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Existence and Stability of Travelling Wave Solutions of Competition Models: A Degree Theoretic Approach

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0. INTRODUCTION

Reaction–diffusion systems of the form

$$\begin{aligned} u_t &= d_1 u_{xx} + f(u, v), \qquad u(0, x) = u_0(x), \qquad x \in \mathbb{R}^1, \\ v_t &= d_2 v_{xx} + g(u, v), \qquad v(0, x) = v_0(x) \end{aligned} \tag{1}$$

arise in various aspects of the sciences and engineering. In particular, they have been used to describe the evolution of interacting and diffusing species in mathematical ecology. This paper is concerned with competitive interactions; the purpose is to obtain large-amplitude, stable travelling wave solutions of (1) with arbitrary constant diffusion coefficients, and which are consistent with the principle of competitive exclusion. The existence and stability of large-amplitude solutions of systems is an important problem in nonlinear diffusion; see, for example, Fife [7]. We obtain such solutions for a robust class of nonlinear terms. This extends previous results of a similar nature which were proved with more restrictive hypotheses on f and g; see Gardner [11]. We also obtain the C^0 stability of such solutions.

The method of proof employs topological degree. We construct a homotopy from a given field (f, g) to a new field, $(\hat{f}, \hat{g})$ which is the gradient of a real-valued function. The results of [11] are then applied to obtain a travelling wave when the field is sufficiently near $(\hat{f}, \hat{g})$. Next, we prove an *a priori* comparison theorem. In particular, we show that the components of the solution of (1) can be wedged between translates of the components of the travelling wave (modulo an exponentially decaying term in t), provided that the initial data are wavelike, (for a precise definition, see Theorem 2.3), and that a travelling front with monotone components exists. To this end, we

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construct a comparison system using a technique developed by Fife and McLeod [8], for scalar equations. The comparison yields C^0-stability of travelling fronts. Moreover, it also provides bounds on the wave speed, θ, which, together with an additional estimate that isolates monotone fronts away from nonmonotone behavior of the travelling wave equations, provides (i) the continuity of θ as a function of the homotopy parameter, and (ii) enough compactness so that a topological degree can be defined and computed at each step of the homotopy. (The velocity θ is implicitly determined by the degree argument.)

In a forthcoming paper [5], a different existence proof will be given which uses the theory of isolated invariant sets and the generalized Morse index. This provides an interesting contrast to the degree theoretic approach used here. However, the implementation of degree theory in the present context[1] involves some novel features (cf. Section 4C), and it is therefore of independent interest.

1. Formulation of the Problem

We shall assume that the nonlinear term in (1) has the form: $F(U) = (uM(U), vN(U))$, where $U = (u, v)$ and $F = (f, g)$; M and N are the growth rates of u and v, respectively. To begin with, we require that

(i) M_v and N_u are negative in the positive quadrant, (competition);

(ii) there exists $K > 0$ such that M and N are negative if either $u \geqslant K$ or $v \geqslant K$ (resource limitation);

in addition, it will be assumed that F has the qualitative properties indicated in Fig. 1 (cf. Theorem 2.1 for a precise statement). There has recently been considerable interest in such equations; see, e.g., Fife and Tang [9], Conway and Smoller [5], Mimura and Kawasaki [13], Mimura and Namba [14], McGehee and Armstrong [15], Brown [1], and Gardner [9–11].

Solutions of (1) will be obtained which are functions of the single variable $\xi = x + \theta t$. If $' = d/d\xi$, then such a solution will satisfy the second-order systems of o.d.e.'s

$$DU'' - \theta U' + F(U) = 0, \tag{2}$$

[1] M. S. Mock [16], has recently applied degree theory to obtain the existence of viscous profiles approximating shock wave solutions of nonlinear conservation laws. However, he treated the codimension zero case (in which the velocity is determined by the states at $\pm\infty$), whereas we treat the codimension one case, in which the velocity is a free parameter. The techniques used here were developed independently of Mock.

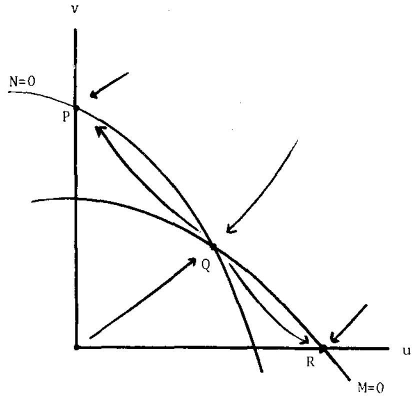

FIGURE 1

where $D = \mathrm{diag}(d_1, d_2)$. Equivalently, $X = (U, U')$ will satisfy the first-order system

$$\begin{aligned} U' &= W, \\ W' &= D^{-1}[\theta W - F(U)]. \end{aligned} \tag{3}$$

In addition, we require X to satisfy the boundary conditions

$$\lim_{\xi \to -\infty} X(\xi) = \tilde{P}, \qquad \lim_{\xi \to +\infty} X(\xi) = \tilde{R}, \tag{4}$$

where $\tilde{P} = (P, \mathbf{0})$, $\tilde{R} = (R, \mathbf{0})$, and P and R are critical points of F which lie on the positive v and u axes, respectively (cf. Fig. 1).

2. STATEMENT OF RESULTS

We first state the existence theorem.

THEOREM 2.1. *Suppose that the* (C^2) *vector field F satisfies* (i) *and* (ii), *and that*

(a) *The zero sets of M and N intersect exactly once in the positive quadrant at Q and are the graphs of monotone decreasing functions $v = k(u)$ and $u = l(v)$, respectively*;

(b) *dF at P, Q, and R (cf. Fig.* 1) *has real eigenvalues, so that P and R are stable nodes and Q is a saddle for the "reaction" flow*: $\dot{U} = F(U)$;

(c) $g_u(P)^2 < 4f_u(P)\, g_v(P), \qquad f_v(R)^2 < 4f_v(R)\, g_u(P).$

Then there exists a solution $X = (U, W)$ of (3), (4) *for which $W(\xi)$ lies in the fourth quadrant of* $\mathbb{R}^2$.

The monotonicity of k and l and hypothesis (c) can be relaxed (cf. [5]). Note that (c) always holds if $F = \nabla H$, since in this case $f_v(P) = g_u(R) = 0$.

We now introduce some notation needed in the statement of the comparison theorem. Let $\nabla f(P) = (a, 0)$ and $(c, d) = \nabla g(P)$. Then by (b) of Theorem 2.1, a and d are negative and c is nonpositive. There exists $Q^- = (Q_1^-, Q_2^-)$ where $Q_1^-, Q_2^- > 0$, such that $\delta^2 = (Q_1^-)^2 + (Q_2^-)^2$ and

$$\begin{aligned} -aQ_1^- &> 2\mu_-(\delta), \\ cQ_1^- - Q_2^- d &> 2\mu_-(\delta), \end{aligned} \qquad (5)_-$$

for some $\mu_-(\delta) > 0$; note that $\mu_-(\delta)$ can be chosen equal to $\mu_-(1)\delta$. Now replace P with R in the above, and define $Q^+ = (Q_1^+, Q_2^+)$ and $\mu_+(1)$ in a similar manner. Let Σ_- and Σ_+ be the rectangles containing P and R respectively, defined by

$$\begin{aligned} \Sigma_+ &= \{(u, v): r - Q_1^+ \leqslant u \leqslant r + Q_1^+, 0 \leqslant v \leqslant Q_2^+\}, \\ \Sigma_- &= \{(u, v): p - Q_2^- \leqslant v \leqslant p + Q_2^-, 0 \leqslant u \leqslant Q_1^-\}. \end{aligned}$$

Finally, suppose that $\delta = \delta_- > 0$ is chosen so that for all U with $|U - P| < 2\delta_-$ we have that

$$\begin{aligned} \nabla f(U) \cdot (-Q_1^-, Q_2^-) &> \mu_-(1)\,\delta_-, \\ \nabla g(U) \cdot (Q_1^-, -Q_2^-) &> \mu_-(1)\,\delta_-. \end{aligned} \qquad (6)_-$$

Similarly, choose $\delta = \delta_+ > 0$ such that for all U with $|U - R| < 2\delta_+$ we have that

$$\begin{aligned} \nabla f(U) \cdot (-Q_1^+, Q_2^+) &> \mu_+(1)\,\delta_+, \\ \nabla g(U) \cdot (Q_1^+, -Q_2^+) &> \mu_+(1)\,\delta_+. \end{aligned} \qquad (6)_+$$

(This is possible since the left-hand sides of $(6)_\pm$ are those of $(5)_\pm$ plus quadratic terms in $\delta_\pm$, whereas the right-hand sides of $(6)_\pm$ are linear in $\delta_\pm$.) Finally, let $\mu = \min(\mu_-(1)\,\delta_-, \mu_+(1)\,\delta_+)$.

THEOREM 2.2. *Suppose that F is as in Fig.* 1 *and that there exists a travelling wave solution $\hat{U}(\xi) = (\hat{u}(\xi), \hat{v}(\xi))$ of* (2), (4) *with monotone components. Let $\Sigma_\pm$ and $\mu > 0$ be chosen as above and suppose that there exists $H > 0$ such that*

$$\begin{aligned} U_0(x) &\in \Sigma_-, & x &< -H, \\ U_0(x) &\in \Sigma_+, & x &> H, \end{aligned}$$

where $U_0(x) = (u_0(x), v_0(x))$ *is the (nonnegative) initial data of* (1). *Then there exist constants* $\xi_1 < \xi_2$ *and* $A, B, \rho > 0$ *such that*

$$\hat{u}(x + \theta t + \xi_1) - Q_1^- e^{-\rho t} < u(x, t) < \hat{u}(x + \theta t + \xi_2) + Ae^{-\rho t},$$

$$\hat{v}(x + \theta t + \xi_2) - Q_2^- e^{-\rho t} < v(x, t) < \hat{v}(x + \theta t + \xi_1) + Be^{-\rho t};$$

(A, B, ξ_1, *and* ξ_2 *are defined in* (18) *of Section* 4); *here* $(u(x, t), v(x, t))$ *is the solution of* (1), *and* $\rho > 0$ *is chosen so small that* $-2Q_i^{\pm}\rho + \mu \geqslant 0$, $i = 1, 2$.

COROLLARY 2.3. *With notation and hypotheses as in Theorem* 2.2, $\hat{U}(\xi)$ *is* C^0*-stable. More precisely, given* $\varepsilon > 0$ *there exist* $(Q_1^{\pm}, Q_2^{\pm})$ *and* $\delta > 0$ *with* $(Q_1^{\pm})^2 + (Q_2^{\pm})^2 = \delta^2$ *such that if*

$$\hat{u}(\xi) - Q_1^- < u_0(\xi) < \hat{u}(\xi) + Q_1^+,$$

$$\hat{v}(\xi) - Q_2^- < v_0(\xi) < \hat{v}(\xi) + Q_2^+,$$

then $\xi_2 - \xi_1 < \varepsilon$ *and the constants* A *and* B *in Theorem* 2.2 *can be chosen equal to* Q_1^+ *and* Q_2^+, *respectively.*

3. PROOF OF THEOREM 1

We divide the proof up into several parts; the preliminaries are contained in A, the a priori estimates in B, and the computation of the degree in C.

A. Preliminaries

We state the following theorems for the convenience of the reader; their proofs are contained in [11].

THEOREM A. *Let* $\hat{F}$ *be as in Theorem* 1, *and in addition, suppose that* $\nabla H = \hat{F}$, D *is scalar, and that* $H(P) = H(R)$, *where* H *is a real-valued function. Then there exists a solution* $\bar{U}$ *of* (2), (4) *with monotone components and with* $\theta = 0$.

THEOREM B. *Let* $\hat{F} = \nabla H$ *be as in Theorem A, and let* L *be the operator on* $L^2(\mathbb{R}^1)^2$ *defined by*

$$L\Sigma = \Sigma'' + d\hat{F}_{\bar{U}}, \qquad \Sigma \in H^2(\mathbb{R}^1)^2,$$

where $\bar{U}(\xi)$ *is the solution of Theorem A, and* $d\hat{F}_{\bar{U}(\xi)}$ *is* $d\hat{F}$ *evaluated at* $\bar{U}(\xi)$. *Then the spectrum of* L *lies on the nonpositive axis; moreover, zero is in the point spectrum of* L *and is a simple eigenvalue. For small* ε, *there exists a solution of* (2), (4) *for the perturbed system with nonlinear term* $F = \nabla H + \varepsilon F_1$. *This solution is locally unique, modulo translations.*

Now let $F(U)$ be the vector field in the statement of Theorem 1. We also assume that $F(U)$ is analytic; this hypothesis will be removed later. There exists a homotopy $F(U, \varepsilon)$, $0 \leqslant \varepsilon \leqslant 1$, such that

(i) $F(U, 0) = \nabla H$, $H(P) = H(R)$,

(ii) $F(U, 1) = F(U)$,

(iii) $F(P, \varepsilon) = F(R, \varepsilon) = 0$, $0 \leqslant \varepsilon \leqslant 1$,

(iv) $F(U, \varepsilon)$ satisfies (a)–(c) in the statement of Theorem 1 for each ε, $0 \leqslant \varepsilon \leqslant 1$,

(v) $F(U, \varepsilon)$ is analytic in U, $0 \leqslant \varepsilon \leqslant 1$, if $F(U)$ is analytic.

This construction is sketched in the Appendix.

We now consider the family of equations

$$D_\epsilon U'' - \theta U' + F(U, \varepsilon) = 0, \tag{2$_\epsilon$}$$

$$\left.\begin{aligned} U' &= W', \\ D_\epsilon W' &= \theta V - F(U, \varepsilon), \end{aligned}\right\} \tag{3$_\epsilon$}$$

together with the boundary conditions (4); here, $D_\epsilon = \varepsilon D + (1 - \varepsilon)I$. In Section B we show that all solutions of $(2)_\epsilon$, (4) with monotone components lie in a compact set of an appropriate function space independent of ε. In Section C, we show that $(2)_\epsilon$, (4) can be transformed to an operator equation of the form: identity + compact, locally, near solutions of $(2)_\epsilon$, (4). The analyticity of F together with the compactness of the set of solutions implies that this is a finite set (modulo translations), so that the indices of the solutions of the transformed equations are defined. We define the degree to be the sum of these indices; we then show that this "degree" is independent of ε, and is nonzero when $\varepsilon = 1$.

B. A priori estimates

The proof of Theorem 2.2 is deferred until Section 4; we shall assume that it is valid for the remainder of this section. Note that an immediate consequence of Theorem 2.2 is that the system (1) cannot admit two travelling wave solutions with different velocities. Now consider the equations

$$u_t = d_1(\varepsilon)\, u_{xx} + f(u, 0, \varepsilon), \tag{8a$_\epsilon$}$$

$$v_t = d_2(\varepsilon)\, v_{xx} + g(0, v, \varepsilon). \tag{8b$_\epsilon$}$$

Note that under the hypotheses of Theorem 2.1 these are equations of Fisher type, provided that we restrict u and v have values in the intervals $[0, r]$ and $[0, p]$, respectively. In particular, $(8a)_\epsilon$ and $(8b)_\epsilon$ each admit a continuum of wave solutions connecting 0 at $-\infty$ to r at $+\infty$, and p at $-\infty$ to 0 at $+\infty$, respectively, which travel with velocities $\theta \geqslant \theta_u$ for some $\theta_u > 0$ for Eq. $(8a)_\epsilon$ and $\theta \leqslant -\theta_v$ for some $\theta_v > 0$, for Eq. $(8b)_\epsilon$. Let $\tilde{u}(\xi)$ and $\tilde{v}(\xi)$ be travellling wave solutions of $(8a)_\epsilon$ and $(8b)_\epsilon$ with velocities θ_u and $-\theta_v$, respectively. (Hence, $\tilde{u}(-\infty) = \tilde{v}(+\infty) = 0$, and $\tilde{u}(+\infty) = r$ and $\tilde{v}(-\infty) = p$.)

LEMMA 3.1. *Suppose that there exists a solution of* $(2)_\epsilon$, (4) *with monotone components and velocity* θ. *Then*

$$|\theta| \leqslant \theta_{\max} = \max_{0 \leqslant \epsilon \leqslant 1} (\theta_u, \theta_v).$$

Proof. Let $U(x,t) = (u,v)(x,t)$ be the solution of $(1)_\epsilon$ with initial data $(\tilde{u}(x), \tilde{v}(x))$, and let $(w(\xi), z(\xi))$ be the wave solution of $(1)_\epsilon$. By Theorem 2.2, there exist ρ, $Q_1 > 0$ and $\xi_0 \in \mathbb{R}$ such that

$$u(x,t) \geqslant w(x + \theta t + \xi_0) - Q_1^- e^{-\rho t}.$$

Since $f_v < 0$, we also have that $\tilde{u}(x + \theta_u t) \geqslant u(x,t)$ by a standard comparison theorem for scalar equations. Thus for fixed $\xi = x + \theta_u t$, we have that

$$r > \tilde{u}(\xi) > w(\xi + (\theta - \theta_u)t + \xi_0) - Q_1^- e^{-\rho t}.$$

If $\theta > \theta_u$, the right-hand side of the above inequality tends to r as t approaches infinity, yielding a contradiction. Thus $\theta \leqslant \theta_u$. Similarly, $-\theta_v \leqslant \theta$. Finally, θ_u and θ_v are bounded independently of ε; see, for example [4, Corollary 2.11], ∎

LEMMA 3.2. *Let* $U(\xi)$ *be a solution of* $(2)_\epsilon$, (4) *with monotone components. Then there exists* $\gamma > 0$ *independent of* ε *and* $U(\xi)$ *such that*

$$\begin{aligned} |U(\xi) - P| &< Ce^{\gamma\xi}, \qquad \xi \leqslant 0, \\ |U(\xi) - R| &< Ce^{-\gamma\xi}, \qquad \xi \geqslant 0, \end{aligned} \tag{9}$$

for some $C > 0$ (*possibly depending on* $U(\xi)$ *and* ε).

Proof. We linearize $(3)_\epsilon$ about $\tilde{P}$ and $\tilde{R}$; the characteristic polynomial, $P_\theta(\lambda)$, is

$$\begin{aligned} P_\theta(\lambda) = \lambda^4 - (r+s)\,\theta\lambda^3 + [\theta^2 rs + (ar + ds)]\,\lambda^2 - (a+d)\,ts\theta\lambda \\ + (ad - bc)\,rs = 0, \end{aligned}$$

where $r = d_1^{-1}(\varepsilon)$, $s = d_2^{-1}(\varepsilon)$, $[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}] = d_U F$ at (P, ε) or (R, ε) and θ is the velocity of U. We rewrite this as

$$(\lambda^2 - \mu_1)(\lambda^2 - \mu_2) + \theta^2 rs\lambda^2 + [-(r+s)\,\theta\lambda^3 - (a+d)\,rs\theta\lambda] = 0, \quad (10)$$

where μ_1 and μ_2 are eigenvalues of $-D_\epsilon^{-1} d_U F$ at (P, ε) or (R, ε). Since all the entries of this matrix are nonnegative, μ_1 and μ_2 are positive. From (10), it is immediately evident that the characteristic polynomial never has a pure imaginary root for any θ. Thus the real parts of the roots P_θ are bounded away from zero for each θ. Since they also depend continuously on θ, we can bound the real parts of the roots of $P_\theta(\lambda)$ away from zero uniformly for $|\theta| \leqslant \theta_{\max}$. Thus, we obtain $C_1 > 0$, $\gamma_1 > 0$ such that $|U(\xi) - P| < C_1(1 + |\xi|) \exp(\gamma_1 \xi)$ for $\xi \leqslant 0$ and $|U(\xi) - R| < C_1(1 + |\xi|) \exp(-\gamma_1 \xi)$ for $\xi \geqslant 0$. The estimate (9) follows for any $0 < \gamma < \gamma_1$. ∎

We remark that when D is the identity, the roots of $P_\theta(\lambda)$ are $\theta/2 \pm \sqrt{\theta^2/4 - \mu}$, where $\mu < 0$ is an eigenvalue of $d_U F$ at (P, ε) or (R, ε). Hence P_θ has two negative and two positive roots. Thus by the above, we see that P_θ has two eigenvalues with positive real part and two eigenvalues with negative real part for any D. It is also evident when D is the identity that the decay rate γ tends to zero as $|\theta|$ tends to infinity.

The next several lemmas show that the constant C in (9) can also be chosen independently of ε and U provided that we first mod out translations. To this end, let L_ϵ be the straight line in the U-plane through Q_ϵ and the origin, and let $\mathscr{S}_\epsilon$ be the collection of solutions U of $(2)_\epsilon$, (4) with monotone components and with $U(0) \in L_\epsilon$. If $X_0 \in \mathbb{R}^4$, let $X_0 \cdot \xi$ be the flow of $(3)_\epsilon$ with $X_0 \cdot 0 = X_0$, and define

$$\chi_\epsilon = \{(U(0), U'(0)) \colon U \in \mathscr{S}_\epsilon\}. \quad (11)$$

We shall denote the various (open) quadrants of $\mathbb{R}^2$ by quad(I), quad(II), quad(III), and quad(IV).

LEMMA 3.3. *Suppose that* $\theta \geqslant 0$. *Then there does not exist an orbit* $X = (U, W)$ *of* $(3)_\epsilon$ *which connects* $\tilde{P}$ *at* $\xi = -\infty$ *to* $\tilde{Q}_\epsilon$ *at* $\xi = +\infty$ *and for which the components of* W *lie in* quad(II) *for all large* $\xi > 0$ *or in* quad(IV) *for all large* $\xi > 0$. *Similarly, if* $\theta < 0$ *there does not exist a connection from* Q_ϵ *at* $\xi = -\infty$ *to* $\tilde{R}$ *at* $\xi = +\infty$ *with* $W(\xi)$ *in* quad(II) *for large* $\xi < 0$ *or with* $W(\xi) \in$ quad(IV) *for large* $\xi < 0$. *These assertions also hold if we replace* $\tilde{Q}_\epsilon$ *in the above with the origin.*

Proof. We suppose that $\theta > 0$ and that $W(\xi) \in$ quad(IV) for large positive ξ, where $(U(\xi), W(\xi))$ connects $\tilde{P}$ at $-\infty$ to $\tilde{Q}_\epsilon$ at $+\infty$. The proofs of the remaining assertions when $\theta \neq 0$ are similar. We linearize $(3)_\epsilon$ about $\tilde{Q}_\epsilon$ and obtain the characteristic polynomial P_θ as before. Since all the entries of

$-D_\epsilon^{-1}\,dF$ at Q_ϵ are positive, μ_1 and μ_2 in (10) will be real; however, they will now have opposite signs. Since $\theta \neq 0$ it follows that P_θ cannot have a purely imaginary root (cf. (10)). We now fix θ and vary the coefficients of D_ϵ until D_ϵ is the identity. In this case, the roots of P_θ are precisely

$$\theta/2 \pm \sqrt{\theta^2/4 - \mu_+}, \qquad \theta/2 \pm \sqrt{\theta^2/4 - \mu_-},$$

where $\mu_+ > 0 > \mu_-$ are the eigenvalues of $d_U F$ at $(Q_\epsilon, \varepsilon)$. Hence for any D_ϵ, P_θ has exactly one negative root and three roots with positive real part. Since $\tilde{Q}_\epsilon$ has precisely one stable direction, the orbit (U, W) must approach $\tilde{Q}_\epsilon$ tangent to this direction for large $\xi > 0$. However, the cone

$$\{(U, W) \colon U = Q_\epsilon + V_1, \text{ where } V_1 \in \overline{\text{quad(I)}} \text{ and } W \in \overline{\text{quad(III)}}\}$$

is negatively invariant, as is the set obtained by interchanging $\overline{\text{quad(I)}}$ and $\overline{\text{quad(III)}}$ in the above. This implies that each of these regions contains an invariant ray of $(3)_\epsilon$. Hence the stable direction at $\tilde{Q}_\epsilon$ has the "wrong" monotonicity, i.e., the first two components have the same monotonicity near $\tilde{Q}_\epsilon$, in contradiction to our hypotheses.

If $\theta = 0$, then the linearization of $(3)_\epsilon$ about $\tilde{Q}_\epsilon$ has two pure imaginary eigenvalues plus one positive and one negative eigenvalue. Once again, the connecting orbit must approach $\tilde{Q}_\epsilon$ tangent to the stable direction, so that the argument of the preceding paragraph still applies.

The (easier) argument when $\tilde{Q}_\epsilon$ is replaced by the origin in $\mathbb{R}^4$ is similar, and will be omitted. ∎

We can now obtain the following lemmas.

LEMMA 3.4. *Suppose that* $\mathscr{S}_\epsilon \neq \varnothing$ *and let* D *and* E *be small neighborhoods of* $\tilde{P}$ *and* $\tilde{R}$, *respectively. Then there exists* $\hat{\xi} > 0$ *such that* $\chi_\epsilon \cdot \hat{\xi} \subseteq E$ *and* $\chi_\epsilon \cdot (-\hat{\xi}) \subseteq D$, *where* χ_ϵ *is as in* (11). (*Here,* $\hat{\xi}$ *may depend on* ε.)

LEMMA 3.5. *Let* $\Gamma = \{\varepsilon \in [0, 1] \colon \mathscr{S}_\epsilon \neq \varnothing\}$. *Then* Γ *is closed and the constant* $\hat{\xi}$ *obtained in Lemma* 3.4 *can be chosen independently of* ε *for all* $\varepsilon \in \Gamma$.

The proofs of Lemmas 3.4 and 3.5 are routine. For brevity's sake, we only sketch the main idea, which is essentially to show that the sets χ_ϵ are uniformly bounded away from $\tilde{Q}_\epsilon$ and the origin. We therefore assume that this is not the case. Noting that the components of elements of $\mathscr{S}_\epsilon$ are monotone, that θ is unique (in the case of Lemma 3.4), and that θ is bounded independently of ε (in the case of Lemma 3.5), we obtain orbits $X_- \cdot \xi$ and $X_+ \cdot \xi$ for some (possibly limiting) value of θ which connect $\tilde{P}$ at $-\infty$ to $\tilde{A}$

at $+\infty$ and $\tilde{A}$ at $-\infty$ to $\tilde{R}$ at $+\infty$ (where $\tilde{A}$ is either $\tilde{Q}_\epsilon$ or the origin), respectively. This violates at least one of the assertions of Lemma 3.3

We remark that another consequence of the uniqueness of θ and Lemmas 3.1 and 3.3 is that θ is a continuous function of ε for $\varepsilon \in \Gamma$. This is proved by an argument similar to that given in the previous paragraph. Since Γ is closed, θ can be extended to a continuous function of ε defined on $[0, 1]$. We choose one such extension if $\Gamma \neq [0, 1]$.

As a consequence of the last two lemmas, we have the following result.

LEMMA 3.6. *The constant C in* (9) *can be chosen independently of $U \in \mathscr{S}_\epsilon$ and of $\varepsilon \in \Gamma$.*

Proof. We choose the neighborhoods D and E of Lemma 3.4 so small that the Hartman linearization theorem holds for each of the systems $(3)_\epsilon$, $0 \leqslant \varepsilon \leqslant 1$, in these neighborhoods. Then for every $X_- \in \partial D$ which lies in the unstable manifold of $\tilde{P}$, there exists $C_1 > 0$ and $\gamma > 0$ such that $|X_- \cdot \xi - \tilde{P}| < C_1 e^{\gamma\xi}$ for $\xi \leqslant 0$, where C_{1_1} and γ are independent of such X_- and of ε. (These are standard theorems in ordinary differential equations; see, for example, Coddington and Levinson, [3, Chap. 4].) Now for any $X_0 \in \chi_\epsilon$ there exists $C_2 > 0$ independent of ε and X_0 such that $|X_0 \cdot \xi - \tilde{P}| < C_2$ for $0 \geqslant \xi \geqslant -\hat{\xi}$. If we let

$$C = e^{\gamma\hat{\xi}} \max(C_1, C_2), \tag{12}$$

the first inequality in (9) holds for all $U \in \mathscr{S}_\epsilon$ and for all $\varepsilon \in \Gamma$, since $X_0 \cdot \xi \in D$ for any $\xi < -\hat{\xi}$, by Lemma 3.5. A similar argument yields the second inequality in (9). ∎

The above estimates were proved by restricting the discussion to the class of solutions of $(2)_\epsilon$, (4) with monotone components. There remains the possibility of the existence of solutions with nonmonotone components. We avoid an explicit discussion of the existence or nonexistence of nonmonotone solutions by obtaining certain estimates of the derivatives of such solutions when they are travelling in the "wrong" direction. Indeed, the central idea of the following lemma is to isolate the possible nonmonotone behavior of the components of the solution away from the rest points in the problem, so that any nonmonotone behavior must be large-amplitude in nature.

LEMMA 3.7. *Suppose that U is a solution of* $(2)_\epsilon$, (4) *with nonmonotone components, and with values in A_δ, where $A_\delta = \{U: U \in \overline{\text{quad(I)}}$ or $|U - P| < \delta$ or $|U - R| < \delta\}$. Then for sufficiently small δ, U behaves in the manner indicated in Fig. 2. More precisely,* (1) $U \in \text{quad(I)}$ *for ξ near $\pm\infty$, and $U'(\xi) \in \text{quad(IV)}$ for ξ near $\pm\infty$;* (2) $U'(\xi) \in \text{quad(IV)}$ *for $\xi \leqslant \xi_0$, where ξ_0 is the smallest ξ for which $U(\xi) \in B_l$ (cf. Fig.* 2), (3) *both*

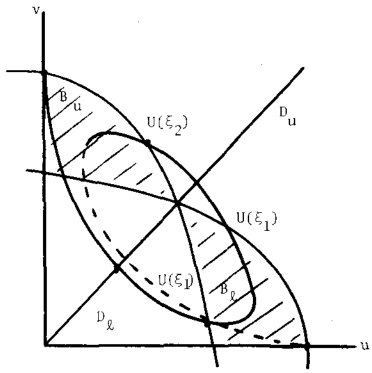

FIGURE 2

components "turn around" while $U \in B_l$, so that $U'(\xi) \in$ quad(II) for some $\xi = \xi_1 > \xi_0$ for which $U(\xi_1) \in B_l$; (4) $U'(\xi) \in$ quad(II) for $\xi_1 \leqslant \xi \leqslant \xi_2$, where ξ_2 is the smallest $\xi \geqslant \xi_1$ for which $U(\xi) \in B_u$.

Proof. The regions B_u and B_l are the two shaded regions in Fig. 2; the regions D_u and D_l consist of those points which lie above and below $(B_u \cup B_l)^{\text{int}}$, respectively. These regions are all closed.

(1) If $U(\xi) \notin \overline{\text{quad(I)}}$ for some ξ near $-\infty$ then there is a finite value of $\xi = \xi^*$ for which $u(\xi)$ has a negative minimum at $\xi = \xi^*$. But for such ξ^*, $d_1(\varepsilon)\, u''(\xi^*) = -uM(U) < 0$, yielding a contradiction. A similar argument can be applied for ξ near $+\infty$.

(2) Since $(U(\xi), U'(\xi))$ approaches $\tilde{P}$ as $\xi \to -\infty$ along the unstable manifold of $\tilde{P}$, we see that $U'(\xi)$ must approach zero tangent so some ray in either the first or fourth quadrant. In the former case, $U(\xi) \in D_u$ for such ξ. However, the set

$$\{(U, U') \colon U \in D_u,\ U' \in \overline{\text{quad(I)}}\}$$

is positively invariant, yielding a contradiction. Thus $U'(\xi)$ must lie in (the interior of) quad(IV) for ξ near $-\infty$. (A similar proof applies when ξ is near $+\infty$.)

(a) We next show that $U'(\xi) \in$ quad(I) until $U(\xi)$ enters B_l. (By (1), this is the case for all ξ near $-\infty$.) (a) Suppose that $u'(\xi^*) = 0$ while $U(\xi^*) \in B_u \cup D_u$, and that ξ^* is the smallest such ξ. Then $v'(\xi^*) \leqslant 0$. Clearly $U(\xi)$ cannot lie in the interior of $B_u \cup D_u$, since in this case $d_1(\varepsilon)\, u''(\xi^*) = -uM(U) > 0$ so that $u(\xi^*)$ is a local minimum. Thus $U(\xi^*)$ lies in $M = 0$, so that $u''(\xi^*) = 0$; hence, $d_1(\varepsilon)\, u'''(\xi^*) = -uM_v(U)\, v'$ at ξ^*. If $v'(\xi^*) < 0$ we see that $u' < 0$ for ξ slightly less than ξ^*, yielding a contradiction. Thus $v'(\xi^*) = 0$. Then either $v''(\xi^*) = 0$ (in which case

$N(U(\xi^*))=0$, and the orbit is identically equal to $\tilde{Q}_\epsilon$), or $v''(\xi^*)\neq 0$. In the latter case, the orbit lies in

$$S_{\pi,\sigma}=\{(U,U')\colon U\in D_\pi,\ U'\in\overline{\mathrm{quad}(\sigma)}\}.$$

$(\pi,\sigma)=(u,\mathrm{I})$ or $(\pi,\sigma)=(l,\mathrm{III})$; since these sets are positively invariant (relative to the positive U-quadrant) we again obtain a contradiction. (b) Suppose that $v'(\xi^*)=0$ while $U(\xi^*)\in B_u\cup D_l$; the argument is similar to that of (a). (c) Suppose that $u'(\xi^*)=0$ while $U(\xi)\in D_l$. In this case, $u(\xi^*)$ may be maximum; however, the orbit must then enter the set $S_{l,\mathrm{III}}$, yielding a contradiction. (d) The case in which $v'(\xi^*)=0$ while $U(\xi^*)\in D_u$ is similar to (c).

(3) Now suppose that $U(\xi_1)\in\partial B_l$ and $U'(\xi_1)\in\mathrm{quad(IV)}$. By hypothesis, at least one of the components of U is nonmonotone; suppose that $v'(\xi^*)=0$ while $U(\xi^*)\in B_l$. Then $v(\xi^*)$ is a local minimum, and for ξ slightly larger than ξ^* we have that $U'(\xi)\in\mathrm{quad(I)}$. Note that v' cannot change sign again while $U(\xi)$ lies above $N=0$ (and hence, while u' remains positive). If u' (and hence v') doesn't change sign for $\xi\geqslant\xi^*$, the solution eventually enters $S_{u,\mathrm{I}}$, yielding a contradiction. Thus u' must change sign, and it must do so before U enters D_u. Thus the solution "turns around" while $U\in B_l$.

(4) The proof is similar to that of (2). ∎

Now let $\mathcal{O}_\epsilon$ be the collection of nonmonotone solutions of $(2)_\epsilon$, (4) for which zero is the smallest value of ξ for which $U(\xi)\in L_\epsilon$ and $U'(\xi)\in\mathrm{quad(II)}$.

LEMMA 3.8. *There exists $\delta>0$ such that $|U'(0)|>\delta$ for all $U\in\mathcal{O}_\epsilon$ and all ε, $0\leqslant\varepsilon\leqslant 1$.*

Again, we only sketch the proof. Suppose that $\theta\geqslant 0$. If $U\in\mathcal{O}_\epsilon$, let ξ_U be the (unique) value of $\xi<0$ for which $U(\xi)\in L_\epsilon$ with $U'(\xi)\in\mathrm{quad(IV)}$; (cf. Fig. 2). It follows that there exists $\delta_1>0$ such that $|U'(\xi_U)|>\delta_1$ for all $U\in\mathcal{O}_\epsilon$ and for all ε, $0\leqslant\varepsilon\leqslant 1$. Otherwise we would obtain a connection (U,U') between $\tilde{P}$ and either $\tilde{Q}_\epsilon$ or the origin with $U'\in\mathrm{quad(IV)}$, in contradiction to Lemma 3.3. If Lemma 3.8 was false, we could in a similar manner obtain a connection (U,U') from $\tilde{P}$ to $\tilde{Q}_\epsilon$ or the origin with $U'(\xi)\in\mathrm{quad(I)}$ for large ξ, again in contradiction to Lemma 3.3.

If $\theta\leqslant 0$, the proof is similar.

C. Existence and Computation of the Degree

We now rewrite $(2)_\epsilon$, (4) as an operator equation. Let $\mathscr{W}=H^2(\mathbb{R}^1)^2$ and $\mathscr{Y}=L^2(\mathbb{R}^1)^2$. By Theorem A there exists a solution $\bar{U}$ of $(2)_0$, (4) with

$\bar{U}(0) \in L_0$ and with monotone components. Let $\mathscr{F}: \mathscr{W} \times [0, 1] \to \mathscr{Y}$ be defined by

$$\mathscr{F}(W, \varepsilon) = D_\varepsilon (W + \bar{U})'' - \theta(\varepsilon)(W + \bar{U})' + F(\bar{U} + W, \varepsilon),$$

where $\theta(\varepsilon)$ is the extension of θ from Γ to $[0, 1]$ chosen earlier. Then $\mathscr{F}(W, \varepsilon) = 0$ if and only if $W + \bar{U}$ is a solution of $(2)_\varepsilon$, (4). (The boundary conditions follow since elements of $H^2(\mathbb{R}^1)$ have C^1 decay at $\pm\infty$.) Let $\mathscr{W}_0$ be the subspace of $\mathscr{W}$ which consists of elements W with $W(0) \in L_0$, and let $\mathscr{F}_0$ be the restriction of $\mathscr{F}$ to $\mathscr{W}_0 \times [0, 1]$. If $\mathscr{C}$ is the maximal subcontinuum of solutions of $\mathscr{F}_0(W, \varepsilon) = 0$ in $\mathscr{W}_0 \times [0, 1]$ which contains the origin, then Lemma 3.8 of Section B implies that if $(W, \varepsilon) \in \mathscr{C}$, then $\bar{U} + W$ has monotone components and hence, by Lemma 3.6, $\mathscr{C}$ is a compact subset of $\mathscr{W}_0 \times [0, 1]$. (The estimate (9) holds for solutions $U(\xi)$ of $(2)_\varepsilon$, (4) for which $U(0) \in L_\varepsilon$. Clearly, for such solutions, $U(\xi) \in L_0$ for some $|\xi| \leqslant \hat{\xi}$, provided that L_0 does not intersect the projections onto the U-plane of the neighborhoods D and E of Lemma 3.4. Hence Lemma 3.6 holds for solutions U with $U(0) \in L_0$, provided that we replace C in (12) by $e^{\gamma\hat{\xi}} C$.)

We now remark that the analyticity of $F(\cdot, \varepsilon)$ implies that the projection $\mathscr{C}(\varepsilon)$ of $\mathscr{C}$ on $\mathscr{W}_0$ at ε is a finite set. This follows from the fact that the stable manifold, $\mathscr{M}_s$, of $(3)_\varepsilon$ at $\tilde{R}$ and the unstable manifold, $\mathscr{M}_u$, of $(3)_\varepsilon$ at $\tilde{P}$ are both two dimensional and analytic. If these manifolds intersect at a point they coincide along a complete connecting orbit, so that if there exist infinitely many distinct connection orbits for some ε, $\mathscr{M}_s$ and $\mathscr{M}_u$ coincide in an open set, and hence, identically. However, it is easily seen that at least one orbit in each of these manifolds must be unbounded, yielding a contradiction.

Let $\lambda = (\bar{W}, \varepsilon)$ and $U_\lambda = \bar{U} + \bar{W}$; consider the mapping $\mathscr{R}$ defined by

$$\mathscr{R}(W, \theta, \lambda) = D_\varepsilon (U_\lambda + W)'' - (\theta(\varepsilon) + \theta)(W + U_\lambda)' + F(U_\lambda + W, \varepsilon).$$

Note that $\mathscr{R}(W, \theta, \lambda) = 0$ if and only if $U_\lambda + W$ is a solution of $(2)_\varepsilon$, (4). By adding and subtracting $F(U_\lambda, \varepsilon)$ to the above expression, it can be seen that $\mathscr{R}$ can be rewritten as

$$\mathscr{R}(W, \theta, \lambda) = D_\varepsilon W'' - (\theta(\varepsilon) + \theta)\, W' + A_\lambda W + g_\lambda(W) - \theta U'_\lambda + \gamma(\lambda),$$

where $A_\lambda = d_U F$ evaluated at (U_λ, ε),

$$\| g_\lambda(W) \|_{\mathscr{Y}} \leqslant c \, \| W \|^2_{\mathscr{Y}},$$

and $\gamma = \gamma(\lambda) \in \mathscr{Y}$; more precisely,

$$\gamma = D_\varepsilon U''_\lambda - \theta(\varepsilon)\, U'_\lambda + F(U_\lambda, \varepsilon).$$

Note that if $\lambda \in \mathscr{C}$ then $\gamma \equiv 0$.

We now find a change of variables which transforms $\mathscr{R}$ into a compact perturbation of the indentity (with respect to W). Let

$$N(W;\theta,\lambda)=D_\epsilon W''-(\theta(\varepsilon)+\theta)\,W'+\bar{A}_\epsilon W+g_\lambda(W),$$

where

$$\bar{A}_\epsilon(\xi)=A_+H(\xi)+A_-(1-H(\xi)),$$

$H(\xi)$ is the Heaviside function, A_+ is d_UF evaluated at (R,ε), and A_- is d_UF at (P,ε). Let $S_{\theta,\lambda}$ be the linear part of $N(\cdot;\theta,\lambda)$. By hypothesis (c) of Theorem 2.1, it follows that there exists $\mu<0$ such that

$$\sup\{W^t\bar{A}_\epsilon W\colon W\in\mathbb{R}^2,\ |W|=1\}<\mu,$$

so that the Lax–Milgram lemma can be invoked to see that $S_{\theta,\lambda}\colon\mathscr{W}\to\mathscr{Y}$ is invertible. Since g_λ is $o(\|W\|^2_{\mathscr{Y}})$ near zero, it follows from the contraction mapping theorem that $N(\cdot;\theta,\lambda)$ is a homeomorphism from a neighborhood C_λ of the origin in $\mathscr{W}$ to a neighborhood D_λ of zero in $\mathscr{Y}$, with $N(0;\theta,\lambda)=0$.

Now let $\mathscr{Y}_\lambda$ be the orthogonal complement of U'_λ in $\mathscr{Y}$ and let $\mathscr{W}_\lambda=S^{-1}_{0,\lambda}\mathscr{Y}_\lambda$. It follows that $U'_\lambda\notin\mathscr{W}_\lambda$; otherwise, we would have that

$$0>\mu>(U'_\lambda,\bar{A}_\epsilon U'_\lambda)-\|U''_\lambda\|^2_{L^2}=(U'_\lambda,S_{0,\lambda}U'_\lambda);$$

if $U'_\lambda\in\mathscr{W}_\lambda$, this would yield a contradiction. Note also that U'_λ, and hence $\mathscr{Y}_\lambda$ varies continuously with λ. Hence we can define a λ-coordinate system on $\mathscr{Y}$ by letting

$$\theta_\lambda=\theta_\lambda(Y)=(-U'_\lambda,Y)_{\mathscr{Y}_\lambda},$$
$$Y_\lambda=Y_\lambda(Y)=Y+\theta_\lambda(Y)\,U'_\lambda\in\mathscr{Y}.$$

Now consider the mapping $\mathscr{G}_\lambda\colon D_\lambda\to\mathscr{Y}$ defined by

$$\mathscr{G}_\lambda(Y)=\mathscr{R}(N^{-1}(Y_\lambda;\theta_\lambda,\lambda),\theta_\lambda,\lambda)=Y+K_\lambda(Y),\tag{13}$$

where $Y_\lambda=Y_\lambda(Y)$, $\theta_\lambda=\theta_\lambda(Y), N^{-1}(W;\theta_\lambda,\lambda)$ is the inverse of N with respect to W, and

$$K_\lambda(Y)=(A_\lambda-\bar{A}_\epsilon)\,N^{-1}(Y_\lambda;\theta_\lambda,\lambda).$$

Note that K_λ is a compact operator on $\mathscr{Y}$.

The task at hand is to define a "degree" for the solutions of $(2)_\epsilon$, (4). The problem is that the operator $\mathscr{G}_\lambda$ is only defined locally; however, a meaningful definition can be given as follows. Let $\mathscr{C}(\varepsilon)=\{W_1,\dots,W_{N(\epsilon)}\}$, let

$\lambda_i = (W_i, \varepsilon)$ and put $\mathscr{G}_i = \mathscr{G}_{\lambda_i}$. Choose small open balls $D_i \subset D_{\lambda_i}$ centered at zero such that

$$(\{W_i\} + S_{0,\lambda_i}^{-1}(\bar{D}_i)) \cap \mathscr{C}(\varepsilon) = \{W_i\}.$$

We claim that $\mathscr{G}_i$ has no zeros on ∂D_i. If $\mathscr{G}_i(Y) = 0$, then $\theta_{\lambda_i}(Y) = 0$, since if this was not the case, $U_{\lambda_i} + W$ would be a solution of $(2)_\varepsilon$, (4) with velocity $\theta(\varepsilon) + \theta_{\lambda_i}(Y) \neq \theta(\varepsilon)$, where $W = N^{-1}(Y; \theta_{\lambda_i}(Y), \lambda_i)$. This is a contradiction, since the velocity, $\theta(\varepsilon)$ of the solution of $(2)_\varepsilon$, (4) is unique. Now suppose that $\mathscr{G}_i(Y) = 0$ with $Y \in \partial D_i \cap \mathscr{Y}_{\lambda_i}$. We obtain a contradiction as follows. The formulations of $\mathscr{R}(W, 0, \lambda_i) = 0$ for small $\| W\|_{\mathscr{W}}$ are precisely those W of the form

$$W = U_{\lambda_i}(\xi + \tau) - U_{\lambda_i}(\xi) \equiv l_i(\tau).$$

The curve $l_i(\tau)$ is tangent to U'_{λ_i} at $\tau = 0$. Since $U'_{\lambda_i} \notin \mathscr{W}_{\lambda_i} = S_{0,\lambda_i}^{-1}(\mathscr{Y}_{\lambda_i})$, it follows that $l_i(\tau)$ is transverse to $\mathscr{W}_{\lambda_i}$ at $\tau = 0$. Now, $N^{-1}(\mathscr{Y}_{\lambda_i}; 0, \lambda_i)$ is a manifold in $\mathscr{W}$ of codimension one which is tangent to $\mathscr{W}_{\lambda_i}$ at the origin. It follows that $l_i(\tau)$ only intersects $N^{-1}(\mathscr{Y}_{\lambda_i}; 0, \lambda_i)$ at the origin. Thus if $W = N^{-1}(Y; 0, \lambda_i)$ with $Y \in \partial D_i$ then $\mathscr{G}_i(Y) = \mathscr{R}(W, 0, \lambda_i) \neq 0$.

We define a degree, $\mathscr{D}(\varepsilon)$, by letting

$$\begin{aligned} \mathscr{D}(\varepsilon) &= \sum_{i=1}^{N(\varepsilon)} \deg(\mathscr{G}_i, D_i, 0), \qquad &\text{if } \mathscr{C}(\varepsilon) \neq \varnothing, \\ &= 0, &\text{if } \mathscr{C}(\varepsilon) = \varnothing. \end{aligned}$$

By the remarks in the preceding paragraph, $\deg(\mathscr{G}_i, D_i, 0)$ is defined, so that the above definition makes sense. It will now be shown that

(i) $\mathscr{D}(\varepsilon)$ is an integer valued, continuous function of ε (and hence constant).

(ii) $\mathscr{D}(0) > 0$.

We prove (i) first. If $\mathscr{C}(\varepsilon_0) = \varnothing$, then the compactness of $\mathscr{C}$ implies that this is the case for all ε near ε_0. (It is here and below that we make essential use of the compactness of $\mathscr{C}$.) Suppose next that $\mathscr{C}(\varepsilon_0) \neq \varnothing$. Without loss of generality, we may assume that $\mathscr{C}(\varepsilon_0) = \{W^0\}$. Let C_0 be a ball of small, fixed radius about zero in $\mathscr{W}$ and let $D_0 = S_{0,\lambda_0} C_0$, where $\lambda_0 = (W^0, \varepsilon_0)$. Again by the compactness of $\mathscr{C}$ we have that $\mathscr{C}(\varepsilon) \subset \{W^0\} + C_0$ for ε sufficiently near ε_0. Suppose first that $\mathscr{C}(\varepsilon) = \{W_1, ..., W_N\} \neq \varnothing$. Let $\gamma_n = (W_n, \varepsilon_n)$, where

$$\begin{aligned} \varepsilon_\eta &= (1-\eta)\,\varepsilon_0 + \eta\varepsilon, && 0 \leqslant \eta \leqslant 1, \\ &= \varepsilon, && 1 \leqslant \eta \leqslant 2; \\ W_\eta &= W^0, && 0 \leqslant \eta \leqslant 1, \\ &= (2-\eta)\,W^0 + (\eta-1)\,W_j, && 1 \leqslant \eta \leqslant 2. \end{aligned}$$

Now consider the homotopy $H(Y, \eta) = \mathscr{G}_{\gamma_\eta}(Y)$. If ε is sufficiently near ε_0 it follows that $0 \notin H(\partial D_0, \eta)$ for $0 \leqslant \eta \leqslant 2$. Thus

$$\mathscr{D}(\varepsilon_0) = \deg(\mathscr{G}_{\gamma_0}, D_0, 0) = \deg(\mathscr{G}_{\gamma_1}, D_0, 0). \tag{14}$$

Now for each $\eta \in [1, 2]$, the zeros of $\mathscr{G}_{\gamma_\eta}$ in D_0 correspond to the zeros of $\mathscr{R}(W; \theta, \gamma_\eta)$ and hence to solutions of $(2)_\varepsilon$, (4). Thus the $N(\varepsilon)$ zeros of $\mathscr{G}_{\gamma_\eta}$ remain distinct throughout this part of the homotopy, so that there exist balls $D_j(\eta) \subset D_0$ which are pairwise disjoint for each η, each of which contains a unique zero of $\mathscr{G}_{\gamma_\eta}$ in its interior. It follows that for each j,

$$\begin{aligned} \deg(\mathscr{G}_{\gamma_1}, D_j(1), 0) &= \deg(\mathscr{G}_{\gamma_2}, D_j(2), 0), \\ &= \deg(\mathscr{G}_{\lambda_j}, D_j(2), 0), \end{aligned}$$

where $\lambda_j = (W_j, \varepsilon)$. Hence from (14) and the above we have that

$$\begin{aligned} \mathscr{D}(\varepsilon_0) &= \sum_{j=1}^{N(\epsilon)} \deg(\mathscr{G}_{\gamma_1}, D_j(1), 0) \\ &= \sum_{j=1}^{N(\epsilon)} \deg(\mathscr{G}_{\lambda_j}, D_j(2), 0) \\ &= \mathscr{D}(\varepsilon). \end{aligned}$$

Thus if $\mathscr{C}(\varepsilon) \neq \varnothing$ for ε near ε_0 we see that (1) holds. Finally, if $\mathscr{C}(\varepsilon) = \varnothing$ for ε near ε_0, then it follows from (14) that $D(\varepsilon_0) = 0$.

We now prove (ii). Let $\mathscr{C}(0) = \{0, W_1, \ldots, W_{N(0)}\}$. We compute $\deg(\mathscr{G}_0, D, 0)$, where D is a small neighborhood of $\{0\}$; the computation of the remaining indices is similar. Since $\theta(0) = 0$, it follows that

$$d\mathscr{G}_0(0)Y = Y + (A_0 - \bar{A}_0)\, S_{0,0}^{-1} Y_0 = Y + CY,$$

where C is a compact linear operator. The claim is that the spectrum of $I + C$ is contained in the positive half-line; Let $W = S_{0,0}^{-1} Y$, where $Y + CY = \mu Y$ and $Y_0 = Y_0(Y)$. Then

$$W'' + dF_{\bar{U}} W_0 - \mu S_{0,0} W_0 + (\mu - 1)\, \theta_0(Y)\, \bar{U}' = 0,$$

where $dF_{\bar{U}}$ is the differential of F at $(\bar{U}, 0)$. Since the left-hand side is symmetric, if follows that μ and W are real valued. We see that $\mu > 0$ as follows. First, it is easily seen that $\mu \neq 0$. If this were the case, we would have that $W_0 = \theta_0(Y) = 0$, since $(\bar{U}')^{\perp} =$ range $(\partial_\xi^2 + dF_{\bar{U}}0)$ by the symmetry of $F_{\bar{U}}$. (Recall that $\bar{U}' \notin S_{0,0}^{-1}(\bar{U}')^{\perp}$.) If $\mu < 0$, it can be seen that $\theta_0(Y) = 0$ by multiplying the above equation by $\bar{U}'$ and integrating. Now multiply the above equation by W_0 and integrate to obtain

$$\int_{-\infty}^{\infty} (-\dot{W}_0^2 + W_0^t\, dF_{\bar{U}} W_0)\, dx - \mu(W_0, S_{0,0} W_0)_{L^2} = 0.$$

Since W_0 is not a multiple of $\bar{U}'$, it follows from Theorem B of Section 3A that the first integral is negative. However, $S_{0,0}$ is also a symmetric, negative operator. If $\mu < 0$ and $W_0 \neq 0$, we obtain the desired contradiction.

If F is only C^2 we can approximate F by analytic expressions which also satisfy the hypotheses of Theorem 2.1, that is

$$F(U) = \lim_{\varepsilon \downarrow 0} F(U, \varepsilon)$$

uniformly in U, where $F(U, \varepsilon)$ is analytic in U for each ε. Since there exists a solution of $(2)_\varepsilon$, (4) for each $\varepsilon > 0$ we use the fact that the continuum $\mathscr{C}$ is compact (which was proved independently of the analyticity of $F(U, \varepsilon)$) to include that a solution of $(2)_0$, (4) exists. ∎

4. Proofs of Theorem 2.2 and Corollary 2.3

(A) We begin with a proof of Theorem 2.2. We adapt a technique used by Fife and McLeod [8] to study the stability of monotone travelling wave solutions of scalar equations with values in the interval [0, 1]. In addition to extending this technique to systems, we also weaken the restrictions on the initial data of the p.d.e.'s. In particular, we allow arbitrary nonnegative data on compact x-intervals; (since Fife and McLeod are concerned with problems arising in population genetics, they restrict the values of the data to the interval [0, 1]).

We first change (x, t) in (1) to the new variables (ξ, t) where $\xi = x + \theta t$; again denoting the solution by (u, v), we obtain

$$\begin{aligned} u_t &= d_1 u_{\xi\xi} - \theta u_\xi + f(u, v), \qquad & u(\xi, 0) &= u_0(\xi), \\ v_t &= d_2 v_{\xi\xi} - \theta v_\xi + g(u, v), \qquad & v(\xi, 0) &= v_0(\xi). \end{aligned} \tag{15}$$

Now let $\hat{U}(\xi) = (\hat{u}(\xi), \hat{v}(\xi))$ be the travelling wave solution with monotone components; $\hat{U}$ is a stationary solution of (15). A comparison system is constructed as follows. Let

$$\gamma(\xi, t) = \hat{u}(\zeta) + Q_1(\zeta)\, q(t) + \lambda\phi(\zeta)\, q(t) - u(\xi, t),$$
$$\delta(\xi, t) = v(\xi, t) - (\hat{v}(\zeta) - Q_2(\zeta)\, q(t)),$$

here, $\zeta = \xi + s(t)$ (where $s(t)$ is to be determined), $q(t) = \exp(-\rho t)$, $\phi(\zeta) = \exp(-\zeta^2)$, $\lambda > 0$ is constant, and $Q_i(\zeta)$ is a smooth monotone function of ζ, $i = 1, 2$, which, for some $K > 0$ satisfies

$$Q_i(\zeta) = Q_i^-, \qquad \zeta \leqslant -K, \qquad i = 1, 2$$
$$Q_i(\zeta) = Q_i^+, \qquad \zeta \geqslant K, \qquad i = 1, 2.$$

The constants $Q_i^{\pm}$, $i = 1, 2$, and ρ are as in the statement of Theorem 2.2. Let $L_i = \partial_t - d_i \partial_\xi^2 + \theta \partial_\xi$. The variables (γ, δ) then satisfy the system

$$\begin{aligned} L_1 \gamma &= s'\hat{u}' + [d_1 \lambda(-\phi'' + \theta\phi') + \lambda s'\phi' - \rho\lambda\phi - d_1 Q_1'' + \theta Q_1']q \\ &\quad + Q_1 q' + f(\hat{u}, \hat{v}) - f(u, v) \equiv N_1(\xi, t, \gamma, \delta) \\ L_2 \delta &= -s'\hat{v}' + [-d_2 Q_2'' + \theta Q_2']q + Q_2 q' \\ &\quad + g(u, v) - g(\hat{u}, \hat{v}) \equiv N_2(\xi, t, \gamma, \delta), \end{aligned} \tag{16}$$

together with the initial data

$$\gamma(\xi, 0) = \hat{u}(\xi + s(0)) + Q_1(\xi + s(0)) + \lambda\phi(\xi + s(0)) - u_0(\xi),$$
$$\delta(\xi, 0) = v_0(\xi) - (\hat{v}(\xi + s(0)) - Q_2(\xi + s(0)).$$

Here, $\hat{u}, \hat{v}, \hat{u}', \hat{v}'$, , and Q_i are evaluated at ζ, and u and v are evaluated at (ξ, t). We claim that for suitably chosen $s(t)$ and λ, $\{\gamma \geqslant 0,\ \delta \geqslant 0\}$ is invariant under the flow of (16).

First we note that since $U_0(\xi)$ is nonnegative and lies in $\Sigma_\pm$ for $|\xi| \geqslant H$ (cf. Theorem 2.2), it follows that if $s(0)$ is chosen sufficiently large and positive, then there exists $H_1 > 0$ depending on H, K, and $s(0)$ such that $\gamma(\xi, 0) \geqslant 0$ and $\delta(\xi, 0) \geqslant 0$ for $|\xi| \geqslant H_1$. For large $s(0)$, it follows from the nonnegativity of $v_0(\xi)$ and behavior of $v_0(\zeta)$ for $|\xi| \geqslant H$, that $\delta(\xi, 0) \geqslant 0$ for all ξ. Next, choose λ (depending on H_1) and $s(0)$) so large that $\gamma(\xi, 0) \geqslant 0$ for $|\xi| \leqslant H_1$.

The invariance of $\{\gamma \geqslant 0,\ \delta \geqslant 0\}$ will follow from [2, Theorem 4.1] if it can be shown that $N_1(\xi, t, 0, \delta) \geqslant 0$ whenever $\delta \geqslant 0$ and that $N_2(\xi, t, \gamma, 0) \geqslant 0$ whenever $\gamma \geqslant 0$. By (i) of Section 1, we have that $f_v(u, v) \leqslant 0$ and $g_u(u, v) \leqslant 0$ whenever (u, v) lies in the positive quadrant; an inspection of (16) shows that the above condition on N_1 and N_2 is implied by $N_i(\xi, t, 0, 0) \geqslant 0$, $i = 1, 2$; i.e.,

$$\begin{aligned} N_1(\xi, t, 0, \delta) \geqslant {} & s'\hat{u}' + [\lambda d_1(-\phi'' + \theta\phi') + \lambda s'\phi' - \lambda\rho\phi - d_1 Q_1'' + \theta Q_1']q \\ & + Q_1 q' + f(\hat{u}, \hat{v}) - f(\hat{u} + \lambda\phi q + Q_1 q, \hat{v} - Q_2 q) \\ N_2(\xi, t, \gamma, 0) \geqslant {} & -s'\hat{v}' + [-d_2 Q_2'' + \theta Q_2']q + Q_2 q' \\ & + g(\hat{u} + \lambda\phi q + Q_1 q, \hat{v} - Q_2 q) - g(\hat{u}, \hat{v}). \end{aligned} \tag{17}$$

Since Q_i' vanish for $|\zeta| \geqslant K$ and ϕ, ϕ', and ϕ'' vanish at $\zeta = \pm\infty$, there exists $H_2 > 0$ such that $|\zeta| \geqslant H_2$ implies that

$$B_1(\zeta)\, q(t) + Q_1(\zeta)\, q'(t) > 2Q_1(\zeta)\, q' = -2\rho Q_1(\zeta)\, q(t),$$
$$B_2(\zeta)\, q(t) + Q_2(\zeta)\, q'(t) > 2Q_2(\zeta)\, q' = -2\rho Q_2(\zeta)\, q(t),$$

where B_1 and B_2 are the terms in brackets in (17). Now suppose that $s' > 0$ so that $s'u' > 0$ and $-s'v' > 0$; for $|\zeta| > H_2$ we have that

$$N_1(\xi, t, 0, 0) \geqslant 2Q_1 q' + \nabla f(U^*) \cdot (-Q_1 q - \lambda\phi q, Q_2 q),$$
$$N_2(\xi, t, 0, 0) \geqslant 2Q_2 q' + \nabla g(U^{**}) \cdot (Q_1 q + \lambda\phi q, -Q_2 q),$$

where U^* and U^{**} lie on the line segment which connects $\hat{U}(\zeta)$ to $\hat{U}_1 = U(\zeta) + (Q_1(\zeta)\, q(t) + \lambda\phi(\zeta)\, q(t),\ Q_2(\zeta)\, q(t))$. Let $\delta = \min(\delta_-, \delta_+)$, where $\delta_\pm$ are as in $(6)_\pm$ of Section 2. Since $(\hat{U}, \hat{U}')$ satisfies (4), there exists $H_3 > 0$ such that $|\zeta| > H_3$ implies that $\hat{U}(\zeta)$ lies within a distance of δ of P or R. There also exists $H_4 \geqslant H_3$ such that $\hat{U}_1(\zeta)$ lies within a distance of 2δ from P or R for $|\zeta| \geqslant H_4$; (this estimate is clearly independent of t). Let H^* be the maximum of H, H_1, H_2, H_3, and H_4. Thus for $|\zeta| \geqslant H^*$ it follows from $(6)_\pm$ of Section 2 that

$$N_1(\xi, t, 0, 0) \geqslant 2Q_1 q' + \mu q = (-2Q_1^\pm \rho + \mu) q \geqslant 0,$$
$$N_2(\xi, t, 0, 0) \geqslant 2Q_2 q' + \mu q = (-2Q_2^\pm \rho + \mu) q \geqslant 0,$$

since ρ was chosen so small that $-2Q_i^\pm \rho + \mu \geqslant 0$, $i = 1, 2$. Thus the desired inequalities hold whenever $|\zeta| \geqslant H^*$.

The remaining case, $|\xi| \leqslant H^*$, is handled as follows. There exists $\kappa > 0$ such that

$$N_1(\xi, t, 0, 0) \geqslant s'(t)\, \hat{u}'(\zeta) - \kappa q(t),$$
$$N_2(\xi, t, 0, 0) \geqslant -s'(t)\, \hat{v}'(\zeta) - \kappa q(t);$$

let

$$\beta = \min\{\min\{\hat{u}'(\zeta), -\hat{v}'(\zeta)\} : |\zeta| \leqslant H^*\} > 0.$$

Now let $s(t)$ be such that $s'(t)\beta - \kappa q(t) = 0$; i.e.,

$$s(t) = s(0) + (\kappa/\beta\rho)(1 - q(t)).$$

It follows with the above choice of $s(t)$ that the desired differential inequalities hold when $|\zeta| \leqslant H^*$. Hence $\gamma(\xi, t) \geqslant 0$ and $\delta(\xi, t) \geqslant 0$ for all ξ and for all $t \geqslant 0$; i.e.,

$$u(x, t) = u(\xi, t) \leqslant \hat{u}(x + \theta t + \xi_2) + Ae^{-\rho t},$$
$$\hat{v}(x + \theta t + \xi_2) - Q_2 e^{-\rho t} \leqslant v(\xi, t) = v(x, t),$$

where

$$\begin{aligned} Q_2 &= \max\{Q_2(\zeta): -\infty < \zeta < \infty\}, \\ A &= \max\{Q_1(\zeta) + \lambda\phi(\zeta): -\infty < \zeta < \infty\}, \\ \xi_2 &= s(\infty) = s(0) + \kappa/\beta\rho;\ (s(0) \text{ is large and positive}). \end{aligned} \tag{18}$$

The constants Q_1, ξ_1, B in Theorem 2.2 are defined in an analogous manner, only now $s(0)$ is large and negative. The remaining two inequalities are proved by a similar argument.

(B) Corollary 2.3 is proved as follows. Choose $Q_i(\zeta)$ so that $Q_i(\zeta)$ lies between Q_i^- and Q_i^+ for all ζ, $i = 1, 2$. Choose $U_0(\xi)$ so that

$$\hat{u}(\xi) - Q_1(\xi) \leqslant u_0(\xi) \leqslant \hat{u}(\xi) + Q_1(\xi),$$
$$\hat{v}(\xi) - Q_2(\xi) \leqslant v_0(\xi) \leqslant \hat{v}(\xi) + Q_2(\xi);$$

Then the constants $s(0)$ and λ can both be chosen equal to zero. Hence, the constant κ in (18) is proportional to δ; this is immediate from (17). It follows that $\xi_2 - \xi_1$ is proportional to δ. This completes the proof. ∎

Appendix

We construct $F(U, \varepsilon)$ in such a manner that $F(U, \varepsilon)$ satisfies the hypotheses of Theorem 2.1 for each ε, and such that $F(U, \varepsilon)$ is analytic in U for each ε, provided that $F(U)$ is analytic.

Let $F(U, 3) = F(U)$, where $F(U)$ is as in Theorem 2.1. Deform $M_3 = M$ and $N_3 = N$ through families $M_\varepsilon, N_\varepsilon$, $2 \leqslant \varepsilon \leqslant 3$, where the zero the sets of M_ε and N_ε are given by functions $v = k_\varepsilon(u)$ and $u = l_\varepsilon(v)$, respectively. This can be done so that M_2 and N_2 are as indicated in Fig. 3b. More precisely, $k_2(u)$ is of order $\sqrt{r - u}$ as $u \uparrow r$ and $l_2(v)$ is of order $\sqrt{p - v}$ as $v \uparrow p$. If M_3 and N_3 are analytic, then M_ε and N_ε can be chosen to be analytic for each ε.

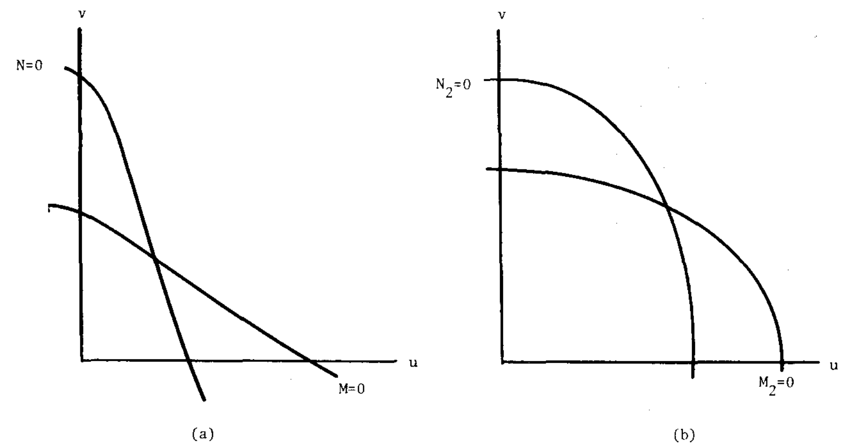

FIGURE 3

Moreover, the homotopy can be performed in such a manner that k_ϵ and l_ϵ are monotone for each ε. Since we are continuing toward a system (when $\varepsilon = 2$) for which $f_v(R_2) = g_u(P_2) = 0$, the homotopy can be performed in such a manner that hypothesis (c) of Theorem 2.1 holds for each ε.

Next, for $1 \leqslant \varepsilon \leqslant 2$ consider the field

$$F(U, \varepsilon) = (\varepsilon - 1) F(U, 2) + (2 - \varepsilon)(u[-v^2 + k_2(u)^2], v[-u^2 + l_2(v)^2]).$$

Note that for each ε, $1 \leqslant \varepsilon \leqslant 2$, the zero sets of $F(U, \varepsilon)$ coincide with those of $F(U, 2)$, and hence, (a) of Theorem 2.1 holds for each ε. Since $k_2(u)$ and $l_2(v)$ are of order $\sqrt{r-u}$ and $\sqrt{p-v}$, it follows that k_2^2 and l_2^2 are of order $r-u$ and $p-v$ as $u \uparrow r$ and $v \uparrow p$. It then follows that $f_u(W)$ and $g_v(W)$ are strictly negative and $f_v(W) = g_u(W) = 0$ for $W = P$ or $W = R$. Hence, conditions (b) and (c) of Theorem 2.1 hold for each $\varepsilon \in [1, 2]$. It also follows that $k_2(u)^2$ and $l_2(v)^2$ (and hence $F(U, \varepsilon)$) are analytic for each such ε, provided that $F(U, 2)$ is analytic. Note that $F(U, 1) = \nabla H_1$ where

$$H_1 = -u^2 v2/2 + \int_0^u s k_2^2(s) d + \int_0^v t l_2^2(t)\, dt.$$

Hence $H_1(P) - H_1(R) = \int_0^p s k_2^2(s)\, ds - \int_0^r t l_2^2(t)\, dt$. We construct a final homotopy from $F(U, 1)$ to $F(U, 0)$, by continuing $k_2(u)$ function $k_1(u)$ through a family (of analytic) $k_{1+\epsilon}(u)$, $0 \leqslant \varepsilon \leqslant 1$, so that (a)–(c) hold for the field

$$F(U, \varepsilon) = (u(-v^2 + k_\epsilon^2(u)), v(-u^2 + l_2^2(v))).$$

If $k_1(0)$ is sufficiently small or large, we have $H_0(P) - H_0(R)$ is alternatively positive and negative. We can therefore $k_1(u)$ in such a manner that $H_0(P) - H_0(R) = 0$.

Finally, rescale ε so that $0 \leqslant \varepsilon \leqslant 1$.

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