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Stability Theory of Solitary Waves in the Presence of Symmetry, I*

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Consider an abstract Hamiltonian system which is invariant under a one-parameter unitary group of operators. By a "solitary wave" we mean a solution the time development of which is given exactly by the one-parameter group. We find sharp conditions for the stability and instability of solitary waves. Applications are given to bound states and traveling waves of nonlinear PDEs such Klein–Gordon and Schrödinger equations.

1. INTRODUCTION

Systems with conserved energy abound in mathematics and physics. In this paper and its sequel we consider abstract Hamiltonian systems of the form

$$\frac{du}{dt} = JE'(u(t)), \tag{1}$$

which are locally well-posed in a space X; here E is a functional (the

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"energy") and J is a skew-symmetric linear operator. We assume that Eq. (1) is invariant under a representation $T(\cdot)$ of a group G on X. For the present we shall assume G is the reals $\mathbb{R}$ under addition. The solutions the stability of which we study are *not* invariant under G but are "translates" of a fixed vector $\phi \in X$; namely,

$$u(t) = T(\omega t)\phi, \tag{2}$$

where $\phi = \phi_\omega$ depends on $\omega \in \mathbb{R}$. They may be interpreted physically as "solitary waves" or "bound states." Let us call the set $\{T(g)\phi \colon g \in G\}$ the *ϕ-orbit*. We say the ϕ-orbit is *stable* if a solution $u(t)$ of (1) exists for all $t \geqslant 0$ and forever remains near the ϕ-orbit in the norm of X provided its initial datum $u(0)$ is sufficiently close to the ϕ-orbit in the norm of X.

If $\omega = 0$ then ϕ is a stationary solution of (1) and $E'(\phi) = 0$. Its stability is determined by the linearized operator $JE''(\phi)$. It is much easier however to analyze the self-adjoint operator $E''(\phi)$. Formally, but formally only, a sufficient condition for stability is that $E''(\phi) \geqslant 0$.

Associated with the group G ($= \mathbb{R}$ in the present paper) is another conserved functional Q (interpreted as the "charge" in certain applications). The vector ϕ is a critical point of $E - \omega Q$. We assume that the "linearized Hamiltonian"

$$H = E''(\phi) - \omega Q''(\phi) \tag{3}$$

has at most one negative eigenvalue. This means there can be at most one unstable direction near ϕ. The invariance implies that zero is in the spectrum of H. If all the rest of the spectrum is positive, then the ϕ-orbit is stable. Our main result is the following (Theorem 2). Assume H has exactly one negative eigenvalue. Then the ϕ-orbit is *stable* if and only if the *scalar* function

$$d(\omega) = E(\phi_\omega) - \omega Q(\phi_\omega) \tag{4}$$

is *convex* at ω. This also turns out to be equivalent to $Q(\phi_\omega)$ decreasing in ω. A geometric way to state this condition is that the hypersurface $\{u \in X \mid Q(\phi_\omega)\}$ does not locally meet the "cone" $\{u \in X \mid E(u) - \omega Q(u) < E(\phi_\omega) - \omega Q(\phi_\omega)\}$. Equivalently again, the hyperplane $Q'(\phi)^\perp$ does not meet the cone $\{y \in H \mid \langle Hy, y\rangle < 0\}$. For precise statements, see the next section.

In Section 2 we state the assumptions and the main result. In Section 3 we prove the stability. The main intermediate step is to show that ϕ minimizes E subject to constant Q. In Section 4 we prove the instability. To do this we construct a kind of Liapunov functional A in a neighborhood U_ε of the ϕ-orbit which is strictly monotone on trajectories in U_ε. We basically follow the method of [13]. Some extensions are given in Section 5.

In [4, 11, 12, 13] the stability and instability of the bound states $\exp(i\omega t)\phi(x)$ of lowest energy of the Klein–Gordon and Schrödinger equations

$$u_{tt} - \Delta u + f(u) = 0, \qquad iu_t - \Delta u + f(u) = 0,$$

where $x \in \mathbb{R}^n$, were studied. In these cases the group $\mathbb{R}$ acts as multiplication by $\exp(i\theta)$, $\theta \in \mathbb{R}$. In Section 6 we indicate how to recover the same results from our abstract theorem. Our proof of stability in Section 3 is however much simpler than that in [11]. Our proof of instability is to a large extent an abstraction of the one in [13]. For special types of nonlinear terms satisfying convexity-like conditions, alternative proofs are possible (as in [2]), but we are interested in the general case. The stable and the critical (borderline) cases have also recently been studied in [14–16]. The spectrum of the linearized equation has also been studied in [6]. Some problems of this type have also been studied in [8, 9] in terms of the linearized equation.

In Section 6 we give a series of examples:

(A) traveling waves of nonlinear wave equations,

(B) Klein–Gordon and Schrödinger bound states from [13],

(C) bound states in the presence of a potential,

(D) standing waves in an optical wave guide, and

(E) solitary waves of generalized KdV equations.

The goal of our second paper (II) is to study a general group G. The setup is the following. Let G be a Lie group of dimension m with Lie algebra $\mathfrak{g}$. Let T be a unitary representation of G on X which leaves E and J invariant. Let $Q: \mathfrak{g} \times X \to \mathbb{R}$ be the invariant associated to T. That is, $Q'(\omega, u) \equiv \partial Q/\partial u = J^{-1}T'_e(\omega)u$ for $\omega \in \mathfrak{g}$ and $u \in D(T'_e(\omega)) \subset X$. Consider now solutions ϕ_ω of the equation

$$E'(\phi) = Q'(\omega, \phi_\omega)$$

for $\omega \in \mathfrak{g}$. Then the "solitary wave" $u(t) = T(\exp(t\omega))\phi_\omega$ satisfies Eq. (1). We study its stability as a solution of (1).

2. Main Results

Let X be a real Hilbert space with inner product (,). If X^* is its dual, there is a natural isomorphism $I: X \to X^*$ defined by

$$\langle Iu, v \rangle = (u, v),$$

where $\langle\ ,\ \rangle$ denotes the pairing between X and X^*. In this paper we will use I explicitly, but we will always identify X^{**} with X in the natural way. Warning: when we refer to adjoints of linear operators we will mean with respect to $\langle\ ,\ \rangle$ and not $(\ ,\ )$.

Let J be a closed linear operator from X^* to X with dense domain $D(J) \subset X^*$. We *assume* that J is skew symmetric; that is,

$$\langle Ju, v\rangle = -\langle u, Jv\rangle \qquad \text{for} \quad u, v \in D(J) \tag{2.1}$$

and also that

$$J \text{ is onto.} \tag{2.2}$$

[We do not need J to be onto but only that ϕ_ω and χ_ω (defined later) belong to the range of J.]

Let $E: X \to \mathbb{R}$ be a C^2 functional defined on all of X. We write its derivative as $\langle E'(u), v\rangle$, where $E': X \to X^*$, and its second derivative as $\langle E''(u)w, v\rangle$.

Let T be a one-parameter group of unitary operators on X. Thus $T(s)$ is a unitary operator from X onto X for each $s \in \mathbb{R}$; that is, $\|T(s)u\| = \|u\|$; which is strongly continuous and satisfies $T(s)\,T(r) = T(s+r)$ for all real s and r. Let $T'(0)$ denote the infinitesimal generator, an operator, $X \to X$, which is skew-adjoint with respect to the inner product $(\ ,\ )$ with dense domain. Using our definition of adjoint the unitarity of T can be expressed as

$$T^*(s)I = IT(-s) \qquad \text{for} \quad s \in \mathbb{R},$$

where $T^*(s): X^* \to X^*$.

We *assume* that E is invariant under T; that is,

$$E(T(s)u) = E(u) \qquad \text{for} \quad s \in \mathbb{R},\ u \in X. \tag{2.3}$$

Differentiating (2.3) with respect to u, we get

$$T(s)^* E'(T(s)u) = E'(u). \tag{2.4}$$

Differentiating again, we get

$$T(s)^* E''(T(s)u)\,T(s) = E''(u). \tag{2.5}$$

Differentiating (2.3) with respect to s at $s=0$, we get

$$\langle E'(u), T'(0)u\rangle = 0 \qquad \text{for} \quad u \in D(T'(0)). \tag{2.6}$$

We *assume* that J "commutes" with T, in the sense that

$$T(s)J = JT^*(-s). \tag{2.7}$$

This can be written equivalently as $T(s)\,JT^*(s) = J$ or as $JIT(s) = T(s)JI$. In particular (2.7) implies that $T^*(s)[D(J)] = D(J)$. Formally, if we differentiate (2.7) with respect to s at $s = 0$, we get $T'(0)J = -J(T'(0))^*$. Hence $J^{-1}T'(0) = -(T'(0))^*J^{-1} = +(J^{-1}T'(0))^*$. In order to make this precise, we make the further *assumption* that

$$\text{there is a bounded linear operator } B\colon X \to X^* \text{ such that } B^* = B \text{ and the operator } JB \text{ is an extension of } T'(0). \tag{2.8}$$

We *define* another functional $Q\colon X \to \mathbb{R}$ by

$$Q(u) = \tfrac{1}{2}\langle Bu, u\rangle. \tag{2.9}$$

It follows that Q is also invariant under T:

$$Q(T(s)u) = Q(u) \qquad \text{for} \quad s \in \mathbb{R},\, u \in X. \tag{2.10}$$

In order to prove (2.10), first let $u \in D(T'(0))$. Then $T(s)u \in D(T'(0)) \subset D(JB)$ and

$$\begin{aligned} \frac{d}{ds} Q(T(s)u) &= \langle Q'(T(s)u),\, T'(0)\, T(s)u\rangle \\ &= \langle BT(s)u,\, JBT(s)u\rangle = 0 \end{aligned}$$

by (2.8) and (2.9). This proves (2.10) for a dense class of u. For general $u \in X$, we simply approximate by elements of $D(T'(0))$.

Differentiating (2.9) and (2.10) we have $Q'(u) = Bu$ and $Q''(u) = B$ for all $u \in X$. Furthermore,

$$\begin{aligned} &\text{(a)} \quad T(s)^*Q'(T(s)u) = Q'(u) \\ &\text{(b)} \quad T^*(s)\, BT(s) = B \\ &\text{(c)} \quad BT'(0) = -T'(0)^*B \\ &\text{(d)} \quad B[D(T'(0))] = D(T'(0)^*). \end{aligned} \tag{2.11}$$

The evolution equation which we shall study is

$$\frac{du}{dt} = JE'(u(t)), \qquad u(t) \in X. \tag{2.12}$$

Note that E and Q are formally conserved under the flow of (2.12). Namely,

$$\frac{dE(u)}{dt} = \left\langle E'(u), \frac{du}{dt} \right\rangle = \langle E'(u), JE'(u) \rangle = 0$$

and

$$\frac{dQ(u)}{dt} = \left\langle Q'(u), \frac{du}{dt} \right\rangle = \langle Bu, JE'(u) \rangle = -\langle JBu, E'(u) \rangle$$
$$= -\langle T'(0)u, E'(u) \rangle = 0 \qquad \text{by (2.6).}$$

The equation will be considered only in a weak sense.

DEFINITION. By a *solution* of (2.12) in a time interval $\mathscr{I}$, we mean a function

$$u \in C(\mathscr{I}; X) \qquad \text{(continuous with values in } X\text{)}$$

such that

$$\frac{d}{dt} \langle u(t), \psi \rangle = \langle E'(u(t)), -J\psi \rangle \tag{2.13}$$

in $\mathscr{D}'(\mathscr{I})$ for all $\psi \in D(J) \subset X^*$.

Assumption 1 (Existence of Solutions). For each $u_0 \in X$ there exists $t_0 > 0$ depending only on μ, where $\|u_0\| \leqslant \mu$, and there exists a solution u of Eq. (2.12) in the interval $\mathscr{I} = [0, t_0)$ such that

(a) $u(0) = u_0$ and

(b) $E(u(t)) = E(u_0)$, $Q(u(t)) = Q(u_0)$ for $t \in \mathscr{I}$.

This assumption can be weakened, say by introducing another Banach space, but we refrain from doing so here.

We remark that if $u(t)$ is a solution of (2.12), so is $T(s)\,u(t)$ for all $s \in \mathbb{R}$. Indeed, by (2.4) and (2.7)

$$\begin{aligned}
&\langle E'(T(s)\,u(t)), J\psi \rangle \\
&\quad = \langle E'(u(t)), T(-s)\,J\psi \rangle = \langle E'(u(t)), JT^*(s)\psi \rangle \\
&\quad = -\langle JE'(u(t)), T^*(s)\psi \rangle = -\frac{d}{dt} \langle u(t), T^*(s)\psi \rangle \\
&\quad = -\frac{d}{dt} \langle T(s)\,u(t), \psi \rangle \qquad \text{for all} \quad \psi \in D(J).
\end{aligned}$$

DEFINITION. By a *bound state* we mean a solution of the evolution equation of the special form

$$u(t) = T(\omega t)\phi, \tag{2.14}$$

where $\omega \in \mathbb{R}$ and $\phi \in X$.

If $\phi \in D(T'(0))$ satisfies the "stationary" equation

$$E'(\phi) = \omega Q'(\phi), \tag{2.15}$$

then $T(\omega t)\phi$ is a bound state. Indeed,

$$\begin{aligned} \frac{d}{dt} T(\omega t)\phi &= \omega T'(0)\, T(\omega t)\phi = \omega JBT(\omega t)\phi \\ &= \omega JT^*(-\omega t)\, Q'(\phi) = JT^*(-\omega t)\, E'(\phi) \\ &= JE'(T(\omega t)\phi) \end{aligned}$$

by (2.8), (2.11), and (2.4).

Assumption 2 (Existence of Bound States). There exist real $\omega_1 < \omega_2$ and a mapping

(a) $\omega \to \phi_\omega$ from the open interval (ω_1, ω_2) into X which is C^1 such that for each $\omega \in (\omega_1, \omega_2)$

(b) $E'(\phi_\omega) = \omega Q'(\phi_\omega)$,
(c) $\phi_\omega \in D(T'(0)^3) \cap D(JIT'(0)^2)$,
(d) $T'(0)\phi_\omega \neq 0$.

We *define* the scalar

$$d(\omega) = E(\phi_\omega) - \omega Q(\phi_\omega) \tag{2.16}$$

and the operator from X to X^*

$$H_\omega = E''(\phi_\omega) - \omega Q''(\phi_\omega). \tag{2.17}$$

Observe that H_ω is self-adjoint in the sense that $H_\omega^* = H_\omega$. This means that $I^{-1}H_\omega$ is a bounded self-adjoint operator on X in the standard sense, since $(I^{-1}Hu, v) = \langle Hu, v \rangle = \langle Hv, u \rangle = (I^{-1}Hu, v)$. The "spectrum" of H_ω consists of the real numbers λ such that $H_\omega - \lambda I$ is not invertible. We claim that $\lambda = 0$ belongs to the spectrum of H_ω. Indeed, from (2.4), (2.11a) and (2.15), we have

$$E'(T(s)\phi_\omega) - \omega Q'(T(s)\phi_\omega) = T^*(-s)[E'(\phi_\omega) - \omega Q'(\phi_\omega)] = 0. \tag{2.17a}$$

Differentiating with respect to s at $s=0$, we deduce that

$$H_\omega(T'(0)\phi_\omega)=0. \tag{2.18}$$

Thus $T'(0)\phi_\omega$ is an eigenvector with eigenvalue 0.

DEFINITION. The ϕ_ω-orbit $\{T(\omega t)\phi_\omega, t\in\mathbb{R}\}$ is *stable* if for all $\varepsilon>0$ there exists $\delta>0$ with the following property. If $\|u_0-\phi_\omega\|<\delta$ and $u(t)$ is a solution of (2.12) in some interval $[0, t_0)$ with $u(0)=u_0$, then $u(t)$ can be continued to a solution in $0\leqslant t<\infty$ and

$$\sup_{0<t<\infty}\inf_{s\in\mathbb{R}}\|u(t)-T(s)\phi_\omega\|<\varepsilon.$$

Otherwise the ϕ_ω-orbit is called *unstable*; in particular, this would happen if solutions ceased to exist after a finite time.

The following is an easy result.

THEOREM 1. *Given Assumptions* 1 *and* 2, *if the operator* H_ω *has its kernel spanned by* $T'(0)\phi_\omega$ *and the rest of its spectrum is positive and bounded away from zero, then the* ϕ_ω*-orbit is stable.*

Assumption 3. For each $\omega\in(\omega_1,\omega_2)$, H_ω has exactly one negative simple eigenvalue and has its kernel spanned by $T'(0)\phi_\omega$ and the rest of its spectrum is positive and bounded away from zero.

THEOREM 2. *Given Assumptions* 1, 2, *and* 3, *let* $\omega_1<\omega<\omega_2$. *Then the* ϕ_ω*-orbit is stable if and only if the function* $d(\cdot)$ *is convex in a neighborhood of* ω.

Remark. For the sufficiency (stability), Assumption 2(c) can be weakened to $\phi\in D(T'(0)^2)$ and (2.2) is not needed at all.

Here is a more detailed result in the case of strict convexity or concavity of d.

THEOREM 3. *Given Assumptions* 1, 2, *and* 3, *let* $\omega_1<\omega<\omega_2$ *and* $d''(\omega)\neq 0$. *Then the following conditions are equivalent.*

(A) $d''(\omega)>0$.

(B) *If* $\langle Q'(\phi_\omega), y\rangle=0$, *then* $\langle H_\omega y, y\rangle\geqslant 0$.

(C) $E(u)$ *is minimized at* $u=\phi_\omega$, *for* u *in a neighborhood of* ϕ_ω *with* $Q(u)=Q(\phi_\omega)$.

(D) *The* ϕ_ω*-orbit is stable.*

We split the proof of Theorem 3 into a series of smaller theorems.

In Section 3 we prove that (A) implies (B), (C), and (D). In Section 4 we prove that (B), (C), and (D) each imply (A). In Section 5 we prove Theorems 1 and 2.

We note that if $d''(\omega) > 0$ then H_ω has at least some negative spectrum. Indeed, differentiating (2.16) we have

$$H_\omega \phi'_\omega = Q'(\phi_\omega), \qquad \text{where} \quad \phi'_\omega = d\phi_\omega/d\omega. \tag{2.19}$$

Differentiating (2.15) twice we get

$$d'(\omega) = -Q(\phi_\omega) \tag{2.20}$$

and

$$d''(\omega) = -\langle Q'(\phi_\omega), \phi'_\omega \rangle = -\langle H\phi'_\omega, \phi'_\omega \rangle. \tag{2.21}$$

Thus $\langle H\phi'_\omega, \phi'_\omega \rangle < 0$ if $d''(\omega) > 0$.

3. Stability

Often, when the parameter ω remains fixed, we will drop the subscript ω. Thus we will write ϕ for ϕ_ω, H for H_ω, and so on. A tubular neighborhood, or simply *tube*, around the orbit $\{T(s)\phi \mid s \in \mathbb{R}\}$ is defined by

$$U_\varepsilon = \{u \in X \colon \inf_{s \in \mathbb{R}} \|u - T(s)\phi\| < \varepsilon\}.$$

We first show that this orbit cannot keep coming back arbitrarily to itself, unless it is periodic.

Lemma 3.1. *Under Assumptions* 2 *and* 3, either (i) $T(s)\phi = \phi$ *for some* $s > 0$ or (ii) $T(s_n)\phi \to \phi$ *implies* $s_n \to 0$.

Proof. Consider the set of critical points of $L = E - \omega Q$ in a neighborhood of ϕ. If u is a critical point, then

$$0 = L'(u) - L'(\phi) = H(u - \phi) + O(\|u - \phi\|^2)$$

since $H = L''(\phi)$. Therefore the set of critical points is locally isomorphic to the nullspace of H. Now ϕ is a critical point and so is $T(s)\phi$ for every s by (2.17a). So there is a neighborhood N of ϕ and a number $\delta > 0$ such that

$$\{u \in N \mid L'(u) = 0\} = \{T(r)\phi \mid |r| < \delta\}.$$

Now suppose (ii) is false. Then there is a sequence $|s_n| \geqslant \delta$ with $T(s_n)\phi \in N$. Fix n. We have just proved that there exists $|r_n| < \delta$ with $T(s_n)\phi = T(r_n)\phi$. So $T(s_n - r_n)\phi = \phi$, which means (i) is valid.

In the next lemma we shall "factor out" the group action within a tube, in the sense that $T(s)u$ is orthogonal to $T'(0)\phi$ for some $s = s(u)$.

LEMMA 3.2. *Given Assumptions* 2 *and* 3, *there exist* $\varepsilon > 0$ *and a* C^2 *map*

$$s\colon U_\varepsilon \to \mathbb{R} \qquad (\mathbb{R}/\textit{period, if the orbit is periodic})$$

such that, for all $u \in U_\varepsilon$ *and all* $r \in \mathbb{R}$,

(i) $\|T(s(u))u - \phi\| \leqslant \|T(r)u - \phi\|$,

(ii) $(T(s(u))u, T'(0)\phi) = 0$,

(iii) $s(T(r)u) = s(u) - r$, *modulo the period if the orbit is periodic*,

(iv) $s'(u) = IT(-s(u))\, T'(0)\phi/(T'(0)^2\phi, T(s(u))u)$,

(v) s' *maps* U_ε *into* $D(J)$ *and* Js' *is a* C^1 *function from* U_ε *into* X.

Proof. The idea is to define $s(u)$ as the minimum of $\rho(s) = \|T(s)u - \phi\|^2$ for u close to the orbit of ϕ. We calculate

$$\rho'(s) = 2(T(s)u - \phi, T'(0)\, T(s)u) = +2(T(s)u, T'(0)\phi)$$

$$\rho''(s) = -2(T'(0)\, T(s)u, T'(0)\phi) = 2(T(s)u, T'(0)^2\phi).$$

At $u = \phi$ and $s = 0$, $\rho'(0) = 0$ and $\rho''(0) = 2\,\|T'(0)\phi\|^2 > 0$. By the implicit function theorem there are an open ball V around ϕ, an interval I around $s = 0$, and a C^2 map $s\colon V \to I$ such that the equation $\rho'(s) = 0$ has a unique solution $s = s(u) \in I$ for all $u \in V$. Thus $s(u)$ is the unique minimum of $\rho(s)$ in I for a given $u \in V$. By Lemma 3.1, for all $\delta > 0$ (δ less than half the period in the periodic case) there exists $\eta(\delta) > 0$ such that if $\|T(s)\phi - \phi\| < \eta(\delta)$, then $|s| < \delta$ (or s lies within δ of some multiple of the period in the periodic case). We choose δ less than half the period in the periodic case, and we choose $I = (-\delta, \delta)$ and $V = \{v\colon \|v - \phi\| < \eta(\delta)/3\}$. If $u \in V$, $r \in \mathbb{R}$, and $\|T(r)u - \phi\| < \|T(s(u))u - \phi\|$, then

$$\|T(r)\phi - \phi\| \leqslant \|T(r)u - \phi\| + \|T(r)(u - \phi)\| < 2\,\|u - \phi\| < \eta(\delta).$$

Therefore $r = s(u)$, plus a multiple of the period in the periodic case. This proves (i) and (ii) for $u \in V$. To show (iii) within V, note that

$$\|T(s(u) - r)\phi - \phi\| \leqslant \|T(r)u - \phi\| + \|T(s(u))u - \phi\| + \|u - \phi\|.$$

So if $T(r)u \in V$ and $u \in V$, we have $s(u) - r \in I$ (modulo the period). By uniqueness, $s(T(r)u) = s(u) - r$ (modulo the period). To show (iv), we differentiate (ii) with respect to $u \in X$ to obtain

$$(T(s(u))w, T'(0)\phi) + \langle s'(u), w\rangle (T''(0)\, T(s(u))u, T'(0)\phi) = 0.$$

Since $T'(0)$ is skew with respect to the inner product,

$$\langle s'(u), w \rangle = \frac{(T(-s(u))\, T'(0)\, \phi, w)}{(T'(0)^2 \phi, T(s(u))u)}$$

for all $w \in X$. This implies (iv). If this formula is differentiated once more with respect to u and if we make use of the assumption that $\phi \in D(T'(0)^3) \cap D(JIT'(0)^2)$, then (v) follows. Finally, we extend the definition of $s(u)$ to $u \in U_\varepsilon$, where $\varepsilon = \eta(\delta)/3$, as follows. If $\|u - T(s_0)\phi\| < \varepsilon$ for some $s_0 \in \mathbb{R}$, we define

$$s(u) \equiv s(T(-s_0)u) - s_0.$$

This definition is independent of the choice of s_0 for the following reason. If $\|u - T(s_0)\phi\| < \varepsilon$ and $\|u - T(s_1)\phi\| < \varepsilon$, then $T(-s_0)u$ and $T(-s_1)u$ belong to V. Since (iii) has already been proved within V, we have

$$s(T(s_0 - s_1)\, T(-s_0)u) = s(T(-s_0)u) - (s_0 - s_1)$$

plus a multiple of the period if the orbit is periodic, where $r = s_0 - s_1$. Thus

$$s(T(-s_1)u) - s_1 = s(T(-s_0)u) - s_0 \qquad \text{(in } \mathbb{R}/\text{period).}$$

Therefore $s(u)$ is defined for all $u \in U_\varepsilon$ and satisfies properties (i)–(v).

From now on we make Assumptions 2 and 3 as well as fix the parameter ω. Recall that $T'(0)\phi$ generates the kernel of H. *Denote* by $\chi = \chi_\omega$ a negative eigenvector of H:

$$H_\omega \chi_\omega = -\lambda_\omega^2\, I\chi_\omega, \qquad \|\chi_\omega\| = 1. \tag{3.1}$$

Denote by $P = P_\omega$ the positive subspace of H. Thus there exists $\delta = \delta_\omega > 0$ such that

$$\langle Hp, p \rangle \geqslant \delta\, \|p\|^2 \qquad \text{for} \quad p \in P. \tag{3.2}$$

THEOREM 3.3. *Let* $d''(\omega) > 0$. *If* $\langle Q'(\phi), y \rangle = (T'(0)\phi, y) = 0$ *and* $y \neq 0$, *then* $\langle Hy, y \rangle > 0$.

Proof. By (2.21) we have $\langle H\phi', \phi' \rangle < 0$. Make a spectral decomposition $\phi' = a_0\chi + b_0 T'(0)\phi + p_0$, where $p_0 \in P$. Then $-a_0^2\lambda^2 + \langle Hp_0, p_0 \rangle < 0$. Now let $y \in X$ with $\langle Q'(\phi), y \rangle = 0$ and $(T'(0)\phi, y) = 0$. Decompose

$$y = a\chi + p \qquad \text{with} \quad p \in P.$$

By (2.19) we have

$$0 = \langle H\phi', y \rangle = -a_0 a\lambda^2 + \langle Hp_0, p \rangle.$$

Therefore

$$\langle Hy, y\rangle = -a^2\lambda^2 + \langle Hp, p\rangle \geqslant -a^2\lambda^2 + \frac{\langle Hp, p_0\rangle^2}{\langle Hp_0, p_0\rangle}$$

$$> -a^2\lambda^2 + \frac{(a_0 a\lambda^2)^2}{a_0^2\lambda^2} = 0.$$

COROLLARY 3.3.1. *If* $\langle Q'(\phi), y\rangle = 0$, *then*

$$\langle Hy, y\rangle \geqslant c\,\|\Pi y\|^2$$

for some $c > 0$ *where* Π *is the orthogonal projection onto* $[T'(0)\phi]^\perp$

THEOREM 3.4. *If* $d''(\omega) > 0$, *there exist* $c > 0$ *and* $\varepsilon > 0$ *such that*

$$E(u) - E(\phi) \geqslant c\,\|T(s(u))u - \phi\|^2$$

for $u \in U_\varepsilon$, $Q(u) = Q(\phi)$.

Proof. Let $q = I^{-1}Q'(\phi)$ and decompose

$$T(s(u))u - \phi = aq + y,$$

where $(y, q) = 0$ and a is a scalar. Then

$$\begin{aligned} Q(\phi) = Q(u) &= Q(T(s(u))u) \\ &= Q(\phi) + \langle Q'(\phi), T(s(u))u - \phi\rangle + O(\|T(s(u))u - \phi\|^2) \\ &= Q(\phi) + a\,\|q\|^2 + O(\|T(s(u))u - \phi\|^2). \end{aligned}$$

Hence $a = O(\|T(s(u))u - \phi\|^2)$. Let $L(u) = E(u) - \omega Q(u)$. Another Taylor expansion gives

$$L(u) = L(T(s(u))u) = L(\phi) + \langle L'(\phi), v\rangle + \tfrac{1}{2}\langle L''(\phi)v, v\rangle + o(\|v\|^2),$$

where $v = T(s(u))u - \phi = aq + y$. Since $Q(u) = Q(\phi)$, $L'(\phi) = 0$, and $L''(\phi) = H$, this can be written as

$$\begin{aligned} E(u) - E(\phi) &= \tfrac{1}{2}\langle Hv, v\rangle + o(\|v\|^2) \\ &= \tfrac{1}{2}\langle Hy, y\rangle + O(a^2) + O(a\,\|v\|) + o(\|v\|^2) \\ &= \tfrac{1}{2}\langle Hy, y\rangle + o(\|v\|^2). \end{aligned} \tag{3.3}$$

Now $0 = (q, y) = \langle Q'(\phi), y\rangle$ and

$$(y, T'(0)\phi) = (T(s(u))u - \phi - aq, T'(0)\phi) = 0$$

by Lemma 3.2 (ii). Therefore by Corollary 3.3.1

$$E(u)-E(\phi)\geqslant \tfrac{1}{2}c\ \|y\|^2+o(\|v\|^2).$$

Finally $\|y\|=\|v-aq\|\geqslant\|v\|-|a|\ \|q\|\geqslant\|v\|-O(\|v\|^2)$. Therefore for $\|v\|$ small

$$E(u)-E(\phi)\geqslant \tfrac{1}{4}c\ \|v\|^2,$$

as we wanted to prove.

THEOREM 3.5. *Given Assumptions* 1, 2, *and* 3 *and* $d''(\omega)>0$, *the* ϕ*-orbit is stable.*

Proof. If it is unstable, there exists a sequence of initial data $u_n(0)$ and $\delta>0$ such that

$$\inf_{s\in\mathbb{R}}\|u_n(0)-T(s)\phi\|\to 0 \qquad \text{but} \qquad \sup_{t>0}\inf_{s\in\mathbb{R}}\|u_n(t)-T(s)\phi\|\geqslant\delta,$$

where $u_n(t)$ is a solution with initial datum $u_n(0)$. By continuity in t, we can pick the first time t_n so that

$$\inf_{s\in\mathbb{R}}\|u_n(t_n)-T(s)\phi\|=\delta, \tag{3.4}$$

the solution u_n existing at least in the time interval $[0,t_n]$. By Assumption 1,

$$E(u_n(t_n))=E(u_n(0))\to E(\phi),$$
$$Q(u_n(t_n))=Q(u_n(0))\to Q(\phi).$$

Choose a sequence $\{v_n\}$ so that $Q(v_n)=Q(\phi)$ and $\|v_n-u_n(t_n)\|\to 0$. By continuity of E, $E(v_n)\to E(\phi)$. Choosing δ sufficiently small, we may apply Theorem 3.4 to obtain

$$0\leftarrow E(v_n)-E(\phi)\geqslant c\ \|T(s(v_n))v_n-\phi\|^2=c\ \|v_n-T(-s(v_n))\phi\|^2.$$

Hence $\|u_n(t_n)-T(-s(v_n))\phi\|\to 0$, which contradicts (3.4).

4. INSTABILITY

THEOREM 4.1. *If* $d''(\omega)<0$, *then*

(a) *$E(u)$ is not locally minimized at ϕ with the constraint $Q(u)=Q(\phi)$.*

(b) *There exists $y\in D(T'(0)^2)$ such that $\langle Hy,y\rangle<0$ and $\langle Q'(\phi),y\rangle=0$.*

Proof. We use the notations (3.1) and (3.2) as before. Consider ϕ_Ω for Ω near ω. Consider the function $q(s, \Omega) = Q(\phi_\Omega + s\chi_\omega)$. Then

$$\frac{\partial q}{\partial \Omega}(0, \omega) = \langle Q'(\phi_\omega), \phi'_\omega \rangle = -d''(\omega) < 0$$

by (2.21). By the implicit function theorem there exists a C^1 function $\Omega(s)$ such that $\Omega(0) = \omega$ and

$$Q(\phi_{\Omega(s)} + s\chi_\omega) = Q(\phi_\omega). \tag{4.1}$$

Now expand $L_\Omega(u) = E(u) - \Omega Q(u)$ near $u = \phi_\Omega$ to get

$$L_\Omega(\phi_\Omega + s\chi_\omega) = L_\Omega(\phi_\Omega) + s\langle L'_\Omega(\phi_\Omega), \chi_\Omega \rangle + \tfrac{1}{2}s^2 \langle L''_\Omega(\phi_\Omega)\, \chi_\omega, \chi_\omega \rangle + o(s^2).$$

This may be written, for $\Omega = \Omega(s)$, as

$$E(\phi_{\Omega(s)} + s\chi_\Omega) - \Omega Q(\phi_\omega) = d(\Omega(s)) + \tfrac{1}{2}s^2 \langle H_{\Omega(s)}\chi_\omega, \chi_\omega \rangle + o(s^2). \tag{4.2}$$

But $d''(\omega) < 0$ so that

$$d(\Omega) < d(\omega) + (\Omega - \omega)\, d'(\omega) = E(\phi_\omega) - \Omega Q(\phi_\omega)$$

for Ω near ω, by (2.16) and (2.20). Furthermore

$$\langle H_\Omega \chi_\omega, \chi_\omega \rangle \leqslant \tfrac{1}{2} \langle H_\omega \chi_\omega, \chi_\omega \rangle < 0$$

for Ω near ω by continuity with respect to Ω. Altogether from (4.2) we have the Taylor expansion

$$E(\phi_{\Omega(s)} + s\chi_\omega) < E(\phi_\omega) + \tfrac{1}{4}s^2 \langle H_\omega \chi_\omega, \chi_\omega \rangle + o(s^2). \tag{4.3}$$

Let

$$z = (d/ds)(\phi_{\Omega(s)} + s\chi_\omega)|_{s=0}.$$

By (4.1), $\langle Q'(\phi_\omega), z \rangle = 0$. Furthermore $E(\phi_{\Omega(s)} + s\chi_\omega) - E(\phi_\omega)$ vanishes to second order at $s = 0$, so that

$$\langle H_\omega z, z \rangle = \frac{d^2}{ds^2}\bigg|_{s=0} E(\phi_{\Omega(s)} + s\chi_\omega) < \tfrac{1}{2} \langle H_\omega \chi_\omega, \chi_\omega \rangle < 0.$$

This proves (a). It also proves (b) except that z might not belong to $D(T'(0)) = D$.

Now D is dense in X and $1-T'(0)^2$ is a positive operator on X since

$$(1-T'(0)^2)v, u) = \|v\|^2 + \|T'(0)v\|^2 \geq \|v\|^2$$

for $v \in D(T'(0)^2)$. Therefore

$$\xi \equiv [1-T'(0)^2]^{-1} I^{-1} Q'(\phi_\omega)$$

belongs to $D(T'(0)^2)$ and

$$\langle Q'(\phi_\omega), \xi \rangle = (I^{-1}Q'(\phi_\omega), [I-T'(0)^2]^{-1} I^{-1} Q'(\phi_\omega)) > 0.$$

Given $z \in X$ with $\langle Q'(\phi_\omega), z \rangle = 0$ and $\langle H_\omega z, z \rangle < 0$, we pick $x \in D(T'(0)^2)$ so that $\|x-z\| < \varepsilon$. Then we let

$$y = x - \frac{\langle Q'(\phi_\omega), x \rangle}{\langle Q'(\phi_\omega), q \rangle} \xi.$$

Then $\langle Q'(\phi_\omega), y \rangle = 0$ and $\|y-z\| = O(\varepsilon)$. If ε is small enough, $\langle Hy, y \rangle < 0$. This completes the proof.

LEMMA 4.2. *For ε sufficiently small there exists a C^1 functional $A: U_\varepsilon \to \mathbb{R}$ such that*

(i) $A(T(s)u) = A(u)$,

(ii) *[Range of $A'(u)$]* $\subset D(J)$,

(iii) $JA'(\phi) = -y$ *with y given in Theorem* 4.1,

(iv) $\langle Q'(u), JA'(u) \rangle = 0$

for all $u \in U_\varepsilon$ and $s \in \mathbb{R}$,

(v) $JA': U_\varepsilon \to X$ *is* C^1.

Proof. With y given in Theorem 4.1, let $Y \in D(J)$ such that $JY = y$. With $s(u)$ given in Lemma 3.1, we define

$$A(u) = -\langle Y, T(s(u))u \rangle. \tag{4.4}$$

Invariance (i) is clearly satisfied. The derivative of (4.4) is

$$A'(u) = -T^*(s(u))Y - \langle Y, T(s(u))\, T'(0)u \rangle\, s'(u). \tag{4.5}$$

By (2.7) the first term belongs to D(J). By Assumption 2, $IT'(0)\phi \in D(J)$. By (2.7) $IT(-s)\, T'(0)\phi \in D(J)$. By Lemma 3.1(iii), $s'(u) \in D(J)$ and therefore $A'(u) \in D(J)$. Putting $u = \phi$,

$$A'(\phi) = -Y - \langle Y, T'(0)\phi \rangle\, s'(\phi).$$

The last term vanishes because

$$\langle Y, T'(0)\phi\rangle = \langle Y, JB\phi\rangle = -\langle B\phi, y\rangle = -\langle Q'(\phi), y\rangle = 0$$

by (2.8) and (2.1). Therefore $A'(\phi) = -Y$. Next, from (i) we have for $u \in D(T'(0)) \cap U_\varepsilon$

$$0 = \frac{d}{ds}\bigg|_{s=0} A(T(s)u) = \langle A'(u), T'(0)u\rangle = \langle A'(u), JQ'(u)\rangle$$
$$= -\langle Q'(u), JA(u)\rangle.$$

By a passage to a limit, the same is true for all $u \in U_\varepsilon$. Finally in order to prove (v), we shall apply J to (4.5) and differentiate once more. To justify this procedure, recall that $T'(0)J = -JT'(0)^*$ with the same domain and that $JY = y \in D(T'(0)^2)$. Thus $Y \in D(JT'(0)^{*2})$. Now the derivative of J applied to (4.5) has several terms. The first term $-JT^*(s(u))Y$ is differentiable because $Y \in D(JT'(0)^*)$. The factor $\langle Y, T(s(u))\, T'(0)u\rangle$ is differentiable because $Y \in D(T'(0)^{*2})$. The last factor $Js'(u)$ is differentiable by Lemma 3.2.

DEFINITION. Solve the differential equation

$$\frac{du}{d\lambda} = -JA'(u) \tag{4.6}$$

with the initial condition $u(0) = v \in U_\varepsilon$. Call the solution $u = R(\lambda, v)$. It exists in some interval $|\lambda| < \lambda_0(v)$ with values in U_ε. It satisfies

$$T(s)\, R(\lambda, v) = R(\lambda, T(s)v) \tag{4.7}$$

$$\frac{d}{d\lambda} Q(R(\lambda, v)) = \langle Q'(u), -JA'(u)\rangle = 0 \tag{4.8}$$

$$\frac{dR(\lambda, \phi)}{d\lambda}\bigg|_{\lambda=0} = -JA'(\phi) = y. \tag{4.9}$$

LEMMA 4.3. *There exists a C^1 functional*

$$\Lambda\colon \{v \in U_\varepsilon : Q(v) = Q(\phi)\} \to \mathbb{R}$$

such that

$$E(R(\Lambda(v), v)) > E(\phi) \tag{4.10}$$

for all $v \in U_\varepsilon$ with $Q(v) = Q(\phi)$ and $v \notin \{T(s)\phi : s \in \mathbb{R}\}$.

Proof. Letting $L = E - \omega Q$, and $M(u) = T(s(u)u)$, we have $L(M(u)) = L(u)$. So Taylor's expansion gives

$$L(u) = L(\phi) + \tfrac{1}{2}\langle H[M(u) - \phi], M(u) - \phi\rangle + o(\|M(u) - \phi\|^2). \quad (4.11)$$

Recall that $M(u) - \phi$ is orthogonal to $T'(0)\phi$. We define $\lambda = \Lambda(v)$ as the unique solution of the equation

$$f(\lambda, v) = (M(R(\lambda, v)) - \phi, \chi) = 0, \quad (4.12)$$

where χ is the negative eigenvector of H. Indeed, $f(0, \phi) = (M(\phi) - \phi, \chi) = 0$ and

$$\begin{aligned}\frac{\partial f}{\partial \lambda}(0, \phi) &= \left(M'(\phi)\frac{\partial R}{\partial \lambda}(0, \phi), \chi\right) = (M'(\phi)y, \chi)\\ &= (y, \chi) + (T'(0)\phi, \chi)\langle s'(\phi), y\rangle = (y, \chi) \neq 0\end{aligned}$$

because $\langle Hy, y\rangle < 0$. By the implicit function theorem, $\lambda = \Lambda(v)$ exists in a neighborhood of $v = \phi$. Since

$$f(\lambda, T(r)v) = (M(T(r)\,R(\lambda, v)) - \phi, \chi) = f(\lambda, v),$$

the function Λ extends to all v in a tube U_ε for some $\varepsilon > 0$.

Into (4.11) we substitute $u = R(\lambda, v) = R(\Lambda(v), v)$. Thus $M(u) - \phi$ is orthogonal to both $T'(0)\phi$ and χ. By (3.2)

$$L(u) \geqslant L(\phi) + \frac{c}{2}\|M(u) - \phi\|^2 + o(\|M(u) - \phi\|^2).$$

Hence $L(u) \geqslant L(\phi)$. Since $Q(u) = Q(v) = Q(\phi)$, we have $E(u) \geqslant E(\phi)$. Equality occurs only if $M(u) = \phi$. That is, u is in the ϕ-orbit and so is v.

LEMMA 4.4. *For $v \in U_\varepsilon$ with $Q(v) = Q(\phi)$ and $v \notin \{T(s)\phi \mid s \in \mathbb{R}\}$ we have*

$$E(\phi) < E(v) + \Lambda(v)\,P(v), \quad (4.13)$$

where we define

$$P(v) = \langle E'(v), -JA'(v)\rangle. \quad (4.14)$$

Proof. We note that

$$\left.\frac{d}{d\lambda}\right|_{\lambda=0} E(R(\lambda, v)) = \left\langle E'(v), \left.\frac{du}{d\lambda}\right|_{\lambda=0}\right\rangle = P(v)$$

and

$$\frac{d^2}{d\lambda^2} E(R(\lambda, v))\bigg|_{\lambda=0, v=\phi} = \left\langle H\frac{du}{d\lambda}, \frac{du}{d\lambda}\right\rangle + \left\langle L'(\phi), \frac{d^2u}{d\lambda^2}\right\rangle$$
$$= \langle Hy, y\rangle < 0.$$

So the second derivative is negative for v in a neighborhood of ϕ and the Taylor expansion in λ gives

$$E(R(\lambda, v)) \leqslant E(v) + \lambda P(v)$$

for all small λ. Combining this inequality with (4.10),

$$E(\phi) < E(R(\Lambda(v), v)) \leqslant E(v) + \Lambda(v)\, P(v).$$

LEMMA 4.5. *There is a C^2 curve $\psi: (-\delta, \delta) \to U_\varepsilon$ such that $\psi(0) = \phi$, $\psi'(0) = y$, $Q(\psi(s)) = Q(\phi)$, $P(\psi(s))$ changes sign at $s = 0$, and $E(\psi(s))$ has a strict maximum at $s = 0$.*

Proof. Since y is a vector tangent to the smooth manifold $\{v \mid Q(v) = Q(\phi)\}$, pick a curve through ϕ tangent to y lying in this manifold. We have to show that P changes sign along this curve. Now

$$\frac{d}{ds}\bigg|_{s=0} E(\psi(s)) = \frac{d}{ds}\bigg|_{s=0} L(\psi(s)) = \langle L'(\phi), y\rangle = 0$$

by (2.15) and

$$\frac{d^2}{ds^2}\bigg|_{s=0} E(\psi(s)) = \langle L''(\phi)y, y\rangle + \langle L'(\phi), \psi''(0)\rangle = \langle Hy, y\rangle < 0.$$

So $E(\psi(s))$ has a strict local maximum at $s = 0$. By (4.13),

$$0 < E(\phi) - E(\psi(s)) \leqslant \Lambda(\psi(s))\, P(\psi(s)) \tag{4.15}$$

for small s. The inequality is strict because $\psi'(0) = y \neq T'(0)\phi$. So it suffices to show that $\Lambda(\psi(s))$ changes sign at $s = 0$. Differentiating the defining equation (4.12) for $\Lambda(\psi(s))$, we get

$$\left(M'(R(\Lambda(\psi(s)), \psi(s))\left\{\frac{\partial R}{\partial \lambda}\frac{d\Lambda}{ds} + \frac{\partial R}{\partial v}\frac{d\psi}{ds}\right\}, \chi\right) = 0.$$

If $s = 0$, then $\psi(0) = \phi$, $\Lambda(\psi(0)) = \Lambda(\phi) = 0$, $\partial R(0, v)/\partial v = \text{identity}$, $\psi'(0) = y$, and $\partial R(0, \phi)/\partial\lambda = y$. Hence

$$\left(M'(\phi)\left\{y\frac{d\Lambda}{ds}\bigg|_{s=0} + y\right\}, \chi\right) = 0.$$

However,

$$(M'(\phi)y, \chi) = (y + \langle s'(\phi), \chi\rangle\, T'(0)\phi, \chi) = (y, \chi) \neq 0,$$

whence

$$\left.\frac{dA(\psi(s))}{ds}\right|_{s=0} = -1 \neq 0. \tag{4.16}$$

We shall need a technical lemma.

LEMMA 4.6. *Let X and W be real Banach spaces with W densely embedded in X^*. Let $u \in C(\mathscr{I}, X) \cap C^1(\mathscr{I}; W^*)$, where $\mathscr{I}$ is an open interval of $\mathbb{R}$. Let $A \in C^1(X, \mathbb{R})$ with $A' \in C(X, W)$. Then $A \circ u \in C^1(\mathscr{I})$ and*

$$\frac{dA(u(t))}{dt} = \left\langle \frac{du}{dt}(t), A'(u(t)) \right\rangle. \tag{4.17}$$

Proof. Since $W \subset X^*$, it follows that $X \subset X^{**} \subset W^*$. The last pairing is between W^* and W. First we cut off and mollify in the time variable. Let $\rho \in C_c^\infty(\mathbb{R})$ be a positive function with integral 1. Let $\zeta_n \in C_c^\infty(\mathscr{I})$ with $\zeta_n \to 1$ in $\mathscr{I}$. Let

$$u_n(t) = \int n\rho(n(t-\tau))\, \zeta_n(\tau)\, u(\tau)\, d\tau.$$

Then $u_n \in C^1(\mathscr{I}, X)$ and $u_n \to u$ in $C(\mathscr{I}, X)$ and $u_n' \to u'$ in $C(\mathscr{I}, W^*)$. Now

$$\frac{d}{dt} A(u_n(t)) = \left\langle \frac{du_n}{dt}(t), A'(u_n(t)) \right\rangle,$$

the pairing being between X and X^*. Since $A' \in C(X, W)$, the pairing may also be regarded between W^* and W. As $n \to \infty$, $A' \circ u_n \to A' \circ u$ in $C(\mathscr{I}, W)$ and $A \circ u_n \to A \circ u$ in $C(\mathscr{I})$. Therefore a passage to the limit yields (4.17).

THEOREM 4.7. *If $d''(\omega) < 0$, the ϕ-orbit is unstable.*

Proof. Recall that $J\colon D(J) \subset X^* \to X$ is a closed linear operator. Let $W = D(J)$ with the graph norm $\|v\|_W^2 = \|v\|^2 + \|Jv\|^2$. Then W is a Hilbert space, and $J\colon W \to X$ and $J^*\colon X^* \to W^*$ are continuous. By definition (2.13) a solution of the evolution equation is a function $u \in C(\mathscr{I}, X)$ such that

$$\frac{d}{dt}\langle u(t), \psi\rangle = \langle E'(u(t)), -J\psi\rangle = \langle J^*E'(u(t)), \psi\rangle$$

for all $\psi \in W$. Therefore $u \in C^1(\mathscr{I}, W^*)$ and

$$\frac{du}{dt} = -J^*E'(u(t)) \qquad \text{for} \quad t \in \mathscr{I}. \tag{4.18}$$

Now fix ω and $\phi = \phi_\omega$. Fix the tube width ε so small that Lemma 4.4 is valid. Let $u_0 = \psi(s)$ be given in Lemma 4.5 so that u_0 is arbitrarily near ϕ, $Q(u_0) = Q(\phi)$, $E(u_0) < E(\phi)$, and $P(u_0) > 0$. ($P(u_0) < 0$ would also do.) According to Assumption 1, there is an interval $[0, t_0)$ and a solution $u(t)$ of (4.18) which satisfies $u(0) = u_0$ and

$$Q(u(t)) = Q(u_0) = Q(\phi), \qquad E(u(t)) = E(u_0) < E(\phi). \tag{4.19}$$

Since t_0 depends only on μ where $\|u_0\| \leqslant \mu$, either the solution $u(t)$ can be continued to a solution for all time $0 \leqslant t < \infty$ which satisfies (4.18) or else it can be continued until it blows up at a finite time T: $u(t) \to \infty$ as $t \uparrow T$. In the latter case we surely have instability. In the former case we argue as follows.

In any interval $[0, t_1)$ in which $u(t) \in U_\varepsilon$ we may apply Lemma 4.4 and (4.19) to obtain

$$0 < E(\phi) - E(u_0) = E(\phi) - E(u(t)) < A(u(t))\, P(u(t)).$$

Therefore $P(u(t)) > 0$. Taking ε smaller if necessary, we may assume $A(u(t)) < 1$. Therefore

$$P(u(t)) > E(\phi) - E(u_0) = \varepsilon_0 > 0. \tag{4.20}$$

By Lemma 4.6, $A \circ u$ is differentiable and

$$\begin{aligned} \frac{d}{dt} A(u(t)) &= \left\langle \frac{du}{dt}, A'(u) \right\rangle = \langle -J^*E'(u), A'(u) \rangle \\ &= \langle E'(u), -JA'(u) \rangle = P(u) > \varepsilon_0 \end{aligned}$$

in the interval $[0, t_1)$. But

$$|A(v)| \leqslant \|Y\|_{X^*}(\|\phi\| + \varepsilon) \qquad \text{for} \quad v \in U_\varepsilon.$$

Therefore

$$t_1 \leqslant \frac{2\,\|Y\|_{X^*}(\|\phi\| + \varepsilon)}{E(\phi) - E(u_0)} < \infty.$$

So the solution must exit from the tube and the ϕ-orbit is unstable. This completes the proof of Theorem 3.

5. Extensions

Most of this section is taken up with the proof of Theorem 2, that is, the extension to the case when $d''(\omega)=0$. After that, we prove Theorem 1 and then extend our results to an arbitrary Banach space X.

COROLLARY 5.1. *Given Assumptions* 1, 2, *and* 3, *if* $d(\cdot)$ *is not convex in a neighborhood of* ω, *then the* ϕ_ω*-orbit is unstable.*

Proof. Let $S=\{\omega \mid \omega_1<\omega<\omega_2$, the ϕ_ω-orbit is stable$\}$. The curve $\omega \to \phi_\omega$ is continuous. Therefore from the definition of stability, the set S is open. Let $R=\{\omega \mid \omega_1<\omega<\omega_2$, $d(\cdot)$ is convex in a neighborhood of $\omega\}$. Obviously R is open. If $d(\cdot)$ is not convex in a neighborhood of ω, then $\omega \notin R$. Hence there is a sequence $\omega_n \to \omega$ such that $d''(\omega_n)<0$. By Theorem 3, already proved, $\omega_n \notin S$. Therefore $\omega \notin S$.

THEOREM 5.2. *Fix* ω *and suppose* $d''(\omega)=0$. *Then* $\langle Hy, y\rangle>0$ *for all* $y\in X$ *such that* $y\neq 0$ *and* y *is orthogonal in* X *to each of the three vectors* $I^{-1}Q'(\phi)$, ϕ', *and* $T'(0)\phi$.

Proof. By (2.8) and Assumption 2, $JB\phi=T'(0)\phi\neq 0$. So by (2.19), $H\phi'=Q'(\phi)=B\phi\neq 0$ where $\phi'=d\phi/d\omega$. By (2.21) $\langle H\phi', \phi'\rangle=d''(\omega)=0$. It follows that ϕ' must have nontrivial components in both χ and P. As in the proof of Theorem 3.2, we spectrally decompose

$$\phi'=a_0\chi+b_0T'(0)\phi+p_0 \qquad (p_0\in P). \tag{5.1}$$

Then $a_0\neq 0$ and $p_0\neq 0$ and

$$Q'(\phi)=H\phi'=a_0H\chi+Hp_0. \tag{5.2}$$

The proof of Theorem 3.3 is repeated exactly except that the inequality is not strict. So if y is orthogonal to both $I^{-1}Q'(\phi)$ and $T'(0)\phi$, then $\langle Hy, y\rangle\geq 0$. Suppose now that $\langle Hy, y\rangle=0$. From the proof of Theorem 3.3, we would then have Schwarz's *equality*

$$\langle Hp, p_0\rangle^2=\langle Hp, p\rangle\langle Hp_0, p_0\rangle,$$

where y has the spectral decomposition $y=a\chi+p$, $p\in P$. Therefore p and p_0 are linearly dependent, so that

$$y=a\chi+cp_0, \tag{5.3}$$

for some scalars a, c. We want to show that $a=c=0$ if y is orthogonal to ϕ' as well. Thus

$$0=(I^{-1}Q'(\phi), y)=\langle aH\chi+Hp_0, a\chi+cp_0\rangle$$

and

$$0 = (\phi', y) = (a_0\chi + p_0, a\chi + cp_0).$$

Hence a and c satisfy the linear system with matrix

$$\begin{pmatrix} a_0\langle H\chi, \chi\rangle & \langle Hp_0, p_0\rangle \\ a_0 & \|p_0\|^2 \end{pmatrix}.$$

This matrix is nonsingular since $\langle H\chi, \chi\rangle$ and $\langle Hp_0, p_0\rangle$ have opposite signs (strictly). Hence $a = c = 0$.

COROLLARY 5.3. *Suppose $d''(\omega_0) = 0$. If ω is sufficiently close to ω_0 and if $y \neq 0$ is orthogonal to $I^{-1}Q'(\phi_\omega)$, ϕ'_{ω_0}, and $T'(0)\phi_\omega$, then $\langle H_\omega y, y\rangle > 0$.*

Proof. This is obvious by continuity in the variable ω.

THEOREM 5.4. *Let $d''(\omega) = 0$ where ω is fixed. Let $d'' \geq 0$ in an open interval containing ω. Then there exists $\varepsilon > 0$ such that $E(u) > E(\phi)$ for all $u \in U_\varepsilon$ with $Q(u) = Q(\phi)$ and $u \neq T(s)\phi$ for $s \in \mathbb{R}$.*

Proof. Recall that we have abbreviated $\phi = \phi_\omega$ with ω fixed. Given $u \in U_\varepsilon$ we define

$$y = T(s)u - \phi_\Omega - aI^{-1}Q'(\phi_\Omega). \tag{5.4}$$

We claim that the three parameters s, Ω, and a can be chosen depending on u so that y is orthogonal to the three vectors $T'(0)\phi_\Omega$, ϕ'_ω, and $I^{-1}Q'(\phi_\Omega)$. This can be accomplished in some neighborhood of ϕ_ω by the implicit function theorem provided a certain 3×3 determinant does not vanish. First note that when $s = 0$, $\Omega = \omega$, and $a = 0$, we have $y = u - \phi_\omega$. Now the three orthogonality conditions

$$0 = (y, T'(0)\phi_\Omega) = (y, \phi'_\omega) = (y, I^{-1}Q'(\phi_\Omega)) \tag{5.5}$$

with y given by (5.4) are three scalar equations for s, Ω, and a. The Jacobian, evaluated at $s = 0$, $\Omega = \omega$, $a = 0$, and $u = \phi_\omega = \phi$, is

$$\begin{pmatrix} \|T'(0)\phi\|^2 & 0 & -\langle Q'(\phi), T'(0)\phi\rangle = 0 \\ (T'(0)\phi, \phi') & -\|\phi'\|^2 & -\langle Q'(\phi), \phi'\rangle = 0 \\ \langle Q'(\phi), T'(0)\phi\rangle = 0 & -\langle Q'(\phi), \phi'\rangle = 0 & -\|I^{-1}Q'(\phi)\|^2 \end{pmatrix}$$

This is a triangular matrix with nonzero diagonal entries and so it is nonsingular. This proves the claim. Furthermore

$$|s| + |\Omega - \omega| + |a| = O(\|u - \phi_\omega\|) = O(\|y\|) \tag{5.6}$$

as $u \to \phi_\omega$. Denoting $v = T(s)u$, a Taylor expansion yields

$$\begin{aligned} Q(u) = Q(v) &= Q(\phi_\Omega) + \langle Q'(\phi_\Omega), v - \phi_\Omega \rangle + O(\|v - \phi_\Omega\|^2) \\ &= Q(\phi_\omega) + \langle Q'(\phi_\omega), \phi'_\omega \rangle (\Omega - \omega) + O(\Omega - \omega)^2 \\ &\quad + \langle Q'(\phi_\Omega), y + aI^{-1}Q'(\phi_\Omega) \rangle + O(\|v - \phi_\Omega\|^2) \\ &= Q(\phi_\omega) + O((\Omega - \omega)^2) + a\, \|I^{-1}Q'(\phi_\Omega)\|^2 + O(\|v - \phi_\omega\|^2). \end{aligned}$$

Since $Q(u) = Q(\phi_\omega)$ by assumption and since $Q'(\phi_\omega) \neq 0$, we deduce that

$$|a| = O((\Omega - \omega)^2 + \|v - \phi_\Omega\|^2) = O(\|y\|^2), \tag{5.7}$$

the last bound coming from (5.6).

Now we expand $E(v) - \Omega Q(v)$ around $v = \phi_\Omega$, noting that $E'(\phi_\Omega) - \Omega Q'(\phi_\Omega) = 0$ by (2.15) and $E''(\phi_\Omega) - \Omega Q''(\phi_\Omega) = H_\Omega$ by definition. Thus

$$\begin{aligned} E(u) - \Omega Q(u) &= E(v) - \Omega Q(v) \\ &= d(\Omega) + \tfrac{1}{2}\langle H_\Omega (v - \phi_\Omega), v - \phi_\Omega \rangle + o(\|v - \phi_\Omega\|^2) \end{aligned}$$

We substitute $v - \phi_\Omega = y + aI^{-1}Q'(\phi_\Omega)$ to get

$$d(\Omega) + \tfrac{1}{2}\langle H_\Omega y, y \rangle + O(a\, \|y\|) + O(a^2) + o(\|v - \phi_\Omega\|^2).$$

By (5.7) all the error terms may be written as $o(\|y\|^2)$. By (5.5) the conditions of Corollary 5.3 are satisfied with ω_0 replaced by ω and ω by Ω. Therefore we deduce

$$E(u) - \Omega Q(u) \geqslant d(\Omega) + \delta\, \|y\|^2 + O(\|y\|^2).$$

This is strictly greater than $d(\Omega)$ provided $y \neq 0$. By assumption $u \neq T(r)\phi_\omega$ for all $r \in \mathbb{R}$. Taking a small enough tube U_ε, we have $u \neq T(r)\phi_\Omega$ for all $r \in \mathbb{R}$, so that $y \neq 0$. Therefore

$$\begin{aligned} E(u) - \Omega Q(u) &> d(\Omega) \\ &\geqslant d(\omega) + d'(\omega)(\Omega - \omega) \\ &= E(\phi_\omega) - \omega Q(\phi_\omega) - Q(\phi_\omega)(\Omega - \omega) \\ &= E(\phi_\omega) - \Omega Q(\phi_\omega) \end{aligned}$$

because d is convex. Since $Q(u) = Q(\phi_\omega)$, we conclude that $E(u) > E(\phi_\omega)$.

Proof of Theorem 2. The cases d not convex and $d''(\omega) > 0$ were treated in Corollary 5.1 and Theorem 3.4. Let d be convex near ω and let

$d''(\omega)=0$. Mimicking the proof of Theorem 3.4, we have $v_n \in U_\varepsilon$, $Q(v_n)=Q(\phi)$, and $E(v_n)\to E(\phi)$. By Theorem 5.4 we must have

$$\inf_{s\in\mathbb{R}} \|v_n - T(s)\phi\| \to 0.$$

Therefore $\inf_{s\in\mathbb{R}} \|u_n(t_n)-T(s)\phi\| \to 0$, which contradicts (3.4). This completes the proof.

Proof of Theorem 1. By the assumption on H, it is obvious that $\langle Hy, y\rangle > 0$ for any nonzero vector y orthogonal to the kernel $T'(0)\phi$. By a simple Taylor expansion we have exactly the conclusion of Theorem 3.4. Finally the stability is proved exactly as in Theorem 3.5.

Extension to Banach spaces. Let X be any real Banach space. Let J and E be as in Section 2. Let T be a one-parameter strongly continuous group of isometries of X onto X. Assume (2.3), (2.7), and (2.8), except that B is merely symmetric: $\langle Bu, v\rangle = \langle Bv, u\rangle$ for $u, v\in X$. Define $Q(u)$ by (2.9).

Assumptions 1 and 2 are unchanged with an exception noted below. Without the inner product we have to rephrase Assumption 3 as follows.

Assumption 3B. For each $\omega\in(\omega_1,\omega_2)$, let $H_\omega = E''(\phi_\omega)-\omega Q''(\phi_\omega)$. We assume

(i) There exists $\chi\in X$ such that $\langle H\chi, \chi\rangle < 0$.

(ii) There exists a closed subspace $P\subset X$ such that

$$\langle Hp, p\rangle \geqslant \delta\, \|p\|^2 \qquad \text{for} \quad p\in P.$$

(iii) For all $u\in X$, there exist unique constants a, b, and a unique $p\in P$ such that

$$u = a\chi + bT'(0)\phi + p.$$

Then we *define* $\Pi_p(u)=p$, $\Pi_0(u)=b$, and $\Pi_N(u)=a$. These operators are continuous projecions. We replace Assumption 2(c) by

(c_1) the functional $u\to \Pi_0(T'(s)u)$ belongs to $D(J)$, and

(c_2) the functional $u\to \Pi_0(T'(s)^2u)$ belongs to X^* for all $s\in\mathbb{R}$.

THEOREM 5.5. *If X is a Banach space and these assumptions hold, then Theorems 2 and 3 are valid.*

LEMMA 5.6. *Let $M=\{\phi + a\chi + p\colon p\in P, a\in\mathbb{R}\}$. There exists $\varepsilon>0$ such that, for all $u\in U_\varepsilon$, there are unique $s=s(u)\in\mathbb{R}$ and $m=m(u)\in M$ such that $u=T(-s)m$.*

Proof. Consider the mapping $s, m \to T(-s)m$ from $\mathbb{R} \times M$ into X. Its derivative at $s=0$, $m=\phi$ is

$$(\tilde{s}, \tilde{m}) \to -\tilde{s}T'(0)\phi + \tilde{m}.$$

This is an isomorphism since $X = \mathbb{R}(T'(0)\phi) \oplus M$. By the implicit function theorem, the mapping $u \to (s, m)$ is a local isomorphism.

COROLLARY. *$u \to (s(u), m(u))$ is smooth,*

$$s(T(r)u) = s(u) - r \qquad \textit{for} \quad r \in \mathbb{R},$$

$s'(u) \in D(J)$, $Js'(u)$ is differentiable, and

$$\langle s'(u), v\rangle = -\frac{\Pi_0[T(s(u))v]}{\Pi_0[T'(s(u))u]}. \tag{5.8}$$

Proof. Writing $u = T(-s)m$ as above, we have $T(r)u = T(-(s-r)m)$ so that $s(T(r)u) = s(u) - r$ since $s(\cdot)$ is uniquely defined. Now we have

$$T(s(u))u = m(u) \in M.$$

Therefore $\Pi_0[T(s(u))u - \phi] = 0$. Differentiating with respect to u in the direction v, we get

$$\Pi_0[T(s(u))v] + \Pi_0[T'(s(u))u]\langle s'(u), v\rangle.$$

Thus we have formula (5.8). It follows from (c_1) and (c_2) above that $s'(u) \in D(J)$.

Only the following changes are necessary in the succeeding discussion. In Theorem 3.3, the condition $(T'(0)\phi, y) = 0$ is replaced by $\Pi_0(y) = 0$. In Theorem 3.4 let $Q'(\phi)^{\perp} = \{y \in X\colon \langle Q'(\phi), y\rangle = 0\}$ and let y be the nearest point of the subspace $Q'(\phi)^{\perp}$ to $m(u) - \phi = T(s(u))u - \phi$. Theorem 4.1 (b) is not difficult to modify. In Lemma 4.3 the equation (4.12) which defines $\Lambda(v)$ is replaced by

$$f(\lambda, v) = \Pi_N[m(R(\lambda, v)) - \phi] = 0$$

and $m(u) = M(u)$. It is this equation which is differentiated in Lemma 4.5.

In Theorem 5.4, the three "orthogonality" conditions become the following. $\Pi_0(y) = 0$, $\langle Q'(\phi_\Omega), y\rangle = 0$, and y is the vector satisfying these two conditions which is nearest $T(s)u - \phi_\Omega$.

6. Examples

In this section we present some examples that are applications of the abstract theory.

A. *Traveling Waves of Nonlinear Wave Equations*

Traveling wave solutions of the nonlinear wave equation

$$u_{tt} - u_{xx} + f(u) = 0 \tag{6.1}$$

are generated by the invariance of the equation under space translation. Equation (6.1) can be written in the form (2.12),

$$\frac{d\mathbf{u}}{dt} = JE'(\mathbf{u}),$$

where

$$\mathbf{u} = \begin{pmatrix} u \\ v \end{pmatrix}, \qquad J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

$$E(\mathbf{u}) = \int \left(\tfrac{1}{2}v^2 + \tfrac{1}{2}u_x^2 + F(u)\right) dx \tag{6.2}$$

$$F' = f, \qquad F(0) = 0.$$

If we define the space $X = H^1(R) \times L^2(R)$, then $X^* = H^{-1}(R) \times L^2(R)$ and we have the isomorphism $I: X \to X^*$, where

$$I = \begin{bmatrix} -\Delta + 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

The above initial value problem is well-defined on the space X. $T(s)$ is a well-defined unitary group on X with

$$T'(0) = \begin{pmatrix} \partial_x & 0 \\ 0 & \partial_x \end{pmatrix}, \qquad D(T'(0)) = H^2(R) \times H^1(R) \subset X$$

and

$$J^{-1}T(s) = T^*(s)J^{-1}.$$

The invariance generates the conserved quantity (momentum)

$$Q(\mathbf{u}) = \tfrac{1}{2}\langle B\mathbf{u}, \mathbf{u}\rangle = \int u_x v \, dx, \tag{6.3}$$

where

$$B = \begin{pmatrix} 0 & -\partial_x \\ \partial_x & 0 \end{pmatrix}.$$

Now for traveling waves to exist we should find nonzero solutions of the equation

$$E'(\boldsymbol{\phi}) - \omega Q'(\boldsymbol{\phi}) = 0.$$

The components of $\boldsymbol{\phi} = (\phi\ \psi)^t$ will satisfy the equations

$$-(1-\omega^2)\phi_{xx} + f(\phi) = 0, \tag{6.4}$$

$$\psi = \omega\phi_x. \tag{6.5}$$

Suppose that f satisfies the following conditions.

(i) $f'(0) > 0$.

(ii) $\exists\eta$ such that $F(\eta) < 0$. (Consequently from (i), $F(u)$ has at least one zero u_0.)

(iii) If u_0 is the zero of F with smallest nonzero absolute value, then $f(u_0) \neq 0$.

LEMMA 6.1. *If f satisfies* (i)–(iii), *then the equation*

$$-p_{xx} + f(p) = 0$$

has a unique solution that satisfies

(a) $p(x) > 0$, $p(x) = p(-x)$, $p(0) = u_0$.

(b) $p(x)$ *decays exponentially like* $e^{-c|x|}$ *with* $c > 0$.

Denote $\phi_\omega(x) = p(x/\sqrt{1-\omega^2})$, $\omega \in (-1, 1)$. Then ϕ_ω satisfies (6.4) and we have a nontrivial traveling wave. The linearized operator of (6.4) is

$$L_\omega = -(1-\omega^2)\,\partial_x^2 + f'(\phi_\omega). \tag{6.6}$$

The kernel of L_ω is spanned by $\partial_x\phi_\omega$. Moreover since $\partial_x\phi_\omega$ has a simple zero at $x=0$, L_ω has exactly one strictly negative eigenvalue $-\alpha_\omega^2$, with an eigenfunction χ_ω,

$$L_\omega\chi_\omega = -\alpha_\omega^2\chi_\omega. \tag{6.7}$$

In order to verify Assumption 3 of the theorem we shall compute the spectrum of the operator

$$H_\omega = E''(\boldsymbol{\phi}_\omega) - \omega Q''(\boldsymbol{\phi}_\omega) = \begin{pmatrix} L_0 & \omega\partial_x \\ -\omega\partial_x & 1 \end{pmatrix}. \tag{6.8}$$

LEMMA 6.2. *The spectrum of the operator H_ω is as follows*:

(1) *There is one negative simple eigenvalue.*

(2) *The kernel is spanned by $T'(0)\boldsymbol{\phi}_\omega$.*

(3) *The positive spectrum of H_ω is bounded away from zero.*

Proof. Let $\boldsymbol{\psi} = (\psi_1 \psi_2)'$ be an eigenfunction of H_ω with eigenvalue λ. Writing the components of the equation $H_\omega \boldsymbol{\psi} = \lambda \boldsymbol{\psi}$, we have

$$-\partial_x^2 \psi_1 + f'(\phi_\omega)\psi_1 + \omega \partial_x \psi_2 = \lambda \psi_1,$$
$$-\omega \partial_x \psi_1 + \psi_2 = \lambda \psi_2.$$

For $\lambda \neq 1$ we can rewrite the above equations as

$$-(1-\omega^2)\,\partial_x^2 \psi_1 + f'(\phi_\omega)\psi_1 = \frac{\lambda(1-\omega^2)-\lambda^2}{1-\lambda}\psi_1 \tag{6.9}$$

and

$$\psi_2 = \frac{\omega}{\lambda - 1}\,\partial_x \psi_1. \tag{6.10}$$

If $\lambda < 0$ then by (6.7) we have

$$\frac{\lambda(1-\omega^2)-\lambda^2}{1-\lambda} = -\alpha_\omega^2$$

or

$$\lambda^2 - (1-\omega^2-\alpha_\omega^2)\lambda - \alpha_\omega^2 = 0,$$

which has exactly one negative root. Therefore H_ω has exactly one negative eigenvalue, $\lambda_-(\omega)$.

Next we substitute $\lambda = 0$ in (6.3) and we get the kernel of H_ω to be spanned by $T'(0)\phi_\omega$.

By Weyl's theorem on the essential spectrum, the rest of the spectrum of H_ω is bounded away from zero. Q.E.D.

Since H_ω satisfies the hypotheses of the theorem, the stability of the traveling wave is determined by the sign of $d''(\omega)$. But

$$d'(\omega) = -Q(\boldsymbol{\phi}_\omega) = -\omega \int |\partial_x \phi_\omega|^2\,dx.$$

Therefore

$$d''(\omega) = (\sqrt{1-\omega^2})'' \int |\partial_x p|^2 \, dx < 0.$$

Hence all traveling waves are unstable.

Remark. The well-known kink solutions $\phi_\omega(x - \omega t)$ travel monotonically from one zero of f to another. Therefore, $\partial_x \phi_\omega$ does not vanish and is the lowest eigenfunction of the operator L_ω with eigenvalue zero. Hence $L_\omega \geqslant 0$. Now H_ω is again given by formula (6.8). By (6.9), H_ω cannot have any negative eigenvalues. By Theorem 1, the kinks are always stable.

B. *Standing Waves*

Equations that are invariant under phase transformation have standing wave solutions or bound states. These solutions are generated by the gauge group $T(s)u = e^{is}u$. Stability properties of these solutions were studied in [13] for the nonlinear Klein–Gordon equation

$$u_{tt} - \Delta u + g(|u|^2)u = 0, \qquad x \in \mathbb{R}^n,$$

and the nonlinear Schrödinger equation

$$-iu_t - \Delta u + g(|u|^2)u = 0, \qquad x \in \mathbb{R}^n.$$

Some are stable and some are unstable. It is easy to verify that both of these equations satisfy Assumptions 1–3 for the case of the state of lowest energy. In particular the linearized operator H_ω satisfies Assumption 3 by Lemma 8 of [13]. Therefore the previously proved stability and instability results follow directly from our abstract theorems.

C. *Nonlinear Schrödinger Equation with a Potential*

In this example we study the stability properties of bound states of the equation

$$ihu_t = -h^2 u_{xx} + V(x)u + f(u)$$

in one dimension where $f(u) = g(|u|^2)u$. By changing variables $t \to t/h$, $y = (x - x_0)/h$, the equation takes the form

$$iu_t = -u_{yy} + V(x_0 + hy)u + f(u). \qquad (6.11)$$

As in Example B, the bound states are solutions of the form $e^{i\omega t}\phi(y)$, generated by the invariance of the equation under the phase transformations $T(s)u = \exp(is)u$. Let

$$E(u) = \int \left(\tfrac{1}{2}\,|u_y|^2 + \tfrac{1}{2}V(x_0 + hy)\,|u|^2 + F(|u|)\right) dy,$$

where $F'(u) = f(u)$ and $F(0) = 0$. Equation (6.11) can be written in the form (2.12) where $J = -i$. The space X where this equation is well-defined is the complex $H^1(\mathbb{R})$ with the real inner product. Now $T(s)$ is a unitary representation on X with a generator $T'(0) = i$, and $D(T'(0)) = X$. Moreover, $B = -1$ and the conserved quantity (charge) is given by

$$Q(u) = \tfrac{1}{2}\langle Bu, u\rangle = -\tfrac{1}{2}\int |u|^2\, dy.$$

Bound states of (6.11) satisfy the equation $E'(\phi) - \omega Q'(\phi) = 0$. For simplicity we look for real solutions ϕ; that is,

$$-h^2\phi_{xx} + V(x)\phi + f(\phi) + \omega\phi = 0 \qquad (6.12)$$

or equivalently

$$-\phi_{yy} + V(x_0 + hy)\phi + f(\phi) + \omega\phi = 0. \qquad (6.13)$$

We will assume that the potential V is bounded, with $V^* \leqslant V(x)$. The following theorem is due to Floer and Weinstein [5].

THEOREM 6.3. *Let $V \in C^2(R)$ have a nondegenerate critical point x_0 and let $\omega > -V^*$. There exists $h_0 > 0$ such that for $0 < h < h_0$ Eq. (6.13) has a nonzero solution $\phi_h(\omega, y)$ "concentrated" around x_0 in the sense that*

$$\phi_h\left(\omega, \frac{x - x_0}{h}\right) = \phi_0\left(\omega, \frac{x - x_0}{h} - z(h)h\right) + \rho_h\left(\omega, \frac{x - x_0}{h}\right),$$

where

(a) *$\phi_0(\omega, y)$ is the unique solution of the equation $-d^2\phi_0/dy^2 + f(\phi_0) + [V(x_0) + \omega]\phi_0 = 0$, which has its maximum at $y = 0$.*

(b) *$z(\cdot)$ is a $C^2([0, h_0])$ function,*

(c) *$\rho_h(\omega, y)$ is C^2 in h and ω with values in $H^2(R)$, $\|\rho_h(\omega, \cdot)\|_{L^2} = O(h)$ as $h \to 0$,*

(d) *$\int_{-\infty}^{\infty} \rho_h(\omega, y) \cdot \partial_y \phi_0(\omega, y - z(h)h)\, dy = 0.$*

From now on we will suppress the subscript h or ω when convenient. We also take $x_0 = 0$. The linearized operator $H_\omega = E''(\phi) - \omega Q''(\phi)$, where $\phi = \phi_h(\omega, \cdot)$ is the bound state, is

$$H_\omega \chi = \mathscr{Re}(R_h \chi) + i\mathscr{Im}(S_h \chi),$$

where

$$R_h = -\partial_y^2 + V(hy) + f'(\phi_h) + \omega,$$
$$S_h = -\partial_y^2 + V(hy) + f(\phi_h)/\phi_h + \omega.$$

Observe that $S\phi = 0$; therefore ϕ is in the kernel of S. Moreover, $\phi(y)$ is positive. It is easy to show that the kernel of S is spanned by ϕ. Therefore for H_ω to satisfy Assumption 3 it is sufficient to show that R_h has one negative simple eigenvalue and no kernel. Note that R_h converges to R_0 in the strong resolvent sense. By the same argument as in [10, p. 34], one can show that R_h has simple eigenvalues in the neighborhoods of the negative and the zero eigenvalues of R_0. The eigenvalues are C^2 functions of h. (The difference between our case and [10] is that the potential is $V(hy)$ rather than $hV(y)$.)

THEOREM 6.4. *If $x_0 = 0$ is a strict local minimum, i.e., $V''(0) > 0$, then R_h has a small positive eigenvalue for h small.*

Proof. In this proof derivatives with respect to h are denoted by primes and derivatives with respect to y by subscripts. Let $E(h)$ be the eigenvalue of R_h close to zero, i.e., $E(0) = 0$, and let ψ_h be the normalized eigenvector. We want to show that $E(h) > 0$ for $h > 0$. Differentiating Eq. (6.13) with respect to y and evaluating at $h = 0$, we get

$$R_0 \phi_{0,y} = 0 \tag{6.14}$$

$$R_0 \phi_{0,yy} = -f''(\phi_0)\, \phi_{0,y}^2. \tag{6.15}$$

Differentiating with respect to h, we get

$$R_0 \phi_0' = 0 \tag{6.16}$$

$$R_0 \phi_0'' + R_0' \phi_0' = -y^2 V''(0) \phi_0. \tag{6.17}$$

Since the zero eigenvalue of R_0 is simple with eigenvector ψ_0, (6.14) and (6.16) imply that there are scalars α and β so that $\phi_0' = \alpha \phi_{0,y}$ and $\psi_0 = \beta \phi_{0,y}$. Let us compute $E'(0)$ and $E''(0)$. We have $E(h) = \langle \psi_n, R_h \psi_h \rangle$ and

$$E'(h) = \langle \psi_n, R_h' \psi_h \rangle + 2E(h) \langle \psi_h', \psi_h \rangle = \langle \psi_h, R_h' \psi_h \rangle \tag{6.18}$$

since ψ_h is normalized. For $h=0$,

$$E'(0)=\langle\psi_0, R_0'\psi_0\rangle=\alpha\beta\langle\phi_{0,y}, f''(\phi_0)\,\phi_{0,y}^2\rangle=0$$

because ϕ_0 is an even function of y. Now differentiating (6.18) with respect to h and evaluating at $h=0$, we get

$$E''(0)=2\langle\phi_0', R_0'\psi_0\rangle+\langle\psi_0, R_0''\psi_0\rangle, \tag{6.19}$$

where $R_0''=f''(\phi_0)\phi_0''+f'''(\phi_0)(\phi_0')^2+y^2V''(0)$.

The first term of (6.19) equals twice

$$\begin{aligned}\langle\psi_0', R_0'\psi_0\rangle&=\langle\psi_0', f''(\phi_0)\,\phi_0'\psi_0\rangle=\alpha\beta\langle\psi_0', f''(\phi_0)\,\phi_{0,y}^2\rangle\\&=-\alpha\beta\langle\psi_0', R_0\phi_{0,yy}\rangle=-\alpha\beta\langle R_0\psi_0', \phi_{0,yy}\rangle\end{aligned}$$

by (6.15). Since $-R_0\psi_0'=R_0'\psi_0=f''(\phi_0)\,\phi_0'\psi_0$,

$$\langle\psi_0', R_0'\psi_0\rangle=\alpha^2\beta^2\langle f''(\phi_0)\,\phi_{0,y}^2, \phi_{0,yy}\rangle. \tag{6.20}$$

The second term of (6.19) is

$$\begin{aligned}\langle\psi_0, R_0''\psi_0\rangle=\langle y^2V''(0)\psi_0, \psi_0\rangle+\langle\psi_0, f'''(\phi_0)(\phi_0')^2\psi_0\rangle\\+\langle f''(\phi_0)\,\phi_0''\psi_0, \psi_0\rangle.\end{aligned} \tag{6.21}$$

Moreover by (6.15)

$$\begin{aligned}\langle f''(\phi_0)\,\phi_0''\psi_0, \psi_0\rangle&=\beta^2\langle\phi_0'', f''(\phi_0)\,\phi_{0,y}^2\rangle=-\beta^2\langle\phi_0'', R_0\phi_{0,yy}\rangle\\&=-\beta^2\langle R_0\phi_0'', \phi_{0,yy}\rangle.\end{aligned}$$

By (6.17) we have

$$\begin{aligned}\langle f''(\phi_0)\,\phi_0''\psi_0, \psi_0\rangle=\beta^2\langle y^2V''(0)\phi_0, \phi_{0,yy}\rangle\\+\alpha^2\beta^2\langle f''(\phi_0)\,\phi_{0,y}^2, \phi_{0,yy}\rangle.\end{aligned} \tag{6.22}$$

Substituting (6.20), (6.21), and (6.22) into (6.19), we get

$$\begin{aligned}E''(0)=3\alpha^2\beta^2\langle f''(\phi_0)\,\phi_{0,y}^2, \phi_{0,yy}\rangle+\alpha^2\beta^2\langle\phi_{0,y}, f'''(\phi_0)\,\phi_{0,y}^3\rangle\\+\beta^2\langle y^2V''(0)\,\phi_{0,y}, \phi_{0,y}\rangle+\beta^2\langle y^2V''(0)\,\phi_0, \phi_{0,yy}\rangle.\end{aligned}$$

Integrating by parts we obtain

$$E''(0)=\beta^2V''(0)\langle\phi_0, \phi_0\rangle. \tag{6.23}$$

Therefore if $V''(0)>0$ we have $E''(0)>0$ and $E(h)>0$ for small $h>0$.

By Theorem 6.4, H_ω satisfies Assumption 3. By Theorem 2 these bound states are stable if $d''(\omega) > 0$. By (2.20) we have the following conclusion.

THEOREM 6.5. *Let $V \in C^2(\mathbb{R})$ with a nondegenerate minimum at x_0. Let $\omega > -V^*$. There exists $h_1 > 0$ such that, if $0 < h < h_1$, then $\phi_h(\omega, \cdot) \exp(i\omega t)$ is stable if*

$$\frac{\partial}{\partial \omega} \int [\phi_0(\omega, y)]^2 \, dy > 0$$

and is unstable if this derivative is <0.

For example if $f(u) = -|u|^{p-1}u$, it is stable if $1 < p < 5$ and unstable if $5 < p < \infty$.

This example generalizes without difficulty to n dimensions. Replace ∂_x^2 by the Laplacian. Theorem 6.3 is true provided f satisfies the usual conditions for the existence of a ground state $\phi_0(\omega, |y|)$ as in [13]. Theorem 6.4 is true if $\partial_i \partial_j V(0)$ is positive definite. For the proof, let $\psi_h^1, ..., \psi_h^n$ be the small eigenfunctions of R_h for h small. Let

$$E(h) = \frac{\langle R_h \psi_h, \psi_h \rangle}{\langle \psi_h, \psi_h \rangle}, \qquad \psi_h = \sum_{j=1}^{n} \alpha_j \psi_h^j$$

for arbitrary $\alpha_1, ..., \alpha_n$. Then we calculate $E'(0) = 0$ and

$$E''(0) = \left[\sum_{i,j} (\partial_i \partial_j V(0)) \, \alpha_i \alpha_j \right] \left[\int \phi_0^2(y) \, dy \right] \left[\int \psi_0^2(y) \, dy \right]^{-1}.$$

Therefore the zero eigenvalue of H_ω migrates to n *positive* eigenvalues of H_ω for h small. Theorem 6.5 is true without change.

D. *Optical Wave Guide*

Suppose we have three layered madia where the outside two are nonlinear and the sandwiched one is linear. The index of refraction is given by

$$g(x, |u|^2) = \begin{cases} \eta_1 + \alpha \, |u|^2, & |x| > d \\ \eta_0, & |x| \leq d, \end{cases} \tag{6.24}$$

where $\eta_1 < \eta_0$, $-\infty < x < \infty$. See Fig. 1. The differential equation that governs $u(x, t)$ is

$$iu_t = -u_{xx} - g(x, |u|^2)u. \tag{6.25}$$

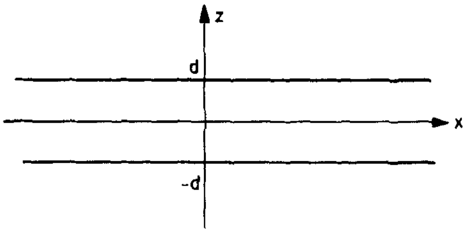

FIGURE 1

This equation is again invariant under phase change, and the conserved quantity is $Q(u) = \frac{1}{2}\langle Bu, u\rangle$, where $B = J^{-1}T'(0) = i^2 = -1$. Therefore

$$E(u) = \int \{\tfrac{1}{2}|u_x|^2 + F(x, |u|)\}\, dx, \qquad Q(u) = -\tfrac{1}{2}\int |u|^2\, dx,$$

where

$$F(x, |u|) = -\int_0^{|u|} g(x, |s|^2)s\, ds.$$

Standing wave solutions generated by the symmetry are solutions of the equation $E'(\phi) - \omega Q'(\phi) = 0$ or explicitly

$$-\phi_{xx} - g(x, |\phi|^2)\phi + \omega\phi = 0. \tag{6.26}$$

Akhmediev [1] has studied the bifurcation diagram for positive solutions of (6.26) (see Fig. 2). Curve AB corresponds to a solution which is positive and symmetric with respect to x. At $\omega = \omega_c$, a bifurcation occurs and BE corresponds to a symmetric solution while BCD is a double curve corresponding to two asymmetric solutions u_1 and u_2, where $u_1(x) = u_2(-x)$. The bifurcation diagram is not complete, i.e., it does not contain all possible solutions. For the complete diagram see [1]. The correspondence with Akhmediev's notation is as follows: $\omega \leftrightarrow n_x$, $1 \leftrightarrow k_0$, $\|u\|_{L^2} \leftrightarrow S$. The equations Akhmediev is considering are Maxwell's, not Schrödinger's, but the results are analogous.

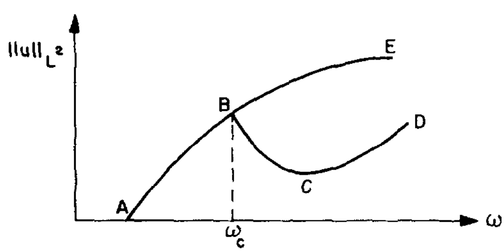

FIGURE 2

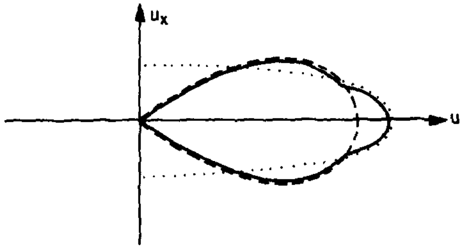

FIGURE 3

We give a brief outline of the qualitative properties of the standing wave solutions. Define $\psi = \phi_x$ and write (3) as an ODE:

$$\begin{aligned} \phi' &= \psi \\ \psi' &= -g(x, |\phi|^2)\phi + \omega\phi. \end{aligned} \tag{6.27}$$

If $|x| \leqslant d$, then $g(x, |\phi|^2) = \eta_0$ and (6.27) becomes

$$\phi' = \psi, \qquad \psi' = (\omega - \eta_0)\phi, \tag{6.27a}$$

which is a center if $\omega < \eta_0$.

If $|x| > d$, then $g(x, |\phi|^2) = \eta_1 + \alpha\, |\phi|^2$ and (4) becomes

$$\phi' = \psi, \qquad \psi' = (\omega - \eta_1)\phi - \alpha\phi^3. \tag{6.27b}$$

The quantity $H(\phi, \psi) = \frac{1}{2}\psi^2 - \frac{1}{2}(\omega - \eta_1)\phi^2 + (\alpha/4)\phi^4$ is conserved, so that if $\omega > \eta_1$ we have a homoclinic orbit. We are interested in the situation where $\eta_1 < \omega < \eta_0$. In this case we have to combine the center with the homoclinic orbit in order to produce solutions of (6.27). The following analysis is from [7].

Case (i). $\omega_c > \omega > \eta_1$. In this case we get only one solution, given by

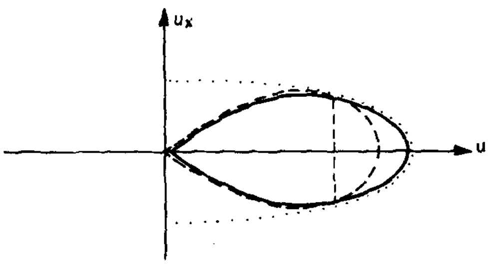

FIGURE 4

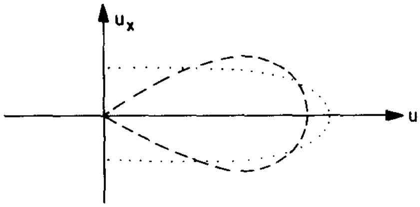

FIGURE 5

the solid line in Fig. 3. The solution $\phi(x)$ is symmetric with respect to x. The linearized operator is

$$R \equiv -\frac{d^2}{dx^2} - \frac{\partial g(x, |\phi|^2)}{\partial \phi} \phi - g(x, |\phi|^2) + \omega. \tag{6.28}$$

A comparison argument shows that R has exactly one negative eigenvalue and no zero eigenvalue.

Case (ii). $\omega = \omega_c$. At the critical ω the solution looks like Fig. 4. The solution is given by the solid line and in this case R has one negative and one zero eigenvalue.

Case (iii). $\eta_0 > \omega > \omega_0$. After we pass the ω_c we have the situation shown in Fig. 5 and we have the three solutions, one symmetric and two asymmetric, shown in Fig. 6.

Again a comparison argument shows that for the symmetric mode R has two strictly negative eigenvalues, while for the asymmetric modes R has only one negative eigenvalue.

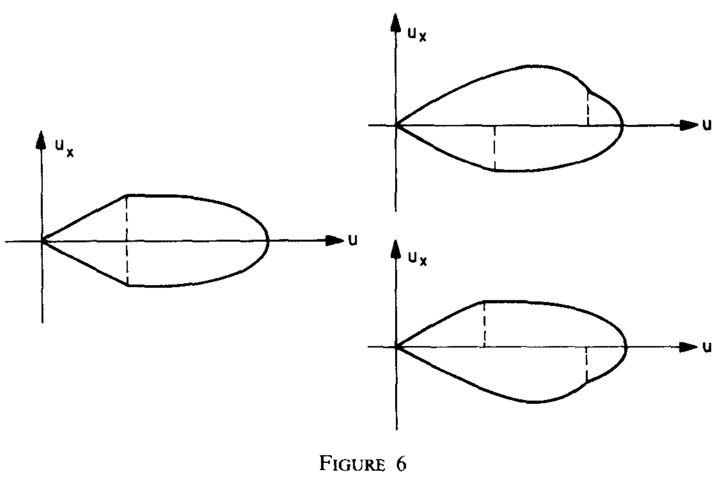

FIGURE 6

Therefore when we are on the branch ABCD of the bifurcation diagram the hypotheses of the theory are satisfied; consequently AB and CD are stable while BC is unstable. It can be proved by a different method that the branch BE is unstable [7].

E. *Generalized KdV Equation*

Consider

$$u_t + u_{xxx} - f(u)_x = 0 \qquad (x \in \mathbb{R}),$$

where f satisfies the same conditions as in Example A. Consider the classical solitary waves of the form $u = \phi_\omega(x - \omega t)$, where $0 < \omega < \infty$ and $\phi_\omega(\pm\infty) = 0$. The group action is translation, $X = H^1(\mathbb{R})$, E is the same functional as in Example 1, and $Q(u) = -\frac{1}{2}\int u^2\,dx$, but

$$J = \partial/\partial x.$$

Our theorem does not apply but almost does. The single difficulty is that J is not onto. But as in Section 4 we would still like to consider $J^{-1}y$, which here is a function which does not vanish at infinity. This difficulty can be surmounted only by use of the third invariant $I(u) = \int u\,dx$. We refer to [3] for details. The results in [3] were found after the present paper was essentially complete. For the case $f(u) = -u^p$, the conclusion is that the solitary wave is stable if $1 < p < 5$ and unstable if $5 \leqslant p < \infty$.

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