t,x) = \psi(x-ct)$ for $c \in (-1,1)$, where

$$\left| \begin{array}{l} \varphi(y) = 2\sqrt{1-c^2}\, sech\left(\frac{y}{\sqrt{1-c^2}}\right) \\ \psi(y) = -2 sech^2\left(\frac{y}{\sqrt{1-c^2}}\right). \end{array} \right. \tag{9}$$

In fact, there exists a two-parameter system of solitary-traveling waves of which the family displayed above is a particular case, [7]. In the same paper, the author shows orbital stability for such waves, provided $1 > |c| > \frac{\sqrt{2}}{2}$. Below, we solve completely the question for linear stability. Namely, for $|c| \in [\frac{\sqrt{2}}{2}, 1)$ we have linear stability, whereas for $|c| \in [0, \frac{\sqrt{2}}{2})$ we have linear instability.

1.1.3. *The nonlinear beam equation.* Another relevant example to consider is the so-called beam equation

$$u_{tt} + \Delta^2 u + u - |u|^{p-1}u = 0, \quad (t,x) \in \mathbf{R}^1 \times \mathbf{R}^d \text{ or } (t,x) \in \mathbf{R}^1 \times [-L,L]^d, \tag{10}$$

where $p > 1, L > 0$ and we either require periodic boundary conditions (in the case $x \in [-L,L]$) or vanishing at infinity for $x \in \mathbf{R}^d$.

This equation has been studied extensively in the literature. It seems that the earliest work on the subject goes back to [30], where (10) is proposed as a model of a suspension bridge. In this paper, the authors have also showed the existence of some traveling wave solutions, that is, solutions in the form $\varphi(x + \vec{c}t)$.

[2]Here the coefficient $\frac{1}{2}$ in front of the $(|u|^2)_{xx}$ can be taken to be arbitrary by rescaling, but the particular choice of $\frac{1}{2}$ will be convenient in the spectral analysis

[3]namely, the complex u describes the fast scale component of the electric field, while the real valued n stands for the deviation of ion density.

In [24] it was proved that traveling wave solutions exist, in the whole space context, whenever[4] $|\vec{c}| \in (0, \sqrt{2})$. The proof is variational in nature, and this allowed the author to derive conditions for orbital stability/instability of such waves for different speeds. In short, the conclusion was that such traveling waves are orbitally unstable for small speeds, while an orbitally stable solutions are observed for values of the parameter $|\vec{c}| \sim \sqrt{2}, |\vec{c}| < \sqrt{2}$. Various other works have explored the decay and scattering properties of the linear beam equation $u_{tt} + \Delta^2 + u = 0$, [25, 26, 27].

1.2. **Some general features of the operators H.** In this section, we construct the operators H that arise in (4) and discuss some of their general features. Of particular importance will be the fact that they are self-adjoint and the fact that they have exactly one negative eigenvalue.

1.2.1. *The beam equation - spectral picture.* Clearly, such a function φ will satisfy the ODE

$$\Delta^2\varphi + \sum_{i,j=1}^{d} c_i c_j \varphi_{x_i x_j} + \varphi - |\varphi|^{p-1}\varphi = 0, \quad x \in (-L, L)^d, \tag{11}$$

which is supplied by the usual periodic boundary conditions at $-L, L$ (or vanishing at infinity for the whole line case). Note that we do not necessarily look for positive solutions of (11), nor do we expect to find any in general (because of the lack of maximum principle for Δ^2, the ground state are not positive), thus the presence of the absolute value in the non-linear term. In the one dimensional case, which will be the main subject of our investigation, (11) reduces to

$$c^2\varphi'' + \varphi'''' + \varphi - |\varphi|^{p-1}\varphi = 0, \quad -L < x < L. \tag{12}$$

Clearly, the question for the existence of such $2L$ periodic solutions φ is a non-trivial one, but we refer the reader to the upcoming publication [10], where it will be addressed. Note though that if φ is constructed in the space $L^2_{per.}[-L, L]$, then one can automatically bootstrap its smoothness to infinity, thanks to the form of (12).

Even less obvious issue is the linear stability of such waves. In order to simplify matters, let us assume henceforth that $d = 1$ and p is an odd integer (in particular, one should keep in mind the model case $p = 3$). Now, for any such periodic wave φ satisfying (12), set $u(t, x) = \varphi(x + ct) + v(t, x + ct)$, which we plug in (10). After ignoring all the quadratic terms in v, taking into account (12) and changing variables $x - ct \to x$, we obtain the following *linearized equation* around the traveling wave profile φ

$$v_{tt} + 2cv_{tx} + v_{xxxx} + c^2 v_{xx} + v - p\varphi^{p-1}v = 0, \quad -L < x < L. \tag{13}$$

This is then a proper time to introduce the self-adjoint operator $\mathcal{H}$

$$\mathcal{H} := \partial_x^4 + c^2\partial_x^2 + 1 - p\varphi^{p-1}, \tag{14}$$

with the domain $D(\mathcal{H}) := H^4_{per.}[-L, L]$. Note that $\mathcal{H} = \mathcal{H}(c, p, \varphi)$ depends implicitly on the speed c, the parameter p and the wave φ itself.

Another fact that should be stated at this point is that $\mathcal{H}$ has zero on its spectrum, with eigenfunction φ'. Indeed, taking a spatial derivative in (12) (under the assumption that p is an odd integer) implies $\mathcal{H}[\varphi'] = 0$. Finally, the bottom of the spectrum of $\mathcal{H}$ is a negative number[5]. Indeed,

$$\langle \mathcal{H}\varphi, \varphi \rangle = -(p-1)\int_{-L}^{L} \varphi^{p+1}(x)dx < 0,$$

[4] as we shall see in our arguments later on, this is a natural space of parameters

[5] note that Weyl's theorem, in the whole line case $\sigma_{a.c.}(\mathcal{H}) = [1, \infty)$

hence $\sigma(\mathcal{H})\cap(-\infty,0)\neq\emptyset$. Unfortunately, we cannot verify that this eigenvalue is simple, although this is easily seen in numerical simulations, for all values of the parameter c.

Note that the computations above for the beam equation in the periodic case apply equally well for the whole line homoclinic solution produced by Levandosky, [24].

1.2.2. *The "good" Boussinesq model: spectral picture.* Here we outline the relevant operators for the Boussinesq model. As we shall see though, the scheme outlined in (4) does not quite work and one needs a little trick to make it work.

In any case, if we impose the ansatz $u=\varphi_c(x+ct)+v(t,x+ct)$ and we get the equation $v_{tt}+2cv_{tx}+Tv=0$, where

$$Tv=\partial_x^4 v-(1-c^2)\partial_x^2 v+p(\varphi_c^{p-1}v)_{xx}$$

Note that the operator T is not self-adjoint and thus unsuitable for our method. However, if we introduce the variable $z: z_x=v$, we get the following linearized equation in terms of z,

$$z_{ttx}+2cz_{txx}+T[z_x]=0. \tag{15}$$

Note that $T[z_x]=\partial_x[H[z]]$, where

$$Hz=\partial_x^4 z-(1-c^2)\partial_x^2 z+p(\varphi_c^{p-1}z_x)_x \tag{16}$$

Thus, the linearized equation becomes $\partial_x[z_{tt}+2cz_{tx}+Hz]=0$, which in view of the conditions $\lim_{|x|\to\infty}|\partial_x^\alpha z(x)|=0$, implies $z_{tt}+2cz_{tx}+Hz=0$. Thus, the linearized equation to consider is again $z_{tt}+2cz_{tx}+Hz=0$, where H is defined in (16). As we prove later on, H has one simple eigenvalue at zero and one simple negative eigenvalue.

Note that while it is clear the presence of unstable mode for $z_{tt}+2cz_{tx}+Hz=0$ implies linear instability for (15) (via $v=z_x$), the reverse it is not immediately clear. The reason for that is that the operator ∂_x is not invertible and one may potentially have a solution $e^{\lambda t}V(x)$ of (15), while the corresponding "solution" $e^{\lambda t}Z(x)=e^{\lambda t}\partial_x^{-1}V(x)$ may not be an L^2 function.

2. Main Results

In this section, we present our results. We start first with a abstract form of the linearized problem, which involves the theory of quadratic pencils.

2.1. **Stability/instability results for quadratic pencils.** In connection with (13), we may write it schematically in the form

$$v_{tt}+2cv_{tx}+\mathcal{H}v=0.$$

Definition 1. *We say that the periodic wave φ is linearly unstable, if there exists $\lambda:\Re\lambda>0$, and a function ψ, so that then the following equation is satisfied*

$$\lambda^2\psi+2c\lambda\psi_x+\mathcal{H}\psi=0 \tag{17}$$

Otherwise, we say that the traveling wave φ is linearly stable.

In order to study the stability of such waves, we actually consider the much more general framework of stability/instability of operator pencils of the form

$$L(\lambda)=\lambda^2 Id+2c\lambda\partial_x+H,$$

where H is a fixed self-adjoint operator, acting on L^2, which satisfies certain mild constraints. This question has been considered before, mainly in a pure operator-theoretic sense. We refer the reader to the work of Shkalikov, [31] and Azizov-Iokhvidov, [3] for a review of the available results. We also develop some of the theory below, as we shall need it for our purposes, see Section 3.1 below.

Going back to the problem at hand, we consider linear, second-order in time equations in the general form

$$u_{tt} + 2\omega u_{tx} + Hu = 0, (t,x) \in \mathbf{R}^1 \times \mathbf{R}^1 \quad \text{or} \quad \mathbf{R}^1 \times [-L, L] \tag{18}$$

where $H = H_c$ is a self-adjoint operator acting on L^2, with domain $D(H)$ and ω is a real parameter. Note that it is better at this point to consider ω as an independent parameter, and to ignore the fact that in the applications $\omega = c$.

Based on our examples, we saw that it is reasonable to assume that the self-adjoint operator H has one simple negative eigenvalue, a simple eigenvalue at zero (which is naturally generated by φ') and the rest of the spectrum is contained in $(0, \infty)$, with a spectral gap. Even though our approach is quite systematic and should be able to handle a more general situation[6], we choose to implement these assumptions for simplicity of the exposition. Thus, we require

$$\begin{cases} \sigma(H) = \{-\delta^2\} \cup \{0\} \cup \sigma_+(H), \sigma_+(H) \subset (\sigma^2, \infty), \sigma > 0 \\ H\phi = -\delta^2 \phi, \ \dim[Ker(H+\delta^2)] = 1 \\ H\psi_0 = 0, \ \ \dim[Ker(H)] = 1 \\ \|\phi\| = \|\psi_0\| = 1 \end{cases} \tag{19}$$

We shall need the following spectral projection operators

$$P_0 : L^2 \to \{\phi\}^\perp; P_0 h := h - \langle h, \phi \rangle \phi$$
$$P_1 : L^2 \to \{\phi, \psi_0\}^\perp; P_1 h := h - \langle h, \phi \rangle \phi - \langle h, \psi_0 \rangle \psi_0$$

Our next assumption is essentially that H_1 is of order higher than one. We put it in the following form: for all $\tau >> 1$, we require

$$(H+\tau)^{-1/2} \partial_x (H+\tau)^{-1/2}, (H+\tau)^{-1} \partial_x \in \mathcal{B}(L^2), \tag{20}$$

Note that the quantities in (20) are well-defined, since for all $\tau >> 1$, we have that $H + \tau \geq (\tau - \delta^2) Id > 0$.

An easy consequence of (20) is that $H^{-1} P_1 \partial_x P_1 \in \mathcal{B}(L^2)$ as well. This follows easily from the resolvent identity, since $H^{-1} P_1 \partial_x P_1 = P_1 (H+\tau)^{-1} \partial_x P_1 + \tau H^{-1} P_1 (H+\tau)^{-1} \partial_x P_1$. In addition, the following non-degeneracy condition is also required

$$\langle \phi', \psi_0 \rangle \neq 0. \tag{21}$$

Lastly, we assume that H has real coefficients. We formulate as follows

$$\overline{Hh} = H\bar{h}. \tag{22}$$

An important observation, that we would like to make right away (and which will be used repeatedly in our arguments later on) is that for every $\lambda > 0$, the operator $(H+\lambda^2) : \{\phi\}^\perp \to \{\phi\}^\perp$ is invertible, since $(H+\lambda^2)|_{\{\phi\}^\perp} \geq \lambda^2 Id$.

The following theorem is the main result of this paper.

[6]for example multiple eigenvalues at zero, which is the case in higher dimension

Theorem 1. *Let H be a self-adjoint operator on L^2. Assume that it satisfies the structural assumption* (19), (20) *as well as the non-degeneracy assumption* (21) *and* (22).

Then, if $\langle H^{-1}[\psi_0'], \psi_0' \rangle \geq 0$, we have instability (in the sense of Definition 1) for all values of $\omega \in \mathbf{R}^1$.

Otherwise, supposing $\langle H^{-1}[\psi_0'], \psi_0' \rangle < 0$, we have

- *the problem* (18) *is unstable if ω satisfies the inequality*

$$0 \leq |\omega| < \frac{1}{2\sqrt{-\langle H^{-1}[\psi_0'], \psi_0' \rangle}} =: \omega^*(H) \tag{23}$$

- *the problem* (18) *is stable, if ω satisfies the reverse inequality*

$$|\omega| \geq \omega^*(H) \tag{24}$$

Remarks:

- Note that the result is a complete characterization of the stability and instability properties of the abstract quadratic pencil problem (18). In essence, it says that there is a critical number $\omega^*(H)$, below which the problem is unstable and above which, there is stability.
- Note that the critical number $\omega^*(H)$ may also depend on[7] c, so that the equality $c = \omega^*(H, c)$ (as in (17)) will be an implicit one. In any case, the conclusions (23) and (24) hold true, and should be used to determine stability, regardless of this implicit dependence.
- A more succinct way of defining $\omega^*(H)$ is the following

$$\omega^*(H) = \begin{cases} +\infty & \text{if } \langle H\psi_0', \psi_0' \rangle \geq 0 \\ \frac{1}{2\sqrt{-\langle H^{-1}[\psi_0'], \psi_0' \rangle}} & \text{if } \langle H\psi_0', \psi_0' \rangle < 0 \end{cases}$$

Then, we can characterize stability in terms of the inequality as follows: $|\omega| \geq \omega^*(H)$.

An immediate corollary is the following "spectral stability/instability" type of statement. We can write (18) in the form

$$\begin{pmatrix} u \\ u_t \end{pmatrix}_t = \begin{pmatrix} 0 & 1 \\ -H & -2\omega\partial_x \end{pmatrix} \begin{pmatrix} u \\ u_t \end{pmatrix} =: \mathcal{T} \begin{pmatrix} u \\ u_t \end{pmatrix}$$

Corollary 1. *In the statement of Theorem 1, assume in addition that $[Hh(-\cdot)](x) = (Hh)(-x)$. Then, in the cases of instability, there is $\lambda > 0$, so that $\lambda, -\lambda$ are both eigenvalues of $\mathcal{T}$ and moreover*

$$\sigma(\mathcal{T}) \subset \{\lambda\} \cup \{-\lambda\} \cup i\mathbf{R}^1.$$

If on the other hand, there is stability, we have $\sigma(\mathcal{T}) \subset i\mathbf{R}^1$.

2.2. **Applications.**

2.2.1. *Stability for traveling waves of the "good" Boussinesq equation.* The following result appears in the literature, for the case $p = 2$, [2]. The idea behind the proof is based on an Evans function calculations, but the authors needed some computer assistance towards the end of the proof[8].

Theorem 2. *The traveling wave φ_c of the Boussinesq equation* (10) *is linearly unstable, if $p \geq 5$. If $2 \leq p < 5$, then it is linearly unstable if $0 \leq |c| < \frac{\sqrt{p-1}}{2}$ and linearly stable, when $\frac{\sqrt{p-1}}{2} \leq |c| < 1$.*

[7]and in fact, for the applications in mind, H will have the form of (14), where the dependence on c is pretty transparent

[8]We were also informed, [1], that J. Alexander and R. Pego have a completely rigorous proof, which is unpublished.

Remarks: Note that the result in Theorem 2 matches precisely the results for orbital stability for the "good" Boussinesq model. Recall that orbital stability was shown by Bona-Sachs for $|c| > \frac{\sqrt{p-1}}{2}$ and the nonlinear instability by Liu, [23].

The critical case $|c| = \frac{\sqrt{p-1}}{2}$ provides a (marginally) linearly stable situation, according to our result, but note that Liu, [23] shows nonlinear instability there. This is due to the fact that in this particular case, there is an additional eigenvalue at zero (which is unaccounted for in terms of symmetries), which is responsible for a secular nonlinear instability.

2.2.2. *Stability for the Klein-Gordon-Zakharov system.*

Theorem 3. *Let $c \in (-1,1)$. Then, the traveling wave solution $(\varphi(x-ct), \psi(x-ct))$ described in* (9) *is spectrally/linearly stable for $|c| \in [\frac{\sqrt{2}}{2}, 1)$ and linearly/spectrally unstable for $|c| \in [0, \frac{\sqrt{2}}{2})$.*

Remarks: Note that the linear stability results match precisely the orbital stability results in [7], except at the endpoints $|c| = \frac{\sqrt{2}}{2}$. At this point, we have linear stability, according to Theorem 3, but it is unclear whether the wave is orbitally stable or not.

2.2.3. *Stability for the beam equation.* We state the relevant results for the beam equation. In it, we need to make some assumptions regarding the existence of such waves and the spectrum of the corresponding operator $\mathcal{H}$, defined in (14). We should note that the existence of ground states, in a sense to be made precise below, was proven by Levandosky, [24]. Note however that our results apply to both the periodic and the whole line cases, with appropriate boundary conditions in each case.

Theorem 4. *Let $p \geq 3$ be an odd integer and $I \subset (-\sqrt{2}, \sqrt{2})$ be an open interval. Assume that there exists a one parameter family $\{\varphi_c\}_{c \in I}$ of solutions to* (12) *(where $L \geq 0$ could be finite so that*

- *$\varphi_c \in H^1$ and $c \to \|\varphi_c'\|_{L^2}$ is a differentiable function on I.*
- *The operator $\mathcal{H}_c = \partial_x^4 + c^2\partial_x^2 + 1 - p\varphi_c^{p-1}$ satisfies* (19) *and* (21).

Then, the wave φ_c is linearly stable if and only if $\partial_c \|\varphi_c'\| < 0$ and

$$|c| \geq \frac{\|\varphi_c'\|}{-2\partial_c\|\varphi_c'\|}.$$

That is, if $\partial_c\|\varphi_c'\| \geq 0$ or $\partial_c\|\varphi_c'\| < 0$, but $|c| < \frac{\|\varphi_c'\|}{-2\partial_c\|\varphi_c'\|}$, the wave φ_c is unstable.

Remarks:

(1) Levandosky, [24] has shown (in the case of the whole line) that a family of "ground state" solutions may be constructed via a variational procedure as follows. More precisely, one first solves the variational problem

$$\left| \begin{array}{l} J(z) = \int_{-\infty}^{\infty} (z_{xx}^2 - c^2 z_x^2 + z^2) dx \to \min \\ \text{subject to } \ I(z) = \int_{-\infty}^{\infty} z^{p+1}(x) dx = 1. \end{array} \right. \tag{25}$$

Then, if one sets $\varphi_c := J(z_c)^{\frac{1}{p-1}} z_c$, where z_c is the solution to the minimization problem, then φ_c solves precisely (12). Note here that the restriction $|c| < \sqrt{2}$ assures that such a problem has a solution and in fact $J(z) \geq 0$. Similar approach works in the periodic case as well, [10].

(2) It should be noted however, that the ground state solutions produced via (25) need not be the only way of producing the family $\{\varphi_c\}$ and in fact it is not, [10]. One reason for that is the apparent lack of uniqueness in solving this type of minimization problem. Second, one

may have "excited" states (or local minima of (25)), which are also interesting solutions to consider. We have not checked out their stability, but it is likely that some of them are stable. We also point out that the variational methods for studying the (orbital) stability of ground states are inapplicable for these excited states, whereas Theorem 4 gives an exhaustive answer.

(3) We have shown in Section 1.2.1 that all reasonable families of solutions must have the property that $\mathcal{H}_c$ has a negative eigenvalue and a zero eigenvalue, with the eigenvector φ_c'. The simplicity of such eigenvalues is however difficult to prove theoretically, due in particular to the lack of explicit formulas for φ_c. Numerically however, one can easily check that $\mathcal{H}_c$ has simple negative and simple zero eigenvalue, even for most of excited states, [10]. In fact, in all numerical runs in [10], $\mathcal{H}_c$ never failed to satisfy (19) and (21).

(4) The restriction that p is odd is technical and it is likely to be unnecessary. More precisely, the issue is only the differentiability of the function $x \to |\varphi(x)|^{p-1}\varphi(x)$.

The paper is organized as follows. In Section 3.1, we first present the relevant basic results of the Shkalikov's theory for quadratic pencils, after which we introduce our main tool, the function $\mathcal{G}$, list its main properties and derive the proof of our main result, Theorem 1. This is done modulo the somewhat technical Lemma 3 (which establishes the linear stability at $\omega = \omega^*(H)$), whose proof is given in Section 4. To achieve this, we study and establish the Laurent expansion of the function $\mathcal{G}$ at zero, which may be of independent use in future investigations.

In the remaining sections, we apply the main result to obtain sharp results for linear stability/instability to the various examples announced above. In Section 5 we consider the Boussinesq model and prove Theorem 2. In Section 6, we set up the Klein-Gordon-Zakharov system and find the set of speeds that yields stable traveling waves as stated in Theorem 3. Finally, in Section 7, we establish Theorem 4 regarding the stability of the traveling waves for the beam equation.

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3. Proof of Theorem 1

We first start with a preliminary material, which is the Shkalikov theory for quadratic pencils, [31].

3.1. Some aspects of the Shkalikov's theory for quadratic pencils.

We follow the presentation of Shkalikov, [31]. We are especially interested in his index formula, which relates the number of unstable eigenvalues of H with the number of unstable eigenvalues of the pencil defined in (17).

Following Shkalikov, [31], introduce a quadratic operator pencil in the form

$$A(\lambda) = \lambda^2 F + (D + iG)\lambda + T,$$

where the coefficients F, D, G, T are operators on a Hilbert space H, satisfying the following conditions

(i) F - bounded, invertible and self-adjoint
(ii) $(T, Dom(T))$ - self-adjoint, invertible
(iii) $D \geq 0, G$ are symmetric; $Dom(D), Dom(G) \subset Dom(T)$ and D, G are $T-$bounded operators.

We say that an operator M is $T-$bounded, if

$$|T|^{-1/2} M |T|^{-1/2} \in \mathcal{B}(H)$$

We say that $\xi \in \rho(A)$, if $A(\xi)$, with domain $Dom(T)$ is invertible.

Clearly, in our application, we will be interested in the case $F = Id$, $D = 0$, $G = -i\omega\partial_x$, $T = H$. However, **note that $T = H$ is not invertible in our case of interest**. Nevertheless, we introduce the associated quadratic pencil

$$\hat{A}(\lambda) := \lambda^2 \hat{F} + \lambda(\hat{D} + i\hat{G}) + J$$

where $\hat{F} = |T|^{-1/2} F |T|^{-1/2}$, $\hat{G} = |T|^{-1/2} G |T|^{-1/2}$. We will be also interested in the spectrum with respect to a smoother space. Namely, introduce the space H_{-1} with a norm

$$\|x\|_{H_{-1}} := \||T|^{1/2} x\|_H$$

It is shown in [31] that the spectrum of A in H_{-1} coincides with the spectrum of $\hat{A}$ considered on the space H. Note that in Definition 1, we only consider smooth enough solutions ψ anyway. Thus, we need to count the number of "unstable" eigenvalues of $\hat{A}$.

The main result in the work of Shkalikov is Theorem 3.7. The following statement is a corollary of it. Here we have just presented a weaker version of the result, which will suffice for our purposes.

Theorem 5. *(Theorem 3.7,* [31]*) Suppose the coefficients of the pencil A, F, D, G, T satisfy conditions $(i)-(iii)$ listed above. Let the numbers of negative eigenvalues of F and T, $\nu(F), \nu(T)$ respectively is finite.*

Then, the spectrum of $A(\lambda)$ in the open right-half plane $\mathcal{C}_r = \{z : \Re z > 0\}$, considered upon the space H_{-1} consists of normal eigenvalues only. Moreover, the total algebraic multiplicity of all eigenvalues lying in $\mathcal{C}_r$ satisfies

$$k(\hat{A}) \leq \nu(T) + \nu(F).$$

Recall that in our case however, the operator $T = H$ is not invertible. This case is also covered by Shkalikov, see Theorem 4.2,[31] under the structural assumption (19). Our proof below just retraces his argument.

Indeed, since $Ker(H) \neq \{0\}$, one needs to consider $H_\tau := H + \tau Id$ for $0 < \tau << 1$, so that $Ker(H_\tau) = \{0\}$.

In order to apply Theorem 5, we need to check that $F = Id, G = -2i\omega\partial_x$ are H_τ bounded. This amounts to showing that $|H_\tau|^{-1/2} \partial_x |H_\tau|^{-1/2} \in \mathcal{B}(L^2)$. This is a direct consequence of (20).

We can now apply Theorem 5 to $\hat{A_\tau}$ to conclude

$$k(\hat{A_\tau}) \leq \nu(H_\tau) + \nu(Id) = 1,$$

for all small enough $\tau > 0$. Since the eigenvalues depend continuously on τ (see [19], Chapter 7), we take a limit as $\tau \to 0+$ to get the desired inequality $k(\hat{A}) \leq 1$.

To recapitulate, we have shown that for fixed real ω and under (19), the equation

$$\lambda^2 \psi + 2\lambda\omega\psi' + H\psi = 0, \quad (\lambda, \psi) \in \mathbf{R}^1_+ \times L^2, \tag{26}$$

has at most one solution with $\Re\lambda > 0$.

Finally, it is easy to see that in this case, the solution λ, if it exists, must be real. Indeed, suppose $\lambda \neq \bar{\lambda}$ is an eigenvalue for the pencil. That is, there is $\psi \in D(H) : \lambda^2\psi + 2\lambda\omega\psi' + H\psi = 0$. Taking a complex conjugate and taking into account (22), we see that

$$\bar{\lambda}^2 \bar{\psi} + 2\bar{\lambda}\omega\bar{\psi}' + H\bar{\psi} = 0$$

Thus, $(\bar{\lambda}, \bar{\psi})$ is another solution to (26),with $\Re\bar{\lambda} > 0$ and hence $k(A) \geq 2$, in contradiction with the inequality $k(A) \leq 1$. Thus, λ must be real and we have established

Corollary 2. *For $\omega \in \mathbf{R}^1$, the equation $\lambda^2\psi + 2\omega\lambda\psi' + H\psi = 0$, $\lambda \in \mathcal{C}, \psi \in D(H)$ has at most one solution (λ, ψ) with $\Re\lambda > 0$. Moreover, such a pair will have λ real, $\lambda > 0$.*

3.2. **Detecting instabilities: Proof of** (23)**.** Clearly, if $\omega = 0$, then (18) is unstable, in the sense of Definition 1, as in this case $\lambda = \delta > 0$ and $\psi = \phi$ will be a solution to (17). So, assume henceforth that $\omega \neq 0$.

We start with an useful Proposition, which appears as Theorem in [5], but it may in fact be a well-known result.

Proposition 1. *Assume that A is a closed, densely defined (not necessarily self-adjoint) operator on a Hilbert space, which is bounded from below (*$\inf_{u\in D(A):\|u\|=1} \langle Au, u\rangle > -\infty$*). Define its self-adjoint part* $H = \Re A = \frac{1}{2}[A + A^*]$*. Then*

$$\inf\{\Re\lambda : \lambda \in \sigma(A)\} \geq \inf \sigma(H).$$

In particular, if $H > 0$, then A is invertible.

In (18), we use the ansatz $u(t, x) = e^{\lambda t}\psi(x)$, whence one gets the following equation for ψ

$$\lambda^2\psi + 2\omega\lambda\psi' + H\psi = 0 \tag{27}$$

Our first observation is that any nontrivial solution to (27) will have a non-trivial projection onto ϕ. Indeed, assuming that $\psi \perp \phi$ and taking into account that $\{\phi\}^\perp$ is invariant under H, we can apply the projection operator P_0 on (27). We get

$$(H + \lambda^2 + 2\omega\lambda P_0\partial_x P_0)\psi = 0 \tag{28}$$

Considering now $(H + \lambda^2 + 2\omega\lambda P_0\partial_x P_0) : \{\phi\}^\perp \to \{\phi\}^\perp$, we see that $(H + \lambda^2)|_{\{\phi\}^\perp} \geq \lambda^2$, whereas $2\omega\lambda P_0\partial_x P_0$ is skew-adjoint operator on $\{\phi\}^\perp$. It follows that $\Re((H+\lambda^2+2\omega\lambda P_0\partial_x P_0) = (H + \lambda^2)|_{\{\phi\}^\perp} \geq \lambda^2 ID$, whence $(H + \lambda^2 + 2\omega\lambda P_0\partial_x P_0)$ is invertible, by Proposition 1. It follows from (28) that $\psi = 0$. Thus, $\langle\psi, \phi\rangle \neq 0$ for any non-trivial solution of (27).

Since solutions of (27) are up to multiplicative constant, we may take $\psi : \langle\psi, \phi\rangle = 1$. That is $\psi := \phi + v$, where $v \perp \phi$. We get the following equation for v

$$(\lambda^2 + 2\omega\lambda\partial_x + H)v = (\delta^2 - \lambda^2)\phi - 2\omega\lambda\phi' \tag{29}$$

Taking dot product with ϕ (and taking into account $\langle\phi', \phi\rangle = 0$ and $\langle(H + \lambda^2)v, \phi\rangle = 0$) yields

$$2\omega\lambda \langle v', \phi\rangle = \delta^2 - \lambda^2$$
$$\langle v, \phi'\rangle = \frac{\lambda^2 - \delta^2}{2\omega\lambda}$$

Taking P_0 in (29) (and observing that $P_0 v = v$) on the other hand implies

$$[H + \lambda^2 + 2\omega\lambda P_0\partial_x P_0]v = -2\omega\lambda\phi'.$$

We now observe that the operator $\mathcal{L}_\lambda = H + \lambda^2 + 2\omega\lambda P_0\partial_x P_0 : \{\phi\}^\perp \to \{\phi\}^\perp$ is invertible by Proposition 1 and the representation

$$\mathcal{L}_\lambda = (H + \lambda^2) + \text{skewsymmetric} : \{\phi\}^\perp \to \{\phi\}^\perp, \quad H + \lambda^2|_{\{\phi\}^\perp} \geq \lambda^2 Id > 0.$$

Thus, $v = -2\omega\lambda[H + \lambda^2 + 2\omega\lambda P_0\partial_x P_0]^{-1}[\phi'] \in \{\phi\}^\perp$, since $\phi' \in \{\phi\}^\perp$.

From this analysis, we can say that the spectral problem (27) has a real solution λ <u>if and only if</u>

$$\mathcal{G}(\omega; \lambda) := \langle[H + \lambda^2 + 2\omega\lambda P_0\partial_x P_0]^{-1}[\phi'], \phi'\rangle + \frac{\lambda^2 - \delta^2}{4\omega^2\lambda^2} \tag{30}$$

has a positive root[9] $\lambda_0 > 0$. In view of Corollary 2, we have that this condition is necessary and sufficient for instability of (18). We have proved the following

[9]Often times, we will omit the dependence of $\mathcal{G}$ on ω, since ω will be mostly fixed

Proposition 2. *If H satisfies the condition of theorem 1 and $\mathcal{G}$ is as in* (30)*, then a* **necessary and sufficient condition for instability is that the function $\mathcal{G}$ vanishes for some $\lambda_0 > 0$.**

Our next result claims the continuity of the function $\mathcal{G}(\omega, \lambda)$

Lemma 1. *Let $\{(\omega_n, \lambda_n)\} \in \mathbf{R}^1_+ \times \mathbf{R}^1_+$, so that $(\omega_n, \lambda_n) \to (\omega_0, \lambda_0) \in \mathbf{R}^1_+ \times \mathbf{R}^1_+$. Then,*

$$\lim_n \mathcal{G}(\omega_n; \lambda_n) = \mathcal{G}(\omega_0; \lambda_0)$$

We provide the straightforward, but somewhat technical proof of Lemma 1 in the Appendix. We now continue with a verification of the fact that $\mathcal{G}(\omega, \cdot)$ changes sign in $(0, \infty)$ (and hence vanishes somewhere there), provided either $-\delta^2 \left\langle H^{-1}[\psi_0'], \psi_0' \right\rangle = |\left\langle \phi, \psi_0' \right\rangle|^2 - \delta^2 \|H^{-1/2} P_0[\psi_0']\|^2 < 0$ or $0 \leq |\omega| < \omega^*(H)$.

3.2.1. *Analysis close to $\lambda = \infty$.* We will show that under appropriate assumptions on H, we have

$$\lim_{\lambda \to \infty} \mathcal{G}(\lambda) = \frac{1}{4\omega^2} > 0. \tag{31}$$

Indeed, $\lim_{\lambda \to \infty} \frac{\lambda^2 - \delta^2}{4\omega^2 \lambda^2} = \frac{1}{4\omega^2}$, so it remains $\lim_{\lambda \to \infty} \left\langle [H + \lambda^2 + 2\omega\lambda P_0 \partial_x P_0]^{-1}[\phi'], \phi' \right\rangle = 0$.

This requires a bit of elementary analysis of the operator $\mathcal{L}$ defined above, in the limit of $\lambda \to \infty$. We have already established that $\mathcal{L}_\lambda$ is invertible on $\{\phi\}^\perp$. We now need an estimate on the norm of its inverse.

Proposition 3. *For every $\lambda > 0$, we have $H + \lambda^2 + 2\omega\lambda P_0 \partial_x P_0 : \{\phi\}^\perp \to \{\phi\}^\perp$. This operator has an inverse (in the said co-dimension one subspace) and its inverse satisfies the estimate*

$$\|(H + \lambda^2 + 2\omega\lambda P_0 \partial_x P_0)^{-1}\|_{\{\phi\}^\perp \to \{\phi\}^\perp} \leq \lambda^{-2} \tag{32}$$

Proof. We have already checked the invertibility. Let $g \in \{\phi\}^\perp$ is real-valued arbitrary function and $f = (H + \lambda^2 + 2\omega\lambda P_0 \partial_x P_0)^{-1}[g] \in \{\phi\}^\perp$, (note f is real valued as well), so that

$$(H + \lambda^2 + 2\omega\lambda P_0 \partial_x P_0) f = g,$$

Taking dot product with f yields

$$\lambda^2 \|f\|^2 \leq \left\langle (H + \lambda^2) f, f \right\rangle = \left\langle (H + \lambda^2 + 2\omega\lambda P_0 \partial_x P_0) f, f \right\rangle = \langle f, g \rangle \leq \|f\| \|g\|,$$

whence $\|(H + \lambda^2 + 2\omega\lambda P_0 \partial_x P_0)^{-1} g\| = \|f\| \leq \lambda^{-2} \|g\|$, as stated. □

Going back to the formula for $\mathcal{L}_\lambda^{-1}$ (which now makes sense), we observe that

$$\limsup_{\lambda \to \infty} |\left\langle [H + \lambda^2 + 2\omega\lambda P_0 \partial_x P_0]^{-1}[\phi'], \phi' \right\rangle| \leq \|\phi'\|^2 \limsup_{\lambda \to \infty} \lambda^{-2} = 0$$

whence we have established (31).

3.2.2. *Analysis close to $\lambda = 0$.* Regarding the behavior close to $\lambda = \varepsilon \sim 0$, we observe first that

$$\frac{\varepsilon^2 - \delta^2}{4\omega^2 \varepsilon^2} = -\frac{\delta^2}{4\omega^2 \varepsilon^2} + O(1),$$

We will show that

$$\left\langle [H + \lambda^2 + 2\omega\lambda P_0 \partial_x P_0]^{-1}[\phi'], \phi' \right\rangle = \frac{1}{\varepsilon^2} \frac{\left\langle \phi', \psi_0 \right\rangle^2}{1 + 4\omega^2 \|H^{-1/2} P_0[\psi_0']\|^2} + O(\varepsilon^{-1}). \tag{33}$$

Before we prove (33), let us pause for the moment to show that it implies all our statements for instabilities. First, if $\delta^{-2}|\langle\phi,\psi_0'\rangle|^2 - \|H^{-1/2}P_0[\psi_0']\|^2 = -\left\langle H^{-1}[\psi_0'],\psi_0'\right\rangle \leq 0$, then the coefficient in front of ε^{-2} for $\mathcal{G}(\varepsilon)$ becomes

$$\frac{\langle\phi',\psi_0\rangle^2}{1+4\omega^2\|H^{-1/2}P_0[\psi_0']\|^2} - \frac{\delta^2}{4\omega^2} = \frac{4\omega^2(\langle\phi,\psi_0'\rangle^2 - \delta^2\|H^{-1/2}P_0[\psi_0']\|^2) - \delta^2}{4\omega^2(1+4\omega^2\|H^{-1/2}P_0[\psi_0']\|^2)} < 0,$$

whence $\lim_{\varepsilon\to 0+}\mathcal{G}(\varepsilon) = -\infty$ and hence $\mathcal{G}$ changes sign in $(0,\infty)$ and it has a root $\lambda_0 > 0$, hence instability.

If, on the other hand one has$|\langle\phi,\psi_0'\rangle|^2 - \delta^2\|H^{-1/2}P_0[\psi_0']\|^2 = -\delta^2\left\langle H^{-1}[\psi_0'],\psi_0'\right\rangle > 0$, the solution to the inequality

$$\frac{\langle\phi',\psi_0\rangle^2}{1+4\omega^2\|H^{-1/2}P_0[\psi_0']\|^2} - \frac{\delta^2}{4\omega^2} < 0,$$

in terms of ω reads as $0 < |\omega| < \omega^*(H)$, that is (23). But we observed that this implies change of sign for $\mathcal{G}$. Thus there is $\lambda_0 > 0$, so that $\mathcal{G}(\lambda_0) = 0$, hence instability.

Back to the proof of (33), we need to compute the leading order term in

$$\left\langle (H+\varepsilon^2+2\omega\varepsilon P_0\partial_x P_0)^{-1}[\phi'],\phi'\right\rangle.$$

Let $z \in \{\phi\}^\perp$ solves $(H+\varepsilon^2+2\omega\varepsilon P_0\partial_x P_0)z = \phi'$. Decompose $z = a\psi_0 + q$, $q \in \{\phi,\psi_0\}^\perp$. Observe that $q = P_1 q$. The equation for z becomes

$$a\varepsilon^2\psi_0 + 2a\omega\varepsilon P_0[\psi_0'] + (H+\varepsilon^2+2\omega\varepsilon P_0\partial_x P_0)q = \phi' \tag{34}$$

The equation again has two different components - along ψ_0 and perpendicular to it.

Along the direction of ψ_0, we take dot product with ψ_0 to obtain

$$a\varepsilon^2 + \left\langle (H+\varepsilon^2+2\omega\varepsilon P_0\partial_x P_0)q,\psi_0\right\rangle = \left\langle \phi',\psi_0\right\rangle$$

which implies (note $\langle q,\psi_0\rangle = 0$, since $q = P_1 q \perp \psi_0$)

$$a\varepsilon^2 - 2\omega\varepsilon\left\langle q,\psi_0'\right\rangle = \left\langle \phi',\psi_0\right\rangle \tag{35}$$

Along the orthogonal direction, we take P_1 in (34) - note that $P_1P_0 = P_1$ and $P_1P_0[\psi_0'] = P_0P_1[\psi_0'] = P_0[\psi_0']$). We get

$$(H+\varepsilon^2+2\omega\varepsilon P_1\partial_x P_1)q = P_1[\phi'] - 2a\omega\varepsilon P_0[\psi_0']$$

Denoting the self-adjoint operator $H_1 = H_1P_1 = P_1HP_1 \geq \sigma^2$ and consider $\mathcal{L}_\varepsilon = H_1 + \varepsilon^2 + 2\omega\varepsilon P_1\partial_x P_1$. Since $H_1 + \varepsilon^2 \geq \sigma^2$, we may write

$$\mathcal{L}_\varepsilon = H_1[Id + H_1^{-1}(\varepsilon^2 + 2\omega\varepsilon P_1\partial_x P_1)],$$

whence by (20), $\lim_{\varepsilon\to 0+}\|H_1^{-1}(\varepsilon^2+2\omega\varepsilon P_1\partial_x P_1)\|_{L^2\to L^2} = 0$, we can construct $\mathcal{L}_\varepsilon^{-1}$ in terms of Neumann series. In fact

$$\mathcal{L}_\varepsilon^{-1} = H_1^{-1} + O(\varepsilon).$$

Since $H_1^{-1} = O(1)$, it follows that

$$q = \mathcal{L}_\varepsilon^{-1}[-2a\omega\varepsilon P_0[\psi_0'] + P_1[\phi']] = -2a\omega\varepsilon H_1^{-1}P_0[\psi_0'] + O(1) + a\varepsilon O(\varepsilon). \tag{36}$$

Expressing q from (36) back in (35) yields

$$a\varepsilon^2 - 2\omega\varepsilon(-2a\omega\varepsilon\left\langle H_1^{-1}P_0[\psi_0'],\psi_0'\right\rangle) + O(\varepsilon)(1+a\varepsilon^2) = \left\langle \phi',\psi_0\right\rangle$$

Note

$$\left\langle H_1^{-1}P_0[\psi_0'],\psi_0'\right\rangle = \left\langle H_1^{-1}P_0[\psi_0'],P_0[\psi_0']\right\rangle = \|H_1^{-1/2}[P_0[\psi_0']]\|_{L^2}^2.$$

As a consequence,

$$a\varepsilon^2(1+4\omega^2\|H_1^{-1/2}[P_0[\psi_0']]\|_{L^2}^2+O(\varepsilon))=\langle\phi',\psi_0\rangle+O(\varepsilon),$$

whence

$$a=\frac{1}{\varepsilon^2}\frac{\langle\phi',\psi_0\rangle}{1+4\omega^2\|H_1^{-1/2}[P_0[\psi_0']]\|_{L^2}^2}+O(\varepsilon^{-1})$$

Going back to (36), we conclude that $q=O(\varepsilon^{-1})$. Now

$$\begin{aligned}\langle(H+\varepsilon^2+2\omega\varepsilon P_0\partial_x P_0)^{-1}\phi',\phi'\rangle &= a\langle\psi_0,\phi'\rangle+\langle q,\phi'\rangle= \\ &= \frac{1}{\varepsilon^2}\frac{\langle\phi',\psi_0\rangle^2}{1+4\omega^2\|H_1^{-1/2}[P_0[\psi_0']]\|_{L^2}^2}+O(\varepsilon^{-1}).\end{aligned}$$

which establishes (33).

3.3. Proof of (24).

We now progress to the proof of (24). Introduce the notation

$$\begin{aligned}\mathcal{A}^{unstable} &:= \{\omega : (18) \text{ is unstable}\}\\ \mathcal{A}^{stable} &:= \{\omega : (18) \text{ is stable}\}\end{aligned}$$

We have already shown that $\mathcal{A}^{unstable}\neq\emptyset$, in fact $(-\omega^*(H),\omega^*(H))\subset\mathcal{A}^{unstable}$.

Our first lemma in this section is an assertion regarding the openness of the set $\mathcal{A}^{unstable}$.

Lemma 2. *Under the assumptions in Theorem 1, the set $\mathcal{A}^{unstable}$ is open.*

Proof. By our analysis in Section 3.2, a point $\omega_0\in\mathcal{A}^{unstable}$, exactly when one can produce a solution to (29) , $v=v(\omega_0)\in\{\phi\}^\perp\cap D(H)$. Define $g:\mathbf{R}^1_+\times\mathbf{R}^1\times\{\phi\}^\perp\to L^2$,

$$g(\lambda,\omega,v)=(\lambda^2+2\omega\lambda P_0\partial_x P_0+H)v+(\lambda^2-\delta^2)\phi+2\omega\lambda\phi'.$$

Fix such a $\omega_0\in\mathcal{A}^{unstable},\lambda_0>0,v_0\in\{\phi\}^\perp\cap D(H)$. We obviously have $g(\lambda_0,\omega_0,v_0)=0$. We will show, that there exists a neighborhood Ω of ω_0 and continuous mappings $\lambda:\Omega\to\mathbf{R}^1_+$ and $v:\Omega\to\{\phi\}^\perp\cap D(H)$, so that $g(\lambda(\omega),\omega,v(\omega))=0$, whence we would conclude that $\Omega\subset\mathcal{A}^{unstable}$. **Note that for the function $\lambda(\omega)$ constructed in this way, we have $\mathcal{G}(\omega,\lambda(\omega))=0$.**

Thus, it remains to check that the implicit function theorem applies. Compute

$$\langle D_{\lambda,v}|_{\lambda=\lambda_0,\omega=\omega_0,v=v_0},[\mu,\zeta]\rangle=2\mu((\lambda_0+\omega_0P_0\partial_xP_0)v_0+\lambda_0\phi+\omega_0\phi')+(\lambda_0^2+2\omega_0\lambda_0P_0\partial_xP_0+H)\zeta.$$

In order to show that this map is a bijection between $\mathbf{R}^1\times\{\phi\}^\perp\to L^2$, take an arbitrary element $z=\tilde{\mu}\phi+h\in L^2$, where $\tilde{\mu}\in\mathbf{R}^1$ and $h\in\{\phi\}^\perp$. We need to show that for each pair $(\tilde{\mu},h)\in\mathbf{R}^1\times\{\phi\}^\perp$, there is unique pair $(\mu,\zeta)\in\mathbf{R}^1\times\{\phi\}^\perp\cap D(H)$, so that

$$\langle D_{\lambda,v}|_{\lambda=\lambda_0,\omega=\omega_0,v=v_0},[\mu,\zeta]\rangle=\tilde{\mu}\phi+h \tag{37}$$

Taking into account that $(\lambda_0+\omega_0P_0\partial_xP_0)v_0+\omega_0\phi'\in\{\phi\}^\perp$, $(\lambda_0^2+2\omega_0\lambda_0P_0\partial_xP_0+H)\zeta\in\{\phi\}^\perp$, we conclude that (37) is equivalent to

$$\begin{aligned}&2\mu\lambda_0=\tilde{\mu}\\ &2\mu((\lambda_0+\omega_0P_0\partial_xP_0)v_0+\omega_0\phi')+(\lambda_0^2+2\omega_0\lambda_0P_0\partial_xP_0+H)\zeta=h.\end{aligned}$$

Thus, recalling that $(\lambda_0^2+2\omega_0\lambda_0 P_0\partial_x P_0+H)$ is invertible on the subspace $\{\phi\}^\perp$, whenever $\lambda_0>0$, we find that it has the unique solution

$$\mu = \frac{\tilde{\mu}}{2\lambda_0}$$
$$\zeta = (\lambda_0^2+2\omega_0\lambda_0 P_0\partial_x P_0+H)^{-1}[h-\frac{\tilde{\mu}}{\lambda_0}((\lambda_0+\omega_0 P_0\partial_x P_0)v_0+\omega_0\phi')].$$

Thus, the implicit function theorem applies and we have shown that some neighborhood of c_0 belongs to $\mathcal{A}^{unstable}$, which was the claim. □

Our next claim concerns the stability of (18) at $\omega_0=\omega^*(H)$.

Lemma 3. *The problem* (18) *is stable at* $\omega^*(H)$, *i.e.* $\omega^*(H)\in\mathcal{A}^{stable}$.

We postpone the proof of Lemma 3 for Section 4. We now assume its validity to finish the proof of the linearized stability statements in Theorem 1. For the proof of (24), given Lemma 3, it will suffice to establish that $\{\omega:\omega>\omega^*(H)\}\subset\mathcal{A}^{stable}$.

Let $\omega_0:\omega_0>\omega^*(H)$. As we saw, this condition implies that the function $\mathcal{G}(\omega_0;\lambda)$, defined in (30), has

$$\lim_{\lambda\to\infty}\mathcal{G}(\omega_0;\lambda)=\frac{1}{4\omega_0^2}>0, \lim_{\lambda\to 0+}\mathcal{G}(\omega_0;\lambda)=+\infty.$$

By the continuity of the function $\lambda\to\mathcal{G}(\omega_0;\lambda)$, there are three options. The first one is that $\mathcal{G}(\omega_0;\lambda)>0$ for all $\lambda>0$. By Proposition 2, this would mean that $\mathcal{G}(\omega_0,\lambda)$ never vanishes and hence, (27) does not have solutions, whence stability for ω_0 follows. We claim that this is the only viable option, by disproving the other two.

The second option is that $\mathcal{G}(\omega_0;\lambda)$ has at least two different roots in $(0,\infty)$, say $0<\lambda_1<\lambda_2<\infty$. This implies that (27) will have two different solutions, $(\lambda_1,\psi_1),(\lambda_2,\psi_2)$, thus contradicting Corollary 2 of the Shkalikov index theory.

The third possibility is that $\mathcal{G}(\omega_0;\lambda)$ has a double root, say $\lambda_0=\lambda_0(\omega_0)>0$, but otherwise is a non-negative function. If we manage to rule out this possibility, we will be done.

In order to clarify our situation, let us reformulate the problem. We need to show that the set $\{\omega>\omega^*(H)\}\cap\mathcal{A}^{unstable}=\emptyset$. On the other hand, we have already established that the set $\{\omega>\omega^*(H)\}\cap\mathcal{A}^{unstable}$, if non-empty, must consist of ω_0, so that the function $\mathcal{G}(\omega_0,\lambda)$ has a double root (at some λ_0).

Thus, we fix one such ω_0 (and the corresponding λ_0). We have

$$\mathcal{G}(\omega_0;\lambda_0)=0=\frac{\partial\mathcal{G}}{\partial\lambda}(\omega_0;\lambda_0) \tag{38}$$

Denoting by $R(\omega,\lambda)=(H+\lambda^2+2\omega\lambda P_0\partial_x P_0)^{-1}$, we see that $\mathcal{G}(\omega_0;\lambda_0)=0$ is the same as

$$\left\langle R(\omega_0,\lambda_0)[\phi'],\phi'\right\rangle=\frac{\delta^2-\lambda_0^2}{4\omega_0^2\lambda_0^2}. \tag{39}$$

whereas $\frac{\partial\mathcal{G}}{\partial\lambda}(\omega_0;\lambda_0)=0$ is equivalent to

$$\left\langle R(\omega_0,\lambda_0)[\lambda_0+\omega_0 P_0\partial_x P_0]HR(\omega_0,\lambda_0)[\phi'],\phi'\right\rangle=\frac{\delta^2}{4\omega_0^2\lambda_0^3}. \tag{40}$$

Next, introduce an extra parameter $\nu:|\nu-1|<<1$. It is clear that the operator νH is still self-adjoint with one negative eigenvalue $-\nu\delta^2$, an eigenvalue at zero and the same eigenfunctions as H. Thus, we may consider the following function

$$\tilde{\mathcal{G}}(\omega,\nu;\lambda) := \left\langle [\nu H + \lambda^2 + 2\omega\lambda P_0\partial_x P_0]^{-1}[\phi'], \phi' \right\rangle + \frac{\lambda^2 - \nu\delta^2}{4\omega^2\lambda^2}$$

for all values of $\lambda > 0$. Note $\tilde{\mathcal{G}}(\omega, 1; \lambda) = \mathcal{G}(\omega, \lambda)$. By an argument identical to the one in Lemma 2, we deduce the existence of a smooth function $\lambda(\nu, \omega)$, in some small neighborhood of $\nu = 1, \omega = \omega_0$, so that $\tilde{\mathcal{G}}(\omega, \nu; \lambda(\nu, \omega)) = 0$ and $\lambda(1, \omega_0) = \lambda_0$. Taking a derivative in ν yields

$$\frac{\partial \tilde{\mathcal{G}}}{\partial \nu}(\omega,\nu;\lambda(\nu,\omega)) + \frac{\partial \tilde{\mathcal{G}}}{\partial \lambda}(\omega,\nu;\lambda(\nu,\omega))\frac{\partial \lambda}{\partial \nu}(\nu,\omega) = 0$$

for all (ν, ω) close to $(1, \omega_0)$. Setting $\nu = 1, \omega = \omega_0$ and taking into account (by (38)) $\frac{\partial \tilde{\mathcal{G}}}{\partial \lambda}(\omega_0, 1; \lambda(1, \omega_0)) = \frac{\partial \mathcal{G}}{\partial \lambda}(\omega_0; \lambda_0) = 0$, implies

$$\frac{\partial \tilde{\mathcal{G}}}{\partial \nu}(\omega_0, 1; \lambda_0) = 0.$$

We compute

$$0 = \frac{\partial \tilde{\mathcal{G}}}{\partial \nu}(\omega_0, 1; \lambda_0) = -\left\langle R(\omega_0,\lambda_0) H R(\omega_0,\lambda_0)[\phi'], \phi' \right\rangle - \frac{\delta^2}{4\omega_0^2\lambda_0^2},$$

whence

$$\left\langle R(\omega_0,\lambda_0) H R(\omega_0,\lambda_0)[\phi'], \phi' \right\rangle = -\frac{\delta^2}{4\omega_0^2\lambda_0^2}. \tag{41}$$

We now use (38), (39) and (40) to get a formula for $\left\langle R(\omega_0,\lambda_0)^2[\phi'], \phi' \right\rangle$, which will be useful in the sequel. We have

$$\begin{aligned}
\frac{\delta^2 - \lambda_0^2}{4\omega_0^2\lambda_0^2} &= \left\langle R(\omega_0,\lambda_0)[\phi'], \phi' \right\rangle = \left\langle R(\omega_0,\lambda_0)[H + \lambda_0^2 + 2\omega_0\lambda_0 P_0\partial_x P_0] R(c_0,\lambda_0)[\phi'], \phi' \right\rangle = \\
&= \left\langle R(\omega_0,\lambda_0) H R(\omega_0,\lambda_0)[\phi'], \phi' \right\rangle + 2\lambda_0 \left\langle R(\omega_0,\lambda_0)[\lambda_0 + \omega_0 P_0\partial_x P_0] H R(\omega_0,\lambda_0)[\phi'], \phi' \right\rangle - \\
&- \lambda_0^2 \left\langle R(\omega_0,\lambda_0)^2[\phi'], \phi' \right\rangle = -\frac{\delta^2}{4\omega_0^2\lambda_0^2} + 2\lambda_0 \frac{\delta^2}{4\omega_0^2\lambda_0^3} - \lambda_0^2 \left\langle R(\omega_0,\lambda_0)^2[\phi'], \phi' \right\rangle,
\end{aligned}$$

whence

$$\left\langle R(\omega_0,\lambda_0)^2[\phi'], \phi' \right\rangle = \frac{1}{4\omega_0^2\lambda_0^2}. \tag{42}$$

Let us recapitulate again what we learned so far. We have that $\omega^*(H) \in \mathcal{A}^{stable} \neq \emptyset$ and by the results of Section 3.2, $\mathcal{A}^{stable} \subset \{\omega \geq \omega^*(H)\}$. On the other hand, the statement that we need to contradict is that $\{\omega > \omega^*(H)\} \cap \mathcal{A}^{unstable} \neq \emptyset$. In addition, for each such $\omega \in \{\omega > \omega^*(H)\} \cap \mathcal{A}^{unstable}$, we have established (39), (40), (41) and (42).

Under these circumstances, we may find $\tilde{\omega} \in \mathcal{A}^{stable}$, so that $\tilde{\omega}$ is an accumulation point of $\mathcal{A}^{unstable}$. This can be constructed as follows: write the open set $\mathcal{A}^{unstable} \cap (\omega^*(H), \infty)$ as a union of open intervals and take any endpoint of these to be $\tilde{\omega}$ (it could very well happen, at least in principle, that $(\omega^*(H), \infty) \subset \mathcal{A}^{unstable}$, in which case, we must pick $\tilde{\omega} := \omega^*(H) \in \mathcal{A}^{stable}$).

Thus, we may find $\omega_j \in \mathcal{A}^{unstable} \cap \{\omega > \omega^*(H)\} : \omega_j \to \tilde{\omega}$. Per our analysis of the set $\mathcal{A}^{unstable} \cap \{\omega > \omega^*(H)\}$, there exists an uniquely defined $\lambda_j = \lambda(\omega_j) > 0$, so that (38), and hence (39), (40), (41) and (42) all hold true.

We first show that $\{\lambda_j\}$ is a bounded sequence. Otherwise, there is a subsequence $\{n_j\}$, so that $\lambda_{n_j} \to \infty$. Note $\mathcal{G}(\omega_{n_j}, \lambda_{n_j}) = 0$. Since $\omega_{n_j} \to \tilde{\omega}$ and by the continuity of the function $\mathcal{G}$ (Lemma 1), we show in the same way as the proof of (31) that

$$0 = \lim_j \mathcal{G}(\omega_{n_j}, \lambda_{n_j}) = \frac{1}{4\tilde{\omega}^2},$$

a contradiction.

Next, we show that in fact $\lim_j \lambda_j = 0$. Otherwise, since $\{\lambda_j\}_j$ is a bounded positive sequence, there is a subsequence $\{n_j\}$, so that $\lambda_{n_j} \to \tilde{\lambda} > 0$. But then again by the continuity of $\mathcal{G}$

$$0 = \lim_j \mathcal{G}(\omega_{n_j}, \lambda_{n_j}) = \mathcal{G}(\tilde{\omega}, \tilde{\lambda}),$$

which implies that $\tilde{\omega} \in \mathcal{A}^{unstable}$, a contradiction.

Now that we know that $\lambda_j \to 0, \omega_j \in \mathcal{A}^{unstable} \cap \{\omega > \omega^*(H)\}$, we have by (42) that

$$\frac{1}{4\omega_j^2\lambda_j^2} = \left\langle R(\omega_j, \lambda_j)^2[\phi'], \phi' \right\rangle = \left\langle R(\omega_j, \lambda_j)[\phi'], R(\omega_j, -\lambda_j)[\phi'] \right\rangle \tag{43}$$

where in the last equality, we have used that

$$R(\omega, \lambda)^* = ((H + \lambda^2 + 2\omega\lambda P_0\partial_x P_0)^{-1})^* = (H + \lambda^2 - 2\omega\lambda P_0\partial_x P_0)^{-1} = R(\omega, -\lambda).$$

As in the proof of (33), set $z_{\pm j} = (H + \lambda_j^2 \pm 2\omega_j\lambda_j P_0\partial_x P_0)^{-1}[\phi']$. That is,

$$\begin{aligned} z_{\pm j} \quad : \quad & (H + \lambda_j^2 \pm 2\omega_j\lambda_j P_0\partial_x P_0) z_j = \phi' \\ & z_{\pm j} = a_{\pm j}\psi_0 + q_{\pm j}, q_{\pm j} \in \{\phi\}^\perp \end{aligned}$$

In the course of the proof of (33), we have established the following formulas for the leading order terms of $a_{\pm j}, q_{\pm j}$,

$$\begin{aligned} a_{\pm j} &= \frac{1}{\lambda_j^2} \frac{\langle \phi', \psi_0 \rangle}{1 + 4\omega_j^2 \|H_1^{-1/2}[P_0[\psi_0']]\|_{L^2}^2} + O(\lambda_j^{-1}) \\ q_{\pm j} &= O(\lambda_j^{-1}) \end{aligned}$$

Now by $q_{\pm j} \in \{\phi\}^\perp$,

$$\left\langle R(\omega_j, \lambda_j)[\phi'], R(\omega_j, -\lambda_j)[\phi'] \right\rangle = \langle a_j\psi_0 + q_j, a_{-j}\psi_0 + q_{-j} \rangle = a_j a_{-j} + \langle q_j, q_{-j} \rangle = \frac{M_j^2}{\lambda_j^4} + O(\lambda_j^{-2}),$$

where $M_j = \langle \phi', \psi_0 \rangle (1 + 4\omega_j^2 \|H_1^{-1/2}[P_0[\psi_0']]\|_{L^2}^2)^{-1} \neq 0$, by (21). Combining this with (43) yields

$$\frac{1}{4\omega_j^2\lambda_j^2} = \frac{M_j^2}{\lambda_j^4} + O(\lambda_j^{-2}),$$

or

$$\frac{\lambda_j^2}{4\omega_j^2} = M_j^2 + O(\lambda_j^2).$$

Taking $j \to \infty$ in the last identity and taking into account that $\lambda_j \to 0, \omega_j \to \tilde{\omega} \neq 0$, $M_j \to \tilde{M} \neq 0$ yields a contradiction.

We have established a contradiction, which arose out of the assumption $\mathcal{A}^{unstable} \cap \{\omega : \omega > \omega^*(H)\} \neq \emptyset$. Thus, $\mathcal{A}^{unstable} \cap \{\omega : \omega > \omega^*(H)\} = \emptyset$ and hence $\mathcal{A}^{stable} = \{\omega : \omega \geq \omega^*(H)\}$.

3.4. **Proof of Corollary 1.** In the course of our exposition for the Shkalikov's theory, we have proved that the spectrum of the pencil (and hence the spectrum of the operator $\mathcal{T}$) is invariant under complex conjugation. More specifically, if (λ, ψ) solves (18), then so does $(\bar{\lambda}, \bar{\psi})$.

Now, under the assumptions in Corollary 1, we can also show that the spectrum is invariant under the transformation $\lambda \to -\lambda$. Indeed, if (λ, ψ) solves (18), then so does $(-\lambda, \psi(-\cdot))$.

Thus, in the unstable situation, we have shown that there is unique positive eigenvalue λ of $\mathcal{T}$ and hence there is an unique negative one as well. Moreover, this argument excludes the possibility for more negative spectrum, since this would imply more positive (unstable) eigenvalues for the pencil, which is forbidden by the Shkalikov's theory.

In the stable situation, one shows by contradiction argument, similar to the one in the previous paragraph, that there is no spectrum off the imaginary axes and thus $\sigma(\mathcal{T}) \subset i\mathbf{R}^1$.

4. Proof of Lemma 3

We now proceed to show Lemma 3. Denote in this section $\omega_0 := \omega^*(H)$ for sake of brevity. We shall need several propositions.

Proposition 4. *For any ω, the function $\varepsilon \to \varepsilon^2 \mathcal{G}(\omega, \varepsilon)$ is real analytic (close to zero) and it has the Laurent expansion*

$$\mathcal{G}(\omega, \varepsilon) = \varepsilon^{-2} D_{-2}(\omega) + \sum_{j=0}^{\infty} D_j(\omega) \varepsilon^j, \tag{44}$$

where $D_{-1}(\omega) = 0$,

$$D_{-2}(\omega) = \frac{\langle \phi', \psi_0 \rangle^2}{1 + 4\omega^2 \| H^{-1/2} P_0[\psi_0'] \|^2},$$

and the functions $\{D_j\}_{j=1}^{\infty}$ are smooth functions of ω. Finally, the radius of analyticity $r(\omega)$ satisfies

$$\inf_{\omega \in [a,b] \subset (0,\infty)} r(\omega) \geq r_{a,b} > 0.$$

That is, on every compact interval $I = [a.b]$, one may choose a common radius of analyticity.

Proof. Most of the statements in Proposition 4 were in fact considered before, see for example the formula for $D_{-2}(\omega)$, derived in (33).

Let us first indicate the real analyticity of the function $\varepsilon^2 \mathcal{G}(\varepsilon)$. We have, from the defining equations and in our previous notations,

$$\mathcal{G}(\omega, \varepsilon) = a_\varepsilon \langle \psi_0, \phi' \rangle + \langle q_\varepsilon, \phi' \rangle + \frac{1}{4\omega^2} - \frac{\delta^2}{4\omega^2 \varepsilon^2}.$$

where $a = a_\varepsilon$ and $q = q_\varepsilon$ are determined from the pair of equations

$$a\varepsilon^2 - 2\omega\varepsilon \langle q, \psi_0' \rangle = \langle \phi', \psi_0 \rangle \tag{45}$$

$$q_\varepsilon = \mathcal{L}_\varepsilon^{-1}(-2a\varepsilon\omega P_0[\psi_0'] + P_1[\phi']), \tag{46}$$

where $\mathcal{L}_\varepsilon = (H + \varepsilon^2 + 2\omega\varepsilon P_1 \partial_x P_1)$. As we observed before, by the von Neumann expansion,

$$\mathcal{L}_\varepsilon^{-1} = \sum_{k=0}^{\infty} (-\varepsilon)^k (H^{-1}(\varepsilon + 2\omega P_1 \partial_x P_1))^k H_1^{-1} = H_1^{-1} + O(\varepsilon)$$

and is real-analytic in ε in a region $|\varepsilon| < \varepsilon_0 << 1$. Substituting the expression for q in the expression for a yields the formula

$$a\varepsilon^2 + 4\omega^2\varepsilon^2 \langle \mathcal{L}_\varepsilon^{-1} P_0[\psi_0'], [\psi_0'] \rangle = 2\omega\varepsilon \langle \mathcal{L}_\varepsilon^{-1} P_1[\phi'], [\psi_0'] \rangle + \langle \phi', \psi_0 \rangle$$

and

$$a\varepsilon^2 = \frac{2\omega\varepsilon \left\langle \mathcal{L}_\varepsilon^{-1} P_1[\phi'], [\psi_0'] \right\rangle + \langle \phi', \psi_0 \rangle}{1 + 4\omega^2 \left\langle \mathcal{L}_\varepsilon^{-1} P_0[\psi_0'], [\psi_0'] \right\rangle}.$$

Observe that $\left\langle \mathcal{L}_\varepsilon^{-1} P_0[\psi_0'], \psi_0' \right\rangle = \|H^{-1/2} P_0[\psi_0']\|^2 + O(\varepsilon)$, whence the denominator stays away from zero, whence the real analyticity of $\varepsilon \to \varepsilon^2 a(\varepsilon)$. Furthermore, the coefficients $D_j(\omega)$ (which can be written by means of the Cauchy formula) are smooth functions of ω. Substituting now this back in the formula (46) yields that $\varepsilon \to \varepsilon q(\varepsilon)$ is real analytic too. Observe also that the radius of analyticity of both functions may be estimated below in terms of ω^{-1}. Therefore, it is uniformly bounded away from zero, when ω belongs to some compact interval, away from zero.

We now compute $D_{-2}(\omega), D_{-1}(\omega)$. To that end, introduce A, B, so that

$$a(\varepsilon) = \varepsilon^{-2} A + \varepsilon^{-1} B + O(1).$$

We compute q up to order $O(1)$. We have $\mathcal{L}_\varepsilon^{-1} = H_1^{-1} + O(\varepsilon)$ and therefore

$$\begin{aligned} q &= (H_1^{-1} + O(\varepsilon))(-2\varepsilon^{-1} A\omega P_0[\psi_0'] - 2B\omega P_0[\psi_0'] + P_1[\phi']) = \\ &= -2A\omega\varepsilon^{-1} H^{-1} P_0[\psi_0'] - 2B\omega H^{-1} P_0[\psi_0'] + H^{-1} P_1[\phi'] + O(1) \end{aligned}$$

Plugging this in (45), we get two equations - at the scale of ε^0 and ε^1

$$\begin{aligned} &A - 2\omega(-2A\omega \left\langle H_1^{-1} P_0[\psi_0'], \psi_0' \right\rangle) = \langle \phi', \psi_0 \rangle \\ &B - 2\omega(\left\langle H^{-1} P_1[\phi'], \psi_0' \right\rangle - 2B\omega \left\langle H^{-1} P_0[\psi_0'], \psi_0' \right\rangle) = 0. \end{aligned}$$

From those, we get

$$A = \frac{\langle \phi', \psi_0 \rangle}{1 + 4\omega^2 \|H^{-1/2} P_0[\psi_0']\|^2}$$

$$B = \frac{2\omega \left\langle H^{-1} P_1[\phi'], \psi_0' \right\rangle}{1 + 4\omega^2 \|H^{-1/2} P_0[\psi_0']\|^2}$$

We are now ready to compute $\mathcal{G}(\omega, \varepsilon)$ up to order $O(1)$. We have

$$\begin{aligned} \mathcal{G}(\omega, \varepsilon) &= a_\varepsilon \langle \phi', \psi_0 \rangle + \langle q_\varepsilon, \phi' \rangle + \frac{1}{4\omega^2} - \frac{\delta^2}{4\omega^2 \varepsilon^2} = \\ &= \varepsilon^{-2} \left(A \langle \phi', \psi_0 \rangle - \frac{\delta^2}{4\omega^2 \varepsilon^2} \right) + \varepsilon^{-1} \left(B \langle \phi', \psi_0 \rangle - 2A\omega \left\langle H^{-1} P_0[\psi_0'], \phi' \right\rangle \right) + O(1) \end{aligned}$$

It remains to observe that

$$\begin{aligned} D_{-2}(\omega) &= A \langle \phi', \psi_0 \rangle - \frac{\delta^2}{4\omega^2} = \frac{\langle \phi', \psi_0 \rangle^2}{1 + 4\omega^2 \|H^{-1/2} P_0[\psi_0']\|^2} - \frac{\delta^2}{4\omega^2}, \\ D_{-1}(\omega) &= B \langle \phi', \psi_0 \rangle - 2A\omega \left\langle H^{-1} P_0[\psi_0'], \phi' \right\rangle = 0. \end{aligned}$$

□

Several things to note. First, recall that the definition of $\omega_0 = \omega^*(H)$ is so that $D_{-2}(\omega_0) = 0$. In fact, $D_{-2}(\omega) < 0, \omega < \omega_0$ and $D_{-2}(\omega) > 0, \omega > \omega_0$.

Now, consider the real-analytic function

$$\mathcal{G}(\omega_0, \varepsilon) = \varepsilon^{-2} D_{-2}(\omega_0) + \sum_{j=0}^\infty D_j(\omega_0) \varepsilon^j = \sum_{j=0}^\infty D_j(\omega_0) \varepsilon^j.$$

By the analyticity, we have that there is k, so that $D_k(\omega_0) \neq 0$. Indeed, otherwise, the real-analytic function $\varepsilon \to \mathcal{G}(\omega_0, \varepsilon)$ will be identical to zero, at least for all ε small enough. This is however a contradiction with the meaning of the function $\mathcal{G}(\omega_0, \cdot)$. Indeed, not only it implies instability at ω_0, but it gives eigenvalues $\lambda = \varepsilon \in (0, \varepsilon_0)$, clearly in contradiction with our earlier conclusion (basically a consequence of Shkalikov theory) that there is at most one eigenvalue. Thus, there is $k : D_k(\omega_0) \neq 0$. Let $k_0 = \min\{k \geq 0 : D_k(\omega_0) \neq 0\}$.

Claim: $D_{k_0}(\omega_0) > 0$.

Indeed, take $\omega_j \to \omega_0+$. By our arguments earlier, for $\omega > \omega_0 = \omega^*(H)$, whether[10] $\omega \in \mathcal{A}^{stable}$ or $\omega \in \mathcal{A}^{unstable}$, $\mathcal{G}(w, \lambda) \geq 0$. Thus, taking any $\varepsilon : 0 < \varepsilon < r_0 = \inf r(\omega_j)$ (which is strictly positive by Proposition 4),

$$
\begin{aligned}
0 \leq \limsup_j \mathcal{G}(\omega_j, \varepsilon) &= \limsup_j (\varepsilon^{-2} D_{-2}(w_j) + (\sum_{m=0}^{k_0-1} D_m(\omega_j)\varepsilon^j) + D_{k_0}(\omega_j)\varepsilon^{k_0} + O(\varepsilon^{k_0+1})) = \\
&= D_{k_0}(\omega_0)\varepsilon^{k_0} + O(\varepsilon^{k_0+1})
\end{aligned}
$$

Clearly, the last inequality will be contradictory for small $\varepsilon > 0$, unless $D_{k_0}(\omega_0) \geq 0$, which was the Claim (recall $D_{k_0}(\omega_0) \neq 0$). Having established this claim, we will show

Proposition 5. *There exists a sequence $\{\omega_\varepsilon\}_{\varepsilon>0}$, so that*

(1) $\omega_\varepsilon < \omega_0 = \omega^*(H)$ *(and hence $\omega_\varepsilon \in \mathcal{A}^{unstable}$ and $\mathcal{G}(\omega_\varepsilon, \lambda_\varepsilon) = 0$ for some $\lambda_\varepsilon = \lambda(\omega_\varepsilon) > 0$),*
(2) $\lim_{\varepsilon \to 0+} \omega_\varepsilon = \omega_0$
(3) $\lambda_\varepsilon = \lambda(\omega_\varepsilon) \to 0+$.

We claim that the existence of such a sequence implies that $\omega_0 = \omega^*(H) \in \mathcal{A}^{stable}$. Indeed, assuming that $\omega_0 = \omega^*(H) \in \mathcal{A}^{unstable}$ implies that there is $\lambda_0 = \lambda(\omega_0) > 0$. By the continuity of the the mapping $\omega \to \lambda(\omega)$ established in Lemma 2, we must have

$$\lim_{\varepsilon \to 0} \lambda(\omega_\varepsilon) = \lambda(\omega_0) = \lambda_0 > 0,$$

in contradiction with property (3) above. Thus, it remains to prove Proposition 5.

Proof. (Proposition 5) Define $\omega_\varepsilon := \omega_0 - \varepsilon^{k_0+3}$. We will show that this sequence satisfies the requirements of Proposition 5, for all small enough ε. We have for all
$\varepsilon : 0 < \varepsilon < \min(\inf_{\omega \in [\omega_0/2, \omega_0]} r(\omega), 1, \omega_0/2)$

$$
\begin{aligned}
\mathcal{G}(\omega_\varepsilon, \varepsilon) &= \varepsilon^{-2} D_{-2}(w_\varepsilon) + (\sum_{m=0}^{k_0-1} D_m(\omega_\varepsilon)\varepsilon^j) + D_{k_0}(\omega_\varepsilon)\varepsilon^{k_0} + O(\varepsilon^{k_0+1}) = \\
&= \varepsilon^{-2}(D_{-2}(w_0) + O(\varepsilon^{k_0+3})) + (\sum_{m=0}^{k_0-1} D_m(\omega_0)\varepsilon^j + O(\varepsilon^{k_0+3})) + D_{k_0}(\omega_0)\varepsilon^{k_0} + O(\varepsilon^{k_0+1}) \\
&= D_{k_0}(\omega_0)\varepsilon^{k_0} + O(\varepsilon^{k_0+1})
\end{aligned}
$$

where we have used that $D_{-2}(w_0) = D_0(\omega_0) = \ldots = D_{k_0-1}(\omega_0) = 0$. Clearly, the last expression is positive (due to $D_{k_0}(\omega_0) > 0$) for all small enough $\varepsilon > 0$.

Now that we have established $\mathcal{G}(\omega_\varepsilon, \varepsilon) > 0$ for all $\varepsilon \in (0, \varepsilon_0)$, we still have $D_{-2}(\omega_\varepsilon) < 0$ (since $\omega_\varepsilon < \omega$) and hence

$$\lim_{\lambda \to 0+} \mathcal{G}(\omega_\varepsilon, \lambda) = \lim_{\lambda \to 0+} \left(\lambda^{-2} D_{-2}(\omega_\varepsilon) + O(1)\right) = -\infty.$$

Therefore, the continuous function $\lambda \to \mathcal{G}(\omega_\varepsilon, \lambda)$ experiences a sign change in $(0, \varepsilon)$, whence the unique root $\lambda_\varepsilon = \lambda(\omega_\varepsilon) \in (0, \varepsilon)$. This establishes (3) in Proposition 5 and hence Lemma 3.

[10]For $\omega \in \mathcal{A}^{stable}$ and for all $\lambda > 0$, $\mathcal{G}(w, \lambda) > 0$, whereas for $\omega \in \mathcal{A}^{unstable}$, $\mathcal{G}(w, \lambda) > 0$ for all $\lambda \neq \lambda_0$ and $\mathcal{G}(w, \lambda_0) = 0$

□

5. Linear stability analysis for the Boussinesq model

The proof of Theorem 2 follows directly from Theorem 1, applied to the linearized problem $z_{tt} + 2cz_{tx} + Hz = 0$, where H is as defined in (16). In order to follow this program, we need to verify that the conditions in Theorem 1 for the operator H are met, then we need to compute the critical index $\omega^*(H)$ and finally, we need to establish that the linear instability for $z_{tt} + 2cz_{tx} + Hz = 0$ is equivalent to the linear instability for the actual problem (15). We will do this in a sequence of several propositions.

Proposition 6. *(Spectral properties of H) The operator H, defined in* (16) *satisfies* (19), (20), (21), (22).

Next,

Proposition 7. *The critical index* $\omega^*(H) = \frac{\sqrt{(p-1)(1-c^2)}}{\sqrt{5-p}}$.

Regarding the existence of the unstable solutions, we have already explained that an instability in $z_{tt} + 2cz_{tx} + Hz = 0$ implies instability in (15). We need to show then the reverse.

Proposition 8. *Suppose that $V \in H^4(\mathbf{R}^1)$ solves the equation*

$$\lambda^2 V + 2c\lambda V' + T[V] = 0,$$

for some $\lambda > 0$. Then there exists a function $Z \in H^4(\mathbf{R}^1)$, so that $Z' = V$ and so that Z solves

$$\lambda^2 Z + 2c\lambda Z' + H[Z] = 0$$

Let us show now that Theorem 2 is a simple consequence of Propositions 6, 7 and 8. Indeed, since we can apply Theorem 1, we only need to find the intervals, in which the speeds yield stable traveling waves. To that end, we set up the inequality

$$1 > |c| \geq \omega^*(H) = \frac{\sqrt{(p-1)(1-c^2)}}{\sqrt{5-p}}.$$

The solution to this inequality is

$$1 > |c| \geq \frac{\sqrt{p-1}}{2}.$$

In the complementary set, $0 \leq |c| < \frac{\sqrt{p-1}}{2}$, we have instability. This finishes the proof of Theorem 2, modulo the claims of the Propositions.

5.1. **Proof of Proposition 6.** We start off by noting that (22) is obvious by inspection. Similarly, (20) follows from the fact that H has one negative eigenvalue[11] and thus, for $\tau >> 1$, we have that $(H + \tau) > Id$, hence is invertible and moreover

$$(H + \tau)^{-1} : L^2 \to H^4,$$

and more generally for all $s \in [0, 1]$, $(H + \tau)^{-s} : L^2 \to H^{4s}$. Therefore, we conclude $(H + \tau)^{-1/2}\partial_x(H + \tau)^{-1/2}, \partial_x(H + \tau)^{-1} : L^2 \to L^2$ are a bounded operators.

[11]to be proved below in this proposition

5.1.1. *Proof of* (19). To that end, note first that H can be written in the following form

$$H = -\partial_x \mathcal{L} \partial_x, \quad \mathcal{L} := -\partial_x^2 + (1-c^2) - p\varphi_c^{p-1}$$

The operator $\mathcal{L}$ is a second order differential operator, which appears in the study of the KdV equation in exactly the same way - as a linearization around the traveling wave solution φ_c. Its spectral properties are well-documented and we just remind the ones that are of concern to us. Namely it has a simple eigenvalue at zero (with eigenvector φ_c'), single and simple negative eigenvalue and it has a spectral gap, that is the rest of the spectrum lies in a set $[\kappa, \infty), \kappa > 0$, see for example [6], Section 3.

We will now be able to infer the same structure for the spectrum of H, mainly due to the representation $H = -\partial_x \mathcal{L} \partial_x$. First, it has an eigenvalue at zero, with eigenvector φ_c ($H\varphi_c = -\partial_x \mathcal{L}[\varphi_c'] = 0$). This eigenvalue is simple (otherwise a contradiction with the simplicity of the zero eigenvalue for $\mathcal{L}$).

Next, we check that H has a negative eigenvalue. Taking two spatial derivatives in (6) we get

$$\varphi_c'''' - (1-c^2)\varphi_c'' + p\varphi_c^{p-1}\varphi'' + p(p-1)\varphi_c^{p-2}(\varphi_c')^2 = 0 \tag{47}$$

taking dot product with φ_c'' and taking into account that $-\mathcal{L}\varphi_c'' = \varphi_c'''' - (1-c^2)\varphi_c'' + p\varphi_c^{p-1}\varphi''$, we get

$$-\left\langle \mathcal{L}\varphi_c'', \varphi_c'' \right\rangle + p(p-1)\int \varphi_c^{p-2}(\varphi_c')^2 \varphi_c'' dx = 0$$

But $\int \varphi_c^{p-2}(\varphi_c')^2\varphi_c'' dx = -\frac{p-2}{3}\int \varphi_c^{p-3}(\varphi_c')^2 dx$. Thus

$$\left\langle H\varphi_c', \varphi_c' \right\rangle = \left\langle \mathcal{L}\varphi_c'', \varphi_c'' \right\rangle = -\frac{p(p-1)(p-2)}{3}\int \varphi_c^{p-3}(\varphi_c')^2 dx.$$

Note that if $p > 2$, the expression is negative, thus yielding directly (by the Ritz-Reileigh criteria) that the minimal eigenvalue

$$\lambda_0(H) = \inf_{\psi:\|\psi\|=1} \langle H\psi, \psi \rangle \leq \|\varphi_c'\|^{-2}\left\langle H\varphi_c', \varphi_c' \right\rangle < 0.$$

Even in the case $p = 2$, this argument implies the existence of a negative eigenvalue. Indeed, assuming that $H \geq 0$ (and since we already know that φ_c is a simple eigenvalue), it follows that $H|_{\{\varphi_c\}^\perp} > 0$. Thus, since $\varphi_c' \in \{\varphi_c\}^\perp$, it should be that $\langle H\varphi_c', \varphi_c' \rangle > 0$, a contradiction with $\langle H\varphi_c', \varphi_c' \rangle = 0$, in the case $p = 2$. This shows that $\lambda_0(H) < 0$.

Finally, we need to establish that $\lambda_0(H)$ is simple. This would follow, if we manage to show that $\lambda_1(H) \geq 0$ (and indeed, since 0 is an eigenvalue, it would follow that $\lambda_1(H) = 0$). Denote the (smooth) eigenvector of $\mathcal{L}$ by ζ. One may apply the Courant maxmin principle for the first eigenvalue, which states

$$\lambda_1(H) = \sup_{z \neq 0} \inf_{u \perp z} \frac{\langle Hu, u \rangle}{\|u\|^2}$$

Taking $z = \phi'$ above yields

$$\lambda_1(H) = \sup_{z \neq 0} \inf_{u \perp z} \frac{\langle Hu, u \rangle}{\|u\|^2} \geq \inf_{u \perp \phi'} \frac{\langle Hu, u \rangle}{\|u\|^2} = \inf_{u \perp \phi'} \frac{\langle \mathcal{L}u', u' \rangle}{\|u\|^2} = \inf_{u' \perp \phi} \frac{\langle \mathcal{L}u', u' \rangle}{\|u\|^2}$$

where in the last identity, we have used that $u \perp \phi'$ exactly when $u' \perp \phi$. It remains to observe now that since $\mathcal{L}$ has a simple negative eigenvalue, with eigenvector ϕ, we have $\mathcal{L}|_{\{\phi\}^\perp} \geq 0$ and hence $\langle \mathcal{L}u', u' \rangle \geq 0$ (since $u' \in \{\phi\}^\perp$), whence

$$\lambda_1(H) \geq \inf_{u' \perp \phi} \frac{\langle \mathcal{L}u', u' \rangle}{\|u\|^2} \geq 0.$$

Thus, property (19) is fully established.

5.1.2. *Proof of* (21). Note first that in our notations, $\psi_0 = \varphi_c/\|\varphi_c\|$ and denote the negative eigenvector of H by $\phi, \|\phi\| = 1$. We need to show $\langle \phi', \varphi_c \rangle \neq 0$. We separate the proof in the cases $p > 2$ and $p = 2$. We have by our computations in Section 5.1.1 above that

$$\left\langle H\varphi_c', \varphi_c' \right\rangle < 0$$

It follows that

$$0 > \left\langle H\varphi_c', \varphi_c' \right\rangle = \lambda_0 \left\langle \varphi_c', \phi \right\rangle^2 + \left\langle H(\varphi_c' - \left\langle \varphi_c', \phi \right\rangle \phi), (\varphi_c' - \left\langle \varphi_c', \phi \right\rangle \phi) \right\rangle$$

Note that $(\varphi_c' - \langle \varphi_c', \phi \rangle \phi) \in \{\phi\}^\perp$. Since $\lambda_0(H)$ is the only negative eigenvalue for H and it is simple, it follows that $H|_{\{\phi\}^\perp} \geq 0$ and in particular

$$\left\langle H(\varphi_c' - \left\langle \varphi_c', \phi \right\rangle \phi), (\varphi_c' - \left\langle \varphi_c', \phi \right\rangle \phi) \right\rangle \geq 0.$$

The last two inequalities imply that

$$0 > \lambda_0 \left\langle \varphi_c', \phi \right\rangle^2,$$

whence $\langle \varphi_c', \phi \rangle \neq 0$. But then $\langle \phi', \varphi_c \rangle = - \langle \varphi_c', \phi \rangle \neq 0$.

In the case $p = 2$, as we have shown $\langle H\varphi_c', \varphi_c' \rangle = 0$, so we need to be more precise in the arguments above. In particular, we need to observe that since $\varphi_c' \perp \varphi_c$ and $\phi \perp \varphi_c$ (as eigenvectors corresponding to different eigenvalues), it follows that actually

$$\varphi_c' - \left\langle \varphi_c', \phi \right\rangle \phi \in span[\phi, \varphi_c]^\perp. \tag{48}$$

Moreover, by (47) (in the case $p = 2$), we have that $\mathcal{L}[\varphi_c''] = 2(\varphi_c')^2$, whence

$$\mathcal{H}[\varphi_c'] = -\partial_x \mathcal{L}[\varphi_c''] = -4\varphi_c'\varphi_c'' \neq \lambda_0 \varphi_c'.$$

The last computation shows that φ_c' and ϕ are linearly independent and hence, we can upgrade (48) to

$$0 \neq \varphi_c' - \left\langle \varphi_c', \phi \right\rangle \phi \in span[\phi, \varphi_c]^\perp. \tag{49}$$

By (19), in particular the spectral gap that we have established for H, and (49), it follows that

$$\left\langle H(\varphi_c' - \left\langle \varphi_c', \phi \right\rangle \phi), (\varphi_c' - \left\langle \varphi_c', \phi \right\rangle \phi) \right\rangle \geq \kappa \|\varphi_c' - \left\langle \varphi_c', \phi \right\rangle \phi\|^2 > 0,$$

where κ is the size of the spectral gap. We can conclude

$$\begin{aligned} \lambda_0 \left\langle \varphi_c', \phi \right\rangle^2 &= \left\langle H\varphi_c', \varphi_c' \right\rangle - \left\langle H(\varphi_c' - \left\langle \varphi_c', \phi \right\rangle \phi), (\varphi_c' - \left\langle \varphi_c', \phi \right\rangle \phi) \right\rangle = \\ &= -\left\langle H(\varphi_c' - \left\langle \varphi_c', \phi \right\rangle \phi), (\varphi_c' - \left\langle \varphi_c', \phi \right\rangle \phi) \right\rangle < 0, \end{aligned}$$

whence $\langle \phi', \varphi_c \rangle = - \langle \varphi_c', \phi \rangle \neq 0$.

5.2. **Proof of Proposition 7.** Recall $H = -\partial_x \mathcal{L} \partial_x$. In our notations, we need to compute $\left\langle H^{-1}\psi_0', \psi_0' \right\rangle = \frac{1}{\|\varphi_c\|^2} \left\langle H^{-1}\varphi_c', \varphi_c' \right\rangle$.

Again, starting from (6), take a derivative in the parameter c. We obtain,

$$(\partial_c \varphi)'' - (1 - c^2)\partial_c \varphi + p\varphi^{p-1}\partial_c \varphi + 2c\varphi = 0.$$

In terms of the operator $\mathcal{L}$, we have $\mathcal{L}[\partial_c \varphi] = 2c\varphi$ or $\frac{1}{2c}\partial_c \varphi = \mathcal{L}^{-1}[\varphi_c]$. Here recall that $\mathcal{L}$ is invertible on $Ker[\mathcal{L}]^\perp = \{\varphi_c'\}^\perp$ and $\varphi_c \in \{\varphi_c'\}^\perp$.

Introduce z, so that $\varphi_c' = H[z]$. We know that such a z exists, since the operator H is invertible on the subspace $Ker[H]^\perp = \{\varphi_c\}^\perp$ and $\varphi_c' \in \{\varphi_c\}^\perp$. We have

$$\varphi_c' = H[z] = -\partial_x \mathcal{L}[z']$$

Thus $\varphi_c = -\mathcal{L}[z'] + const$ and the constant turns out to be zero by testing this identity at $x \to \infty$. Thus, z is such that $\mathcal{L}[z'] = -\varphi_c$ and hence

$$z' = -\mathcal{L}^{-1}[\varphi_c] = -\frac{1}{2c}\partial_c \varphi.$$

Now

$$-\left\langle H^{-1}\varphi_c', \varphi_c'\right\rangle = -\left\langle z, \varphi_c'\right\rangle = \left\langle z', \varphi_c\right\rangle = -\frac{1}{2c}\left\langle \partial_c\varphi_c, \varphi_c\right\rangle = -\frac{1}{4c}\partial_c\|\varphi_c\|^2.$$

All in all

$$-\left\langle H^{-1}\psi_0', \psi_0'\right\rangle = -\frac{1}{\|\varphi_c\|^2}\left\langle H^{-1}\varphi_c', \varphi_c'\right\rangle = -\frac{1}{4c}\frac{\partial_c[\|\varphi_c\|^2]}{\|\varphi_c\|^2]}$$

It remains to compute $\|\varphi_c\|^2$ and perform the elementary calculus operations. We have

$$\begin{aligned} \|\varphi_c\|^2 &= \left(\frac{p+1}{2}(1-c^2)\right)^{\frac{2}{p-1}} \int_{-\infty}^{\infty} sech^{\frac{4}{p-1}}\left(\frac{\sqrt{1-c^2}(p-1)}{2}\xi\right) d\xi = \\ &= m_p(1-c^2)^{\frac{2}{p-1}-\frac{1}{2}} \end{aligned}$$

Thus,

$$-\frac{1}{4c}\frac{\partial_c[\|\varphi_c\|^2]}{\|\varphi_c\|^2]} = \frac{5-p}{4(p-1)(1-c^2)}.$$

Clearly, for $p \geq 5$, we have $\left\langle H^{-1}\psi_0', \psi_0'\right\rangle = \frac{p-5}{4(p-1)(1-c^2)} \geq 0$ and hence instability. If $p < 5$, we get

$$\omega^*(H) = \frac{1}{2\sqrt{-\left\langle H^{-1}\psi_0', \psi_0'\right\rangle}} = \frac{\sqrt{(p-1)(1-c^2)}}{\sqrt{5-p}}.$$

5.3. **Proof of Proposition 8.** Proposition 8 is a simple exercise. Writing down the particular form of the operator T, we see that V satisfy

$$(\partial_x^4 - (1-c^2)\partial_x^2 + 2c\lambda\partial_x + \lambda^2)V = -p\partial_x^2(\varphi_c^{p-1}V).$$

Observe that the symbol of the operator $M = \partial_x^4 - (1-c^2)\partial_x^2 + 2c\lambda\partial_x + \lambda^2$

$$\xi^4 + (1-c^2)\xi^2 - 2ci\lambda\xi + \lambda^2$$

has a positive real part, bounded away from zero and is thus invertible on L^2. In particular, we may write

$$V = -pM^{-1}\partial_x^2[\varphi_c^{p-1}V].$$

Now, it suffices to set

$$Z := -pM^{-1}\partial_x[\varphi_c^{p-1}V],$$

so that $Z_x = V$ (and hence it satisfies $\lambda^2 Z + 2c\lambda Z + H[Z] = 0$) and check that such a function is well-defined and belongs[12] to $L^2(\mathbf{R}^1)$. Since $\varphi_c^{p-1}V \in L^2$, it suffices to show that $\partial_x M^{-1}$ defines a bounded operator on L^2 or

$$\sup_{\xi}\left|\frac{\xi}{\xi^4 + (1-c^2)\xi^2 - 2ci\lambda\xi + \lambda^2}\right| < \infty$$

The last inequality follows by inspection.

[12]further regularity of course holds and can be inferred in the standard bootstrap fashion

6. Linear stability analysis for the Klein-Gordon-Zakharov system

In this section, we will provide the complete details of the proof of Theorem 3. We need to first setup the problem at hand in the form (17). This is not straight forward, due to the requirement that $\mathcal{H}$ be self-adjoint. In fact, the choice of $\frac{1}{2}$ that we have made in the formulation of (7) helps us accomplish exactly that.

We take the ansatz $u(t,x) = \varphi(x-ct) + v(t,x-ct), n(t,x) = \psi(x-ct) + w(t,x-ct)$ for real-valued functions v, w. We plug it in (7) and ignore all terms $O(v^2+w^2)$. We get the following linear equation for (v,w),

$$\left\{ \begin{array}{l} v_{tt} - 2cv_{tx} - (1-c^2)v_{xx} + v + \psi v + \varphi w = 0 \\ w_{tt} - 2cw_{tx} - (1-c^2)w_{xx} - (\varphi v)_{xx} = 0. \end{array} \right. \tag{50}$$

At this point, we introduce the convenient quantity $\mu = \sqrt{1-c^2}$. Setting $w = z_x$ and taking off one derivative from the second equation, we obtain the following system for $\vec{\Phi} = \left(\begin{array}{c} v \\ z \end{array} \right)$

$$\vec{\Phi}_{tt} - 2c\vec{\Phi}_{tx} + \mathcal{H}\vec{\Phi} = 0, \quad \mathcal{H} := \left(\begin{array}{cc} H_1 & A \\ A^* & H_2 \end{array} \right), \tag{51}$$

where

$$\begin{aligned} H_1 &= -(1-c^2)\partial_{xx} + 1 + \psi = -\mu^2\partial_{xx} + 1 - \frac{\varphi^2}{2\mu^2} \\ H_2 &= -(1-c^2)\partial_{xx} = -\mu^2\partial_{xx} \\ Az &= \varphi z_x, A^* v = -(\varphi v)_x \end{aligned}$$

Clearly, the operator $\mathcal{H}$ is self-adjoint. It now remains to show that $\mathcal{H}$ satisfies the requirements of Theorem 1, after which, we will compute the quantity of interest $\langle \mathcal{H}^{-1}\psi_0', \psi_0' \rangle$. We formulate the needed results in a series of Propositions.

Proposition 9. *The self-adjoint operator $\mathcal{H}$ defined in* (51) *has a simple eigenvalue at zero, with an eigenvector $(\varphi', -\frac{\varphi^2}{2\mu^2})$.*

Proposition 10. *The self-adjoint operator $\mathcal{H}$ defined in* (51) *has one simple negative eigenvalue.*

6.1. **Proof of Proposition 9.** We need to solve $\mathcal{H}\left(\begin{array}{c} f \\ g \end{array} \right) = 0$. This is equivalent to

$$\left| \begin{array}{l} -\mu^2 f'' + f - \frac{\varphi^2}{2\mu^2} f + \varphi g' = 0 \\ -(\varphi f)' - \mu^2 g'' = 0. \end{array} \right.$$

Integrating the second equation yields $g' = -\frac{\varphi f}{\mu^2}$, which we plug in the first equation. We get

$$-\mu^2 f'' + f - \frac{3\varphi^2}{2\mu^2} f = 0. \tag{52}$$

Recall now the equation (8), which defines φ. In fact, taking a derivative ∂_x in it yields

$$-\mu^2\varphi''' + \varphi' - \frac{3\varphi^2}{2\mu^2}\varphi' = 0.$$

By comparing the last two formulas, we see that $f = \varphi'$ is a solution of (52). Moreover, by the standard theory for one-dimensional Hill's operators[13] (see for example [6], Section 3), φ' is the unique solution to (52), up to a multiplicative constant.

It remains to observe that since $g' = -\frac{\varphi f}{\mu^2} = -\frac{\varphi\varphi'}{\mu^2}$, we have

$$g = -\frac{\varphi^2}{2\mu^2}$$

and thus the eigenvector, corresponding to the zero eigenvalue is $(\varphi', -\frac{\varphi^2}{2\mu^2})$.

6.2. **Proof of Proposition 10.** The proof of Proposition 10 is less standard than the proof of Proposition 9. We set up the eigenvalue problem in the form $\mathcal{H}\begin{pmatrix} f \\ g \end{pmatrix} = -a^2 \begin{pmatrix} f \\ g \end{pmatrix}$ for some $a \in (0, \infty)$. That is, we need to show that there exists an unique $a_0 > 0$, so that the eigenvalue problem has an unique (up to a multiplicative constant) solution $\begin{pmatrix} f \\ g \end{pmatrix}$. Write the eigenvalue problem in the form

$$\left| \begin{array}{l} -\mu^2 f'' + f - \frac{\varphi^2}{2\mu^2} f + \varphi g' = -a^2 f \\ -\mu^2 g'' - (\varphi f)' = -a^2 g. \end{array} \right.$$

From the second equation, we express $g = \partial_x (a^2 - \mu^2 \partial_{xx})^{-1} [\varphi f]$. This is possible, since $a^2 - \mu^2 \partial_{xx} \geq a^2 Id$ and hence is invertible. Thus,

$$g' = \partial_{xx}(a^2 - \mu^2 \partial_{xx})^{-1}[\varphi f] = -\frac{\varphi f}{\mu^2} + \frac{a^2}{\mu^2}(a^2 - \mu^2 \partial_{xx})^{-1}[\varphi f].$$

Plugging this in the first equation, we obtain

$$-\mu^2 \partial_{xx} f + (1 + a^2) f - \frac{3\varphi^2}{2\mu^2} f + \frac{a^2}{\mu^2}[\varphi(a^2 - \mu^2 \partial_{xx})^{-1}(\varphi f)] = 0. \tag{53}$$

To recapitulate, we have obtained the equation (53) to be equivalent to the eigenvalue problem $\mathcal{H}\begin{pmatrix} f \\ g \end{pmatrix} = -a^2 \begin{pmatrix} f \\ g \end{pmatrix}$. That is, to prove Proposition 10, we need to show that there exists an unique $a_0 > 0$, so that the operator

$$M_a = -\mu^2 \partial_{xx} + (1 + a^2) - \frac{3\varphi^2}{2\mu^2} + \frac{a^2}{\mu^2}[\varphi(a^2 - \mu^2 \partial_{xx})^{-1}(\varphi \cdot)],$$

has an eigenvalue at zero for $a = a_0$ and in addition, this eigenvalue is simple.

Several things to note for the one-parameter family of operators M_a. It is clear that $M_a, a \geq 0$ are self-adjoint and in addition,

Claim: If $a \geq b \geq 0$, then $M_a \geq M_b + (a^2 - b^2) Id \geq M_b$.

Let us finish the proof of Proposition 10, based on this Claim, whose proof we postpone for the end of this section. Denote

$$\lambda(a) = \inf\{\lambda : \lambda \in \sigma(M_a)\} = \inf_{\|f\|=1} \langle M_a f, f \rangle .$$

Clearly, $a \to \lambda(a)$ is a continuous function and in view of Claim, $a \to \lambda(a)$ is an increasing function of its argument.

[13]Note that this particular operator is in fact the ubiquitous L_-, which appears in the linearization of standing waves for the cubic Schrödinger equation, which is known to have one-dimensional kernel

Next, we consider the easy case $a >> 1$. Observe that if $a^2 > \sup_x \frac{3\varphi^2(x)}{2\mu^2} - 1$, M_a becomes a positive operator (the operator $[\varphi(a^2 - \mu^2\partial_{xx})^{-1}(\varphi\cdot)]$ is positive by its Fourier transform representation, see the proof of the Claim below) and thus $\lambda(a) > 0$.

The case $a = 0$ presents another interesting observation. Namely, since

$$M_0 = -\mu^2\partial_{xx} + 1 - \frac{3\varphi^2}{2\mu^2},$$

we already know that $M_0[\varphi'] = 0$ and hence by Sturm-Liouville theory, there is a negative eigenvalue for M_0, that is $\lambda(0) < 0$. Alternatively, one may directly compute

$$\langle M_0\varphi, \varphi\rangle = -\frac{1}{\mu^2}\int_{-\infty}^{\infty} \varphi^3(y)dy < 0,$$

whence again $\lambda(0) < 0$. Thus, the continuous and increasing function $a \to \lambda(a)$ is negative at $a = 0$ and positive for large a, whence it has exactly one zero, say a_0. Thus, the eigenvalue that we are looking for is $\lambda = -a_0^2 < 0$. We still need to check that this eigenvalue is simple, which by the equivalences that we have established means that we have to show that 0 is an isolated (simple) eigenvalue of M_{a_0}.

This is however an easy consequence of the fact that $M_{a_0} \geq M_0 + a_0^2 Id$ (from the Claim). Indeed, denote by ϕ_0 the eigenvector for M_0, which corresponds to its unique and simple negative eigenvalue ([6], Section 3). Then, $M_0|_{\{\phi_0\}^\perp} \geq 0$, whence by the Courant maxmin principle

$$\lambda_1(M_{a_0}) = \sup_{z\neq 0} \inf_{u\perp z:\|u\|=1} \langle M_{a_0}u, u\rangle \geq a_0^2 + \inf_{u\perp\phi_0:\|u\|=1} \langle M_0u, u\rangle \geq a_0^2,$$

Thus, we have that $\lambda(a_0) = \lambda_0(M_{a_0}) = 0$, while $\lambda_1(M_{a_0}) \geq a_0^2$, which finishes the proof of Proposition 10.

6.2.1. *Proof of Claim.* By the particular form of the operators M_a, it suffices to establish that for all test functions f,

$$a^2\left\langle[\varphi(a^2 - \mu^2\partial_{xx})^{-1}(\varphi f)], f\right\rangle \geq b^2\left\langle[\varphi(b^2 - \mu^2\partial_{xx})^{-1}(\varphi f)], f\right\rangle, \tag{54}$$

whenever $a \geq b$. This is easily seen by the Plancherel's theorem. Denoting $h = \varphi f$, we have

$$a^2\left\langle[\varphi(a^2 - \mu^2\partial_{xx})^{-1}(\varphi f)], f\right\rangle = \int_{-\infty}^{\infty} \frac{a^2}{a^2 + 4\pi^2\mu^2\xi^2}|\hat{h}(\xi)|^2 d\xi \geq$$

$$\geq \int_{-\infty}^{\infty} \frac{b^2}{b^2 + 4\pi^2\mu^2\xi^2}|\hat{h}(\xi)|^2 d\xi = b^2\left\langle[\varphi(b^2 - \mu^2\partial_{xx})^{-1}(\varphi f)], f\right\rangle,$$

where we have used the elementary inequality $\frac{a^2}{a^2+4\pi^2\mu^2\xi^2} \geq \frac{b^2}{b^2+4\pi^2\mu^2\xi^2}$, if $a \geq b$.

6.3. **Proof of Theorem 3.** Now that we have checked most of the required conditions on $\mathcal{H}$, let us verify the remaining ones and the quantity $\langle\mathcal{H}^{-1}\psi_0', \psi_0'\rangle$. Here $\psi_0 = m\begin{pmatrix}\varphi' \\ -\frac{\varphi^2}{2\mu^2}\end{pmatrix}$, where m is so that $\|\psi_0\| = 1$. Thus, we need to compute $\mathcal{H}^{-1}\begin{pmatrix}\varphi' \\ -\frac{\varphi^2}{2\mu^2}\end{pmatrix}$. Recall that $\mathcal{H}^{-1}$ exists on $\{\psi_0\}^\perp$ and since $\psi_0' \in \{\psi_0\}^\perp$, it is possible to compute $\mathcal{H}^{-1}\psi_0'$.

This could be done almost explicitly, in any case explicit enough, so that we may compute $\langle \mathcal{H}^{-1}\psi_0', \psi_0' \rangle$ precisely in terms of c. We have

$$\left| \begin{array}{l} -\mu^2 f'' + f - \frac{\varphi^2}{2\mu^2} f + \varphi g' = \varphi'' \\ -\mu^2 g'' - (\varphi f)' = -(\frac{\varphi^2}{2\mu^2})'. \end{array} \right.$$

Integrating once in the second equation yields

$$g' = \frac{1}{\mu^2}(\frac{\varphi^2}{2\mu^2} - \varphi f).$$

Plugging this in the first equation yields

$$-\mu^2 f'' + f - \frac{3\varphi^2}{2\mu^2} f + \frac{\varphi^3}{2\mu^4} = \varphi''. \tag{55}$$

On the other hand, recalling equation (8) in the form

$$-\mu^2 \varphi'' + \varphi - \frac{\varphi^3}{2\mu^2} = 0, \tag{56}$$

take a derivative in the parameter μ in (56). We get

$$-\mu^2(\partial_\mu \varphi)'' + \partial_\mu \varphi - \frac{3\varphi^2}{2\mu^2}\partial_\mu \varphi + \frac{\varphi^3}{\mu^3} = 2\mu\varphi''.$$

Dividing by 2μ yields the relation

$$-\mu^2 \left(\frac{\partial_\mu \varphi}{2\mu}\right)'' + \left(\frac{\partial_\mu \varphi}{2\mu}\right) - \frac{3\varphi^2}{2\mu^2}\left(\frac{\partial_\mu \varphi}{2\mu}\right) + \frac{\varphi^3}{\mu^3} = \varphi''. \tag{57}$$

Comparing the equations (57) and (55) clearly implies (since we know that there is an unique solution by the invertibility of $\mathcal{H}$ on $\{\psi_0\}^\perp$) that

$$f = \frac{\partial_\mu \varphi}{2\mu}.$$

Thus, it follows that

$$g' = \frac{1}{\mu^2}(\frac{\varphi^2}{2\mu^2} - \varphi f) = g' = \frac{1}{\mu^2}(\frac{\varphi^2}{2\mu^2} - \varphi\frac{\partial_\mu \varphi}{2\mu})$$

Now,

$$\left\langle \mathcal{H}\begin{pmatrix} \varphi'' \\ -(\frac{\varphi^2}{2\mu^2})' \end{pmatrix}, \begin{pmatrix} \varphi'' \\ -(\frac{\varphi^2}{2\mu^2})' \end{pmatrix} \right\rangle = \left\langle \begin{pmatrix} f \\ g \end{pmatrix}, \begin{pmatrix} \varphi'' \\ -(\frac{\varphi^2}{2\mu^2})' \end{pmatrix} \right\rangle = \left\langle \frac{\partial_\mu \varphi}{2\mu}, \varphi'' \right\rangle + \left\langle g', \frac{\varphi^2}{2\mu^2} \right\rangle =$$

$$= -\frac{\partial_\mu[\int |\varphi'(y)|^2 dy]}{4\mu} - \frac{\partial_\mu[\int \varphi^4(y)dy]}{16\mu^5} + \frac{\int \varphi^4(y)dy}{4\mu^6}.$$

Now since

$$\int |\varphi'(y)|^2 dy = 4\int (sech'(y/\mu))^2 dy = 4\mu \int (sech'(x))^2 dx$$

$$\int \varphi^4(y)dy = 16\mu^5 \int sech(x)^4 dx,$$

we have

$$\left\langle \mathcal{H}\begin{pmatrix} \varphi'' \\ -(\frac{\varphi^2}{2\mu^2})' \end{pmatrix}, \begin{pmatrix} \varphi'' \\ -(\frac{\varphi^2}{2\mu^2})' \end{pmatrix} \right\rangle = -\frac{1}{\mu}\left(\int (sech'(x))^2 dx + \int sech(x)^4 dx\right).$$

Recall however, that we have to also compute the normalization factor m^2. We have

$$\begin{aligned} \frac{1}{m^2} &= \left\| \begin{pmatrix} \varphi' \\ -\frac{\varphi^2}{2\mu^2} \end{pmatrix} \right\|^2 = \int |\varphi'(y)|^2 dy + \frac{1}{4\mu^4} \int \varphi^4(y) dy = \\ &= 4\mu(\left(\int (sech'(x))^2 dx + \int sech(x)^4 dx \right). \end{aligned}$$

Thus, putting the last two formulas together

$$\left\langle \mathcal{H}^{-1}\psi_0', \psi_0' \right\rangle = m^2 \left\langle \mathcal{H} \begin{pmatrix} \varphi'' \\ -(\frac{\varphi^2}{2\mu^2})' \end{pmatrix}, \begin{pmatrix} \varphi'' \\ -(\frac{\varphi^2}{2\mu^2})' \end{pmatrix} \right\rangle = -\frac{1}{4\mu^2}. \tag{58}$$

From this identity, we get that $\left\langle \mathcal{H}^{-1}\psi_0', \psi_0' \right\rangle < 0$ for all values of the parameters, so in particular we claim that (21) is satisfied.

Indeed, if we assume that $0 = \langle \phi', \psi_0 \rangle = -\langle \phi, \psi_0' \rangle$, it would follow that $\psi_0' \in \{\phi\}^\perp$. Since $\mathcal{H}|_{\{\phi\}^\perp} \geq 0$, it follows that

$$\left\langle \mathcal{H}^{-1}\psi_0', \psi_0' \right\rangle \geq 0,$$

which is a contradiction with (58). Properties (20) and (22) are obvious and hence, we may apply Theorem 1. Since

$$\omega^*(\mathcal{H}) = \frac{1}{2\sqrt{-\left\langle \mathcal{H}^{-1}\psi_0', \psi_0' \right\rangle}} = \mu,$$

we have linear stability for all c satisfying $1 > |c| \geq \mu = \sqrt{1-c^2}$, and instability for $0 \leq |c| < \mu = \sqrt{1-c^2}$. Solving the inequalities yields stability for $c : |c| \in [\frac{\sqrt{2}}{2}, 1)$ and instability for $c : |c| \in [0, \frac{\sqrt{2}}{2})$. Theorem 3 is established.

7. Proof of Theorem 4

Since we cannot verify theoretically the assumptions[14] of Theorem 1, we have put the appropriate assumptions, essentially requiring them, so that Theorem 1 applies.

First, the case $c = 0$ always presents instability, since $\mathcal{H}_c$ has a negative eigenvalue. So, assume $c > 0$, the cases $c < 0$ being symmetric.

Next, we need to compute $\mathcal{H}^{-1}[\varphi_c'']$. This is done by just taking a derivative with respect to the parameter c in the defining equation (12). We have

$$c^2(\partial_c\varphi)'' + 2c\varphi_c'' + (\partial_c\varphi)'''' + (\partial_c\varphi) - p\varphi^{p-1}(\partial_c\varphi) = 0$$

or equivalently $\mathcal{H}[\partial_c\varphi] = -2c\varphi_c''$. Assuming $c \neq 0$, we conclude $\mathcal{H}^{-1}[\varphi_c''] = -\frac{1}{2c}\partial_c\varphi$. Now,

$$\left\langle \mathcal{H}^{-1}\psi_0', \psi_0' \right\rangle = -\frac{1}{2c\|\varphi'\|^2} \left\langle \partial_c\varphi_c, \varphi'' \right\rangle = \frac{1}{2c\|\varphi'\|^2} \left\langle \partial_c\varphi', \varphi' \right\rangle = \frac{\partial_c(\|\varphi_c'\|^2)}{4c\|\varphi'\|^2}.$$

According to Theorem 1, we have instability for all c such that

$$\partial_c\|\varphi_c'\| \geq 0$$

and moreover, if $\partial_c\|\varphi_c'\| < 0$, we have instability for all $c > 0$, satisfying

$$c < \frac{1}{2\sqrt{-\frac{\partial_c\|\varphi_c'\|^2}{4c\|\varphi_c'\|^2}}}$$

[14]but, as we have alluded to before, they hold nice and steady in numerical simulations, [10]

or

$$\sqrt{c} < \frac{1}{\sqrt{-\frac{\partial_c \|\varphi_c'\|^2}{\|\varphi_c'\|^2}}}.$$

In the complementary range, we have spectral stability.

Appendix A. Proof of Lemma 1

The term $\frac{\lambda^2-\delta^2}{4\omega^2\lambda^2}$ is clearly continuous in both ω, λ, so we concentrate on the continuity of the mapping $(\omega, \lambda) \to \left\langle [H + \lambda^2 + 2\omega\lambda P_0 \partial_x P_0]^{-1}[\phi'], \phi' \right\rangle$.

We now need to show the continuity of the map stated above. Taking a sequence $(\omega_n, \lambda_n) \to (\omega_0, \lambda_0)$ and denoting

$$\begin{aligned} R_n &= (H + \lambda_n^2 + 2\omega_n\lambda_n P_0\partial_x P_0)^{-1} \\ R_0 &= (H + \lambda_0^2 + 2\omega_0\lambda_0 P_0\partial_x P_0)^{-1}, \end{aligned}$$

we need to show that

$$\left\langle R_n[\phi'], \phi' \right\rangle \to \left\langle R_0[\phi'], \phi' \right\rangle,$$

which follows from

$$\|(R_n - R_0)P_0\|_{L^2 \to L^2} \to 0,$$

which remains to be proved. By the resolvent identity, we have

$$R_n - R_0 = -R_n(\lambda_n^2 - \lambda_0^2 + 2(\omega_n\lambda_n - \omega_0\lambda_0)P_0\partial_x P_0)R_0$$

and since $|\lambda_n^2 - \lambda_0^2 + 2(\omega_n\lambda_n - \omega_0\lambda_0)| \leq C(|\omega_n| + |\omega_0| + |\lambda_0| + |\lambda_n|)(|\omega_n - \omega_0| + |\lambda_n - \lambda_0|)$, it will suffice to prove

$$\limsup_n \|R_n R_0 P_0\|_{\mathcal{B}(L^2)} + \|R_n(P_0\partial_x P_0)R_0 P_0\|_{\mathcal{B}(L^2)} < \infty. \tag{59}$$

The first estimate follows from Proposition 3

$$\limsup_n \|R_n R_0 P_0\|_{\mathcal{B}(L^2)} \leq \limsup_n \|R_n P_0\|_{\mathcal{B}(L^2)} \|R_0 P_0\|_{\mathcal{B}(L^2)} \leq \limsup_n (\lambda_n^{-2}\lambda_0^{-2}) = \lambda_0^{-4}.$$

For the second term, we further use the resolvent identity to write

$$R_n(P_0\partial_x P_0)R_0 P_0 = (H + \lambda_n^2)^{-1} P_0 \partial_x R_0 P_0 - 2\omega_n\lambda_n R_n P_0 \partial_x P_0 (H + \lambda_n^2)^{-1} \partial_x R_0 P_0,$$

and we do similar expansion on the right-side with $R_0 = (H + \lambda_0^2)^{-1} + ...$ Clearly, in the above formula, we can pair the operators ∂_x with various resolvents of the form $(H + \lambda_n^2)^{-1}, (H + \lambda_0^2)^{-1}$, through (20) to obtain the desired boundedness results.

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