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Linear stability analysis for periodic travelling waves of the Boussinesq equation and the Klein–Gordon–Zakharov system

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The question of the linear stability of spatially periodic waves for the Boussinesq equation (in the cases $p = 2, 3$) and the Klein–Gordon–Zakharov system is considered. For a wide class of solutions, we completely and explicitly characterize their linear stability (instability) when the perturbations are taken with the same period T. In particular, our results allow us to completely recover the linear stability results, in the limit $T \to \infty$, for the whole-line case.

1. Introduction

In this paper we are interested in the stability of spatially periodic waves for certain models, which involve the second-order derivative in time. Our interest is mainly in two partial differential equations, the Boussinesq equation and the Klein–Gordon–Zakharov (KGZ) system, although the methods that we develop here will certainly find applications in other related models.

The Cauchy problem for the Boussinesq equation, with periodic boundary conditions, is

$$u_{tt} + u_{xxxx} - u_{xx} + (f(u))_{xx} = 0, \quad (t, x) \in \mathbb{R}^1_+ \times [0, T], \tag{1.1}$$

where $f(u)$ is, for the most part, $f(u) = u^p$, $p > 1$. This is a model that was derived by Boussinesq [7] for $p = 2$, but was subsequently studied by many authors, in both the periodic and whole-line contexts. We now review the current results regarding the well-posedness properties of the Boussinesq equation. While we have a very satisfactory theory for the local solutions (see below), the global well-posedness does not hold. More precisely, even if one requires smooth compactly supported data, the solutions may develop singularities in finite time [6]. This makes the

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stability question, which is the main subject of this paper, even more relevant and interesting.

In the whole-line scenario, local well-posedness was established by Bona and Sachs [6] in the Sobolev spaces $H^{(5/2)+}(\mathbb{R}^1) \times H^{(3/2)+}(\mathbb{R}^1)$. Further contributions were made by Tsutsumi and Mathashi [25] and Linares [19] (who also showed global existence for small data). Farah [11] showed well-posedness in $H^s(\mathbb{R}^1) \times \tilde{H}^{s-2}(\mathbb{R}^1)$ when $s > -1/4$ and the space $\tilde{H}^\alpha$ is defined via $\tilde{H}^\alpha = \{u\colon u_x \in H^{\alpha-1}(\mathbb{R}^1)\}$. Kishimoto and Tsugava [18] finally showed well-posedness for all $s > -1/2$, which is likely to be sharp.

Regarding the case of periodic boundary conditions, we refer the reader to Fang and Grillakis [10], who established local well-posedness in $H^s(\mathbb{T}) \times H^{s-2}(\mathbb{T})$, $s > 0$ (when $1 < p < 3$ in (1.1)). This result was improved for $p = 2$ to $s > -1/4$ by Farah and Scialom [12]. Oh and Stefanov [21] recently showed local well-posedness in $H^s(\mathbb{T}) \times H^{s-2}(\mathbb{T})$, $s > -3/8$.

Our other main object of investigation is the Klein–Gordon–Zakharov system, which is given by[1]

$$\left.\begin{aligned} u_{tt} - u_{xx} + u + uv = 0, \quad (t,x) \in \mathbb{R}^1 \times \mathbb{R}^1 \text{ or } (t,x) \in \mathbb{R}^1 \times [0,T],\\ n_{tt} - n_{xx} = \tfrac{1}{2}(|u|^2)_{xx}. \end{aligned}\right\} \tag{1.2}$$

This system describes the interaction of a Langmuir wave and an ion sound wave in plasma. More precisely, u is the fast-scale component of the electric field, whereas n denotes the deviation in ion density [26]. The system (1.2) is locally well posed in various function spaces (see [13, 22]). In our previous paper [17], we showed that the Cauchy problem (1.2) is locally well posed (in both periodic and whole-line contexts) in $H^\alpha \times H^{\alpha-1} \times H^{\alpha-1} \times H^{\alpha-2}$ whenever $\alpha > 1/2$. In [22, 23], Ozawa *et al.* showed that small initial data values produce solutions that persist globally, whereas large solutions are generally expected to blow up in finite time.

The stability of periodic travelling waves has been studied extensively in the last decade. The nonlinear stability of periodic waves for the Korteweg–de Vries equation based on the Jacobi elliptic functions of cnoidal type was considered in [2], and for modified Korteweg–de Vries and nonlinear Schrodinger equations of dnoidal type in [1]. In [15], Hakkaev *et al.* considered the nonlinear stability of periodic waves for the generalized Benjamin–Bona–Mahony equation. Recently, Arruda [3] considered the nonlinear stability of periodic travelling waves for the Boussinesq equation with $p = 2$. The approach is based on the theory developed in [4, 5, 14] for the stability of solitary waves. It should be noted that his results are only about stability of the waves (which happens when the Grillakis–Shatah–Strauss (GSS) functional d is strictly convex), while the cases of instability are left open. This is due to the well-known limitations of the GSS approach for proving instability, namely, that it is required that the skew-symmetric operator J in the linearization must be onto. This condition is not met in Arruda's analysis, so one cannot say anything about instability based on the classical GSS theory.

In this paper, we completely characterize the linearized stability of periodic travelling waves for the Boussinesq model (the cases $p = 2, 3$) and the Klein–Gordon–

[1]The coefficient $\frac{1}{2}$ in front of the nonlinear term $(|u|^2)_{xx}$ is non-standard, but rather adopted for convenience of presentation. In particular, it helps create a self-adjoint linearized operator, which otherwise can be achieved via a simple change of the time variable.

Zakharov system. These are the cases of second-order-in-time models, for which we can explicitly write the solutions in elliptic functions (and, moreover, we can explicitly compute the relevant portion of the spectrum of the linearized operators). While this certainly helps in the analysis, we believe that our results should be generalizable to all values of $p > 1$. The main tool is the theory for linearized stability for such models, recently developed by the second and third authors [24]. In particular, the theory in [24] captures both the stability and instability regions up to and including the turning points (in which there is linear stability, but it may be nonlinearly secularly unstable). We note that, while there are nonlinear stability results available for the KGZ system in the whole-line case (see [8]), theorem 3.4 seems to be the first result to deal with the periodic case. Indeed, we completely characterize the linearly stable and unstable periodic waves. In addition, we believe that, while it is possible to generalize the nonlinear stability results to the periodic cases, the same difficulties with the instabilities will persist within the framework of the standard GSS theory, due to the non-invertibility of the skew-symmetric operator J.

The paper has the following structure. In § 2 we present the construction of our main object of study: periodic travelling waves. This is not new material by any means, but we present it in order to single out the solutions of interest (note that there are solutions, which are not considered herein), and to introduce some notation. In § 3 we set up the linear stability problem, after which we present the main results. In § 4 we outline the theory for linearized stability for second-order partial differential equations from [24], and point out the relevant spectral theoretic results for their linearized operators. In § 5 we prove the main results, theorems 3.2, 3.3 for the Boussinesq model, while in § 6 we prove theorem 3.4 for the KGZ system.

2. Construction of the periodic travelling waves

In this section, we show a glimpse of the construction of the periodic waves: in § 2.1 for the Boussinesq equation (when $p = 2, 3$), and in § 2.2 for the KGZ system.

2.1. Construction of the travelling wave solutions for the Boussinesq equation

Applying the travelling wave ansatz, one sees that there is a one-parameter family of travelling waves of the form $\varphi(x - ct)$, $|c| \in (-1, 1)$, that obey the equation $\partial_{xx}[c^2\varphi + \varphi'' - \varphi + f(\varphi)] = 0$, $0 \leqslant x \leqslant T$, whence there exist a, C such that

$$c^2\varphi + \varphi'' - \varphi + f(\varphi) = Cx + a, \quad 0 \leqslant x \leqslant T.$$

By the periodicity of φ, we conclude that $C = 0$, and thus we have a family of waves satisfying

$$\varphi'' - (1 - c^2)\varphi + f(\varphi) = a, \quad 0 \leqslant x \leqslant T. \tag{2.1}$$

We now construct solutions of (2.1) in various cases of interest, most notably $p = 2$ and $p = 3$. This material is not new, but in order to introduce the particular parametrization that is convenient for us we include a sketch of the construction for completeness.

2.1.1. *The case* $p = 2$

We consider only the symmetric case $a = 0$ and define $w = 1 - c^2$. For the nonlinearity, $f(u) = u^2/2$, we have that

$$-w\varphi + \tfrac{1}{2}\varphi^2 + \varphi'' = 0. \tag{2.2}$$

Therefore,

$$\varphi'^2 - w\varphi^2 + \tfrac{1}{3}\varphi^3 = b. \tag{2.3}$$

Hence, the periodic solutions are given by the trajectories $H(\varphi, \varphi') = b$ of the Hamiltonian

$$H(x, y) = y^2 + \tfrac{1}{3}x^3 - wx^2.$$

The level set $H(x, y) = b$ contains two periodic trajectories if $w > 0$, $b \in (-\frac{2}{3}w^3, 0)$, and a unique periodic trajectory if $b > 0$. We consider here the cases where $b < 0$ and $\varphi_c > 0$. To express φ_c through elliptic functions, we denote by $\varphi_1 > \varphi_0 > 0$ the positive solutions of $\frac{1}{3}\rho^3 - w\rho^2 - b = 0$. Then $\varphi_0 \leqslant \varphi_c \leqslant \varphi_1$, and one can rewrite (2.3) as

$$\varphi_c'^2 = \tfrac{1}{3}(\varphi_c - \varphi_0)(\varphi_1 - \varphi_c)(\varphi_c + \varphi_0 + \varphi_1 - 3w). \tag{2.4}$$

Introducing a new variable $s \in (0, 1)$ via $\varphi_c = \varphi_0 + (\varphi_1 - \varphi_0)s^2$, we transform (2.4) into

$$s'^2 = \alpha^2(1 - s^2)(k'^2 + k^2 s^2),$$

where α, k, k' are positive constants ($k^2 + k'^2 = 1$) given by

$$\alpha^2 = \frac{2\varphi_1 + \varphi_0 - 3w}{12}, \qquad k^2 = \frac{\varphi_1 - \varphi_0}{2\varphi_1 + \varphi_0 - 3w}.$$

Therefore,

$$\varphi_c(x) = \varphi_0 + (\varphi_1 - \varphi_0)\,\mathrm{cn}^2(\alpha x; k). \tag{2.5}$$

By the above formulae,

$$\left.\begin{aligned} \varphi_1 - \varphi_0 &= 12\alpha^2\kappa^2, \\ \varphi_1 &= 4\alpha^2(1 + \kappa^2) + w, \\ \varphi_0 &= 4\alpha^2(1 - 2\kappa^2) + w, \\ w^2 &= 16\alpha^4(1 - \kappa^2 + \kappa^4). \end{aligned}\right\} \tag{2.6}$$

The fundamental period of the cnoidal wave φ_c in (2.5) is

$$T = \frac{2K(\kappa)}{\alpha} = \frac{4K(\kappa)\sqrt[4]{1 - \kappa^2 + \kappa^4}}{\sqrt{w}}, \quad T \in \left(\frac{2\pi}{\sqrt{w}}, \infty\right). \tag{2.7}$$

Here and below, $K(\kappa)$ and $E(\kappa)$ denote the elliptic integrals of the first and second kind in a Legendre form. Furthermore, we use the following relations:

$$K'(\kappa) = \frac{E(\kappa) - (1 - \kappa^2)K(\kappa)}{\kappa(1 - \kappa^2)}, \qquad E'(\kappa) = \frac{E(\kappa) - K(\kappa)}{\kappa}.$$

Lemma 2.1. *For any $w > 0$ and $T \in (2\pi/\sqrt{w}, \infty)$, there exists a constant $b = b(w)$ such that the periodic travelling solution (2.5) has period T. The function $b(w)$ is differentiable.*

Proof. It is easily seen that the period T is a strictly increasing function of k:

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}k}(\sqrt[4]{1-\kappa^2+\kappa^4}K(\kappa)) &= \frac{\kappa(2\kappa^2-1)K(\kappa)+2(1-\kappa^2+\kappa^4)K'(\kappa)}{4\sqrt[4]{1-\kappa^2+\kappa^4}^3}\\ &= \frac{2(1-\kappa^2+\kappa^4)E(\kappa)+(1-\kappa^2)(\kappa^2-2)K(\kappa)}{4\kappa(1-\kappa^2)\sqrt[4]{1-\kappa^2+\kappa^4}^3}\\ &> 0.\end{aligned}$$

Given w and b in their range, consider the functions $\varphi_0(w,b)$, $\varphi_1(w,c)$, $k(w,b)$ and $T(w,b)$ given by (2.7) and (2.6). We obtain

$$\frac{\partial T}{\partial b} = \frac{\mathrm{d}T}{\mathrm{d}k}\frac{\mathrm{d}k}{\mathrm{d}b} = \frac{1}{2k}\frac{\mathrm{d}T}{\mathrm{d}k}\frac{\mathrm{d}(k^2)}{\mathrm{d}b}.$$

Furthermore, using that $\frac{1}{3}\varphi_0^3 - w\varphi_0^2 = \frac{1}{3}\varphi_1^3 - w\varphi_1^2$, we have

$$\begin{aligned}\frac{\mathrm{d}(k^2)}{\mathrm{d}b} &= \frac{3(\varphi_0-w)(\partial\varphi_1/\partial b)-3(\varphi_1-w)(\partial\varphi_0/\partial c)}{(2\varphi_1+\varphi_0-3w)^2}\\ &= \frac{3w^2(\varphi_0-\varphi_1)}{(\varphi_0^2-2w\varphi_0)(\varphi_1^2-2w\varphi_1)(2\varphi_1+\varphi_0-3w)^2}.\end{aligned}$$

We see that $\partial T(w,b)/\partial b \neq 0$, whence the implicit function theorem implies the result. □

2.1.2. *The case $p = 3$*

We consider the 'symmetric' case $a = 0$ only, with $f(u) = u^3$. Define $w = 1 - c^2 \in (0,1)$. Multiplying by φ' and integrating implies that

$$\varphi'^2 = b + w\varphi^2 - \tfrac{1}{2}\varphi^4. \tag{2.8}$$

Hence, the periodic solutions are given by the periodic trajectories $H(\varphi,\varphi') = b$ of the Hamiltonian

$$H(x,y) = y^2 + \tfrac{1}{4}x^4 - \tfrac{1}{2}wx^2.$$

There are then two possibilities.

- (*Outer case.*) For any $b > 0$ the orbit defined by $H(\varphi,\varphi') = b$ is periodic and oscillates around the eight-shaped loop $H(\varphi,\varphi') = 0$ through the saddle at the origin.

- (*Left and right cases.*) For any $b \in (-\frac{1}{2}w^2, 0)$ there exist two periodic orbits defined by $H(\varphi,\varphi') = b$ (the left and right ones). These are located inside the eight-shaped loop and oscillate around the centres at $(\mp\sqrt{w}, 0)$, respectively.

We consider the left and right cases of the Duffing oscillator only. In these cases, denote by $\varphi_1 > \varphi_0 > 0$ the positive roots of the quartic equation $\frac{1}{2}z^4 - wz^2 - b = 0$. Then, up to a translation, we obtain the respective explicit formulae

$$\varphi(z) = \mp\varphi_1 \operatorname{dn}(\alpha z; k), \quad k^2 = \frac{\varphi_1^2 - \varphi_0^2}{\varphi_1^2} = \frac{2\varphi_1^2 - 2w}{\varphi_1^2}, \quad \alpha = \frac{\varphi_1}{\sqrt{2}}, \quad T = \frac{2K(k)}{\alpha}. \tag{2.9}$$

Note that the fundamental period T may be written as

$$T = \frac{2K(\kappa)}{\alpha} = \frac{2K(\kappa)\sqrt{2-\kappa^2}}{\sqrt{w}}, \quad T \in \left(\frac{\sqrt{2}\pi}{\sqrt{w}}, \infty\right). \tag{2.10}$$

Note that this is a two-parameter family of solutions, parametrized explicitly in this case by φ_0, φ_1, although we shall need a different parametrization. In fact, we like to think of this family as being parametrized (implicitly) in terms of T and w, where these two are independent of each other. We have the following lemma.

Lemma 2.2. *For $T > \sqrt{2}\pi/\sqrt{w}$, there exists a constant $b = b(w)$ such that the periodic travelling wave solution (2.9) determined by $H(\varphi, \varphi') = b(w)$ has a period T. In addition, the function $b(w)$ is differentiable.*

For the proof, see [16, lemma 3.1].

2.2. Construction of the travelling wave solutions for the KGZ system

We are looking for T-periodic travelling solutions of the Klein–Gordon–Zakharov system (1.2). Thus, we take the ansatz $u(t,x) = \varphi_c(x - ct)$, $n(t,x) = \psi_c(x - ct)$, where we take the speed $c \in (-1, 1)$. Putting this into (1.2), we obtain the following relation between ψ and φ:

$$(c^2 - 1)\psi'' = \tfrac{1}{2}(\varphi^2)'', \quad 0 \leqslant x \leqslant T.$$

Two integrations in x imply that

$$(c^2 - 1)\psi(x) = \tfrac{1}{2}\varphi^2(x) + bx + a$$

for some constants a, b. By the periodicity we have that $b = 0$, whence

$$\psi(x) = -\frac{\varphi^2(x) + a}{2(1 - c^2)}.$$

For simplicity, we only consider the case $a = 0$. That is,

$$\psi_c = -\frac{\varphi_c^2}{2(1 - c^2)}. \tag{2.11}$$

Returning to the other equation in (1.2) and using (2.11), we obtain the following equation for φ_c:

$$-(1 - c^2)\varphi_c'' + \varphi_c - \frac{\varphi_c^3}{2(1 - c^2)} = 0, \quad 0 \leqslant x \leqslant T. \tag{2.12}$$

Thus, as in § 2.1.2, after multiplying by $\varphi_{\rm c}$ and integrating we get that

$$\varphi_{\rm c}'^2 = b + \frac{\varphi_{\rm c}^2}{w} - \frac{\varphi_{\rm c}^4}{4w^2} = \frac{1}{4w^2}(4w^2 b + 4w\varphi_{\rm c}^2 - \varphi_{\rm c}^4). \tag{2.13}$$

Denote by $\varphi_1 > \varphi_0 > 0$ the positive roots of the polynomial

$$P(z) = z^4 - 4wz^2 - 4w^2 b.$$

Then (2.13) can be written in the form

$$\varphi_{\rm c}'^2 = \frac{1}{4w^2}(\varphi_{\rm c}^2 - \varphi_0^2)(\varphi_1^2 - \varphi_{\rm c}^2), \tag{2.14}$$

and the solution of (2.13) is given by

$$\varphi_{\rm c}(x) = \varphi_1 \,\mathrm{dn}(\alpha x, \kappa), \tag{2.15}$$

where

$$\varphi_1^2 + \varphi_0^2 = 4w, \qquad \alpha = \frac{\varphi_1}{2w}, \qquad \kappa^2 = \frac{\varphi_1^2 - \varphi_0^2}{\varphi_1^2}. \tag{2.16}$$

Moreover,

$$(2-\kappa^2)\varphi_1^2 = 4w, \qquad \alpha = \frac{1}{\sqrt{w(2-\kappa^2)}}, \qquad 4w^2 b = 4w\varphi_1^2 - \varphi_1^4. \tag{2.17}$$

Since dn has fundamental period $2K(\kappa)$, the solution $\varphi_{\rm c}$ has fundamental period $T = 2K(\kappa)/\alpha$. In terms of κ, w, this is given by

$$T = 2K(\kappa)\sqrt{2-\kappa^2}\sqrt{w}, \quad T \in I = (\sqrt{2}\pi\sqrt{w}, \infty). \tag{2.18}$$

LEMMA 2.3. *For any $w > 0$ and $T > \sqrt{2}\pi\sqrt{w}$, there exists a constant $b = b(w)$ such that the periodic travelling solution (2.15) has period T.*

Proof. The period T is a strictly increasing function of κ:

$$\frac{\mathrm{d}}{\mathrm{d}\kappa}[\sqrt{2-\kappa^2}K(\kappa)] = \frac{K'(\kappa) + E'(\kappa)}{\sqrt{2-\kappa^2}} > 0.$$

From (2.16) and (2.17), we have that

$$\frac{\mathrm{d}T}{\mathrm{d}b} = \frac{\mathrm{d}T}{\mathrm{d}\kappa}\frac{\mathrm{d}\kappa}{\mathrm{d}b} = \frac{1}{2\kappa}\frac{\mathrm{d}T}{\mathrm{d}\kappa}\frac{\mathrm{d}\kappa^2}{\mathrm{d}b},$$

$$\frac{\mathrm{d}\kappa^2}{\mathrm{d}b} = \frac{\mathrm{d}\kappa^2}{\mathrm{d}\varphi_1^2}\frac{\mathrm{d}\varphi_1^2}{\mathrm{d}b} = \frac{16w^2}{\varphi_1^4(4w - 2\varphi_1^2)}.$$

The implicit function theorem then implies the result. □

3. Results

3.1. Setting the linear stability problem for the Boussinesq equation

We now set up the linear stability/instability problem for (1.1). Set the ansatz $u = \varphi_{\rm c}(x+ct) + v(t, x+ct)$ and ignore all terms $O(v^2)$. We get $v_{tt} + 2cv_{tx} + Mv = 0$, where

$$Mv = \partial_x^4 v - (1-c^2)\partial_x^2 v + (f'(\varphi_{\rm c})v)_{xx}.$$

Note that this operator M is not self-adjoint. However, if we introduce the variable z: $z_x = v$, we get the following linearized equation in terms of z:

$$z_{ttx} + 2cz_{txx} + M[z_x] = 0. \tag{3.1}$$

Note that $M[z_x] = \partial_x[H[z]]$, where

$$H_c z = \partial_x^4 z - (1 - c^2)\partial_x^2 z + (f'(\varphi_c)z_x)_x. \tag{3.2}$$

Thus, the linearized equation becomes $\partial_x[z_{tt} + 2cz_{tx} + Hz] = 0$. In our considerations we say that the wave φ_c is spectrally unstable, exactly when there is an exponentially growing mode, that is, a pair $\lambda \in \mathcal{C}$: $\operatorname{Re}\lambda > 0$ and a T-periodic function $\psi \in H^4_{\mathrm{per}}(0,T)$ such that $\partial_x[\lambda^2\psi + 2c\lambda\psi' + H\psi] = 0$. This of course implies upon integration that, for some constant a,

$$\lambda^2\psi + 2c\lambda\psi' + H\psi = a.$$

Integrating in $[0,T]$ and taking into account that both ψ' and $H\psi$ are exact derivatives implies that $a = \lambda^2 \int_0^T \psi(x)\,\mathrm{d}x/T$. Thus, letting $\tilde{\psi} := \psi - a/\lambda^2$ implies that $\int_0^T \tilde{\psi}(x)\,\mathrm{d}x = 0$ and

$$\lambda^2\tilde{\psi} + 2c\lambda\tilde{\psi} + H\tilde{\psi} = 0.$$

These arguments motivate the following definition.

Definition 3.1. We say that the travelling wave φ_c is spectrally/linearly unstable if there exists a T-periodic *mean value zero* function $\psi \in D(H_c)$ and λ: $\operatorname{Re}\lambda > 0$ such that

$$\lambda^2\psi + 2c\lambda\psi' + H_c\psi = 0. \tag{3.3}$$

The question of linear stability of equations in the form

$$z_{tt} + 2\omega z_{tx} + \mathcal{H}z = 0, \tag{3.4}$$

or what is equivalent (at least in this case) to the solvability of

$$\lambda^2\psi + 2\omega\lambda\psi' + \mathcal{H}\psi = 0 \quad \text{in } L^2_0(0,T) = \left\{ f \in L^2_{\mathrm{per}}(0,T)\colon \int_0^T f(x)\,\mathrm{d}x = 0 \right\}, \tag{3.5}$$

was addressed in a recent paper by the second and third authors [24]. Note that the self-adjoint operator $\mathcal{H}$ that appears in (3.2) is in the form

$$\mathcal{H} = -\partial_x\mathcal{L}\partial_x, \quad \mathcal{L} = -\partial_x^2 + (1 - c^2) - f'(\varphi_c).$$

Here, $\mathcal{L}$ is the ubiquitous standard second-order Schrödinger operator, which appears in the linearization of the generalized Korteweg–de Vries equation around its travelling wave solution φ_c. This observation is crucial for the spectral properties of the operator $\mathcal{H}$, as the properties of $\mathcal{L}$ are generally well known, at least for the cases under consideration, $p = 2, 3$.

3.2. Setting the linear stability problem for the KGZ system

We linearize the KGZ system as follows. We take $u(t,x) = \varphi_c(x-ct)+v(t,x-ct)$ and $n(t,x) = \psi_c(x-ct)+h(t,x-ct)$ and ignore the contributions of all quadratic and higher-order terms. We obtain the following *linear system* for the corrections v, h:

$$\left.\begin{aligned} v_{tt} - 2cv_{tx} - (1-c^2)v_{xx} + v + \psi_c v + \varphi_c h = 0, \\ h_{tt} - 2ch_{tx} - (1-c^2)h_{xx} - (\varphi_c v)_{xx} = 0. \end{aligned}\right\} \tag{3.6}$$

Furthermore, we introduce a new mean value zero function z such that $h = z_x$ and $w = 1 - c^2$. The second equation in (3.6) becomes

$$\partial_x[z_{tt} - 2cz_{tx} - (1-c^2)z_{xx} - (\varphi_c z_x)_x] = 0,$$

whence integrating in x yields

$$z_{tt} - 2cz_{tx} - (1-c^2)z_{xx} - (\varphi_c z_x)_x = a(t)$$

for some function $a(t)$. Observe, however, that in our choice of z we have required that $\int_0^T z(t,x)\,\mathrm{d}x = 0$. Thus, integrating the last equation in $[0,T]$ yields that all integrals on the left-hand side are 0 (each term is either an exact derivative or z_{tt}, which is mean value zero), $a(t) = 0$, whence $z_{tt} - 2cz_{tx} - (1-c^2)z_{xx} - (\varphi_c z_x)_x = 0$. We have shown that one can rewrite the linear stability problem (3.6) as

$$\boldsymbol{\Phi}_{tt} - 2c\boldsymbol{\Phi}_{tx} + \mathcal{H}\boldsymbol{\Phi} = 0, \tag{3.7}$$

where $\boldsymbol{\Phi} = \binom{v}{z}$ and

$$\mathcal{H} = \begin{pmatrix} H_1 & A \\ A^* & H_2 \end{pmatrix}, \tag{3.8}$$

where

$$H_1 = -(1-c^2)\partial_x^2 + 1 + \psi_c = -w\partial_x^2 + 1 - \frac{\varphi_c^2}{2w}, \qquad H_2 = -w\partial_x^2,$$
$$Az = \varphi_c z_x, \qquad A^* z = -(\varphi_c z)_x.$$

Clearly, the operator $\mathcal{H}$ is self-adjoint and, when considered over the domain

$$D(\mathcal{H}) = H^2[0,T] \times H_0^2[0,T] \subset L^2[0,T] \times L_0^2[0,T],$$

is such that $D(\mathcal{H}) \to L^2[0,T] \times L_0^2[0,T]$. Note that, when considering this spectral problem, our basic Hilbertian space is $L^2[0,T] \times L_0^2[0,T]$, instead of the usual $L^2[0,T] \times L^2[0,T]$.

3.3. Precise formulation of the main results

Our main results are described in the following theorems. Note that, in all theorems, the issue is linear stability for the stated families of spatially periodic solutions, *when the perturbation is taken to be periodic with the same period as the underlying solution*.

The issue of stability/instability, when perturbations are taken to have a period of the form nT (where n is integer) is a more complicated one. Clearly, the instability

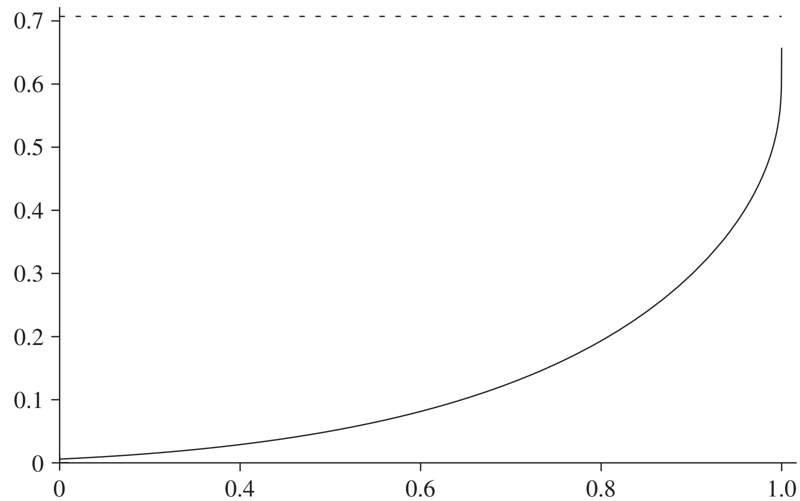

Figure 1. Graph of the positive function $\sqrt{M(\kappa)/4+M(\kappa)}$, together with its terminal value $\sqrt{2}/2$ as a reference.

results continue to apply in this case, but it may very well be that some previously stable waves (in the context of the same period perturbations) become unstable when perturbed by nT-periodic functions.

THEOREM 3.2. *Let the nonlinearity in (1.1) have the form $f(u) = u^3$. The two-parameter family of dnoidal solutions, described in (2.9), is then linearly stable,* if and only if

$$|c| \geqslant \sqrt{\frac{M(\kappa)}{4+M(\kappa)}}, \quad \kappa \in (0,1),$$

where

$$M(\kappa) := \frac{[4E(\kappa) - \pi^2/K(\kappa)][(2-\kappa^2)E(\kappa) - 2(1-\kappa^2)K(\kappa)]}{(2-\kappa^2)(E^2(\kappa) - (1-\kappa^2)K(\kappa))}, \quad \kappa \in (0,1),$$

where $E(\kappa)$, $K(\kappa)$ are elliptic integrals of the first and second kind in a Legendre form.

We now give a different formulation of the main result. Let $T > \sqrt{2}\pi$. The waves described in (2.9) are then a one-parameter family of waves, having a fundamental period T, which can be parametrized by c: $|c| < \sqrt{1-2\pi^2/T^2}$ (note that c, κ are in a one-to-one relationship given by (2.10)). Now, theorem 3.2 asserts that the stable waves in this family are exactly those with $|c| \geqslant c_T$, where $c_T \in (0, \sqrt{1-2\pi^2/T^2})$ is determined as follows. Let κ_T be the unique solution of

$$K(\kappa)\sqrt{2-\kappa^2}\sqrt{4+M(\kappa)} = T.$$

Then $c_T = \sqrt{M(\kappa_T)/(4+M(\kappa_T))}$.

Our next theorem concerns the quadratic case.

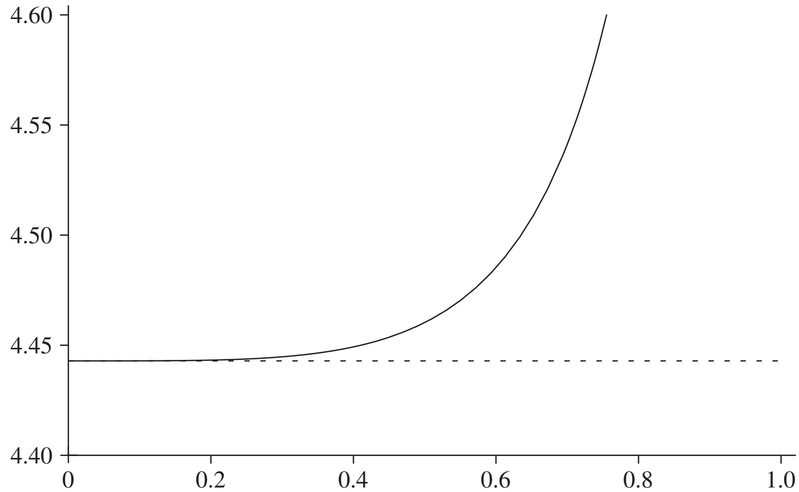

Figure 2. The solid line is the graph of the increasing function $K(\kappa)\sqrt{2-\kappa^2}\sqrt{4+M(\kappa)}\colon (0,1)\to\mathbb{R}^1$. The dashed line is $\sqrt{2}\pi$. Note that the range of this function is $(\sqrt{2}\pi,\infty)$.

THEOREM 3.3. *Let the nonlinearity in (1.1) have the form $f(u)=u^2/2$. The periodic solutions (2.5) are then linearly stable if and only if*

$$|c| \geqslant \sqrt{\frac{\tilde{F}(\kappa)}{4+\tilde{F}(\kappa)}}, \quad \kappa\in(0,1),$$

where

$$\begin{aligned}\tilde{F}(\kappa) = {} & \left[2F(\kappa) - \frac{F^2(\kappa)}{16\sqrt{1-\kappa^2+\kappa^4}K^2(\kappa)}\right] \\ & \times \Bigg(F(\kappa) + 256K^4(\kappa)F'(\kappa)G(\kappa)(1-\kappa^2+\kappa^4) \\ & \qquad + \frac{4096K^6(\kappa)(1-\kappa^2+\kappa^4)^{3/2}(F'(\kappa)G(\kappa))^2}{1-16\sqrt{1-\kappa^2+\kappa^4}K^2(\kappa)F'(\kappa)G(\kappa)}\Bigg)^{-1}\end{aligned}$$

and

$$\begin{aligned}F(\kappa) &= 16K(\kappa)[3E(\kappa) + (\kappa^2 - 2 + \sqrt{1-\kappa^2+\kappa^4})K(\kappa)], \\ G(\kappa) &= \frac{1}{128\,\mathrm{d}[K^4(\kappa)(1-\kappa^2+\kappa^4)]/\mathrm{d}\kappa}.\end{aligned}$$

An alternative formulation is the following. Fix $T>2\pi$ and consider the family of cnoidal solutions, described in (2.5). This is a one-parameter family of solutions (where κ and c are related by (2.7)), having fundamental period T and indexed by c, say, where $c\colon c<\sqrt{1-4\pi^2/T^2}$. The linearly stable waves with period T are then exactly those for which

$$\sqrt{1-\frac{4\pi^2}{T^2}} > |c| \geqslant c_T,$$

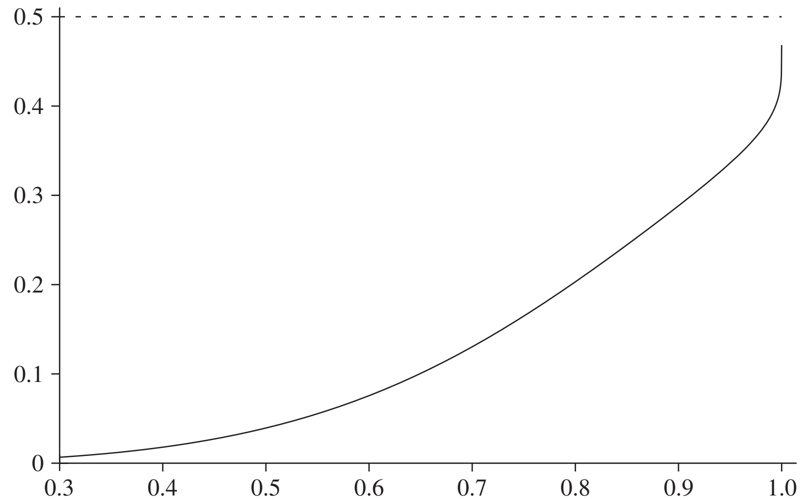

Figure 3. Graph of the positive function $\sqrt{\tilde{F}(\kappa)/(4+\tilde{F}(\kappa))}$, together with its terminal value $\frac{1}{2}$.

where $c_T \in (0, \sqrt{1-4\pi^2/T^2})$ is determined as follows. Take κ_T to be the unique solution of the algebraic equation

$$2K(\kappa)\sqrt[4]{1-\kappa^2+\kappa^4}\sqrt{4+\tilde{F}(\kappa)} = T.$$

Then $c_T = \sqrt{\tilde{F}(\kappa_T)/(4+\tilde{F}(\kappa_T))}$.

Remark. Using the results of theorems 3.2 and 3.3, one can reconstruct the results on linear stability of the whole-line waves [24]:

$$\varphi_c(\xi) = \left[\left(\frac{p+1}{2}\right)(1-c^2)\right]^{1/(p-1)} \operatorname{sech}^{2/(p-1)}\left(\frac{\sqrt{1-c^2}(p-1)}{2}\xi\right). \tag{3.9}$$

Recall that the results of [24] predict linear stability if and only if $|c| \geqslant \sqrt{p-1}/2$.

Take $p=3$. The periodic waves described in (2.9) in the limit $\kappa \to 1-$ then correspond to the whole-line waves described in (3.9). Note that, since $\lim_{\kappa\to 1-} E(\kappa) = 1$, $\lim_{\kappa\to 1-} K(\kappa) = \infty$ and $\lim_{\kappa\to 1-}(1-k^2)K(\kappa) = 0$, we can easily conclude that $\lim_{\kappa\to 1-} M(\kappa) = 4$, whence

$$\lim_{\kappa\to 1-}\sqrt{\frac{M(\kappa)}{4+M(\kappa)}} = \frac{\sqrt{2}}{2}.$$

Similarly, one can check that, for $p=2$ (more precisely if $f(u) = u^2/2$), we have that

$$\lim_{\kappa\to 1-}\sqrt{\frac{\tilde{F}(\kappa)}{4+\tilde{F}(\kappa)}} = \frac{1}{2}.$$

Thus, we obtain the results in [24] for $p=2,3$ as a corollary.

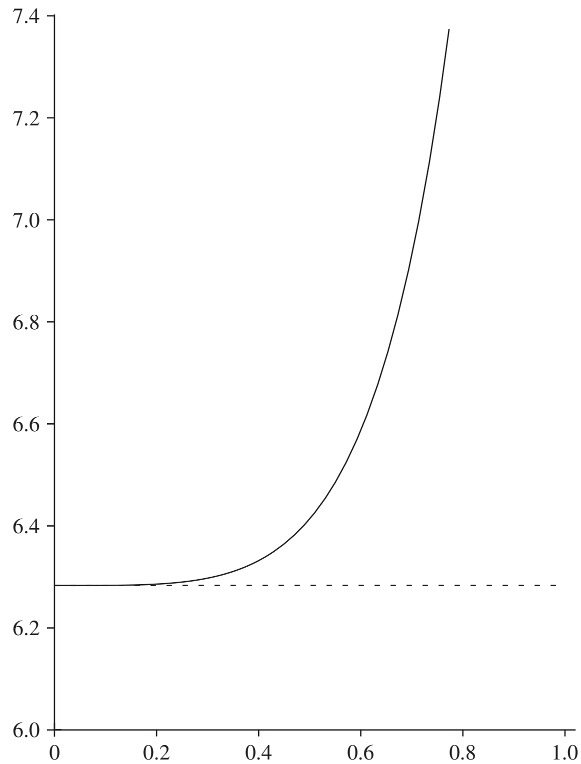

Figure 4. The solid line is the graph of the function $2K(\kappa)\sqrt[4]{1-\kappa^2+\kappa^4}\sqrt{4+\tilde{F}(\kappa)}\colon (0,1) \to \mathbb{R}^1$. The dashed line is 2π. Note that the range of this function is $(2\pi, \infty)$.

Our next result concerns the KGZ system (1.2).

THEOREM 3.4. *The KGZ system (1.2) has a two-parameter family of travelling wave solutions (φ_c, ψ_c), described in (2.11) and (2.15). These waves are stable if and only if $\kappa \in (\kappa_0, 1)$ and*

$$1 > |c| \geqslant \frac{1}{\sqrt{1+4N(\kappa)}},$$

where κ_0 and the function N are defined later.

If we take a limit as $\kappa \to 1$, we have that

$$\lim_{\kappa \to 1} \frac{1}{\sqrt{1+4N(\kappa)}} = \frac{\sqrt{2}}{2}.$$

Since $\kappa \to 1$ corresponds to the case $T = \infty$ or the case of the whole line, this allows us to conclude that the corresponding whole-line solitons are stable, provided that $|c| \geqslant \sqrt{2}/2$. This was the conclusion in [24], so we are able to deduce this result, as a consequence of theorem 3.4.

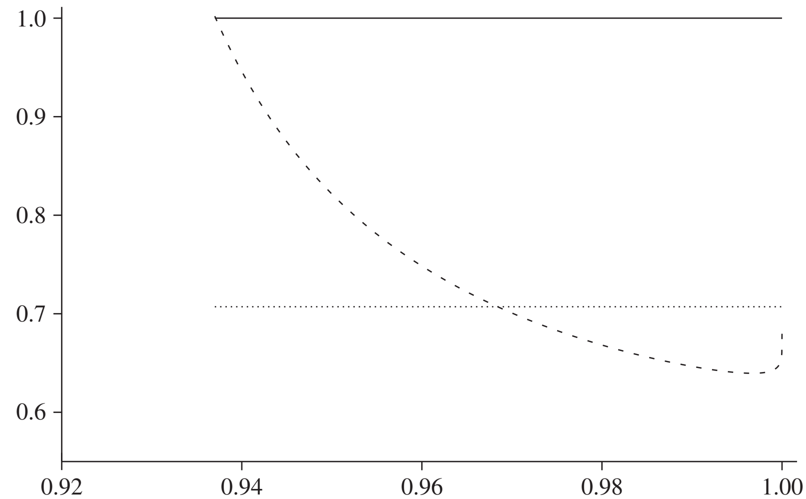

Figure 5. The dashed line is the graph of the function $1/\sqrt{1+4N(\kappa)}$ in $(\kappa_0, 1)$. The dotted line is the terminal value of $\sqrt{2}/2$.

Once again, we provide an alternative interpretation of theorem 3.4. Let $T > 0$ be a fixed period. There then exists a one-parameter family of periodic waves with period T, described in (2.11), (2.15). This family may be parametrized by c with the following restrictions on c: if $T < \sqrt{2}\pi$, then $1 > |c| > \sqrt{1 - T^2/2\pi^2}$; otherwise, if $T \geqslant \sqrt{2}\pi$, $c \in (-1, 1)$. The parameters κ and c are related by (2.18). The stable waves in this family are then given by $|c| \geqslant c_T = 1/\sqrt{1+4N(\kappa_T)}$, where $\kappa_T \in (\kappa_0, 1)$ is found as the unique solution (see figure 6) of the algebraic equation

$$\frac{4K(\kappa)\sqrt{2-\kappa^2}\sqrt{N(\kappa)}}{\sqrt{1+4N(\kappa)}} = T.$$

4. Preliminaries

4.1. Linear stability theory for second-order equations

In this section, we give a precise statement of the results of [24], concerning the linear stability of (3.4) or what is equivalent to the solvability of (3.5). We assume the following about the self-adjoint operator $\mathcal{H}$:

$$\left.\begin{array}{c} \sigma(\mathcal{H}) = \{-\delta^2\} \cup \{0\} \cup \sigma_+(\mathcal{H}), \quad \sigma_+(\mathcal{H}) \subset (\sigma^2, \infty), \quad \sigma > 0, \\ \mathcal{H}\phi = -\delta^2\phi, \qquad \dim[\mathrm{Ker}(\mathcal{H} + \delta^2)] = 1, \\ \mathcal{H}\psi_0 = 0, \qquad \dim[\mathrm{Ker}(\mathcal{H})] = 1, \\ \|\psi_0\| = 1. \end{array}\right\} \tag{4.1}$$

Note that, since $\psi_0' \perp \psi_0$ and $\mathrm{Ker}[\mathcal{H}] = \mathrm{span}[\psi_0]$, we may uniquely define

$$\mathcal{H}^{-1}\colon\ \mathrm{Ker}[\mathcal{H}]^\perp \to \mathrm{Ker}[\mathcal{H}]^\perp$$

and, in particular, the vector $\mathcal{H}^{-1}[\psi_0']$.

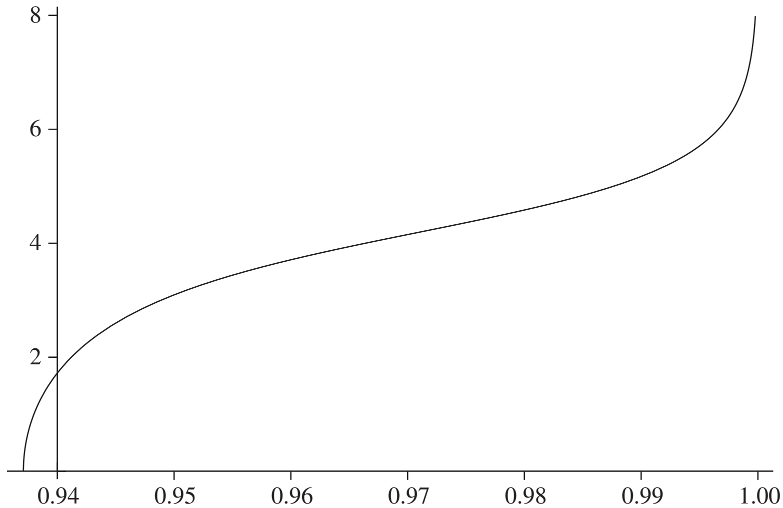

Figure 6. Graph of the increasing function $4K(\kappa)\sqrt{2-\kappa^2}\sqrt{N(\kappa)}/\sqrt{1+4N(\kappa)}\colon (\kappa_0,1)\to\mathbb{R}^1_+$. Note that its range is $(0,\infty)$.

We next require that, for all $\tau \gg 1$ (note that $H+\tau>0$ is invertible),

$$(\mathcal{H}+\tau)^{-1/2}\partial_x(\mathcal{H}+\tau)^{-1/2}, (\mathcal{H}+\tau)^{-1}\partial_x \in \mathcal{B}(L^2). \tag{4.2}$$

Finally, we require that

$$\overline{\mathcal{H}h} = \mathcal{H}\bar{h}. \tag{4.3}$$

Note that the last identity ensures that H maps real-valued functions into real-valued functions. The following theorem, in a more general form, appears as [24, theorem 1].

THEOREM 4.1. *Let $\mathcal{H}$ be a self-adjoint operator on a Hilbert space H. Assume that it satisfies the structural assumptions (4.1), (4.2) as well as the reality assumption (4.3).*

If $\langle \mathcal{H}^{-1}[\psi_0'],\psi_0'\rangle \geqslant 0$, one then has a solution to (3.5) for all values of $\omega\in\mathbb{R}^1$, that is, one has instability in the sense of definition 3.1. Otherwise, supposing that $\langle \mathcal{H}^{-1}[\psi_0'],\psi_0'\rangle < 0$,

- *the problem (3.5) has solutions if ω satisfies the inequality*

$$0 \leqslant |\omega| < \frac{1}{2\sqrt{\langle -\mathcal{H}^{-1}[\psi_0'],\psi_0'\rangle}} =: \omega^*(\mathcal{H}), \tag{4.4}$$

- *the problem (3.5) does not have solutions (i.e. stability) if ω satisfies the reverse inequality*

$$|\omega| \geqslant \omega^*(\mathcal{H}). \tag{4.5}$$

Remark.

(i) Theorem 4.1 appears in [24] as a result about the stability of (3.4), but we state it in its equivalent form for solvability of (3.5).

(ii) In the applications that we consider, we restrict our attention to the Hilbert space $H = L^2_0[0,T]$ or $H = L^2[0,T] \times L^2_0[0,T]$, depending on the situation that we are in.

4.2. Spectral theory for the Schrödinger operators of Boussinesq waves

We review and state the main results regarding the spectral theory for the second-order Schrödinger operators, arising in the linearization around Boussinesq waves. We consider the cases $p = 2$ and $p = 3$ again separately.

4.2.1. *The case* $p = 2$

In this section, we present some spectral results for $\mathcal{L}$ that will be useful in the following. The first is a technical lemma that is used to establish the simplicity of the zero eigenvalue for $\mathcal{H} = -\partial_x \mathcal{L} \partial_x$.

Lemma 4.2. *We have that* $\langle \mathcal{L}^{-1}1, 1\rangle \neq 0$.

Proof. In this case $\mathcal{L} = -\partial_x^2 + w - \varphi_{\mathrm{c}}$ and $\operatorname{Ker} \mathcal{L} = \operatorname{span} \varphi'_{\mathrm{c}}$. The spectral properties of the operator $\Lambda = -\mathrm{d}^2/\mathrm{d}y^2 - 4(1+k^2) + 12k^2 \operatorname{sn}^2(y;k)$ in $[0, 2K(k)]$ are well known [15]. The first three (simple) eigenvalues and corresponding eigenfunctions of Λ are

$$\begin{aligned}
\mu_0 &= \kappa^2 - 2 - 2\sqrt{1-\kappa^2+4\kappa^4} < 0,\\
\psi_0(x) &= \operatorname{dn}(x;\kappa)[1 - (1 + 2\kappa^2 - \sqrt{1-\kappa^2+4\kappa^4}) \operatorname{sn}^2(x;\kappa)] > 0,\\
\mu_1 &= 0,\\
\psi_1(x) &= \operatorname{dn}(x;\kappa) \operatorname{sn}(y;\kappa) \operatorname{cn}(\alpha x;\kappa) = \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}y} \operatorname{cn}^2(y;\kappa),\\
\mu_2 &= \kappa^2 - 2 + 2\sqrt{1-\kappa^2+4\kappa^4} > 0,\\
\psi_2(x) &= \operatorname{dn}(x;\kappa)[1 - (1 + 2\kappa^2 + \sqrt{1-\kappa^2+4\kappa^4}) \operatorname{sn}^2(y;\kappa)].
\end{aligned}$$

Since the eigenvalues of $\mathcal{L}$ and Λ are related by $\lambda_n = \alpha^2 \mu_n$, it follows that the first three eigenvalues of the operator $\mathcal{L}$, equipped with periodic boundary condition on $[0, 2K(k)]$, are simple and $\lambda_0 < 0$, $\lambda_1 = 0$, $\lambda_2 > 0$. The corresponding eigenfunctions are $\psi_0(\alpha x)$, $\psi_1(\alpha x) = \text{const.}\varphi'$ and $\psi_2(\alpha x)$. Note that $1, \varphi_{\mathrm{c}} \perp \operatorname{Ker} \mathcal{L}$ and

$$\mathcal{L}(1) = w - \varphi_{\mathrm{c}}, \tag{4.6}$$

and hence we can take the inverse in (4.6),

$$1 = w\mathcal{L}^{-1}1 - \mathcal{L}^{-1}\varphi_{\mathrm{c}}. \tag{4.7}$$

Taking the dot product with 1 yields the relation

$$\langle \mathcal{L}^{-1}1, 1\rangle = \frac{1}{w}\langle 1, 1\rangle + \frac{1}{w}\langle \mathcal{L}^{-1}\varphi_{\mathrm{c}}, 1\rangle.$$

Differentiating (2.2) with respect to c, we get that $\mathcal{L}[\mathrm{d}\varphi_{\mathrm{c}}/\mathrm{dc}] = 2c\varphi_{\mathrm{c}}$, whence

$$\mathcal{L}^{-1}\varphi_{\mathrm{c}} = \frac{1}{2c}\frac{\mathrm{d}\varphi_{\mathrm{c}}}{\mathrm{dc}}. \tag{4.8}$$

Entering this last formula into the expression for $\langle\mathcal{L}^{-1}1, 1\rangle$, we obtain

$$\langle\mathcal{L}^{-1}1, 1\rangle = \frac{1}{w}T + \frac{1}{2cw}\left(\partial_{\mathrm{c}}\int_0^T \varphi_{\mathrm{c}}\,\mathrm{d}x\right). \tag{4.9}$$

Using (2.6) and

$$\int_0^{2K(\kappa)} \mathrm{cn}^2(y;\kappa)\,\mathrm{d}y = \frac{2}{\kappa^2}[E(\kappa) - (1-\kappa^2)K(\kappa)],$$

we get

$$\int_0^T \varphi_{\mathrm{c}}\,\mathrm{d}x = 8\alpha[3E(\kappa) + (\kappa^2 - 2 + \sqrt{1-\kappa^2+\kappa^4})K(\kappa)] = \frac{1}{T}F(\kappa), \tag{4.10}$$

where

$$F(\kappa) = 16K(\kappa)[3E(\kappa) + (\kappa^2 - 2 + \sqrt{1-\kappa^2+\kappa^4})K(\kappa)].$$

We now need to compute

$$\partial_{\mathrm{c}}F(\kappa) = F'(\kappa)\frac{\mathrm{d}\kappa}{\mathrm{d}w}\frac{\mathrm{d}w}{\mathrm{dc}} = -2cF'(\kappa)\frac{\mathrm{d}\kappa}{\mathrm{d}w}.$$

Thus, to compute $\mathrm{d}\kappa/\mathrm{d}w$, we differentiate with respect to w the relation

$$w^2 = 16\alpha^4(1-\kappa^2+\kappa^4) = 256\frac{K^4(\kappa)(1-\kappa^2+\kappa^4)}{T^4}, \tag{4.11}$$

obtained from (2.6) and (2.7). We obtain

$$\frac{\mathrm{d}\kappa}{\mathrm{d}w} = wT^4G(\kappa), \tag{4.12}$$

where

$$G(\kappa) = \frac{1}{128\,\mathrm{d}[K^4(\kappa)(1-\kappa^2+\kappa^4)]/\mathrm{d}\kappa}.$$

From the above relations, (2.7) and (4.11), we have that

$$\begin{aligned}\langle\mathcal{L}^{-1}1, 1\rangle &= \frac{T}{w} - T^3F'(\kappa)G(\kappa) \\ &= \frac{T}{w}[1 - 16\sqrt{1-\kappa^2+\kappa^4}K^2(\kappa)F'(\kappa)G(\kappa)]. \end{aligned}\tag{4.13}$$

The expression in the brackets above is strictly positive, as seen in figure 7, which verifies the lemma. □

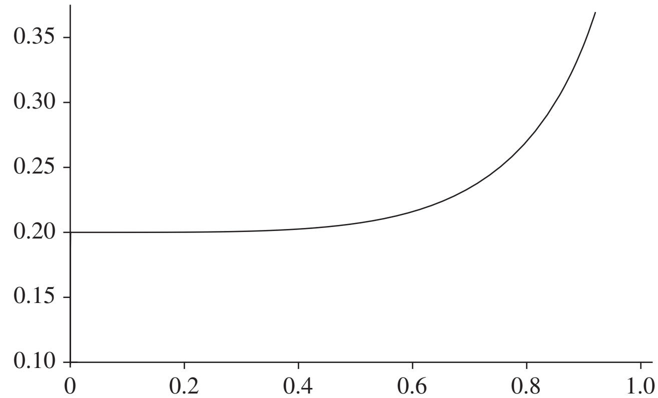

Figure 7. Graph of the positive function $[1-16\sqrt{1-\kappa^2+\kappa^4}K^2(\kappa)F'(\kappa)G(\kappa)]$.

4.2.2. *The case* $p=3$

Consider

$$\mathcal{L}=-\partial_x^2+w-3\varphi^2. \tag{4.14}$$

We use (2.9) to rewrite the operator $\mathcal{L}$ in an appropriate form. From the expression for $\varphi(x)$ from (2.9) and the relations between the elliptic functions $\mathrm{sn}(x)$, $\mathrm{cn}(x)$ and $\mathrm{dn}(x)$, we obtain that

$$\mathcal{L}=\alpha^2[-\partial_y^2+6k^2\,\mathrm{sn}^2(y)-4-k^2],$$

where $y=\alpha x$.

It is well known [15] that the first five eigenvalues of $\Lambda=-\partial_y^2+6k^2\,\mathrm{sn}^2(y,k)$, with periodic boundary conditions on $[0,4K(k)]$, where $K(k)$ is the complete elliptic integral of the first kind, are simple. These eigenvalues, with their corresponding eigenfunctions, are

$$\begin{aligned}
&\nu_0=2+2k^2-2\sqrt{1-k^2+k^4}, && \psi_0(y)=1-(1+k^2-\sqrt{1-k^2+k^4})\,\mathrm{sn}^2(y,k),\\
&\nu_1=1+k^2, && \psi_1(y)=\mathrm{cn}(y,k)\,\mathrm{dn}(y,k)=\mathrm{sn}'(y,k),\\
&\nu_2=1+4k^2, && \psi_2(y)=\mathrm{sn}(y,k)\,\mathrm{dn}(y,k)=-\,\mathrm{cn}'(y,k),\\
&\nu_3=4+k^2, && \psi_3(y)=\mathrm{sn}(y,k)\,\mathrm{cn}(y,k)=-k^{-2}\mathrm{dn}'(y,k),\\
&\nu_4=2+2k^2+2\sqrt{1-k^2+k^4}, && \psi_4(y)=1-(1+k^2+\sqrt{1-k^2+k^4})\,\mathrm{sn}^2(y,k).
\end{aligned}$$

It follows that the first three eigenvalues of the operator $\mathcal{L}$, equipped with periodic boundary condition on $[0,2K(k)]$ (that is, in the case of the left and right families), are simple and $\lambda_0=\alpha^2(\nu_0-\nu_3)<0$, $\lambda_1=\alpha^2(\nu_3-\nu_3)=0$, $\lambda_2=\alpha^2(\nu_4-\nu_3)>0$. The corresponding eigenfunctions are $\chi_0=\psi_0(\alpha x)$, $\chi_1=\varphi'(x)$, $\chi_2=\psi_4(\alpha x)$. Thus, we have proved the following.

Proposition 4.3. *The linear operator* $\mathcal{L}$ *defined by* (4.14) *has the following spectral properties.*

(i) *The first three eigenvalues of* $\mathcal{L}$ *are simple.*

(ii) *The second eigenvalue of* $\mathcal{L}$ *is* $\lambda_1=0$, *which is simple.*

(iii) *The rest of the spectrum consists of a discrete set of eigenvalues, which are strictly positive.*

We next verify the following technical result.

LEMMA 4.4. *The operator $\mathcal{L}$ verifies that $\langle \mathcal{L}^{-1}1, 1\rangle \neq 0$.*

Proof. This statement was needed and proved in [9], but we repeat the short argument for completeness. First we prove that, for $i \neq 0, 4$, $\langle \psi_i, 1\rangle = 0$.

Using the expressions for Λ and ψ_4, we get that

$$\nu_i\langle \psi_i, 1\rangle = 6\kappa^2\langle \mathrm{sn}^2(y;\kappa), \psi_i\rangle = \frac{12\kappa^2}{\nu_4}\langle 1-\psi_4, \psi_i\rangle.$$

It follows that, for $i \neq 4$,

$$0 = \langle \psi_4, \psi_i\rangle = \left(1 - \frac{\nu_i\nu_4}{12\kappa^2}\right)\langle \psi_i, 1\rangle. \tag{4.15}$$

Observe, however, that $\nu_0\nu_4 = 12\kappa^2$, which means that $(1-\nu_i\nu_4/12\kappa^2) \neq 0$ whenever $i \neq 0$ (since $\nu_i \neq \nu_0$). By (4.15), this implies that $\langle \psi_i, 1\rangle = 0$, $i \neq 0, 4$.

From [20, theorem 2.15], the eigenfunctions of the operator $\mathcal{L}$ form an orthonormal basis of $L^2[0,T]$, and hence we compute $\langle \mathcal{L}^{-1}1, 1\rangle$ by expanding 1 in the eigenfunction expansion. Note that all terms corresponding to mean value zero eigenfunctions disappear (since $\langle \psi_i, 1\rangle = 0$, $i \neq 0, 4$). Hence, the expansion for $\langle \mathcal{L}^{-1}1, 1\rangle$ has only two non-zero terms. More precisely, we have that

$$\begin{aligned}\langle \mathcal{L}^{-1}1, 1\rangle &= \frac{\langle 1, \chi_0\rangle^2}{\alpha^2(\nu_0-\nu_3)\|\chi_0\|^2} + \frac{\langle 1, \chi_2\rangle^2}{\alpha^2(\nu_4-\nu_3)\|\chi_2\|^2}\\ &= \frac{2}{\alpha^3}\left[\frac{B_1(\kappa)}{(\kappa^2-2-2\sqrt{1-\kappa^2+\kappa^4})B_3(\kappa)}\right.\\ &\qquad\qquad\left.+\frac{B_2(\kappa)}{(\kappa^2-2+2\sqrt{1-\kappa^2+\kappa^4})B_4(\kappa)}\right],\end{aligned} \tag{4.16}$$

where we have used the formulae

$$\int_0^{2K(\kappa)} \mathrm{sn}^2(y)\,\mathrm{d}y = \frac{2}{\kappa^2}[K(\kappa)-E(\kappa)],$$
$$\int_0^{2K(\kappa)} \mathrm{sn}^4(y)\,\mathrm{d}y = \frac{2}{3\kappa^4}[(2+\kappa^2)K(\kappa)-2(1+\kappa^2)E(\kappa)]$$

and

$$B_1(\kappa) = \left(\frac{\sqrt{1-\kappa^2+\kappa^4}-1}{\kappa^2}K(\kappa) + \frac{1+\kappa^2-\sqrt{1-\kappa^2+\kappa^4}}{\kappa^2}E(\kappa)\right)^2,$$

$$B_2(\kappa) = \left(-\frac{\sqrt{1-\kappa^2+\kappa^4}+1}{\kappa^2}K(\kappa) + \frac{1+\kappa^2+\sqrt{1-\kappa^2+\kappa^4}}{\kappa^2}E(\kappa)\right)^2,$$

$$\begin{aligned}B_3(\kappa) = K(\kappa) &- \frac{2(1+\kappa^2-\sqrt{1-\kappa^2+\kappa^4})}{\kappa^2}[K(\kappa)-E(\kappa)]\\ &+ \frac{(1+\kappa^2-\sqrt{1-\kappa^2+\kappa^4})^2}{3\kappa^4}[(2+\kappa^2)K(\kappa)-2(1+\kappa^2)E(\kappa)],\end{aligned}$$

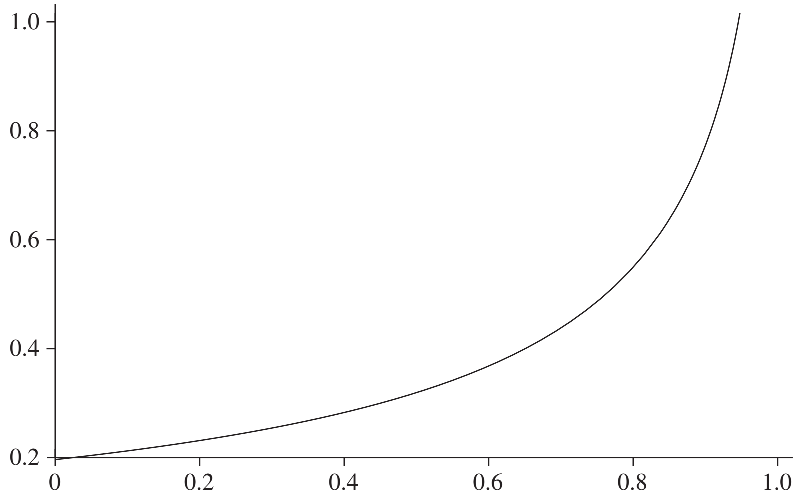

Figure 8. Graph of the positive function
$B_1(\kappa)/(\kappa^2-2-2\sqrt{1-\kappa^2+\kappa^4})B_3(\kappa)+B_2(\kappa)/(\kappa^2-2+2\sqrt{1-\kappa^2+\kappa^4})B_4(\kappa)$.

$$B_4(\kappa)=K(\kappa)-\frac{2(1+\kappa^2+\sqrt{1-\kappa^2+\kappa^4})}{\kappa^2}[K(\kappa)-E(\kappa)] + \frac{(1+\kappa^2+\sqrt{1-\kappa^2+\kappa^4})^2}{3\kappa^4}[(2+\kappa^2)K(\kappa)-2(1+\kappa^2)E(\kappa)].$$

From the graph of the function

$$\frac{B_1(\kappa)}{(\kappa^2-2-2\sqrt{1-\kappa^2+\kappa^4})B_3(\kappa)}+\frac{B_2(\kappa)}{(\kappa^2-2+2\sqrt{1-\kappa^2+\kappa^4})B_4(\kappa)}$$

in figure 8, we realize that $\langle \mathcal{L}^{-1}1,1\rangle>0$, and hence lemma 4.4 is established. □

Lemmas 4.2 and 4.4 allow us to verify an important property about the simplicity of the zero eigenvalue for the operator $\mathcal{H}_c$.

Corollary 4.5. *Let $p=2$ or $p=3$. The operator $\mathcal{H}_c=-\partial_x\mathcal{L}\partial_x$ (corresponding to $f(z)=z^3, z^2/2$) defined in (3.2) then has 0 as a simple eigenvalue in $L_0^2[0,T]$, with an eigenfunction $\psi_0=\varphi_c-(1/T)\int_0^T\varphi_c$.*

Proof. First, $\varphi_c-(1/T)\int\varphi_c(x)\,dx$ is easily seen to be an eigenfunction, since

$$\mathcal{H}_c\left[\varphi_c-\frac{1}{T}\int\varphi_c(x)\,dx\right]=-\partial_x\mathcal{L}[\varphi_c']=0,$$

since φ_c' is an eigenfunction for $\mathcal{L}$.

Regarding uniqueness, let $f\in L_0^2[0,T]$, so $\mathcal{H}_cf=0$. It follows that

$$\mathcal{L}[f']=c=\text{const.}$$

Since $\operatorname{Ker}(\mathcal{L})=\operatorname{span}\{\varphi_c'\}$ and $1\perp\varphi_c'$, we can resolve the last equation as $f'=c\mathcal{L}^{-1}1$. Thus,

$$0=\langle 1,f'\rangle=c\langle 1,\mathcal{L}^{-1}1\rangle,$$

whence $c = 0$, since $\langle 1, \mathcal{L}^{-1}1 \rangle \neq 0$ by lemma 4.4. It follows that f' is an eigenvector for $\mathcal{L}$. Thus, $f' = \mu\varphi_{\mathrm{c}}'$, by proposition 4.3. But, then, $f = \mu[\varphi_{\mathrm{c}} - (1/T)\int \varphi_{\mathrm{c}}(x)\,\mathrm{d}x]$, since we are in the space $L^2_0(0,T)$ and the uniqueness is established. □

4.3. Spectral theory for the Schrödinger operator $\mathcal{H}$ of the KGZ system

First, as in the Boussinesq case, we show that the operator $\mathcal{H}$ has a simple eigenvalue at 0. In addition, we identify the unique (up to a multiplicative constant) eigenfunction of $\mathcal{H}$. Recall that in our considerations we work with the space $L^2_0[0,T]$, that is, the second component contains only functions with mean value 0. This proposition closely mirrors the corresponding statement of [24, proposition 8], with a few notable differences.

PROPOSITION 4.6. *The self-adjoint operator $\mathcal{H}$ introduced in (3.8) has an eigenvalue at* 0, *which is simple. In addition, the unique (up to a multiplicative constant) eigenfunction is given by*

$$\boldsymbol{\psi}_0 = \begin{pmatrix} \varphi_{\mathrm{c}}' \\ -\dfrac{1}{2w}\left(\varphi_{\mathrm{c}}^2 - T^{-1}\displaystyle\int_0^T \varphi_{\mathrm{c}}^2\right) \end{pmatrix}.$$

Proof. Let $\binom{f}{g}$ be an eigenvector corresponding to a zero eigenvalue, that is, let $\mathcal{H}\binom{f}{g} = 0$. In other words,

$$\left.\begin{aligned} -wf'' + f - \frac{\varphi^2}{2w}f + \varphi g' &= 0, \\ -(\varphi f)' - wg'' &= 0. \end{aligned}\right\} \tag{4.17}$$

Integrating the second equation in x implies that, for some constant c_0, we have

$$g' = -\frac{\varphi f}{w} + c_0,$$

whence the equation for f becomes

$$-wf'' + f - \frac{3\varphi^2}{2w}f + c_0\varphi = 0. \tag{4.18}$$

We show that $c_0 = 0$, and then $f = d\varphi_{\mathrm{c}}'$ for some constant d. To that end, recall the defining equation for φ_{c}, namely, (2.12), and differentiate it with respect to x. We get that

$$-w\varphi_{\mathrm{c}}'' + \varphi_{\mathrm{c}}' - \frac{3\varphi^2}{2w}\varphi_{\mathrm{c}}' = 0. \tag{4.19}$$

Following the usual analogy with the Korteweg–de Vries equation, we introduce the second-order differential operator

$$\mathcal{L} = -w\partial_x^2 + 1 - \frac{3\varphi^2}{2w}.$$

Using that $\mathrm{dn}^2 + \kappa^2 \mathrm{sn}^2 = 1$ and $w\alpha^2 = 1/(2-\kappa^2)$, we get

$$\begin{aligned}\mathcal{L} &= -w\partial_x^2 + 1 - \frac{3\varphi_1^2}{2w}(1-\kappa^2 \mathrm{sn}^2(\alpha x,\kappa)) \\ &= -w\partial_x^2 + 1 - 6w\alpha^2 + 6w\alpha^2\kappa^2 \mathrm{sn}^2(\alpha x,\kappa) \\ &= w\alpha^2(-\partial_y^2 + 6\,\mathrm{sn}^2(y,\kappa) - \kappa^2 - 4),\end{aligned}$$

where $y = \alpha x$. It follows that the first three eigenvalues of the operator $\mathcal{L}$, equipped with periodic boundary condition on $[0, 2K(k)]$, are simple, and

$$\lambda_0 = w\alpha^2(\nu_0 - \nu_3) < 0, \qquad \lambda_1 = w\alpha^2(\nu_3 - \nu_3) = 0, \qquad \lambda_2 = w\alpha^2(\nu_4 - \nu_3) > 0.$$

The corresponding eigenfunctions are $\chi_0 = \psi_0(\alpha x)$, $\chi_1 = \varphi'(x)$, $\chi_2 = \psi_4(\alpha x)$, where ν_i and ψ_i are given in § 4.2.2.

In particular, the kernel of $\mathcal{L}$ is spanned by φ_{c}', i.e. $\mathrm{Ker}(\mathcal{L}) = \mathrm{span}[\varphi_{\mathrm{c}}']$. Going back to (4.18), we can rewrite it as

$$\mathcal{L}[f] + c_0\varphi_{\mathrm{c}} = 0.$$

Note that all solutions to this last equation are given by

$$f = d\varphi' - c_0\mathcal{L}^{-1}[\varphi_{\mathrm{c}}],$$

where d is an arbitrary scalar, since $\varphi_{\mathrm{c}} \perp \mathrm{span}[\varphi_{\mathrm{c}}'] = \mathrm{Ker}(\mathcal{L})$. Putting this last formula into the equation for g yields

$$g' = -\frac{\varphi f}{w} + c_0 = -\frac{\varphi}{w}(d\varphi' - c_0\mathcal{L}^{-1}[\varphi]) + c_0 = -\frac{d}{w}\varphi\varphi' + c_0\biggl(\frac{\varphi\mathcal{L}^{-1}[\varphi]}{w} + 1\biggr).$$

Integrating the last expression in $[0, T]$ and using the periodicity yields

$$c_0\biggl(\frac{\langle\varphi, \mathcal{L}^{-1}[\varphi]\rangle}{w} + T\biggr) = 0. \tag{4.20}$$

Thus, if we verify that $\langle\varphi, \mathcal{L}^{-1}[\varphi]\rangle \neq -Tw$, we can conclude from (4.20) that $c_0 = 0$, whence $f = d\varphi'$. Furthermore, $g' = -(d/w)\varphi\varphi'$, whence

$$g = -d\frac{\varphi^2}{2w} + \text{const}.$$

Recall, however, that the constant in the formula above is uniquely determined by the fact that g has mean value 0 (i.e. $g \in L_0^2[0, T]$), whence

$$g = d\biggl(-\frac{\varphi^2}{2w} + \frac{\int_0^T \varphi^2}{2Tw}\biggr).$$

Thus, proposition 4.6 is established modulo the following.

FACT. We have that $\langle\varphi_{\mathrm{c}}, \mathcal{L}^{-1}[\varphi_{\mathrm{c}}]\rangle > -Tw/3$, so, in particular, $\langle\varphi, \mathcal{L}^{-1}[\varphi]\rangle > -Tw$.

From (2.12), we have that $\varphi_{\mathrm{c}}^3 = -2w^2\varphi_{\mathrm{c}}'' + 2w\varphi_{\mathrm{c}}$, and thus

$$\mathcal{L}\varphi = -w\varphi_{\mathrm{c}}'' + \varphi_{\mathrm{c}} - \frac{3}{2w}\varphi_{\mathrm{c}}^3 = 2w\varphi_{\mathrm{c}}'' - 2\varphi_{\mathrm{c}}. \tag{4.21}$$

On the other hand, differentiating $-w^2\varphi_c'' + w\varphi_c - \varphi_c^3/2 = 0$ with respect to w (and dividing by w) and using (4.21) to express φ_c'' results in

$$\mathcal{L}\left[\frac{d\varphi_c}{dw}\right] = 2\varphi'' - \frac{1}{w}\varphi_c = \frac{1}{w}\varphi_c + \frac{1}{w}\mathcal{L}\varphi_c.$$

Taking $\mathcal{L}^{-1}$ in the last identity yields

$$\mathcal{L}^{-1}\varphi_c = w\frac{d\varphi_c}{dw} - \varphi_c.$$

Since $\int_0^{K(\kappa)} \mathrm{dn}^2(y,\kappa)\,dy = E(\kappa)$, we compute that $\int_0^T \varphi^2\,dx = (16w^2/T)E(\kappa)K(\kappa)$. Therefore,

$$\begin{aligned}\langle \mathcal{L}^{-1}\varphi_c, \varphi_c\rangle &= w\left\langle \varphi_c, \frac{d\varphi_c}{dw}\right\rangle - \langle \varphi_c, \varphi_c\rangle \\ &= \frac{w}{2}\partial_w[\|\varphi_c\|^2] - \|\varphi_c\|^2 \\ &= \frac{w}{2}\partial_w\left[\frac{16w^2}{T}E(\kappa)K(\kappa)\right] - \frac{16w^2}{T}E(\kappa)K(\kappa) \\ &= \frac{8w^3}{T}\frac{d}{d\kappa}[K(\kappa)E(\kappa)]\frac{d\kappa}{dw}.\end{aligned}$$

To compute $d\kappa/dw$, note that, from (2.16), we have that $2w\alpha = \varphi_1$, whence

$$4w(2-\kappa^2)K^2(\kappa) = T^2. \tag{4.22}$$

Differentiating (4.22) with respect to w, we get that

$$\frac{d\kappa}{dw} = -\frac{(2-\kappa^2)K^2(\kappa)}{w\,d[(2-\kappa^2)K^2(\kappa)]/d\kappa}. \tag{4.23}$$

Thus, using (4.22),

$$\begin{aligned}\langle \mathcal{L}^{-1}\varphi_c, \varphi_c\rangle &= -\frac{8w^2}{T}\frac{d}{d\kappa}[K(\kappa)E(\kappa)]\frac{(2-\kappa^2)K^2(\kappa)}{d[(2-\kappa^2)K^2(\kappa)]/d\kappa} \\ &= -wT\left[2\frac{d[K(\kappa)E(\kappa)]/d\kappa}{d[(2-\kappa^2)K^2(\kappa)]/d\kappa}\right].\end{aligned}$$

Looking at figure 9, we realize that, since

$$\frac{1}{3} = \lim_{\kappa\to 0} 2\frac{d[K(\kappa)E(\kappa)]/d\kappa}{d[(2-\kappa^2)K^2(\kappa)]/d\kappa} \geqslant 2\frac{d[K(\kappa)E(\kappa)]/d\kappa}{d[(2-\kappa^2)K^2(\kappa)]/d\kappa},$$

we have that

$$\langle \mathcal{L}^{-1}\varphi_c, \varphi_c\rangle \geqslant -\tfrac{1}{3}wT,$$

which establishes the claim. □

The next thing one needs to establish, in order to apply theorem 4.1, is that the operator $\mathcal{H}$ for the KGZ system (defined in (3.8)) has a simple negative eigenvalue. This result should be compared with the corresponding statement in [24, proposition 9] for the whole-line case.

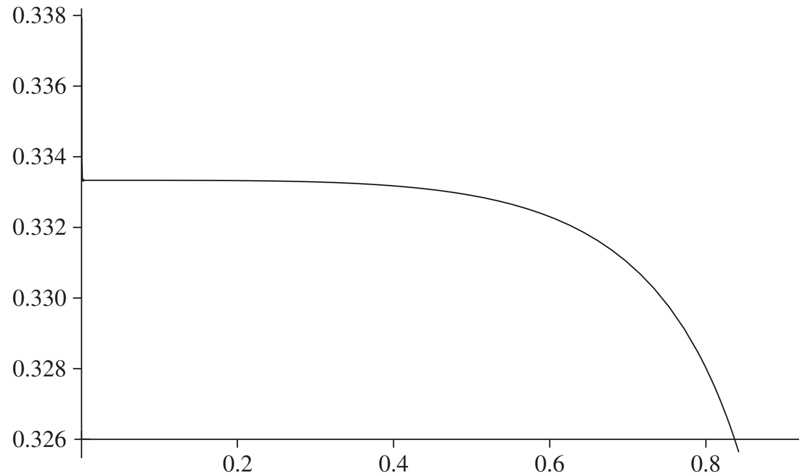

Figure 9. Graph of the function $2(\mathrm{d}[K(\kappa)E(\kappa)]/\mathrm{d}\kappa)/(\mathrm{d}[(2-\kappa^2)K^2(\kappa)]/\mathrm{d}\kappa)$.

Proposition 4.7. *The operator $\mathcal{H}$, defined in (3.8), has a simple negative eigenvalue.*

Proof. Consider the eigenvalue problem in the form

$$\mathcal{H}\begin{pmatrix} f \\ g \end{pmatrix} = -a^2 \begin{pmatrix} f \\ g \end{pmatrix}$$

for some $a \in (0, \infty)$. As in proposition 4.6, this can be rewritten as[2]

$$\left.\begin{aligned} -wf'' + f - \frac{\varphi^2}{2w} f + \varphi g' &= -a^2 f, \\ -(\varphi f)' - wg'' &= -a^2 g. \end{aligned}\right\} \tag{4.24}$$

From the second equation, we may resolve for g that

$$g = (a^2 - w\partial_x^2)^{-1}\partial_x[\varphi_{\mathrm{c}} f]. \tag{4.25}$$

This last formula requires a bit of justification, but, basically, since $\partial_x[\varphi f]$ is guaranteed to have mean value 0, it suffices to define $(a^2 - w\partial_x^2)^{-1}$ (where $a^2 > 0$, $w > 0$) on $L^2[0, T]$ by

$$(a^2 - w\partial_x^2)^{-1}\left[\sum_{n=-\infty}^{\infty} a_n \mathrm{e}^{\mathrm{i}nx}\right] := \sum_{n=-\infty}^{\infty} \frac{a_n}{a^2 + 4\pi^2 wn^2/T^2} \mathrm{e}^{2\pi \mathrm{i}nx/T},$$

whence the formula for g in (4.25) makes sense. In fact, $L_0^2[0, T]$ is invariant under the action of $(a^2 - w\partial_x^2)^{-1}$ and, hence, $g \in L^2[0, T]$. Furthermore, we use (4.25) to deduce the following formula for g':

$$g' = \partial_x^2(a^2 - w\partial_x^2)^{-1}[\varphi_{\mathrm{c}} f] = -\frac{\varphi f}{w} + \frac{a^2}{w}(a^2 - w\partial_x^2)^{-1}[\varphi_{\mathrm{c}} f].$$

[2]Note that the second equation requires that $\int_0^T g(x)\,\mathrm{d}x = 0$. This means, in particular, that the statement for existence of a negative eigenvalue is invalid, unless the second component of the Hilbert space is $L_0^2[0, T]$.

We put this into the first equation of (4.24) to obtain the following equation for f:

$$-wf'' + (1+a^2)f - \frac{3\varphi_c^2}{w} f + \frac{a^2}{w}[\varphi_c(a^2 - w\partial_x^2)^{-1}\varphi_c f] = 0. \tag{4.26}$$

Introduce a one-parameter family of self-adjoint operators:

$$M_a := -w\partial_x^2 + (1+a^2) - \frac{3\varphi^2}{2w} + \frac{a^2}{w}[\varphi_c(a^2 - w\partial_x^2)^{-1}(\varphi_c\cdot)].$$

In order to finish the proof of proposition 4.7, we need to establish that there is a unique $a_0 > 0$ such that the operator M_{a_0} has an eigenvalue 0 and such an eigenvalue is simple. We first show that there exists $a_0 > 0$ such that M_{a_0} has an eigenvalue at 0, and we then show that this eigenvalue 0 is simple for M_{a_0}.

To that end, we establish the following.

Claim. *For $a \geqslant b \geqslant 0$, we have $M_a \geqslant M_b + (a^2 - b^2)\,\mathrm{Id} \geqslant M_b$.*

Assuming the validity of the claim, we complete the proof of proposition 4.7. Define $\lambda(a)$ to be the minimal eigenvalue for M_a, that is,

$$\lambda(a) := \inf\{\lambda\colon \lambda \in \sigma(M_a)\} = \inf_{\|f\|=1} \langle M_a f, f\rangle.$$

Clearly, the function $a \to \lambda(a)$ is continuous (in fact, more generally, $a \to M_a$ is continuous as a function from $\mathbb{R}^1 \to B(L^2[0,T])$). Moreover, as a consequence of the claim, $\lambda(a)$ is a strictly increasing function of its argument. In order to show the existence of a_0, it suffices to show that $a \to \lambda(a)$ changes sign in $[0,\infty)$ (and, hence, vanishes at some $a_0 > 0$). But at $a = 0$ we have that

$$M_0 = -w\partial_x^2 + 1 - \frac{3\varphi^2}{2w} = \mathcal{L},$$

which was considered before. Since we have checked that $\mathcal{L}$ has a (simple) negative eigenvalue, $-\delta^2$ say, it follows that $\lambda(0) < 0$. On the other hand, by the claim,

$$M_a \geqslant M_0 + a^2\,\mathrm{Id} = \mathcal{L} + a^2\,\mathrm{Id}.$$

In particular, for every $a > \delta$, we have $M_a > (a^2 - \delta^2)\,\mathrm{Id}$, whence $\lambda(a) \geqslant a^2 - \delta^2 > 0$. Thus, for any $b > \delta$, the function $\lambda(a)$ changes sign (exactly once) in the interval $(0, b)$. We have shown that there exists $a_0\colon \lambda(a_0) = 0$.

We now have to show the second part of the proposition, namely, that 0 is an isolated eigenvalue for M_{a_0}. Let ϕ_0 be the eigenvector for the simple negative eigenvalue for $M_0 = \mathcal{L}$. Note that, by proposition 5.1, the second eigenvalue of $\mathcal{L}$ is 0, which means that $M_0|_{\phi_0{}^\perp} = \mathcal{L}|_{\phi_0{}^\perp} \geqslant 0$. In addition, by the claim, we have that $M_{a_0} \geqslant M_0 + a_0^2\,\mathrm{Id} = \mathcal{L} + a_0^2\,\mathrm{Id}$, and thus, by the Courant minimax principle for the second eigenvalue,

$$\lambda_1(M_{a_0}) = \sup_{z\neq 0} \inf_{u\perp z\colon \|u\|=1} \langle M_{a_0} u, u\rangle \geqslant a_0^2 + \inf_{u\perp\phi_0\colon \|u\|=1} \langle \mathcal{L}u, u\rangle \geqslant a_0^2 > 0.$$

It follows that $\lambda_0(M_{a_0}) = 0$, while $\lambda_1(M_{a_0}) > 0$, which shows the simplicity of the zero eigenvalue for M_{a_0}. It now remains to establish the claim.

Proof. By the form of the operators M_a, it suffices to show for all trigonometric polynomials $f \in L^2[0,T]$ that, for $a \geqslant b$,

$$a^2\langle\varphi_c(a^2 - w\partial_x^2)^{-1}[\varphi_c f], f\rangle \geqslant b^2\langle\varphi_c(b^2 - w\partial_x^2)^{-1}[\varphi_c f], f\rangle.$$

But, letting a_n be the Fourier coefficients of $\varphi_c f$, that is,

$$\varphi_c f = \sum_n a_n \frac{e^{2\pi i n x/T}}{\sqrt{T}} \quad \text{or} \quad a_n = \frac{1}{\sqrt{T}}\int_0^T \varphi_c(x) f(x) e^{-2\pi i n x/T}\,dx,$$

we see that

$$\begin{aligned} b^2\langle\varphi_c(b^2 - w\partial_x^2)^{-1}[\varphi_c f], f\rangle &= \sum_n \frac{b^2}{b^2 + 4\pi^2 w n^2/T^2}|a_n|^2 \\ &\leqslant \sum_n \frac{a^2}{a^2 + 4\pi^2 w n^2/T^2}|a_n|^2 \\ &= a^2\langle\varphi_c(a^2 - w\partial_x^2)^{-1}[\varphi_c f], f\rangle, \end{aligned}$$

where we have used that

$$\frac{b^2}{b^2 + 4\pi^2 w n^2/T^2} \leqslant \frac{a^2}{a^2 + 4\pi^2 w n^2/T^2}$$

whenever $w > 0$, $a \geqslant b \geqslant 0$. □

5. Linear stability for the Boussinesq equation: proof of theorems 3.2 and 3.3

Now that we have the solutions, we need to check that the operator H_c satisfies the requirements in theorem 4.1, after which we need to compute the index $\omega^*(H_c)$. We collect the necessary results in the following propositions.

PROPOSITION 5.1 (spectral properties of H_c). *The operator $\mathcal{H}_c$, as given in (3.2), satisfies (4.1)–(4.3) for $p = 2, 3$.*

Our next result gives a precise formula for the index $\omega^*(\mathcal{H}_c)$.

PROPOSITION 5.2. *We have the following.*

(1) *For $p = 2$,*

$$\omega^*(\mathcal{H}_c) = \frac{\sqrt{w}}{2}\sqrt{\tilde{F}(\kappa)},$$

where $\tilde{F}(\kappa)$ is defined in (5.6).

(2) *For $p = 3$,*

$$\omega^*(\mathcal{H}_c) = \frac{\sqrt{w}}{2}\sqrt{\frac{[4E(\kappa) - \pi^2/K(\kappa)][(2-\kappa^2)E(\kappa) - 2(1-\kappa^2)K(\kappa)]}{(2-\kappa^2)(E^2(\kappa) - (1-\kappa^2)K(\kappa))}}.$$

REMARK 5.3. The function under the square root is positive for all values of κ : $0 < \kappa < 1$.

We now complete the proof of theorem 3.2, based on the results of propositions 5.1 and 5.2.

Let $p = 3$. We apply theorem 4.1, from which we get stability, provided that $|c| \geqslant \omega^*(\mathcal{H})$. Thus, we need to resolve the inequality

$$|c| \geqslant \frac{\sqrt{1-c^2}}{2}\sqrt{M(\kappa)},$$

where we have taken

$$M(\kappa) := \frac{[4E(\kappa) - \pi^2/K(\kappa)][(2-\kappa^2)E(\kappa) - 2(1-\kappa^2)K(\kappa)]}{(2-\kappa^2)(E^2(\kappa) - (1-\kappa^2)K(\kappa))},$$

and we obtain, for the interval of the stable speeds, that

$$|c| \geqslant \sqrt{\frac{M(\kappa)}{4 + M(\kappa)}},$$

as stated in theorem 3.2.

Similarly, for $p = 2$, according to proposition 5.2 we have that

$$|c| \geqslant \frac{\sqrt{1-c^2}}{2}\sqrt{\tilde{F}(\kappa)},$$

whence we conclude similarly that

$$|c| \geqslant \sqrt{\frac{\tilde{F}(\kappa)}{4 + \tilde{F}(\kappa)}}$$

is a necessary and sufficient condition for stability of the corresponding travelling wave.

5.1. Proof of proposition 5.1

We note that standard arguments imply the validity of (4.2), since $\mathcal{H}$ is a fourth-order operator. The reality condition (4.3) is also trivially satisfied, as all our potentials are real valued. The condition that is hard to check is (4.1). We need to verify that the operator $\mathcal{H}$ has a simple eigenvalue at 0. This was indeed the conclusion of corollary 4.5.

Thus, it remains to show that the operator $\mathcal{H}$ defined in (3.2) has a simple negative eigenvalue, and proposition 5.1 follows. This is a non-trivial fact. Interestingly enough, this was needed (and proved in our paper [17]) when we considered the transverse instability of the same spatially periodic waves in the Kadomtsev–Petviashvili (KP) and modified KP models, that is, exactly for the operators considered here, corresponding to the cases $f(u) = u^2/2$ and $f(u) = u^3$. More precisely, we have derived a necessary condition such that the first two eigenvalues of $\mathcal{H} = -\partial_x \mathcal{L} \partial_x$ satisfy

$$\lambda_0(\mathcal{H}) < \lambda_1(\mathcal{H}) = 0,$$

which was verified for $p = 2, 3$. This is exactly what is needed here. The interested reader may consult [17, § 4.1] for a full and complete proof.

5.2. Proof of proposition 5.2

5.2.1. *The case $p = 2$*

We compute the index of stability $\omega^*(\mathcal{H})$. We have that

$$\langle \mathcal{H}^{-1}\psi_0', \psi_0' \rangle = \frac{1}{\|\varphi_c + A\|^2} \langle \mathcal{H}^{-1}\varphi_c', \varphi_c' \rangle, \tag{5.1}$$

where $A = -(1/T)\int_0^T \varphi_c \, dx$. Let $f\colon \mathcal{H}[f] = \varphi_c'$. It follows that $-\mathcal{L}f' = \varphi_c + b$ for some constant b. Hence,

$$-f' = \mathcal{L}^{-1}\varphi_c + b\mathcal{L}^{-1}1. \tag{5.2}$$

Thus,

$$\langle \mathcal{H}^{-1}\psi_0', \psi_0' \rangle = \frac{1}{\|\varphi_c + A\|^2} \langle f, \varphi_c' \rangle = \frac{1}{\|\varphi_c + A\|^2} \langle \mathcal{L}^{-1}\varphi_c, \varphi_c \rangle + b\langle \mathcal{L}^{-1}1, \varphi_c \rangle. \tag{5.3}$$

From (5.2), we have that $0 = -\langle f', 1 \rangle = \langle \mathcal{L}^{-1}\varphi_c, 1 \rangle + b\langle \mathcal{L}^{-1}1, 1 \rangle$, whence

$$b = -\frac{\langle \mathcal{L}^{-1}\varphi_c, 1 \rangle}{\langle \mathcal{L}^{-1}1, 1 \rangle}. \tag{5.4}$$

Combining (4.8), (5.1) and (5.4) yields

$$\begin{aligned}\langle \mathcal{H}^{-1}\psi_0', \psi_0' \rangle &= \frac{1}{\|\varphi_c + A\|^2}\left(\langle \mathcal{L}^{-1}\varphi_c, \varphi_c \rangle - \frac{\langle \mathcal{L}^{-1}\varphi_c, 1 \rangle \langle \mathcal{L}^{-1}1, \varphi_c \rangle}{\langle \mathcal{L}^{-1}1, 1 \rangle} \right) \\ &= \frac{1}{\|\varphi_c + A\|^2}\left(\frac{1}{4c}\frac{d}{dc}\langle \varphi_c, \varphi_c \rangle - \frac{\langle \mathcal{L}^{-1}\varphi_c, 1 \rangle \langle \mathcal{L}^{-1}1, \varphi_c \rangle}{\langle \mathcal{L}^{-1}1, 1 \rangle} \right).\end{aligned}$$

From (2.2), after integrating

$$\int_0^T \varphi_c^2 \, dx = 2w \int_0^T \varphi_c \, dx = \frac{2w}{T} F(\kappa),$$

we use (4.12) to find that

$$\frac{1}{4c}\frac{d}{dc}\langle \varphi_c, \varphi_c \rangle = \frac{1}{T}[-F(\kappa) - 256K^4(\kappa)F'(\kappa)G(\kappa)(1 - \kappa^2 + \kappa^4)]. \tag{5.5}$$

We next use (4.8) and (4.10)–(4.12) to compute

$$\begin{aligned}\langle \mathcal{L}^{-1}1, \varphi_c \rangle &= \langle \mathcal{L}^{-1}\varphi_c, 1 \rangle \\ &= \frac{1}{2c}\left(\partial_c \int_0^T \varphi_c \, dx \right) \\ &= -wT^3F'(\kappa)G(\kappa) \\ &= -\frac{256}{wT}[K^4(\kappa)F'(\kappa)G(\kappa)(1 - \kappa^2 + \kappa^4)].\end{aligned}$$

Finally, using the formulae for $\int_0^T \varphi_c^2 \, dx$, $\int_0^T \varphi_c \, dx$ allows us to find

$$\|\varphi_c + A\|^2 = \frac{w[2F(\kappa) - F^2(\kappa)/16\sqrt{1 - \kappa^2 + \kappa^4}K^2(\kappa)]}{T}.$$

Putting all these formulae together yields

$$\begin{aligned}\langle \mathcal{H}^{-1}\psi', \psi'\rangle &= \frac{1}{w[2F(\kappa) - F^2(\kappa)/16\sqrt{1-\kappa^2+\kappa^4}K^2(\kappa)]} \\ &\quad \times \Bigg[-F(\kappa) - 256K^4(\kappa)F'(\kappa)G(\kappa)(1-\kappa^2+\kappa^4) \\ &\qquad - \frac{4096K^6(\kappa)(1-\kappa^2+\kappa^4)^{3/2}(F'(\kappa)G(\kappa))^2}{1-16\sqrt{1-\kappa^2+\kappa^4}K^2(\kappa)F'(\kappa)G(\kappa)}\Bigg].\end{aligned}$$

Thus, if we assign the function

$$\begin{aligned}\tilde{F}(\kappa) := \left[2F(\kappa) - \frac{F^2(\kappa)}{16\sqrt{1-\kappa^2+\kappa^4}K^2(\kappa)}\right] \\ \times \Bigg(F(\kappa) + 256K^4(\kappa)F'(\kappa)G(\kappa)(1-\kappa^2+\kappa^4) \\ + \frac{4096K^6(\kappa)(1-\kappa^2+\kappa^4)^{3/2}(F'(\kappa)G(\kappa))^2}{1-16\sqrt{1-\kappa^2+\kappa^4}K^2(\kappa)F'(\kappa)G(\kappa)}\Bigg)^{-1},\end{aligned} \tag{5.6}$$

we get that $\langle \mathcal{H}^{-1}\psi', \psi'\rangle = -1/w\tilde{F}(\kappa)$. Thus, the index formula holds as stated in proposition 5.2, namely,

$$\omega^*(\mathcal{H}) = \frac{w}{2}\sqrt{\tilde{F}(\kappa)}.$$

5.2.2. *The case $p = 3$*

In this section, we compute the index of stability. For this we need to first consider

$$\langle \mathcal{H}^{-1}\psi_0', \psi_0'\rangle = \frac{1}{\|\varphi_c + A\|^2}\langle \mathcal{H}^{-1}\varphi_c', \varphi_c'\rangle,$$

where $A = -(1/T)\int_0^T \varphi_c\,dx = -\alpha\sqrt{2}\pi/2K(\kappa)$. Thus, we need to compute $\mathcal{H}^{-1}[\varphi_c']$. Let $f\colon \mathcal{H}[f] = \varphi_c'$. It follows that $-\mathcal{L}f' = \varphi_c + b$ for some constant b. We conclude that

$$-f' = \mathcal{L}^{-1}\varphi_c + b\mathcal{L}^{-1}1.$$

Note that $\mathcal{L}^{-1}1$ is well defined, since $1 \perp \varphi_c'$, which spans $\mathrm{Ker}(\mathcal{L})$. Thus,

$$\begin{aligned}\langle \mathcal{H}^{-1}\psi_0', \psi_0'\rangle &= \frac{1}{\|\varphi_c + A\|^2}\langle f, \varphi_c'\rangle \\ &= \frac{1}{\|\varphi_c + A\|^2}\langle -f', \varphi_c\rangle \\ &= \frac{\langle \mathcal{L}^{-1}\varphi_c, \varphi_c\rangle + b\langle \mathcal{L}^{-1}1, \varphi_c\rangle}{\|\varphi_c + A\|^2}.\end{aligned} \tag{5.7}$$

Differentiating (2.1) (with $a = 0$) with respect to c yields

$$\mathcal{L}^{-1}\varphi_c = \frac{1}{2c}\frac{d\varphi_c}{dc}. \tag{5.8}$$

From (2.9), we get that $\int_0^T \varphi_c^2 \, \mathrm{d}x = 8K(\kappa)E(\kappa)/T$ and

$$\begin{aligned}\langle \mathcal{L}^{-1}\varphi_c, \varphi_c\rangle &= \frac{1}{2c}\left\langle \frac{\mathrm{d}\varphi_c}{\mathrm{d}c}, \varphi_c\right\rangle \\ &= \frac{1}{4c}\partial_c\left[\int_0^T \varphi_c^2 \, \mathrm{d}x\right] \\ &= -\frac{4}{T}\frac{E^2(\kappa)-(1-\kappa^2)K^2(\kappa)}{\kappa(1-\kappa^2)}\frac{\mathrm{d}\kappa}{\mathrm{d}w}. \end{aligned} \tag{5.9}$$

In addition, note that

$$\langle \mathcal{L}^{-1}1, \varphi_c\rangle = \langle \mathcal{L}^{-1}\varphi_c, 1\rangle = \frac{1}{2c}\partial_c\left[\int_0^T \varphi_c(x;\kappa)\, \mathrm{d}x\right] = 0,$$

and hence

$$\langle \mathcal{H}^{-1}\psi_0', \psi_0'\rangle = \frac{\langle \mathcal{L}^{-1}\varphi_c, \varphi_c\rangle}{\|\varphi_c + A\|^2}.$$

From the relations (2.9), we have

$$w = \frac{2-\kappa^2}{2}\varphi_1^2 = \frac{4K^2(\kappa)(2-\kappa^2)}{T^2},$$

which, after differentiating with respect to w, allows us to write

$$\frac{\mathrm{d}\kappa}{\mathrm{d}w} = \frac{T^2}{8}\frac{1}{(2-\kappa^2)K(\kappa)\,\mathrm{d}K(\kappa)/\mathrm{d}\kappa - \kappa K^2(\kappa)}.$$

Thus,

$$\langle \mathcal{L}^{-1}\varphi_c, \varphi_c\rangle = -\frac{T}{2}\frac{(E^2(\kappa)-(1-\kappa^2)K^2(\kappa))}{\kappa(1-\kappa^2)}\frac{1}{(2-\kappa^2)K(\kappa)\,\mathrm{d}K(\kappa)/\mathrm{d}\kappa - \kappa K^2(\kappa)}.$$

Using that $\mathrm{d}K(\kappa)/\mathrm{d}\kappa = (E(\kappa)-(1-\kappa^2)K(\kappa))/\kappa(1-\kappa^2)$, we obtain that

$$\langle \mathcal{L}^{-1}\varphi_c, \varphi_c\rangle = -\frac{1}{\alpha}\frac{E^2(\kappa)-(1-\kappa^2)K^2(\kappa)}{(2-\kappa^2)E(\kappa)-2(1-\kappa^2)K(\kappa)} = -\frac{1}{\alpha}B(\kappa). \tag{5.10}$$

Since $\int_0^{2K(\kappa)} \mathrm{dn}(y;\kappa)\, \mathrm{d}y = \pi$, we get that

$$\|\varphi_c + A\|^2 = \alpha\left(4E(\kappa) - \frac{\pi^2}{K(\kappa)}\right) = \alpha C(\kappa). \tag{5.11}$$

From the above relations and (5.7), we get that

$$\langle \mathcal{H}^{-1}\psi_0', \psi_0'\rangle = -\frac{2-\kappa^2}{wC(\kappa)}B(\kappa). \tag{5.12}$$

Thus,

$$\begin{aligned}\omega^*(\mathcal{H}) &= \frac{1}{2\sqrt{-\langle \mathcal{H}^{-1}\psi_0', \psi_0'\rangle}} \\ &= \frac{\sqrt{w}}{2}\sqrt{\frac{C(\kappa)}{(2-\kappa^2)B(\kappa)}} \\ &= \frac{\sqrt{w}}{2}\sqrt{\frac{[4E(\kappa)-\pi^2/K(\kappa)][(2-\kappa^2)E(\kappa)-2(1-\kappa^2)K(\kappa)]}{(2-\kappa^2)(E^2(\kappa)-(1-\kappa^2)K(\kappa))}},\end{aligned}$$

which is exactly the claim of proposition 5.2.

6. Linear stability of the KGZ system: proof of theorem 3.4

We have already checked the conditions on the operator $\mathcal{H}$, defined in (3.8) in § 4.3. Namely, we established the simplicity of the eigenvalue at 0 in proposition 4.6, and we then verified the existence and simplicity of a single negative eigenvalue. It now remains to compute the index $\omega^*(\mathcal{H})$, after which we obtain a characterization of the linear stability by theorem 4.1, namely, $|c| \geqslant \omega^*(\mathcal{H})$.

PROPOSITION 6.1. *For $\kappa \in (0,\kappa_0)$, $\kappa_0 = 0.937095\ldots$, $\langle \mathcal{H}^{-1}\psi_0', \psi_0'\rangle < 0$. For $\kappa \in (\kappa_0, 1)$, $\langle \mathcal{H}^{-1}\psi_0', \psi_0'\rangle > 0$, and*

$$\omega^*(\mathcal{H}) = \frac{\sqrt{w}}{2\sqrt{N(\kappa)}},$$

where N is defined in (6.6). Note that $N(\kappa) > 0$, $\kappa \in (\kappa_0, 1)$.

Assuming the validity of proposition 6.1, we now complete the proof of theorem 3.4. To that end, observe that, since $\langle \mathcal{H}^{-1}\psi_0', \psi_0'\rangle < 0$, $\kappa \in (0,\kappa_0)$, we have instability whenever $\kappa \in (0,\kappa_0)$. For $\kappa \in (\kappa_0, 1)$, we need to solve the inequality

$$1 > |c| \geqslant \frac{\sqrt{w}}{2\sqrt{N(\kappa)}} = \frac{\sqrt{1-c^2}}{2\sqrt{N(\kappa)}},$$

which results in the following necessary and sufficient condition for linear stability:

$$1 > |c| \geqslant \frac{1}{\sqrt{1+4N(\kappa)}}, \quad \kappa \in (\kappa_0, 1).$$

This is exactly the statement of theorem 3.4.

6.1. Proof of proposition 6.1

We now estimate the index of stability $\langle \mathcal{H}^{-1}\psi_0', \psi_0'\rangle$,

$$\psi_0 = m\begin{pmatrix} \varphi_c' \\ -\dfrac{\varphi_c^2}{2w} + \dfrac{1}{2wT}\displaystyle\int_0^T \varphi_c^2\,\mathrm{d}x \end{pmatrix},$$

where m is such that $\|\psi_0\| = 1$. Thus, we need to compute

$$\mathcal{H}^{-1}\begin{pmatrix} \varphi_c'' \\ -\left(\dfrac{\varphi_c^2}{2w}\right)' \end{pmatrix}.$$

We have that

$$-wf'' + f - \frac{\varphi_c^2}{2w}f + \varphi_c g' = \varphi_c'', \qquad -(\varphi_c f)' - wg'' = -\left(\frac{\varphi_c^2}{2w}\right)'.$$

Integrating the second equation once yields

$$g' = \frac{\varphi_c^2}{2w^2} + \frac{c_1}{w} - \frac{\varphi_c f}{w}, \tag{6.1}$$

where c_1 is a constant of integration and needs to be determined. The first equation becomes

$$-wf'' + f - \frac{3\varphi_c^2}{2w}f + \frac{\varphi_c^3}{2w^2} + \frac{c_1\varphi_c}{w} = \varphi_c''$$

or

$$\mathcal{L}f + \frac{\varphi_c^3}{2w^2} + \frac{c_1\varphi_c}{w} = \varphi_c''. \tag{6.2}$$

On the other hand, taking the derivative with respect to w in (2.12) yields

$$\varphi_c'' = \mathcal{L}\frac{\mathrm{d}\varphi_c}{\mathrm{d}w} + \frac{\varphi_c^3}{2w^2}. \tag{6.3}$$

From (6.2) and (6.3), we have that

$$\mathcal{L}\left(f - \frac{\mathrm{d}\varphi_c}{\mathrm{d}w}\right) = -\frac{c_1}{w}\varphi_c$$

and, hence,

$$f = \frac{\mathrm{d}\varphi_c}{\mathrm{d}w} - \frac{c_1}{w}\mathcal{L}^{-1}\varphi_c = (1-c_1)\frac{\mathrm{d}\varphi}{\mathrm{d}w} + \frac{c_1}{w}\varphi_c. \tag{6.4}$$

Putting this into (6.1) and integrating, we get

$$\begin{aligned} c_1 &= \frac{\int_0^T \varphi_c(\mathrm{d}\varphi_c/\mathrm{d}w)\,\mathrm{d}x - (1/2w)\int_0^T \varphi_c^2\,\mathrm{d}x}{T + (1/w)\langle\varphi_c, \mathcal{L}^{-1}\varphi_c\rangle} \\ &= \frac{\int_0^T \varphi_c(\mathrm{d}\varphi_c/\mathrm{d}w)\,\mathrm{d}x - (1/2w)\int_0^T \varphi_c^2\,\mathrm{d}x}{T + \int_0^T \varphi_c(\mathrm{d}\varphi_c/\mathrm{d}w)\,\mathrm{d}x - (1/w)\int_0^T \varphi_c^2\,\mathrm{d}x}. \end{aligned} \tag{6.5}$$

Using that $\int_0^{K(\kappa)} \varphi_c^2 = (16w^2/T)E(\kappa)K(\kappa)$ and (4.23), we have that

$$\begin{aligned} \int_0^T \varphi_c\frac{\mathrm{d}\varphi_c}{\mathrm{d}w}\,\mathrm{d}x &= \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}w}\int_0^T \varphi_c^2\,\mathrm{d}x \\ &= \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}w}\left[\frac{16w^2}{T}E(\kappa)K(\kappa)\right] \\ &= \frac{8w}{T}\left[2E(\kappa)K(\kappa) - \frac{(2-\kappa^2)K^2(\kappa)\,\mathrm{d}[E(\kappa)K(\kappa)]/\mathrm{d}\kappa}{\mathrm{d}[(2-\kappa^2)K^2(\kappa)]/\mathrm{d}\kappa}\right]. \end{aligned}$$

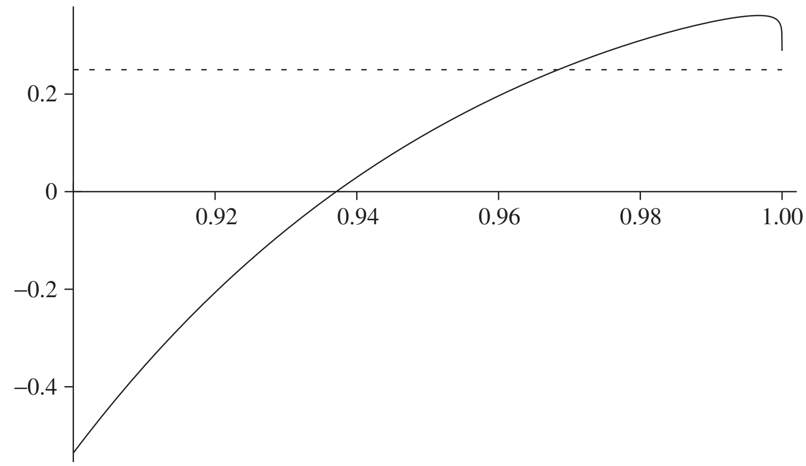

Figure 10. The function $N(\kappa)$, together with $\frac{1}{4}$. Recall that, for stability, one needs $\langle \mathcal{H}^{-1}\psi_0', \psi_0' \rangle < 0$ and, hence, $N(\kappa) > 0$.

The above formula allows us to express c_1 as a function of κ only:

$$c_1 = \frac{2E(\kappa)K(\kappa)\mathrm{d}[(2-\kappa^2)K^2(\kappa)]/\mathrm{d}\kappa - 2(2-\kappa^2)K^2(\kappa)\mathrm{d}[E(\kappa)K(\kappa)]/\mathrm{d}\kappa}{(2-\kappa^2)K^2(\kappa)\mathrm{d}[(2-\kappa^2)K^2(\kappa)]/\mathrm{d}\kappa - 2(2-\kappa^2)K^2(\kappa)\mathrm{d}[E(\kappa)K(\kappa)]/\mathrm{d}\kappa}$$
$$= \frac{(2-\kappa^2)E^2(\kappa) - 8(1-\kappa^2)E(\kappa)K(\kappa) + 2(1-\kappa^2)(2-\kappa^2)K^2(\kappa)}{2(2-\kappa^2)^2E(\kappa)K(\kappa) - 2(1-\kappa^2)(2-\kappa^2)K^2(\kappa) - 2(2-\kappa^2)E^2(\kappa)}.$$

Now,

$$\langle \mathcal{H}^{-1}\psi_0', \psi_0' \rangle = m^2 \left\langle \mathcal{H}\begin{pmatrix} f \\ g \end{pmatrix}, \begin{pmatrix} f \\ g \end{pmatrix} \right\rangle$$
$$= m^2 \left\langle \begin{pmatrix} \varphi_c'' \\ -\left(\dfrac{\varphi_c^2}{2w}\right)' \end{pmatrix}, \begin{pmatrix} f \\ g \end{pmatrix} \right\rangle$$
$$= m^2 \left(\langle \varphi_c'', f \rangle + \left\langle g', \frac{\varphi_c^2}{2w} \right\rangle \right).$$

From (2.12) and the expression for f and g', we get that

$$\langle \varphi_c'', f \rangle = \frac{1}{T}(2J_2 - J_3 - J_4 + J_5), \qquad \left\langle g', \frac{\varphi_c^2}{2w} \right\rangle = \frac{1}{T}(J_1 + J_2 - J_3 - J_4),$$

where

$$I_1 = \int_0^{K(\kappa)} \mathrm{dn}^4(y,\kappa)\,\mathrm{d}y = \frac{4-2k^2}{3}E(k) - \frac{1-k^2}{3}K(k),$$
$$J_1 = \frac{T}{2w^3}\langle \varphi_c^2, \varphi_c^2 \rangle = 16\frac{K(\kappa)}{2-\kappa^2}I_1,$$
$$J_2 = \frac{Tc_1}{2w^2}\langle \varphi_c, \varphi_c \rangle = 8c_1E(\kappa)K(\kappa),$$

$$
\begin{aligned}
J_3 &= \frac{T(1-c_1)\langle \varphi_{\rm c}\,{\rm d}\varphi_{\rm c}/{\rm d}w, \varphi_{\rm c}^2\rangle}{2w^2}\\
&= 8(1-c_1)\left[\frac{3K(\kappa)}{2-\kappa^2}I_1 - \frac{(2-\kappa^2)K^2(\kappa)}{{\rm d}[(2-\kappa^2)K^2(\kappa)]/{\rm d}\kappa}\frac{\rm d}{{\rm d}\kappa}\left[\frac{K(\kappa)}{2-\kappa^2}I_1\right]\right],\\
J_4 &= \frac{Tc_1}{2w^3}\langle \varphi_{\rm c}^2, \varphi_{\rm c}^2\rangle = 32c_1\frac{K(\kappa)}{2-\kappa^2}I_1,\\
J_5 &= \frac{T(1-c_1)}{w}\left\langle \varphi_{\rm c}, \frac{{\rm d}\varphi_{\rm c}}{{\rm d}w}\right\rangle\\
&= 8(1-c_1)\left[2K(\kappa)E(\kappa) - \frac{(2-\kappa^2)K^2(\kappa)\,{\rm d}[E(\kappa)K(\kappa)]/{\rm d}\kappa}{{\rm d}[(2-\kappa^2)K^2(\kappa)]/{\rm d}\kappa}\right].
\end{aligned}
$$

In addition, we have that

$$
\begin{aligned}
\frac{1}{m^2} &= \left\| \begin{pmatrix} \varphi_{\rm c}' \\ -\dfrac{\varphi_{\rm c}^2}{2w} + \dfrac{1}{2wT}\displaystyle\int_0^T \varphi_{\rm c}^2\,{\rm d}x \end{pmatrix} \right\|^2\\
&= \langle \varphi_{\rm c}', \varphi_{\rm c}'\rangle + \frac{1}{4w^2}\langle \varphi_{\rm c}^2, \varphi_{\rm c}^2\rangle - \frac{1}{4w^2T}\left(\int_0^T \varphi_{\rm c}^2\,{\rm d}x\right)^2.
\end{aligned}
$$

Using that ${\rm dn}'(y) = -\kappa^2\,{\rm sn}(y)\,{\rm cn}(y)$ and ${\rm sn}^2(y) + {\rm cn}^2(y) = 1$, we get that

$$
\begin{aligned}
\langle \varphi_{\rm c}', \varphi_{\rm c}'\rangle &= \varphi_1^2\alpha\kappa^4\left[\int_0^{2K(\kappa)} {\rm sn}^2(y,\kappa)\,{\rm d}y - \int_0^{2K(\kappa)} {\rm sn}^4(y,\kappa)\,{\rm d}y\right]\\
&= \frac{8w}{3T}\frac{K(\kappa)}{2-\kappa^2}[2(2-\kappa^2)E(\kappa) - 4(1-\kappa^2)K(\kappa)]
\end{aligned}
$$

and

$$
\begin{aligned}
\frac{1}{m^2} &= \frac{w}{T}\left[\frac{8K(\kappa)}{3(2-\kappa^2)}[2(2-\kappa^2)E(\kappa) - 4(1-\kappa^2)K(\kappa)] + \frac{16K(\kappa)}{2-\kappa^2}I_1 - \frac{16}{2-\kappa^2}E^2(\kappa)\right]\\
&= \frac{16w}{T}\left[E(\kappa)K(\kappa) - \frac{1-\kappa^2}{2-\kappa^2}K^2(\kappa) - \frac{1}{2-\kappa^2}E^2(\kappa)\right].
\end{aligned}
$$

Combining the above relations yields

$$
\begin{aligned}
\langle \mathcal{H}^{-1}\psi_0', \psi_0'\rangle &= \frac{J_1 + 3J_2 - 2J_3 - 2J_4 + J_5}{16w[E(\kappa)K(\kappa) - (1-\kappa^2)K^2(\kappa)/(2-\kappa^2) - E^2(\kappa)/(2-\kappa^2)]}\\
&:= -\frac{N(\kappa)}{w}.
\end{aligned}
\tag{6.6}
$$

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