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COMPUTATION AND STABILITY OF TRAVELING WAVES IN SECOND ORDER EVOLUTION EQUATIONS*

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Abstract. The topic of this paper is nonlinear traveling waves occuring in a system of damped wave equations in one space variable. We extend the freezing method from first to second order equations in time. When applied to a Cauchy problem, this method generates a co-moving frame in which the solution becomes stationary. In addition, it generates an algebraic variable which converges to the speed of the wave, provided the original wave satisfies certain spectral conditions and initial perturbations are sufficiently small. We develop a rigorous theory for this effect by recourse to some recent nonlinear stability results for waves in first order hyperbolic systems. Numerical computations illustrate the theory for examples of Nagumo and FitzHugh–Nagumo type.

Key words. systems of damped wave equations, traveling waves, nonlinear stability, freezing method, second order evolution equations, point spectra, essential spectra

AMS subject classifications. Primary, 65P40, 35L52, 47A25; Secondary, 35B35, 35P30, 37C80

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1. Introduction.

In this paper we study the numerical computation and stability of traveling waves in second order evolution equations. Our model system is a nonlinear wave equation in one space dimension

$$Mu_{tt} = Au_{xx} + f(u, u_x, u_t),\ x \in \mathbb{R},\ t \geqslant 0, u(x,t) \in \mathbb{R}^m \tag{1.1}$$

with matrices $A, M \in \mathbb{R}^{m,m}$ and a smooth nonlinearity $f : \mathbb{R}^{3m} \to \mathbb{R}^m$. We also require M to be invertible and $M^{-1}A$ to be real diagonalizable with positive eigenvalues (positive diagonalizable for short). This ensures that the principal part of (1.1) is well-posed.

Our main concern is traveling wave solutions $u_\star : \mathbb{R} \times [0,\infty) \to \mathbb{R}^m$ of (1.1), i.e.,

$$u_\star(x,t) = v_\star(x - \mu_\star t),\ x \in \mathbb{R},\ t \geqslant 0, \tag{1.2}$$

such that

$$\lim_{\xi\to\pm\infty} v_\star(\xi) = v_\pm \in \mathbb{R}^m \quad \text{and} \quad f(v_\pm, 0, 0) = 0. \tag{1.3}$$

Here $v_\star : \mathbb{R} \to \mathbb{R}^m$ is a nonconstant function and denotes the profile (or pattern) of the wave, $\mu_\star \in \mathbb{R}$ its translational velocity, and $v_\pm$ its asymptotic states. The quantities $v_\star$ and $\mu_\star$ are generally unknown, and explicit formulas are available only for very specific equations. As usual, a traveling wave $u_\star$ is called a traveling pulse if $v_+ = v_-$ and a traveling front if $v_+ \neq v_-$.

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An important subcase of the system (1.1) is the *nonlinear telegraph equation*

$$Mu_{tt} + B(u)u_t = Au_{xx} + g(u), \tag{1.4}$$

where $B(u) \in \mathbb{R}^{m,m}$ plays the role of a damping matrix and $g : \mathbb{R}^m \to \mathbb{R}^m$ is a smooth nonlinearity. Equations of this type arise from various mathematical models. We mention [10], where these models are derived for transmission lines with small inductance and nonlinear shunt conductance as well as for the movement of cell populations. Another source is diffusion laws of Maxwell–Cattaneo type (rather than Fickian law) when applied to chemical reactions or to population dispersal (see, e.g., [25], and see [22] for a discussion of biological relevance). Explicit formulas for traveling waves of the scalar telegraph equation with linear damping and bistable nonlinearity have been derived by several authors. We refer inter alia to [14, 19], [38, Ex. 4.4] and to further references given in [25]. The existence of traveling waves for a nonlinear telegraph equation with a quasi-linear diffusion term is shown in [15].

An important issue in all these results is stability of the traveling wave. Global stability is proved for the scalar bistable case with linear damping in the papers [13, 14], while local stability for the scalar nonlinear case (g bistable and $B(u) = 1 - \tau g'(u)$ for some $\tau > 0$) is shown in the recent paper [25].

Our aims in this paper are twofold. First, we generalize the method of freezing solutions of the Cauchy problem associated with (1.1), from first order to second order equations in time (cf. [4, 7]). Second, we investigate local stability with asymptotic phase for traveling waves of the system (1.1). We prove a general stability theorem and show that under its assumptions the freezing equation inherits the traveling wave and its velocity as a (Lyapunov-)stable equilibrium. This is taken as the theoretical basis for the success of the general method.

To be more specific, the freezing approach replaces the function values $u(x,t)$ by two new unknowns $\gamma(t) \in \mathbb{R}$ and $v(\xi,t) \in \mathbb{R}^m$ via the ansatz

$$u(x,t) = v(\xi,t), \quad \xi := x - \gamma(t), \quad x \in \mathbb{R},\, t \geqslant 0. \tag{1.5}$$

Inserting this into (1.1) leads to the equation (see section 2 for details)

$$Mv_{tt} = (A - \gamma_t^2 M)v_{\xi\xi} + 2\gamma_t M v_{\xi t} + \gamma_{tt} M v_\xi + f(v, v_\xi, v_t - \gamma_t v_\xi). \tag{1.6}$$

As written, (1.6) is not yet a well-posed system for the unknowns v and γ. In section 2 we explain how to turn it into a well-posed system by adding a scalar phase condition. The resulting partial differential algebraic equation (PDAE) is then solved numerically. We also discuss and test two alternative phase conditions: the fixed and the orthogonal phase condition. The general idea of the approach is to split wave-like solutions of evolution equations into a spatial profile that moves as little as possible and into time-dependent algebraic variables that indicate the motion of the profile. The methodology has proved to be useful for quite a variety of wave solutions, including several space dimensions; see [4, 5].

We present two applications of the method in section 2 and defer the proof of well-posedness of the freezing approach to section 4 and the appendix. Our first example is the nonlinear telegraph equation with linear damping and bistable nonlinearity (called the Nagumo wave equation), and the second example derives from the FitzHugh–Nagumo model when extended by a small inductance as in [10]. For both equations we test the method on solutions obtained by extending a well-known relation of traveling waves for the hyperbolic system (1.1) to those of a parabolic system; cf. [14, 19] and section 2.2. Moreover, we compare solutions of different index formulations of the PDAE obtained by differentiating the constraint equation.

Our second main subject is nonlinear stability of traveling waves. For parabolic equations there are quite a few results (e.g., [21, 23, 36]) showing that nonlinear stability with asymptotic phase follows from linearized stability under suitable conditions. Here, linearized stability means that the spectrum of the differential operator obtained from linearizing about the traveling wave lies strictly in the left half plane except for a simple eigenvalue at zero. There are many fewer results of this type for first order evolution equations of hyperbolic and hyperbolic-parabolic type (see [33, 34]) or for second order equations like (1.4), see [25].

As can be seen from (1.6), a traveling wave (1.2) of (1.1) leads to a steady state $v = v_\star$ of the so-called co-moving frame equation

$$Mv_{tt} = (A - \mu_\star^2 M)v_{\xi\xi} + 2\mu_\star M v_{\xi t} + f(v, v_\xi, v_t - \mu_\star v_\xi),\ \xi \in \mathbb{R},\ t \geqslant 0, \tag{1.7}$$

that is,

$$0 = (A - \mu_\star^2 M)v_{\star,\xi\xi}(\xi) + f(v_\star(\xi), v_{\star,\xi}(\xi), -\mu_\star v_{\star,\xi}(\xi)),\ \xi \in \mathbb{R}. \tag{1.8}$$

Linearizing (1.7) about the steady state $v_\star$ yields the differential operator

$$\begin{aligned} \mathcal{P}(\partial_t, \partial_\xi)v = {} & Mv_{tt} - (A - \mu_\star^2 M)v_{\xi\xi} - 2\mu_\star M v_{\xi t} \\ & + (\mu_\star D_3 f(\star) - D_2 f(\star))v_\xi - D_3 f(\star)v_t - D_1 f(\star)v, \end{aligned} \tag{1.9}$$

where arguments are abbreviated by $(\star) = (v_\star, v_{\star,\xi}, -\mu_\star v_{\star,\xi})$. Solving $\mathcal{P}(\partial_t, \partial_\xi)v = 0$ via separation of variables $v(\xi, t) = e^{\lambda t} w(\xi)$ (or Laplace transform) leads to the equation $\mathcal{P}(\lambda, \partial_\xi)w = 0$ for the operator polynomial

$$\begin{aligned} \mathcal{P}(\lambda, \partial_\xi) = {} & \lambda^2 M + \lambda\left(-D_3 f(\star) - 2\mu_\star M \partial_\xi\right) - (A - \mu_\star^2 M)\partial_\xi^2 \\ & + (\mu_\star D_3 f(\star) - D_2 f(\star))\partial_\xi - D_1 f(\star). \end{aligned} \tag{1.10}$$

In section 3 we collect some general facts about the spectrum of this operator polynomial and we provide further details for the numerical examples from section 2. As usual, $\mathcal{P}$ has the eigenvalue zero with associated eigenfunction $v_{\star,\xi}$ due to shift equivariance. An important part of the essential spectrum is determined by the dispersion relation which is the constant coefficient quadratic eigenvalue problem obtained by taking the limit $\xi \to \infty$ in the coefficients of $\mathcal{P}(\lambda, \partial_\xi)$.

Section 4 contains our main stability result (Theorem 4.8) for traveling waves of the system (1.1). The proof employs a special transformation to a first order hyperbolic system and then uses the extensive stability theory developed in [32, 33]. It is deferred to Appendix A since it is somewhat technical and involves a careful use of function spaces and spectral relations between first and second order formulations. Let us note that our result implies the main stability theorem of [25] provided the spectral assumptions have been verified as in [25]. In general, it may be difficult to control extra isolated eigenvalues analytically, and one has to resort to numerical computations such as [2].

Our second main result (Theorem 4.10) concerns the PDAE of the freezing approach under the assumptions of the stability theorem and for the fixed phase condition. It is shown that the pair $(v_\star, \mu_\star)$ becomes an equilibrium of the PDAE which is stable in the classical Lyapunov sense with respect to suitable norms. In this situation the solutions of the freezing PDAE will converge to the traveling wave and its speed when initial data are sufficiently close. This is in accordance with the numerical experiments from section 2 and is taken as a justification of the general approach.

2. Freezing traveling waves in damped wave equations. In this section we extend the freezing method [4, 7] from first to second order evolution equations for the case of translational equivariance. A generalization to several space dimensions and more general symmetries is discussed in [6].

2.1. Derivation of the partial differential algebraic equation. Consider the Cauchy problem associated with (1.1),

$$Mu_{tt} = Au_{xx} + f(u, u_x, u_t), \qquad x \in \mathbb{R},\ t \geq 0, \tag{2.1a}$$

$$u(\cdot, 0) = u_0, \quad u_t(\cdot, 0) = v_0, \qquad x \in \mathbb{R},\ t = 0, \tag{2.1b}$$

for some initial data $u_0, v_0 : \mathbb{R} \to \mathbb{R}^m$. For the new variables (1.5) one obtains

$$u_t = -\gamma_t v_\xi + v_t, \quad u_{tt} = -\gamma_{tt} v_\xi + \gamma_t^2 v_{\xi\xi} - 2\gamma_t v_{\xi t} + v_{tt}, \tag{2.2}$$

which, when inserted into (2.1a), leads to (1.6). Then it is convenient to introduce the real valued time-dependent functions $\mu_1 := \gamma_t$ and $\mu_2 := \mu_{1,t} = \gamma_{tt}$ which transform (1.6) into the coupled PDE/ODE system

$$Mv_{tt} = (A - \mu_1^2 M)v_{\xi\xi} + 2\mu_1 M v_{\xi t} + \mu_2 M v_\xi + f(v, v_\xi, v_t - \mu_1 v_\xi), \tag{2.3a}$$

$$\mu_{1,t} = \mu_2, \quad \gamma_t = \mu_1. \tag{2.3b}$$

The quantity $\gamma(t)$ denotes the position, $\mu_1(t)$ the translational velocity, and $\mu_2(t)$ the acceleration of the wave v at time t. In contrast to the differential equation (2.1a), (2.3) is not a well-posed system, because three new unknowns (γ, μ_1, μ_2) are introduced, but only two new differential equations are added. To compensate this extra degree of freedom, we impose an additional scalar algebraic constraint, also known as a phase condition, of the general form

$$\psi(v, v_t, \mu_1, \mu_2) = 0,\ t \geqslant 0. \tag{2.4}$$

When combined with (2.3) we obtain a PDAE.

We next specify initial data for the system (2.3), (2.4) as follows:

$$v(\cdot, 0) = u_0, \quad v_t(\cdot, 0) = v_0 + \mu_1^0 u_{0,\xi}, \quad \mu_1(0) = \mu_1^0, \quad \mu_2(0) = \mu_2^0, \quad \gamma(0) = 0. \tag{2.5}$$

Starting with $\gamma(0) = 0$ and $\mu_1(0) = \mu_1^0$, the first equation in (2.5) follows from (1.5) and (2.1b), while the second condition in (2.5) can be deduced from (2.2), (2.1b), (2.3b). At first glance, the initial values μ_1^0 and μ_2^0 can be taken arbitrarily and set to zero, for example. However, it is a well-known fact from the theory of DAEs that they are determined by the algebraic constraint (2.4) and so-called hidden constraints. The hidden constraints are obtained by differentiating the constraint equations with respect to time and then replacing time derivatives through the differential equation. Initial data which satisfy these constraints are called consistent. Depending on the numerical solver, it is often not necessary to prescribe consistent initial values for μ_1^0 and μ_2^0.

For the phase condition we require that it vanishes at the traveling wave solution

$$\psi(v_\star, 0, \mu_\star, 0) = 0. \tag{2.6}$$

In essence, this condition singles out one element from the family of shifted profiles $v_\star(\cdot - \gamma), \gamma \in \mathbb{R}$.

In the following we discuss two possible choices for a phase condition.

Type 1: fixed phase condition. Let $\hat{v}:\mathbb{R}\to\mathbb{R}^m$ denote a time-independent and sufficiently smooth template (or reference) function, e.g., $\hat{v}=u_0$. Then we consider the following fixed phase condition:

$$\psi_{\mathrm{fix},3}(v) := \langle v-\hat{v}, \hat{v}_\xi\rangle_{L^2} = 0,\ t\geqslant 0. \tag{2.7}$$

This is equivalent to requiring at each time t the necessary condition $\rho'(0)=0$ for a local minimum of the L^2-distance of the shifted versions of v from the template $\hat{v}$ to occur at $\gamma=0$,

$$\rho(\gamma) := \|v(\cdot,t)-\hat{v}(\cdot-\gamma)\|_{L^2}^2 = \|v(\cdot+\gamma,t)-\hat{v}(\cdot)\|_{L^2}^2 .$$

As is common in the theory of DAEs, it is possible to reduce the index of the resulting PDAE (2.3), (2.7): We differentiate (2.7) w.r.t. t and obtain

$$\psi_{\mathrm{fix},2}(v_t) := \langle v_t, \hat{v}_\xi\rangle_{L^2} = 0,\ t\geqslant 0. \tag{2.8}$$

Finally, differentiating (2.8) once more w.r.t. t and using (2.3a) yields the condition

$$\begin{aligned}\psi_{\mathrm{fix},1}(v,v_t,\mu_1,\mu_2) := \langle (M^{-1}A-\mu_1^2 I_m)v_{\xi\xi} + 2\mu_1 v_{\xi t} + \mu_2\langle v_\xi,\hat{v}_\xi\rangle_{L^2} \\ + M^{-1}f(v,v_\xi,v_t-\mu_1 v_\xi), \hat{v}_\xi\rangle_{L^2} = 0,\ t\geqslant 0.\end{aligned} \tag{2.9}$$

This equation can be solved for μ_2 if the template $\hat{v}$ is chosen such that $\langle v_\xi,\hat{v}_\xi\rangle_{L^2}\neq 0$ holds for any $t\geqslant 0$.

The numbers $j=1,2,3$ in the notation $\psi_{\mathrm{fix},j}$ indicate the index of the resulting PDAE (in a formal sense) as the minimum number of differentiations with respect to t, necessary to obtain an explicit differential equation for the unknowns (v,μ_1,μ_2) (cf. [20, Chap. 1], [9, Chap. 2]). Using a lower index formulation, e.g., complementing (2.3) with (2.9) instead of (2.7), may ease the numerics but does not necessarily improve the accuracy of the method; see, e.g., Figures 3 and 6.

Also note that the index 2 formulation (2.3), (2.8) and the index 1 formulation (2.3), (2.9) explicitly enforce constraints on $\mu_1(0)=\mu_1^0$ and $\mu_2(0)=\mu_2^0$. Namely, setting $t=0$ in (2.8) and using (2.5) yields

$$\mu_1^0\langle u_{0,\xi},\hat{v}_\xi\rangle_{L^2} + \langle v_0,\hat{v}_\xi\rangle_{L^2} = 0, \tag{2.10}$$

which determines μ_1^0. Similarly, setting $t=0$ in (2.9) and using (2.5) leads to

$$0 = \langle (M^{-1}A+(\mu_1^0)^2 I_m)u_{0,\xi\xi} + 2\mu_1^0 v_{0,\xi} + M^{-1}f(u_0,u_{0,\xi},v_0),\hat{v}_\xi\rangle_{L^2} + \mu_2^0\langle u_{0,\xi},\hat{v}_\xi\rangle_{L^2}, \tag{2.11}$$

which determines μ_2^0. These are the hidden constraints, mentioned above.

Type 2: orthogonal phase condition. The orthogonal phase condition reads as follows:

$$\psi_{\mathrm{orth},2}(v,v_t) := \langle v_t, v_\xi\rangle_{L^2} = 0,\ t\geqslant 0. \tag{2.12}$$

For first order evolution equations, condition (2.12) has an immediate interpretation as a necessary condition for minimizing $\|v_t\|_{L^2}$ (cf. [4]). The same interpretation is possible here when applied to a proper formulation as a first order system; see [5, (4.46)]. In any case, this condition expresses orthogonality of v_t to the vector v_ξ tangent to the group orbit $\{v(\cdot-\gamma):\gamma\in\mathbb{R}\}$ at $\gamma=0$. Complementing (2.3) with the condition (2.12) leads to a PDAE of index 2 in the sense above. As before, we can lower the index of the PDAE by differentiating (2.12) w.r.t. t and using (2.3a). This leads to the phase condition

$$\begin{aligned}\psi_{\mathrm{orth},1}(v,v_t,\mu_1,\mu_2) := \langle (M^{-1}A-\mu_1^2 I_m)v_{\xi\xi} + 2\mu_1 v_{\xi t} + \mu_2\langle v_\xi, v_\xi\rangle_{L^2} \\ + M^{-1}f(v,v_\xi,v_t-\mu_1 v_\xi), v_\xi\rangle_{L^2} + \langle v_t, v_{\xi t}\rangle_{L^2} = 0,\ t\geqslant 0,\end{aligned} \tag{2.13}$$

and (2.3), (2.13) yields a PDAE of index 1. Note that (2.13) can be explicitly solved for μ_2, provided that $\langle v_\xi, v_\xi\rangle_{L^2} \neq 0$ for any $t \geqslant 0$.

Similar to the type 1 phase condition, we obtain constraints for consistent initial values when setting $t=0$ in (2.12), (2.13). Condition (2.12) gives an equation for μ_1^0,

$$0 = \mu_1^0\langle u_{0,\xi}, u_{0,\xi}\rangle_{L^2} + \langle v_0, u_{0,\xi}\rangle_{L^2}, \tag{2.14}$$

while (2.13), (2.1b), (1.5) give an equation for μ_2^0,

$$\begin{aligned}0 = \langle 2(\mu_1^0)^2 u_{0,\xi\xi} + 3\mu_1^0 v_{0,\xi} + M^{-1}\left(Au_{0,\xi\xi} + f(u_0,u_{0,\xi},v_0)\right), u_{0,\xi}\rangle_{L^2} \\ + \langle v_0, v_{0,\xi}\rangle_{L^2} + \mu_1^0\langle v_0, u_{0,\xi\xi}\rangle_{L^2} + \mu_2^0\langle u_{0,\xi}, u_{0,\xi}\rangle_{L^2}.\end{aligned} \tag{2.15}$$

Let us summarize the system of equations obtained by the freezing method from the original Cauchy problem (2.1). Combining the differential equations (2.3), the phase condition (2.4), and the initial data (2.5), we arrive at the following PDAE to be solved numerically:

$$\begin{aligned}&Mv_{tt} = (A-\mu_1^2 M)v_{\xi\xi} + 2\mu_1 M v_{\xi,t} + \mu_2 M v_\xi + f(v,v_\xi,v_t-\mu_1 v_\xi), \\ &\mu_{1,t} = \mu_2, \quad \gamma_t = \mu_1,\end{aligned} \tag{2.16a}$$

$$0 = \psi(v,v_t,\mu_1,\mu_2), \tag{2.16b}$$

$$\begin{aligned}&v(\cdot,0) = u_0, \quad v_t(\cdot,0) = v_0 + \mu_1^0 u_{0,\xi}, \\ &\mu_1(0) = \mu_1^0, \quad \mu_2(0) = \mu_2^0, \quad \gamma(0) = 0.\end{aligned} \tag{2.16c}$$

Here (2.16b) is one of (2.7), (2.8), (2.9), (2.12), or (2.13). The system (2.16) is to be solved for (v,μ_1,μ_2,γ) with given initial data $(u_0,v_0,\mu_1^0,\mu_2^0)$. Consistent initial values μ_1^0 for μ_1 and μ_2^0 for μ_2 are computed from the phase condition and the initial data (cf. (2.10), (2.11), (2.14), (2.15)).

The ODE for γ is called the reconstruction equation in [35]. It decouples from the other equations in (2.16) and can be solved in a postprocessing step. The ODE for μ_1 is the new feature of the PDAE for second order systems when compared to first order parabolic and hyperbolic equations; cf. [4, 7, 32].

Finally, note that because of (1.8) and (2.6), the triple $(v,\mu_1,\mu_2) = (v_\star,\mu_\star,0)$ is a stationary solution of (2.16a), (2.16b). Obviously, in this case we have $\gamma(t) = \mu_\star t$. For a stable traveling wave we expect that solutions (v,μ_1,μ_2,γ) of (2.16) show the limiting behavior $(v(t),\mu_1(t),\mu_2(t)) \to (v_\star,\mu_\star,0)$ as $t\to\infty$ if the initial data are close to their limiting values. Our theorems in section 4 will justify this expectation under suitable spectral conditions.

2.2. Traveling waves related to parabolic equations. As a special class of systems for which we will test and validate our numerical method, we consider the case $f(u,u_x,u_t) = g(u) + Cu_x - Bu_t$ in (1.1), i.e.,

$$Mu_{tt} = Au_{xx} + Cu_x - Bu_t + g(u), \quad x\in\mathbb{R},\ t\geq 0,\ u(x,t)\in\mathbb{R}^m, \tag{2.17}$$

with constant matrices $M, A, B, C \in \mathbb{R}^{m,m}$ and smooth $g : \mathbb{R}^m \to \mathbb{R}^m$. The following proposition shows an important relation between traveling waves (1.2) of the damped wave equation (2.17) and traveling waves

$$u_\star(x,t) = w_\star(x - c_\star t),\ x \in \mathbb{R},\ t \geqslant 0, \tag{2.18}$$

with nonvanishing velocity $c_\star$ of the parabolic equation

$$Bu_t = \tilde{A}u_{xx} + \tilde{C}u_x + g(u),\ x \in \mathbb{R},\ t \geq 0. \tag{2.19}$$

The matrices $\tilde{A}, \tilde{C} \in \mathbb{R}^{m,m}$ in (2.19) may differ from A, C in (2.17). This observation goes back to [19] and has also been used in [14]. Note that in this case $w_\star : \mathbb{R} \to \mathbb{R}^m$ solves the traveling wave equation

$$0 = \tilde{A}w_{\star,\zeta\zeta} + c_\star Bw_{\star,\zeta} + \tilde{C}w_{\star,\zeta} + g(w_\star),\ \zeta \in \mathbb{R}. \tag{2.20}$$

Proposition 2.1. (i) *Let* (2.18) *be a traveling wave of the parabolic equation* (2.19). *Then for every* $0 \neq k \in \mathbb{R}$ *and* $A, C, M \in \mathbb{R}^{m,m}$, *satisfying* $\tilde{A} = k^2A - c_\star^2 M$, $\tilde{C} = kC$, (1.2) *with*

$$v_\star(\xi) = w_\star(k\xi), \quad \mu_\star = c_\star k^{-1} \tag{2.21}$$

defines a traveling wave of the damped wave equation (2.17).

(ii) *Conversely, let* (1.2) *be a traveling wave of* (2.17). *Then for every* $0 \neq k \in \mathbb{R}$ (2.18) *with*

$$w_\star(\zeta) = v_\star(\zeta k^{-1}), \quad c_\star = \mu_\star k \tag{2.22}$$

defines a traveling wave of (2.19) *with* $\tilde{A} = k^2(A - \mu_\star^2 M)$, $\tilde{C} = kC$.

Proof. (i) By assumption, $w_\star$ satisfies (2.20). Let $0 \neq k \in \mathbb{R}$ and $A, C, M \in \mathbb{R}^{m,m}$ so that $\tilde{A} = k^2A - c_\star^2 M$, $\tilde{C} = kC$ hold and define $v_\star, \mu_\star$ by (2.21). Then $u_\star(x,t) = v_\star(x - \mu_\star t) = w_\star\big(k(x - \mu_\star t)\big)$ satisfies

$$-Mu_{\star,tt} - Bu_{\star,t} + Au_{\star,xx} + Cu_{\star,x} + g(u_\star) = \tilde{A}w_{\star,\zeta\zeta} + c_\star Bw_{\star,\zeta} + \tilde{C}w_{\star,\zeta} + g(w_\star) = 0.$$

(ii) By assumption, $v_\star, \mu_\star$ from (1.2) satisfy (1.8). Let $0 \neq k \in \mathbb{R}$ and define $\tilde{A} := k^2(A - \mu_\star^2 M), \tilde{C} := kC \in \mathbb{R}^{m,m}$ and $w_\star, c_\star$ by (2.22). Then $u_\star(x,t) = w_\star(x - c_\star t) = v_\star\left(\frac{x - c_\star t}{k}\right)$ satisfies

$$\begin{aligned} &\tilde{A}u_{\star,xx} - Bu_{\star,t} + \tilde{C}u_{\star,x} + g(u_\star) \\ &\quad = (A - \mu_\star^2 M)v_{\star,\xi\xi} + \mu_\star Bv_{\star,\xi} + Cv_{\star,\xi} + g(v_\star) = 0. \end{aligned}$$ □

According to Proposition 2.1, any traveling wave (2.18) of the parabolic equation (2.19) leads to a traveling wave (1.2) of the damped wave equation (2.17) and vice versa.

Remark 2.2. The profiles $v_\star, w_\star$ and the velocities $\mu_\star, c_\star$ coincide if $k = 1$. In this case $\tilde{A} = A - c_\star^2 M$, and the matrices A and $\tilde{A}$ differ (provided $c_\star \neq 0$). If we insist on $A = \tilde{A}$, then the profiles will be different.

In case $C = 0$ both systems (2.17) and (2.19) share a symmetry property: if $v_\star(\xi)(\xi \in \mathbb{R}), c_\star$, resp., $w_\star(\zeta)(\zeta \in \mathbb{R}), \mu_\star$ is a traveling wave, then so is the reflected pair $v_\star(-\xi)(\xi \in \mathbb{R}), -c_\star$, resp., $w_\star(-\zeta)(\zeta \in \mathbb{R}), -\mu_\star$. Thus, choosing $k < 0$ in (2.21), resp., (2.22) will not produce new waves other than those induced by reflection symmetry. Therefore, we assume k to be positive in the following.

It is instructive to consider two limiting cases of the transformation (2.21) when a traveling wave $w_\star$ with velocity $c_\star \neq 0$ is given for the parabolic equation (2.19).

First assume $A = \tilde{A}$ and let $M \to 0$. Then the relation $\tilde{A} = k^2 A - c_\star^2 M$ implies $k \to 1$ and $v_\star \to w_\star$, $\mu_\star \to c_\star$. Thus the profile and the velocity of the traveling waves (1.2) of the system (1.1), (1.4) converge to the correct limit in the parabolic case. Second, consider the scalar case, fix $A > 0$, and let $M \to \infty$. Then the relation $\tilde{A} = k^2 A - c_\star^2 M$ implies $k \to \infty$ and $\mu_\star = \frac{c_\star}{k} \to 0$. Thus a large value of M creates a slow wave for the system (1.1), (1.4) which has steep gradients in its profile due to $v_{\star,\xi}(\xi) = k w_{\star,\zeta}(k\xi)$.

2.3. Applications and numerical examples. In the following we consider a scalar example with a Nagumo nonlinearity and a system with nonlinearity of FitzHugh–Nagumo type. In the first case we have explicit traveling wave solutions which allow us to check the numerics. For the second example existence and good approximations of a wave are known for the parabolic case and thus by Proposition 2.1 also for the damped wave case. We solve the PDAE (2.16) providing us with wave profiles, their positions, velocities, and accelerations. All numerical computations in this paper were done with Comsol Multiphysics 5.2 [1]. Specific data of time and space discretization are given below.

Example 2.3 (Nagumo wave equation). The scalar parabolic Nagumo equation [28, 29],

$$u_t = u_{xx} + g(u),\ x \in \mathbb{R},\ t \geq 0, \quad g(u) = u(1-u)(u-b), \quad 0 < b < 1, \tag{2.23}$$

has a well-known explicit traveling front solution $u_\star(x,t) = w_\star(x - c_\star t)$ given by

$$w_\star(\zeta) = \left(1 + \exp\left(-\frac{\zeta}{\sqrt{2}}\right)\right)^{-1}, \quad c_\star = -\sqrt{2}\left(\frac{1}{2} - b\right),$$

with asymptotic states $w_- = 0$ and $w_+ = 1$. The corresponding Nagumo wave equation

$$\varepsilon u_{tt} + u_t = u_{xx} + g(u),\ x \in \mathbb{R},\ t \geq 0, \tag{2.24}$$

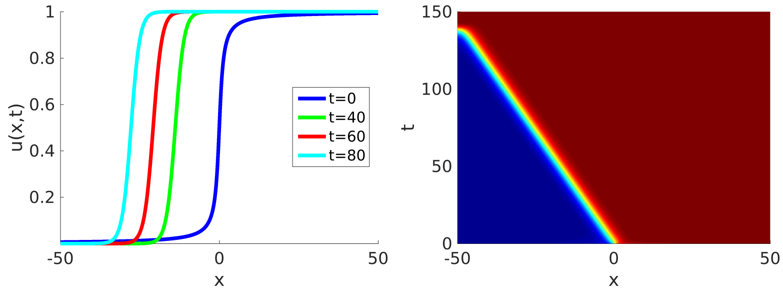
is of the type of the nonlinear telegraph equation (1.4). By Proposition 2.1(i) there exists a traveling wave $u_\star(x,t) = v_\star(x - \mu_\star t)$ given by

$$v_\star(\xi) = w_\star(k\xi), \quad \mu_\star = \frac{-\sqrt{2}\left(\frac{1}{2} - b\right)}{k}, \quad k = \left(1 + 2\varepsilon\left(\frac{1}{2} - b\right)^2\right)^{1/2}. \tag{2.25}$$

As in [38] one may derive this expression also directly from (2.24). Figure 1 shows the result of a direct numerical simulation of (2.24) with $\varepsilon = b = \frac{1}{4}$ on the spatial domain $(-50, 50)$ with homogeneous Neumann boundary conditions and initial data

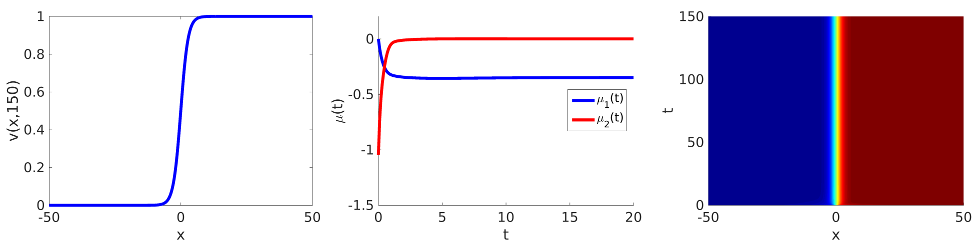
$$u_0(x) = \frac{1}{\pi}\arctan(x) + \frac{1}{2}, \quad v_0(x) = 0, \quad x \in (-50, 50). \tag{2.26}$$

The space is discretized with piecewise linear finite elements on an equidistant grid with stepsize $\triangle x = 0.1$, and for the time discretization we used the BDF method of



(a) (b)

FIG. 1. *Traveling front of Nagumo wave equation* (2.24) *for parameters* $\varepsilon = b = \frac{1}{4}$ *at different time instances* (a) *and its time evolution* (b).



(a) (b) (c)

FIG. 2. *Solution of the frozen Nagumo wave equation* (2.27) *for* $\varepsilon = b = \frac{1}{4}$. *Approximation of profile* $v(x, 150)$ *at the final time* $T = 150$ (a), *time evolutions of velocity* μ_1 *and acceleration* μ_2 (b), *and of the profile* v (c).

order 2 with absolute tolerance atol $= 10^{-3}$, relative tolerance rtol $= 10^{-2}$, temporal stepsize $\triangle t = 0.1$, and final time $T = 150$.

Next we solve with the same data the frozen Nagumo wave equation resulting from (2.16)

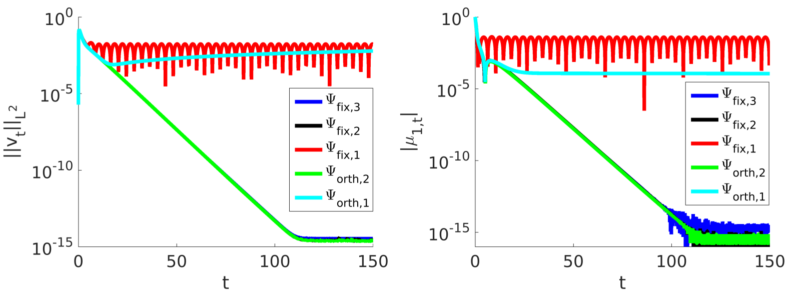
$$\text{(2.27a)}\quad \begin{aligned} \varepsilon v_{tt} + v_t &= (1-\mu_1^2\varepsilon)v_{\xi\xi} + 2\mu_1\varepsilon v_{\xi,t} + (\mu_2\varepsilon + \mu_1)v_\xi + g(v),\\ \mu_{1,t} &= \mu_2, \quad \gamma_t = \mu_1, \end{aligned}$$

$$\text{(2.27b)}\quad 0 = \left\langle v_t(\cdot,t), \hat{v}_\xi \right\rangle_{L^2(\mathbb{R},\mathbb{R})},$$

$$\text{(2.27c)}\quad \begin{aligned} v(\cdot,0) &= u_0, \quad v_t(\cdot,0) = v_0 + \mu_1^0 u_{0,\xi},\\ \mu_1(0) &= \mu_1^0, \quad \mu_2(0) = \mu_2^0, \gamma(0) = 0. \end{aligned}$$

Here we use the index-2 formulation of the fixed phase condition from (2.8) with consistent intial data $\mu_1^0 = 0$, $\mu_2^0 = -1.0312$, which are calculated from (2.10) and (2.11) by using the initial data (2.26).

Figure 2 shows the solution $(v, \mu_1, \mu_2, \gamma)$ of (2.27) on the spatial domain $(-50, 50)$ with homogeneous Neumann boundary conditions, and reference function $\hat{v} = u_0$. The discretization data are taken as before. The diagrams show that after a very short transition phase the profile becomes stationary, the acceleration μ_2 converges to zero, and the speed μ_1 approaches an asymptotic value $\mu_\star^{\text{num}}$ close to the exact value $\mu_\star \approx -0.34816$, given by (2.25). From the value of the function $\gamma(t)$, $t \geqslant 0$ (not



(a) (b)

FIG. 3. *Comparison of phase conditions for the frozen Nagumo wave equation* (2.27) *for parameters* $\varepsilon = b = \frac{1}{4}$*: Time evolution of* $\|v_t\|_{L^2}$ (a) *and* $|\mu_{1,t}|$ (b).

shown), which is obtained by integrating the last equation in (2.27a), one can recover the position of the front in the original system (2.24).

If we replace the phase condition $\psi_{\text{fix},2}$ in (2.27b) by $\psi_{\text{fix},3}$ or $\psi_{\text{orth},2}$, we obtain basically the same results as shown in Figure 2. Since we expect $v_t(t) \to 0$ and $\mu_{1,t}(t) \to 0$ as $t \to \infty$, we use these quantities as an indicator for checking whether the solution has become stationary. Figure 3 shows the time evolution of $\|v_t\|_{L^2}$ and $|\mu_{1,t}|$ when solving (2.27) with different phase conditions. While the phase conditions of indexes 2 and 3 behave as expected, the index 1 formulation yields small but oscillating values for the norms of v_t and $\mu_{1,t}$. We attribute this behavior to the fact that our adaptive solver enforces the differentiated conditions (2.9), (2.13) but does not control v_t, μ_t directly. Further investigations show that the consistency condition for μ_2^0 does not really affect the numerical results for the different phase conditions. Therefore, in the next example we do not compute the expression for μ_2^0 but use the expected limiting value as initial datum $\mu_2^0 = 0$.

Example 2.4 (FitzHugh–Nagumo wave system). This classical system [12] reads in the parabolic case as follows:

$$u_t = \tilde{A}u_{xx} + g(u), \quad \tilde{A} = \begin{pmatrix} 1 & 0 \\ 0 & \rho \end{pmatrix}, \; g(u) = \begin{pmatrix} u_1 - \frac{1}{3}u_1^3 - u_2 \\ \phi(u_1 + a - bu_2) \end{pmatrix}, \tag{2.28}$$

with $u = u(x,t) \in \mathbb{R}^2$ and positive parameters $\rho, a, b, \phi \in \mathbb{R}$. Equation (2.28) is known to exhibit traveling wave solutions in a wide range of parameters, but there are apparently no explicit formulas. For the values

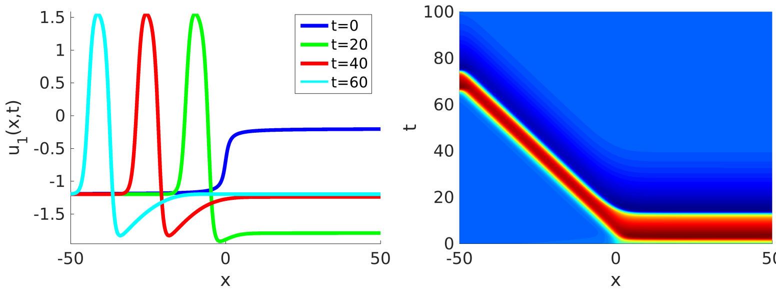
$$\rho = 0.1, \quad a = 0.7, \quad \phi = 0.08, \quad b = 0.8 \tag{2.29}$$

one finds a traveling pulse with

$$w_\pm \approx (-1.19941, \, -0.62426)^\top, \quad c_\star \approx -0.7892. \tag{2.30}$$

For the same ρ, a, ϕ but $b = 3$, there is a traveling front with asymptotic states and velocity given by

$$w_- \approx (1.18779, \, 0.62923)^\top, \quad w_+ \approx (-1.56443, \, -0.28814)^\top, \quad c_\star \approx -0.8557.$$



(a) (b)

FIG. 4. *First component u_1 of traveling pulse for the FitzHugh–Nagumo wave system* (2.31)*, $\varepsilon = 10^{-2}$, $\rho = 0.1$, $a = 0.7$, $\phi = 0.08$, $b = 0.8$: Different time instances* (a) *and space-time plot* (b).

The corresponding FitzHugh–Nagumo wave system

$$Mu_{tt} + Bu_t = Au_{xx} + g(u),\ x \in \mathbb{R},\ t \geq 0, \tag{2.31}$$

again has the form of the nonlinear telegraph equation (1.4). It corresponds to a model of nerve conductance including small effects from inductance; cf. [10]. Applying Proposition 2.1(i) with $M = \varepsilon I_2$ requires the equality $\tilde{A} + c_\star^2 M = k^2 A$, i.e., $1 + c_\star^2 \varepsilon = k^2 A_{11}$, $\rho + c_\star^2 \varepsilon = k^2 A_{22}$, and $A_{12} = A_{21} = 0$. With $A_{11} := 1$ and parameter values from (2.29), Proposition 2.1(i) shows that (2.31) has a traveling pulse (or a traveling front) solution with a scaled profile $v_\star$ and velocity $\mu_\star = \frac{c_\star}{k}$ for the settings

$$M = \varepsilon I_2, \quad B = I_2, \quad A = \operatorname{diag}\left(1, \tfrac{\rho + c_\star^2 \varepsilon}{1 + c_\star^2 \varepsilon}\right), \quad k = \sqrt{1 + c_\star^2 \varepsilon}, \quad \varepsilon > 0.$$

In the following we show the numerical findings for the traveling pulse. Results for the traveling front are very similar and are not displayed here. We choose the parameter values (2.29) and $\varepsilon = 10^{-2}$. Space and time are discretized as in Example 2.3. Figure 4 shows the time evolution of the traveling pulse solution $u = (u_1, u_2)^T$ of (2.31) on the spatial domain $(-50, 50)$ with homogeneous Neumann boundary conditions. The initial data are

$$u_0(x) = (\tfrac{1}{\pi}\arctan(x) + \tfrac{1}{2},\ 0)^\top + v_\pm, \quad v_0(x) = (0,\ 0)^\top, \quad x \in \mathbb{R}, \tag{2.32}$$

where $v_\pm = w_\pm$ is the asymptotic state from (2.30).

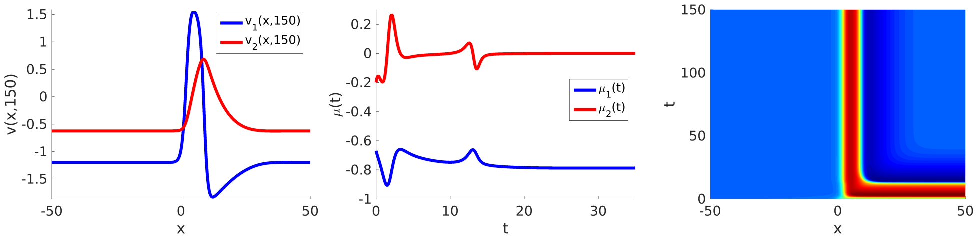
Next consider for the same parameter values the corresponding frozen FitzHugh–Nagumo wave system

$$\begin{aligned} Mv_{tt} + Bv_t &= (A - \mu_1^2 M)v_{\xi\xi} + 2\mu_1 M v_{\xi,t} + (\mu_2 M + \mu_1 B)v_\xi + g(v), \\ \mu_{1,t} &= \mu_2, \quad \gamma_t = \mu_1, \end{aligned} \tag{2.33a}$$

$$0 = \left\langle v_t(\cdot,t), \hat{v}_\xi \right\rangle_{L^2(\mathbb{R},\mathbb{R})}, \tag{2.33b}$$

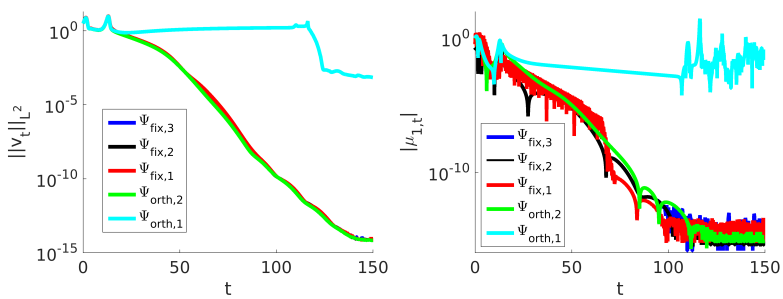
$$v(\cdot,0) = u_0, \quad v_t(\cdot,0) = v_0 + \mu_1^0 u_{0,\xi}, \quad \mu_1(0) = \mu_1^0, \quad \gamma(0) = 0, \tag{2.33c}$$

where we again use the index-2 formulation (2.8) of the fixed phase condition. Figure 5 shows the numerical solution $(v, \mu_1, \mu_2, \gamma)$ of (2.33) on the spatial domain $(-50, 50)$, with homogeneous Neumann boundary conditions, initial data u_0, v_0 from (2.32), and reference function $\hat{v} = u_0$. For the algebraic variables we choose the consistent initial



(a) (b) (c)

FIG. 5. *Solution of the frozen FitzHugh–Nagumo wave system* (2.16), $\varepsilon = 10^{-2}$, $\rho = 0.1$, $a = 0.7$, $\phi = 0.08$, $b = 0.8$*: Profile components* v_1, v_2 *at* $T = 150$ (a), *time evolutions of velocity* μ_1 *and acceleration* μ_2 (b) *and of profile component* v_1 (c).



(a) (b)

FIG. 6. *Comparison of phase conditions for the frozen FitzHugh–Nagumo wave system* (2.33), $\varepsilon = 10^{-2}$, $\rho = 0.1$, $a = 0.7$, $\phi = 0.08$, $b = 0.8$*: Time evolution of* $\|v_t\|_{L^2}$ (a) *and* $|\mu_{1,t}|$ (b).

datum $\mu_1^0 = 0$, obtained by using $v_0 = 0$ in (2.10), and we further set $\mu_2^0 = 0$ which does not satisfy the consistency condition (2.11). Time and space discretizations are done as in the nonfrozen case. Again the profile quickly stabilizes and the velocity and the acceleration reach their asymptotic values.

Finally, Figure 6 shows that similar results are obtained if we replace the phase condition $\psi_{\text{fix},2}$ in (2.33b) by $\psi_{\text{fix},3}$, $\psi_{\text{fix},1}$, or $\psi_{\text{orth},2}$. Contrary to our first example, the fixed phase condition of index 1 provides good results in this case, while the index 1 formulation of the orthogonal phase condition $\psi_{\text{orth},1}$ continues to show small oscillations of the time derivatives.

3. Spectra and eigenfunctions of traveling waves. In this section we recall some standard notions of point and essential spectrum for operator polynomials. In particular, we study the spectrum of the quadratic operator polynomial (cf. (1.10))

$$\mathcal{P}(\lambda) := \lambda^2 P_2 + \lambda P_1 + P_0, \quad \lambda \in \mathbb{C}, \tag{3.1}$$

which is essential for the stability analysis in section 4. Here the differential operators P_j are defined by

$$\begin{aligned} P_2 &= M, \quad P_1 = -D_3 f(\star) - 2\mu_\star M \partial_\xi, \\ P_0 &= -(A - \mu_\star^2 M)\partial_\xi^2 + (\mu_\star D_3 f(\star) - D_2 f(\star))\partial_\xi - D_1 f(\star), \end{aligned} \tag{3.2}$$

where $(\star) = (v_\star, v_{\star,\xi}, -\mu_\star v_{\star,\xi})$ and $v_\star$, $\mu_\star$ denote the profile and velocity of a traveling wave solution $u_\star(x,t) = v_\star(x - \mu_\star t)$ of (1.1). Note that P_j is a differential operator of order $2-j$ for $j = 0,1,2$.

Definition 3.1. *Let $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$ be complex Banach spaces and let $\mathcal{P}(\lambda) = \sum_{j=0}^{q} P_j \lambda^j, \lambda \in \mathbb{C}$ be an operator polynomial with linear continuous coefficients $P_j : Y \to X$, $j = 0, \ldots, q$.*

(a) *The resolvent set $\rho(\mathcal{P})$ is defined by*

$$\rho(\mathcal{P}) = \{\lambda \in \mathbb{C} : \mathcal{P}(\lambda) \text{ is bijective and } \mathcal{P}(\lambda)^{-1} : X \to Y \text{ is bounded}\}$$

and the spectrum by $\sigma(\mathcal{P}) = \mathbb{C} \setminus \rho(\mathcal{P})$.

(b) *$\lambda_0 \in \sigma(\mathcal{P})$ is called isolated if there is $\varepsilon > 0$ such that $\lambda \in \rho(\mathcal{P})$ for all $\lambda_0 \neq \lambda \in \mathbb{C}$ with $|\lambda - \lambda_0| < \varepsilon$.*

(c) *If $\mathcal{P}(\lambda_0)y_0 = 0$ for some $\lambda_0 \in C$ and $y_0 \in Y \setminus \{0\}$, then λ_0 is called an eigenvalue with eigenvector y_0. The eigenvalue λ_0 has finite multiplicity if $\dim(\mathcal{N}(\mathcal{P}(\lambda_0))) < \infty$ and if there is a maximum number $n \in \mathbb{N}$, for which there exist polynomials $y(\lambda) = \sum_{j=0}^{r} (\lambda - \lambda_0)^j y_j$ with coefficients $y_j \in Y$ satisfying*

$$y_0 \neq 0, \quad (\mathcal{P}y)^{(\nu)}(\lambda_0) = 0, \quad \nu = 0, \ldots, n-1. \tag{3.3}$$

This maximum number $n = n(\lambda_0)$ is called the maximum partial multiplicity, and $\dim(\mathcal{N}(\mathcal{P}(\lambda_0)))$ is called the geometric multiplicity of λ_0.

(d) *The point spectrum is defined by*

$$\sigma_{\text{point}}(\mathcal{P}) = \{\lambda \in \sigma(\mathcal{P}) : \lambda \text{ is isolated eigenvalue of finite multiplicity}\}.$$

Points in $\rho(\mathcal{P}) \cup \sigma_{\text{point}}(\mathcal{P})$ are called normal, and the essential spectrum is defined by

$$\sigma_{\text{ess}}(\mathcal{P}) := \{\lambda \in \mathbb{C} : \lambda \text{ is not a normal point of } \mathcal{P}\}.$$

Remark 3.2. There is no loss of generality in assuming the root polynomials in (c) to be of the form $y(\lambda) = \sum_{j=0}^{n-1} (\lambda - \lambda_0)^j y_j$. For if $r < n-1$, we simply set $y_j = 0, j = r+1, \ldots, n-1$. And if $r \geq n$ we subtract from y the term $\sum_{j=n}^{r} (\lambda - \lambda_0)^j y_j$ which has λ_0 as a zero of order at least n and thus does not change the root property (3.3). The eigenvalue λ_0 is simple iff the geometric and the maximum partial multiplicity are equal to 1. In this case $\mathcal{N}(\mathcal{P}(\lambda_0)) = \text{span}(y_0)$ for some $y_0 \neq 0$ and $\mathcal{P}'(\lambda_0)y_0 \notin \mathcal{R}(\mathcal{P}(\lambda_0))$. For more details on root polynomials and partial and algebraic multiplicities we refer to [24, 26, 27]. Our definition of essential spectrum follows [21].

By definition, the spectrum $\sigma(\mathcal{P})$ of $\mathcal{P}$ can be decomposed into its point spectrum and its essential spectrum,

$$\sigma(\mathcal{P}) = \sigma_{\text{ess}}(\mathcal{P}) \,\dot{\cup}\, \sigma_{\text{point}}(\mathcal{P}).$$

The function spaces underlying the definition of spectra are subspaces of $L^2(\mathbb{R}, \mathbb{R}^m)$. Details will be specified in section 4 and proved in Appendix A. In this section we just indicate important ingredients of both types of spectrum.

3.1. Point spectrum on the imaginary axis. Applying ∂_ξ to the traveling wave equation (1.8) yields

$$0 = (A - \mu_\star^2 M)v_{\star,\xi\xi\xi} + D_2 f(\star)v_{\star,\xi\xi} + D_1 f(\star)v_{\star,\xi} - \mu_\star D_3 f(\star)v_{\star,\xi\xi} = -P_0 v_{\star,\xi},\ \xi \in \mathbb{R},$$

provided that $v_\star \in C^3(\mathbb{R}, \mathbb{R}^m)$ and $f \in C^1(\mathbb{R}^{3m}, \mathbb{R}^m)$. Therefore, $w = v_{\star,\xi}$ solves the quadratic eigenvalue problem $\mathcal{P}(\lambda)w = 0$ for $\lambda = 0$, and $w = v_{\star,\xi}$ is an eigenfunction if the wave profile $v_\star$ is not constant.

PROPOSITION 3.3 (the eigenvalue zero). *Let $v_\star \in C^3(\mathbb{R}, \mathbb{R}^m)$, $\mu_\star$ be a nontrivial classical solution of* (1.8)*, and $f \in C^1(\mathbb{R}^{3m}, \mathbb{R}^m)$. Then $\lambda = 0$ is an eigenvalue with eigenfunction $v_{\star,\xi}$ of the quadratic eigenvalue problem $\mathcal{P}(\lambda)w = 0$.*

Note that we did not yet state that zero is an isolated eigenvalue of finite multiplicity and hence belongs to the point spectrum. For such a claim it will be sufficient to know that $\mathcal{P}(0)$ is Fredholm of index 0. In Proposition A.4 of Appendix A we provide sufficient conditions for such a statement by reduction to a first order system. For example, it is suffient to assume that all dispersion curves (discussed below in section 3.2) are to the left of the imaginary axis.

As usual, further isolated eigenvalues are difficult to detect analytically, and we refer to the extensive literature on solving quadratic eigenvalue problems and on locating zeros of the so-called Evans function; see, e.g., [2, 37].

Example 3.4 (Nagumo wave equation). The quadratic eigenvalue problem for the linearization of the Nagumo wave equation (2.24) about the traveling wave solution $u_\star(x,t) = v_\star(\xi)$, $\xi = x - \mu_\star t$, with $v_\star$, $\mu_\star$ and k from (2.25) reads

$$\begin{aligned}\left[\mathcal{P}(\lambda)w\right](\xi) &= \varepsilon\left(\lambda - \mu_\star\partial_\xi\right)^2 w(\xi) + \left(\lambda - \mu_\star\partial_\xi\right) w(\xi) - w_{\xi\xi}(\xi)\\ &\quad + \left(3v_\star^2(\xi) - 2(b+1)v_\star(\xi) - b\right) w(\xi) = 0,\ \xi \in \mathbb{R}.\end{aligned}$$

The above discussion shows that it has the eigenvalue $\lambda = 0$ with eigenvector

$$w(\xi) = v_{\star,\xi}(\xi) = \tfrac{k}{\sqrt{2}} \exp\left(-\tfrac{k\xi}{\sqrt{2}}\right)\left(1 + \exp\left(-\tfrac{k\xi}{\sqrt{2}}\right)\right)^{-2}, \quad \xi \in \mathbb{R}.$$

3.2. Essential spectrum and dispersion relation of traveling waves. An important part of the essential spectrum of $\mathcal{P}$ from (3.1), (3.2) is determined by the constant coefficient operators obtained by letting $\xi \to \pm\infty$ in the variable coefficient operators P_0, P_1 (abbreviating arguments by $(\pm) = (v_\pm, 0, 0)$),

$$\begin{aligned}\mathcal{P}^\pm(\lambda) &= \lambda^2 P_2 + \lambda P_1^\pm + P_0^\pm,\ \lambda \in \mathbb{C}, \quad P_1^\pm = -D_3 f(\pm) - 2\mu_\star M\partial_\xi,\\ P_0^\pm &= -(A - \mu_\star^2 M)\partial_\xi^2 + (\mu_\star D_3 f(\pm) - D_2 f(\pm))\partial_\xi - D_1 f(\pm).\end{aligned} \tag{3.4}$$

We seek bounded solutions w of $\mathcal{P}^\pm(\lambda)w = 0$ by the Fourier ansatz $w(\xi) = e^{i\omega\xi}z$, $z \in \mathbb{C}^m$, $|z| = 1$, and arrive at the quadratic eigenvalue problem

$$\mathcal{A}_\pm(\lambda, \omega)z = \left(\lambda^2 A_2 + \lambda A_1^\pm(\omega) + A_0^\pm(\omega)\right) z = 0$$

with matrices

$$\begin{aligned}A_2 = M, \quad A_1^\pm(\omega) &= -D_3 f(\pm) - 2i\omega\mu_\star M,\\ A_0^\pm(\omega) &= \omega^2(A - \mu_\star^2 M) + i\omega(\mu_\star D_3 f(\pm) - D_2 f(\pm)) - D_1 f(\pm).\end{aligned} \tag{3.5}$$

We claim that every $\lambda \in \mathbb{C}$ satisfying the dispersion relation

$$\det\left(\lambda^2 A_2 + \lambda A_1^\pm(\omega) + A_0^\pm(\omega)\right) = 0 \tag{3.6}$$

for some $\omega \in \mathbb{R}$ and either sign belongs to the essential spectrum of $\mathcal{P}$. A proof of this statement is obtained in the standard way by constructing a singular (or Weyl) sequence $w_n \in H^2(\mathbb{R}, \mathbb{C}^m)$ for $\mathcal{P}(\lambda)$. By definition (see, e.g., [11, Ch. IX, Def. 1.2]) such a sequence has no convergent subsequence in L^2 and satisfies

$$\|w_n\|_{L^2} = 1, \quad \|\mathcal{P}(\lambda) w_n\|_{L^2} \to 0 \quad \text{as} \quad n \to \infty. \tag{3.7}$$

Then $\mathcal{P}(\lambda)$ cannot have a bounded inverse. Moreover, using the reduction to first order problems in A.2 and [11, Ch. IX, Theorem 1.3], the operator $\mathcal{P}(\lambda)$ cannot be semi-Fredholm with finite dimensional kernel. Hence λ belongs to the essential spectrum by Definition 3.1. For the construction of $\{w_n\}_{n\in\mathbb{N}}$ one employs a cutoff function $\chi_n \in C^\infty(\mathbb{R}, [0,1])$ satisfying $\chi_n(\xi) = 0$ for $\xi \notin J_n = [-(2n+1), -(n-1)] \cup [n-1, 2n+1]$ as well as $\chi_n(\xi) = 1$ for $n \le |\xi| \le 2n$. Using this, one defines $w_n = (\|\chi_n w\|_{L^2})^{-1} \chi_n w$ with $w(\xi) = e^{i\omega\xi} z$ from above. The property (3.7) then follows by a lengthy but straightforward computation. The property of "no convergent subsequence" follows from $w_n \rightharpoonup 0$ (see [11, Ch. IX, (1.2)]) which is a consequence of the estimate

$$|\langle u, w_n\rangle_{L^2}| \le \|u\|_{L^2(J_n)} \|w_n\|_{L^2(J_n)} \le \|u\|_{L^2(J_n)} \to 0 \quad \text{as} \quad n \to \infty \quad \forall u \in L^2(\mathbb{R}, \mathbb{C}^m).$$

Summing up, we have the following result.

PROPOSITION 3.5 (dispersion set and essential spectrum). *Let $f \in C^1(\mathbb{R}^{3m}, \mathbb{R}^m)$ with $f(v_\pm, 0, 0) = 0$ for some $v_\pm \in \mathbb{R}^m$. Let $v_\star \in C^2(\mathbb{R}, \mathbb{R}^m)$, $\mu_\star$, be a nontrivial classical solution of* (1.8) *satisfying $v_\star(\xi) \to v_\pm$ as $\xi \to \pm\infty$. Then, the dispersion set*

$$\sigma_{\text{disp}}(\mathcal{P}) := \{\lambda \in \mathbb{C} \mid \lambda \textit{ satisfies } (3.6) \textit{ for some } \omega \in \mathbb{R} \textit{ and some sign } \pm\} \tag{3.8}$$

belongs to the essential spectrum $\sigma_{\text{ess}}(\mathcal{P})$ of $\mathcal{P}$.

In the general matrix case it is not easy to analyze the shape of the algebraic set $\sigma_{\text{disp}}(\mathcal{P})$, since (3.6) amounts to finding the zeroes of a polynomial of degree $2m$. As in [21], Propositions A.3 and A.4 in the appendix will show that there is no essential spectrum in a connected component of $\mathbb{C} \setminus \sigma_{\text{disp}}(\mathcal{P})$ which contains a neighborhood of $+\infty$.

In view of the stability results in Theorems 4.8 and 4.10 our main interest is in finding a spectral gap, i.e., a constant $\beta > 0$ such that

$$\operatorname{Re} \lambda \le -\beta < 0 \quad \forall\, \lambda \in \sigma_{\text{disp}}(\mathcal{P}). \tag{3.9}$$

We discuss this condition for three subcases.

(i) Parabolic case: $M = 0$, $D_2 f(\pm) = 0$, $D_3 f(\pm) = -I_m$. The dispersion relation (3.6) reads

$$\det\left(\tilde{\lambda} I_m + \omega^2 A - D_1 f(\pm)\right) = 0, \quad \tilde{\lambda} = \lambda - i\omega\mu_\star, \tag{3.10}$$

and the corresponding eigenvalue problem may be written as

$$\tilde{\lambda} z = -\left(\omega^2 A - D_1 f(\pm)\right) z, \quad 0 \ne z \in \mathbb{C}^m, \quad \tilde{\lambda} = \lambda - i\omega\mu_\star. \tag{3.11}$$

Let us assume positivity of A and $-D_1 f(\pm)$ in the sense that

$$\operatorname{Re} z^H A z > 0, \quad \operatorname{Re} z^H D_1 f(\pm) z < 0 \quad \forall\, z \in \mathbb{C}^m. \tag{3.12}$$

Multiplying (3.11) by z^H and taking the real part shows that the solutions $\tilde{\lambda}$ of (3.10) have negative real parts and the gap is guaranteed. This is still true if A is nonnegative but has zero eigenvalues. In this case, (2.1) is of mixed hyperbolic-parabolic type and the nonlinear stability theory becomes considerably more involved; see [34].

(ii) Undamped hyperbolic case: $M = I_m$, $D_2 f(\pm) = 0$, $D_3 f(\pm) = 0$. The dispersion relation (3.6) reads

$$\det\left(\tilde{\lambda}^2 I_m + \omega^2 A - D_1 f(\pm)\right) = 0, \quad \tilde{\lambda} = \lambda - i\omega\mu_\star.$$

Whenever $\lambda \in \mathbb{C}, \omega \in \mathbb{R}$ solve this system, so does the pair $-\lambda, -\omega$. Hence, the eigenvalues lie either on the imaginary axis or on both sides of the imaginary axis. Therefore, a spectral gap cannot exist. This is the Hamiltonian case, where one can only expect stability (but not asymptotic stability) of the wave. We refer to the local stability theory developed in [17], [18] (see also [23] for a recent account). Note that in this case the positivity assumption (3.12) only guarantees $\operatorname{Re}\tilde{\lambda}^2 < 0$, i.e., $\frac{\pi}{4} < |\arg(\tilde{\lambda})| \leqslant \frac{\pi}{2}$ for $\tilde{\lambda} = \lambda - i\omega\mu_\star$ and all eigenvalues $\lambda \in \sigma(\mathcal{A}(\cdot,\omega))$.

(iii) Scalar case: $M = 1$, $D_2 f(\pm) = 0$, $D_3 f(\pm) = -\eta$. Now we have $A = a$, $-D_1 f(\pm) = \delta$ with real numbers $a, \eta, \delta > 0$, and the dispersion equation (3.6) becomes

$$\tilde{\lambda}^2 + \eta\tilde{\lambda} + a\omega^2 + \delta = 0, \quad \tilde{\lambda} = \lambda - i\omega\mu_\star. \tag{3.13}$$

This case occurs with the Nagumo wave equation. The solutions of (3.13) are

$$\lambda = i\omega\mu_\star - \frac{\eta}{2} \pm \left(\frac{\eta^2}{4} - \delta - \omega^2 a\right)^{1/2}, \quad \omega \in \mathbb{R}.$$

If $\eta^2 \leq 4\delta$, then all solutions λ of (3.13) lie on the vertical line $\operatorname{Re}\lambda = -\frac{\eta}{2} < 0$. A short discussion shows that they actually cover this line under the assumption $\mu_\star^2 < a$, which corresponds to positivity of the matrix $A - \mu_\star^2 M$ occuring in (1.8). If $\eta^2 > 4\delta$ then the solutions λ of (3.13) lie again on this line (resp., cover it if $\mu_\star^2 < a$) for values $|\omega| \geq \omega_0 := (\frac{1}{a}(\frac{\eta^2}{4} - \delta))^{1/2}$. But for values $|\omega| \leq \omega_0$ they form the ellipse

$$p_1^{-2}\,(\operatorname{Re}\lambda + \frac{\eta}{2})^2 + p_2^{-2}\,(\operatorname{Im}\lambda)^2 = 1, \quad p_1 = a^{1/2}\omega_0, \;\; p_2 = |\mu_\star|\omega_0. \tag{3.14}$$

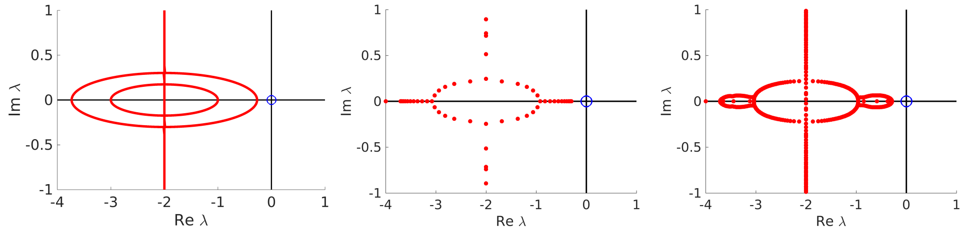
The rightmost point of the ellipse $-\beta := -\frac{\eta}{2} + (\frac{\eta^2}{4} - \delta)^{1/2}$ is still negative and therefore can be taken for the spectral gap (3.9).

Example 3.6 (spectrum of Nagumo wave equation). As in Example 2.3, consider the Nagumo wave equation (2.24) with $M = \varepsilon > 0$, $A = 1$, and $f(u, u_x, u_t) = g(u) - u_t$. The traveling wave solution $u_\star(x,t)$, given by (2.25), connects the asymptotic states $v_- = 0$ and $v_+ = 1$. With $D_1 f(-) = g'(v_-) = -b$, $D_1 f(+) = g'(v_+) = b - 1$, $D_2 f(\pm) = 0$, and $D_3 f(\pm) = -1$ we find the dispersion relation

$$\varepsilon\tilde{\lambda}^2 + \tilde{\lambda} + \omega^2 + b = 0 \quad \text{or} \quad \varepsilon\tilde{\lambda}^2 + \tilde{\lambda} + \omega^2 - b + 1 = 0, \quad \tilde{\lambda} = \lambda - i\omega\mu_\star.$$

The scalar case discussed above applies with the settings $\eta = \frac{1}{\varepsilon} = a$, $\delta_\pm = -\frac{g'(v_\pm)}{\varepsilon}$. The subset $\sigma_{\mathrm{disp}}(\mathcal{P})$ of the essential spectrum lies on the union of the line $\operatorname{Re}\lambda = -\frac{1}{2\epsilon}$ and possibly two ellipses defined by (3.14) with $\omega_0 = \omega_\pm = \left(\frac{1}{4\varepsilon} + g'(v_\pm)\right)^{1/2}$. The ellipse belonging to v_+, resp., v_- occurs if $1 - b < \frac{1}{4\varepsilon}$, resp., $b < \frac{1}{4\varepsilon}$. Since $0 < b < 1$ both ellipses show up in $\sigma_{\mathrm{disp}}(\mathcal{P})$ if $\varepsilon \leq \frac{1}{4}$. In any case, there is a gap beween the essential spectrum and the imaginary axis in the sense of (3.9) with

$$\beta = \frac{1}{2\varepsilon}\left(1 - \left(1 - 4\varepsilon^2 \min(b, 1-b)\right)^{1/2}\right).$$



(a) (b) (c)

FIG. 7. *Dispersion set of the Nagumo wave equation for* $\varepsilon = b = \frac{1}{4}$ (a) *and the numerical spectrum on the spatial domain* $[-R, R]$ *for* $R = 50$ (b) *and* $R = 400$ (c).

Figure 7(a) shows the piece of spectrum guaranteed by our propositions at parameter values $\varepsilon = b = \frac{1}{4}$. It contains the eigenvalue at zero (blue circle) determined by Proposition 3.3 and the dispersion set (red lines) determined by Proposition 3.5. There may be further isolated eigenvalues. The numerical spectrum of the Nagumo wave on the spatial domain $[-R, R]$ and subject to periodic boundary conditions is shown in Figure 7(b) for $R = 50$ and in Figure 7(c) for $R = 400$. Each of them consists of the approximations of the eigenvalue (blue circle) and of the essential spectrum (red dots). The missing line inside the ellipse in Figure 7(b) gradually appears numerically when enlarging the spatial domain; see Figure 7(c). The second ellipse develops only on even larger domains.

Example 3.7 (spectrum of FitzHugh–Nagumo wave system). As in Example 2.4, the FitzHugh–Nagumo wave system (2.31) with $M = \varepsilon I_2$, $A = \operatorname{diag}(1, \frac{\rho + c_\star^2 \varepsilon}{1 + c_\star^2 \varepsilon})$, $f(u, u_x, u_t) = g(u) - u_t$, and parameters from (2.29) has a traveling pulse solution $u_\star(x,t) = v_\star(x - \mu_\star t)$ with velocity $\mu_\star = \frac{c_\star}{k}$, where $k = \sqrt{1 + c_\star^2 \varepsilon}$ and $c_\star \approx -0.7892$. For the subsequent numerics we again choose $\varepsilon = 10^{-2}$ and use the profile $v_\star$ and velocity $\mu_\star$ obtained from the simulation in Example 2.4. For the nonlinearity f we find with g from (2.28)

$$D_1 f(\pm) = Dg(v_\pm) = \begin{pmatrix} 1 - (v_{\pm,1})^2 & -1 \\ \phi & -b\phi \end{pmatrix}, \quad D_2 f(\pm) = 0, \quad D_3 f(\pm) = -I_2.$$

Then every $\lambda \in \mathbb{C}$ satisfying the dispersion relation for the FitzHugh–Nagumo pulse

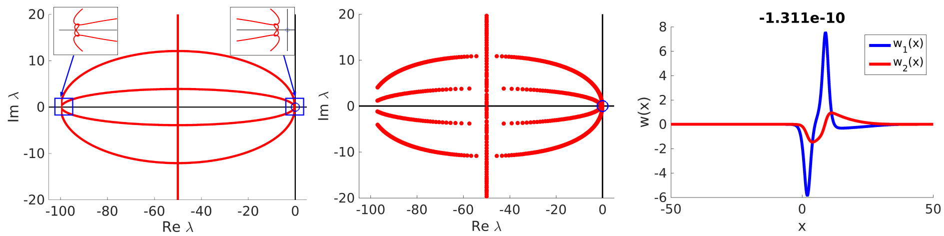
$$\det \begin{pmatrix} \varepsilon\lambda^2 + p(\omega)\lambda + q_1(\omega) & 1 \\ -\phi & \varepsilon\lambda^2 + p(\omega)\lambda + q_2(\omega) \end{pmatrix} = 0 \tag{3.15}$$

for some $\omega \in \mathbb{R}$ belongs to $\sigma_{\text{ess}}(\mathcal{P})$. Here we use the abbreviations

$$p(\omega) = 1 - 2i\omega\mu_\star\varepsilon, \quad q_1(\omega) = \omega^2(1 - \mu_\star^2\varepsilon) - i\omega\mu_\star - (1 - (v_{\pm,1})^2),$$
$$q_2(\omega) = \omega^2 \left(\frac{\rho + c_\star^2 \varepsilon}{1 + c_\star^2 \varepsilon} - \mu_\star^2 \varepsilon \right) - i\omega\mu_\star + b\phi.$$

Equation (3.15) leads to the quartic problem

$$0 = a_4\lambda^4 + a_3\lambda^3 + a_2\lambda^2 + a_1\lambda + a_0 \tag{3.16}$$



(a) (b) (c)

FIG. 8. *Dispersion set of the FitzHugh–Nagumo wave system,* $\varepsilon = 10^{-2}$*, and parameters from* (2.29) (a)*, the numerical spectrum* (b)*, and both components of the eigenfunction belonging to* $\lambda \approx 0$ (c).

with ω-dependent coefficients

$$a_4 = \varepsilon^2, \quad a_3 = 2\varepsilon p, \quad a_2 = \varepsilon(q_1 + q_2) + p^2, \quad a_1 = p(q_1 + q_2), \quad a_0 = q_1 q_2 + \phi.$$

Instead of solving (3.16), we solved numerically the quadratic eigenvalue problem (3.15) using parameter continuation with respect to ω. In this way we obtain analytical information about the dispersion set of the FitzHugh–Nagumo pulse (red lines in Figure 8(a)), which is part of the essential spectrum by Proposition 3.5. Zooming into the essential spectrum shows that the parabola-shaped structure contains at both ends a loop which is already known from the first order limit case; see [3]. From these results and with the help of Proposition A.4 it is obvious that there is again a spectral gap to the imaginary axis, but we have no analytic expression for this gap. The numerical spectrum for periodic boundary conditions is shown in Figure 8(b). It consists of the approximations of the point spectrum (blue circle) and of the essential spectrum (red dots). Figure 8(c) shows the approximation of both components w_1 and w_2 of the eigenfunction $w(\xi) \approx v_{\star,\xi}(\xi)$ belonging to the small eigenvalue $\lambda = 1.311 \cdot 10^{-10}$ close to zero. Note that an approximation of $v_\star = (v_{\star,1}, v_{\star,2})^T$ was provided in Figure 5(a).

4. First order systems and stability of traveling waves. In this section we transform the original second order damped wave equation (1.1) into a first order system of triple size. To the first order system we then apply stability results from [33] and derive asymptotic stability of traveling waves for the original second order problem and the second order freezing method. Transferring regularity and stability between these two systems requires some care, and we will provide details of the proofs in Appendix A.

4.1. Transformation to first order system and stability with asymptotic phase. In the following we impose the following smoothness condition.

Assumption 4.1. The function $f : \mathbb{R}^{3m} \to \mathbb{R}^m$ satisfies $f \in C^3(\mathbb{R}^{3m}, \mathbb{R}^m)$.

We also impose the following well-posedness condition.

Assumption 4.2. The matrix $M \in \mathbb{R}^{m,m}$ is invertible and $M^{-1}A$ is positive diagonalizable.

Assumption 4.2 implies that there is a (not necessarily unique) positive diagonalizable matrix $N \in \mathbb{R}^{m,m}$ satisfying $N^2 = M^{-1}A$. Let $\lambda_1 \geqslant \cdots \geqslant \lambda_m > 0$ denote the real positive eigenvalues of N.

We transform to a first order system by introducing $U = (U_1, U_2, U_3)^\top \in \mathbb{R}^{3m}$ via

$$U_1 = u, \quad U_2 = u_t + Nu_x, \quad U_3 = u_t - Nu_x + cu, \tag{4.1}$$

where $c \in \mathbb{R}$ is an arbitrary constant to be determined later. These variables transform (1.1) into the first order system

$$U_t = EU_x + F(U), \tag{4.2}$$

with $E \in \mathbb{R}^{3m,3m}$ and $F : \mathbb{R}^{3m} \to \mathbb{R}^{3m}$ given by

$$\begin{aligned} &E = \begin{pmatrix} N & 0 & 0 \\ 0 & N & 0 \\ 0 & 0 & -N \end{pmatrix}, \quad F(U) = \begin{pmatrix} -cU_1 + U_3 \\ \tilde{f}(U) \\ \tilde{f}(U) + cU_2 \end{pmatrix}, \\ &\tilde{f}(U) := M^{-1} f\left(U_1, \frac{1}{2}N^{-1}(U_2 - U_3 + cU_1), \frac{1}{2}(U_2 + U_3 - cU_1)\right). \end{aligned} \tag{4.3}$$

Thus we write the second order Cauchy problem (2.1) as a first order Cauchy problem for (4.2),

$$U_t = EU_x + F(U), \quad U(\cdot, 0) = U_0 := (u_0, v_0 + Nu_{0,x}, v_0 - Nu_{0,x} + cu_0)^\top. \tag{4.4}$$

Remark 4.3. The transformation to a first order system has some arbitrariness and does not influence the results for the second order problem (1.1). The current transformation to a system of dimension $3m$ improves an earlier version [5] of our work which was limited to the semilinear case $f(u, u_x, u_t) = -Bu_t + Cu_x + g(u)$. There we used $U_1 = u, U_2 = u_t - Nu_x$ to obtain a system of minimal dimension $2m$. But for this transformation the general nonlinear equation (1.1) does not lead to a semilinear system of type (4.2). The drawback of the nonminimal dimension $3m$ is that extra eigenvalues of the linearized system appear which have no analogue for the linearized second order system. The constant c above will be used in section A.2 to control these extra eigenvalues.

We emphasize that system (4.2) is diagonalizable hyperbolic. More precisely, there is a nonsingular block-diagonal matrix $T \in \mathbb{R}^{3m,3m}$, so that the change of variables $W = T^{-1}U$ transforms (4.2) into diagonal hyperbolic form

$$W_t = \Lambda_E W_x + G(W),\ \Lambda_E = T^{-1}ET = \operatorname{diag}(\Lambda, \Lambda, -\Lambda),\ G(W) = T^{-1}F(TW), \tag{4.5}$$

where $\Lambda = \operatorname{diag}(\lambda_1, \ldots, \lambda_m)$. For systems of type (4.4), (4.5) we have local well-posedness of the Cauchy problem in suitable function spaces such as

$$\mathcal{CH}^k(J; \mathbb{R}^n) = \bigcap_{j=0}^{k} C^{k-j}\big(J, H^j(\mathbb{R}, \mathbb{R}^n)\big), \quad J \subseteq \mathbb{R} \text{ interval},\ k \in \mathbb{N}_0,\ n \in \mathbb{N}; \tag{4.6}$$

see, e.g., [31, sect. 6]. Our regularity condition on the traveling wave is as follows.

Assumption 4.4. The pair $(v_\star, \mu_\star) \in C_b^2(\mathbb{R}, \mathbb{R}^m) \times \mathbb{R}$ satisfies $v_{\star,\xi} \in H^3(\mathbb{R}, \mathbb{R}^m)$ and is a nonconstant solution of the second order traveling wave equation (1.8) with

$$\lim_{\xi \to \pm\infty} v_\star(\xi) = v_\pm, \quad \lim_{\xi \to \pm\infty} v_{\star,\xi}(\xi) = 0, \quad f(v_\pm, 0, 0) = 0.$$

The first order system (4.2) then has a traveling wave

$$U_\star(x,t) = V_\star(x-\mu_\star t), \quad V_\star := \begin{pmatrix} v_\star \\ (N-\mu_\star I_m)v_{\star,\xi} \\ cv_\star - (N+\mu_\star I_m)v_{\star,\xi} \end{pmatrix} \in C_b^2(\mathbb{R},\mathbb{R}^m)\times C_b^1(\mathbb{R},\mathbb{R}^{2m}). \tag{4.7}$$

The profile $V_\star$ solves the equation

$$0 = (E+\mu_\star I_{3m})V_{\star,\xi} + F(V_\star) \tag{4.8}$$

and satisfies

$$\lim_{\xi\to\pm\infty} V_\star(\xi) = V_\pm := (v_\pm, 0, cv_\pm) \quad \text{and} \quad F(V_\pm) = 0. \tag{4.9}$$

Our next assumption is as follows.

Assumption 4.5. The matrix $A - \mu_\star^2 M$ is nonsingular.

It guarantees that (1.8) is a regular second order system and that $v_\star \in C_b^5(\mathbb{R},\mathbb{R}^m)$ holds due to Assumptions 4.1 and 4.4. From $A-\mu_\star^2 M = M(N-\mu_\star I_m)(N+\mu_\star I_m)$ one further infers that the matrix $E+\mu_\star I_{3m}$ in (4.8) is nonsingular. This will enable us to apply the stability results from [33] which hold for hyperbolic systems where the matrix $E+\mu_\star I_{3m}$ is real diagonalizable with nonzero but not necessarily distinct eigenvalues. The condition also ensures that any solution $V_\star \in C_b^1(\mathbb{R},\mathbb{R}^{3m})$ of (4.8) has a first component in $C_b^2(\mathbb{R},\mathbb{R}^m)$ which solves the second order traveling wave equation (1.8). Moreover, using the limits from Assumption 4.4 in (1.8) shows

$$\lim_{\xi\to\pm\infty} v_{\star,\xi\xi}(\xi) = 0. \tag{4.10}$$

Next, recall the dispersion set (3.6) for the original second order problem

$$\sigma_{\mathrm{disp}}(\mathcal{P}) = \Big\{\lambda\in\mathbb{C} : \det\big(\lambda^2 A_2 + \lambda A_1^\pm(\omega) + A_0^\pm(\omega)\big) = 0 \\ \text{for some } \omega\in\mathbb{R} \text{ and some sign } \pm\Big\} \tag{4.11}$$

with $A_0^\pm, A_1^\pm, A_2$ given in (3.5). We require the following.

Assumption 4.6. There is $\delta>0$, such that $\operatorname{Re}(\sigma_{\mathrm{disp}}(\mathcal{P})) < -\delta$.

Finally, we exclude nonzero eigenvalues in the right half plane.

Assumption 4.7. The eigenvalue 0 of $\mathcal{P}$ is simple and there is no other eigenvalue of $\mathcal{P}$ with real part greater than $-\delta$ with δ given by Assumption 4.6.

With these assumptions our first main result reads as follows.

THEOREM 4.8 (stability with asymptotic phase). *Let Assumptions* 4.1–4.7 *hold. Then, for all $0<\eta<\delta$ there is $\rho>0$ such that for all $u_0\in v_\star + H^3(\mathbb{R},\mathbb{R}^m)$, $v_0\in H^2(\mathbb{R},\mathbb{R}^m)$ with*

$$\|u_0 - v_\star\|_{H^3} + \|v_0 + \mu_\star v_{\star,\xi}\|_{H^2} \le \rho, \tag{4.12}$$

the Cauchy problem (2.1) *has a unique global solution $u\in v_\star + \mathcal{CH}^2([0,\infty);\mathbb{R}^m)$. Moreover, there exist $\varphi_\infty = \varphi_\infty(u_0,v_0)$ and $C=C(\eta,\rho)$ satisfying*

$$|\varphi_\infty| \le C\Big(\|u_0-v_\star\|_{H^3} + \|v_0+\mu_\star v_{\star,\xi}\|_{H^2}\Big) \tag{4.13}$$

and

$$\begin{aligned}(4.14)\quad &\|u(\cdot,t)-v_\star(\cdot-\mu_\star t-\varphi_\infty)\|_{H^2}+\|u_t(\cdot,t)+\mu_\star v_{\star,\xi}(\cdot-\mu_\star t-\varphi_\infty)\|_{H^1}\\ &\quad\le C\Big(\|u_0-v_\star\|_{H^3}+\|v_0+\mu_\star v_{\star,\xi}\|_{H^2}\Big)e^{-\eta t}\quad \forall t\ge 0.\end{aligned}$$

The proof will be given in Appendix A. Let us note that the loss of one derivative for the solution when compared to initial data is typical for hyperbolic stability theorems and results from the theory in [33].

4.2. Stability of the freezing method. Let us first apply the freezing method to the first order system (4.4). We introduce new unknowns $\gamma(t)\in\mathbb{R}$ and $V(\xi,t)\in\mathbb{R}^{3m}$ via the ansatz

$$(4.15)\qquad U(x,t)=V(\xi,t),\quad \xi:=x-\gamma(t),\quad x\in\mathbb{R},\ t\ge 0.$$

This formally leads to

$$\begin{aligned}&(4.16a)\qquad V_t=(E+\mu I_{3m})V_\xi+F(V),\\ &(4.16b)\qquad \gamma_t=\mu,\\ &(4.16c)\qquad V(\cdot,0)=V_0:=U_0=(u_0,v_0+Nu_{0,\xi},v_0-Nu_{0,\xi}+cu_0)^\top,\quad \gamma(0)=0,\end{aligned}$$

with E and F from (4.3). In (4.16) we introduced the time-dependent function $\mu(t)\in\mathbb{R}$ for convenience. As before, (4.16b) decouples and can be solved in a postprocessing step. One needs an additional algebraic constraint to compensate the extra variable μ. To relate the second order freezing equation (2.3) and the first order version (4.16), we omit the introduction of μ_2 in (2.3) and write it in the form

$$\begin{aligned}&(4.17a)\qquad Mv_{tt}=(A-\mu^2M)v_{\xi\xi}+2\mu Mv_{\xi t}+\mu_tMv_\xi+f(v,v_\xi,v_t-\mu v_\xi),\\ &(4.17b)\qquad \gamma_t=\mu,\\ &(4.17c)\qquad v(\cdot,0)=u_0,\quad v_t(\cdot,0)=v_0+\mu(0)u_{0,\xi},\quad \gamma(0)=0.\end{aligned}$$

Transforming (4.17) into a first order system by introducing $V=(V_1,V_2,V_3)^\top\in\mathbb{R}^{3m}$,

$$(4.18)\qquad V_1=v,\quad V_2=v_t+(N-\mu I_m)v_\xi,\quad V_3=v_t-(N+\mu I_m)v_\xi+cv,$$

we again find the system (4.16). As a consequence we obtain the equivalence of the freezing systems for the first and the second order formulation. Henceforth, we restrict to the fixed phase condition (2.7) for which we make the following assumption.

Assumption 4.9. The template function $\hat v$ belongs to $v_\star+H^1(\mathbb{R},\mathbb{R}^m)$ and satisfies

$$\begin{aligned}&(4.19a)\qquad \langle\hat v-v_\star,\hat v_\xi\rangle_{L^2}=0,\\ &(4.19b)\qquad \langle v_{\star,\xi},\hat v_\xi\rangle_{L^2}\ne 0.\end{aligned}$$

Condition (4.19a) implies that (2.6) holds for the fixed phase condition (2.7), so that $(v_\star,\mu_\star,0)$ is a stationary solution of (2.16a), (2.16b) (skipping the γ-equation needed for reconstruction only). Condition (4.19b) specifies a suitable nondegeneracy.

Now we are ready to state asymptotic stability (in the sense of Lyapunov) of the steady state $(v_\star,\mu_\star,0)$ for the freezing system (2.16) that belongs to the nonlinear wave equation.

THEOREM 4.10 (stability of the freezing method). *Let Assumptions* 4.1–4.7 *hold and consider the phase condition* $\psi_{\mathrm{fix},3}(v,v_t,\mu_1,\mu_2)=\langle v-\hat{v},\hat{v}_\xi\rangle_{L^2}$ *with a template function* $\hat{v}$ *which fulfills Assumption* 4.9. *Then, for all* $0<\eta<\delta$ *there is* $\rho>0$ *such that for all* $u_0\in v_\star+H^3(\mathbb{R},\mathbb{R}^m)$, $v_0\in H^2(\mathbb{R},\mathbb{R}^m)$ *and* $\mu_1^0\in\mathbb{R}$ *which satisfy*

$$\|u_0-v_\star\|_{H^3}+\|v_0+\mu_\star v_{\star,\xi}\|_{H^2}\le\rho \tag{4.20}$$

and the consistency conditions (2.10), (2.11), $\langle u_0-\hat{v},\hat{v}_\xi\rangle_{L^2}=0$, *the following properties hold. The freezing system* (2.16) *has a unique global solution*

$$(v,\mu_1,\mu_2,\gamma)\in(v_\star+\mathcal{CH}^2([0,\infty);\mathbb{R}^m))\times C^1([0,\infty))\times C([0,\infty))\times C^2([0,\infty)).$$

Moreover, there exists some $C=C(\rho,\eta)>0$ *such that the following exponential stability estimate holds for all* $t\ge 0$*:*

$$\begin{aligned}&\|v(\cdot,t)-v_\star\|_{H^2}+\|v_t(\cdot,t)\|_{H^1}+|\mu_1(t)-\mu_\star|\\&\quad\le C\big(\|u_0-v_\star\|_{H^3}+\|v_0+\mu_\star v_{\star,\xi}\|_{H^2}\big)\,e^{-\eta t}.\end{aligned} \tag{4.21}$$

The proof builds on the fact that the original second order version (2.16) and the first order version (4.16) of the freezing method for traveling waves in (1.1) are equivalent in suitable function spaces. This will be detailed in Appendix A

Appendix A. Proof of Stability theorems.

In this appendix we provide a detailed proof of Theorems 4.8 and 4.10.

A.1. Results for first order systems.

Let us recall the stability result from [33, Thm. 2.5] for first order systems of the general type

$$W_t=\Lambda_E W_x+G(W),\quad x\in\mathbb{R},\ t\ge 0, W(x,t)\in\mathbb{R}^l, \tag{A.1a}$$

$$W(\cdot,0)=W_0. \tag{A.1b}$$

The assumptions are as follows:

(i) The matrix $\Lambda_E\in\mathbb{R}^{l,l}$ is diagonal.

(ii) The nonlinearity G belongs to $C^3(\mathbb{R}^l,\mathbb{R}^l)$.

(iii) There exists a traveling wave solution $W(x,t)=W_\star(x-\mu_\star t)$ of (A.1) such that $W_\star\in C_b^1(\mathbb{R},\mathbb{R}^l)$, $W_{\star,\xi}\in H^2(\mathbb{R},\mathbb{R}^l)$.

(iv) The matrix function $Y(\xi)=DG\big(W_\star(\xi)\big)$ satisfies $\lim_{\xi\to\pm\infty}Y(\xi)=Y_\pm$ and $\lim_{\xi\to\pm\infty}Y'(\xi)=0$.

(v) The matrix $\Lambda_E+\mu_\star I_l\in\mathbb{R}^{l,l}$ is nonsingular.

(vi) There exists $\delta>0$ such that

$$\begin{aligned}\operatorname{Re}\{s\in\mathbb{C}: s\in\sigma\big(i\omega(\Lambda_E+\mu_\star I_l)+Y_\pm\big)\ \text{for some}\ \omega\in\mathbb{R}\\ \text{and some sign}\ \pm\}\le-\delta.\end{aligned}$$

(vii) The operator $\mathcal{Y}_{\mathrm{1st}}=(\Lambda_E+\mu_\star I_l)\partial_\xi+Y(\cdot):H^1(\mathbb{R},\mathbb{R}^l)\to L^2(\mathbb{R},\mathbb{R}^l)$ has the algebraically simple eigenvalue 0 and satisfies $\sigma_{\mathrm{point}}(\mathcal{Y}_{\mathrm{1st}})\cap\{\operatorname{Re}s>-\delta\}=\{0\}$.

Then for every $0<\eta<\delta$ there is $\rho_0>0$ so that for all $W_0\in W_\star+H^2(\mathbb{R},\mathbb{R}^l)$ with $\|W_0-W_\star\|_{H^2}\le\rho_0$ the Cauchy problem (A.1) has a unique global solution W in $W_\star+\mathcal{CH}^1([0,\infty);\mathbb{R}^l)$. Moreover, there is $\varphi_\infty=\varphi_\infty(W_0)\in\mathbb{R}$ and $C=C(\eta,\rho_0)>0$ such that

$$|\varphi_\infty|\le C\|W_0-W_\star\|_{H^2}, \tag{A.2}$$

$$\|W(\cdot,t) - W_\star(\cdot - \mu_\star t - \varphi_\infty)\|_{H^1} \le C\|W_0 - W_\star\|_{H^2} e^{-\eta t} \quad \forall t \ge 0. \tag{A.3}$$

In [33, Thm. 2.5] the eigenvalues of Λ_E are assumed to be in decreasing order. However, this was done for convenience of the proof only, and the result holds verbatim without this ordering. Our goal is to apply the stability result to the system (4.5) where Λ_E is diagonal but the eigenvalues are not ordered. In the following we show the assumptions (ii)–(vii) for the system (4.5). Our first observation is that instead of checking assumptions (ii)–(vii) for the transformed data $W_\star = T^{-1}V_\star$, $\Lambda_E = T^{-1}ET$, and $G = T^{-1}FT$, it is sufficient to check them for the data $V_\star$, E, and F of the original system (4.2).

Condition (ii) follows from Assumption 4.1. Moreover, conditon (iii) is a consequence of (4.7) and Assumption 4.4. From (4.3) we obtain for $Z = DF(V_\star)$

$$Z = \begin{pmatrix} -cI_m & 0 & I_m \\ \Phi_1 & \Phi_2 & \Phi_3 \\ \Phi_1 & \Phi_2 + cI_m & \Phi_3 \end{pmatrix}, \quad \begin{pmatrix} \Phi_1 \\ \Phi_2 \\ \Phi_3 \end{pmatrix} := \begin{pmatrix} M^{-1}D_1 f(\star) - c\Phi_3 \\ \frac{1}{2}M^{-1}(D_2 f(\star)N^{-1} + D_3 f(\star)) \\ \frac{1}{2}M^{-1}(-D_2 f(\star)N^{-1} + D_3 f(\star)) \end{pmatrix}, \tag{A.4}$$

where $(\star) = (v_\star, v_{\star,\xi}, -\mu_\star v_{\star,\xi})$. By Assumption 4.4 the limit $Z_\pm = \lim_{\xi\to\pm\infty} Z(\xi)$ is

$$Z_\pm = \begin{pmatrix} -cI_m & 0 & I_m \\ \Phi_1^\pm & \Phi_2^\pm & \Phi_3^\pm \\ \Phi_1^\pm & \Phi_2^\pm + cI_m & \Phi_3^\pm \end{pmatrix}, \quad \begin{pmatrix} \Phi_1^\pm \\ \Phi_2^\pm \\ \Phi_3^\pm \end{pmatrix} := \begin{pmatrix} M^{-1}D_1 f(\pm) - c\Phi_3^\pm \\ \frac{1}{2}M^{-1}(D_2 f(\pm)N^{-1} + D_3 f(\pm)) \\ \frac{1}{2}M^{-1}(-D_2 f(\pm)N^{-1} + D_3 f(\pm)) \end{pmatrix}, \tag{A.5}$$

where $(\pm) = (v_\pm, 0, 0)$. Differentiating (A.4) w.r.t. ξ and using Assumption 4.4 as well as (4.10) then shows $Z'(\xi) \to 0$ as $\xi \to \pm\infty$. Further, condition (v) follows from Assumption 4.5 as has been noted in section 4. The conditions (vi) and (vii) are discussed in the next subsection.

A.2. Spectral relations of first and second order problems. We transfer the spectral properties of the original second order problem (1.1) to the first order problem (4.2) and vice versa. Throughout this section we impose Assumptions 4.1, 4.2, and 4.4 and define $V_\star$ by (4.7).

By Definition 3.1, the spectral problem for the second order problem (1.1), considered in a co-moving frame, is given by the solvability properties of

$$\mathcal{P}(\lambda) : H^2(\mathbb{R},\mathbb{C}^m) \to L^2(\mathbb{R},\mathbb{C}^m), \quad \text{defined by (3.1).}$$

The analogue for the first order formulation (4.2) is the first order differential operator

$$\begin{aligned} &\mathcal{P}_{1\text{st}}(\lambda) : H^1(\mathbb{R},\mathbb{C}^{3m}) \to L^2(\mathbb{R},\mathbb{C}^{3m}), \\ &\mathcal{P}_{1\text{st}}(\lambda) = \lambda I_{3m} - \mathcal{Z}_{1\text{st}}, \quad \mathcal{Z}_{1\text{st}} = (E + \mu_\star I_{3m})\partial_\xi + Z(\cdot), \end{aligned} \tag{A.6}$$

obtained by linearizing (4.2) in the co-moving frame about the traveling wave $V_\star$. Introducing the first order operators

$$\mathcal{P}_-(\lambda) = \lambda - (N + \mu_\star I_m)\partial_\xi, \quad \mathcal{P}_+(\lambda) = \lambda + (N - \mu_\star I_m)\partial_\xi, \tag{A.7}$$

we may write $\mathcal{P}_{1\text{st}}(\lambda)$ as a block operator

$$\mathcal{P}_{1\text{st}}(\lambda) = \begin{pmatrix} \mathcal{P}_-(\lambda) + cI_m & 0 & -I_m \\ -\Phi_1 & \mathcal{P}_-(\lambda) - \Phi_2 & -\Phi_3 \\ -\Phi_1 & -\Phi_2 - cI_m & \mathcal{P}_+(\lambda) - \Phi_3 \end{pmatrix}. \tag{A.8}$$

Finally, it is convenient to introduce the normalized operator polynomial $\tilde{\mathcal{P}}(\lambda) = M^{-1}\mathcal{P}(\lambda)$, $\lambda \in \mathbb{C}$, which has the same spectrum as $\mathcal{P}(\lambda)$. The key to the relation of spectra is the following factorization:

(A.9)
$$T_1\mathcal{P}_{1\text{st}}(\lambda) = \begin{pmatrix} \tilde{\mathcal{P}}(\lambda) & -\Phi_2 - cI_m & \mathcal{P}_+(\lambda) - \Phi_3 \\ 0 & \mathcal{P}_-(\lambda) + cI_m & -\mathcal{P}_+(\lambda) \\ 0 & 0 & -I_m \end{pmatrix} \begin{pmatrix} I_m & 0 & 0 \\ -\mathcal{P}_+(\lambda) & I_m & 0 \\ -\mathcal{P}_-(\lambda) - cI_m & 0 & I_m \end{pmatrix},$$

where

$$T_1 = \begin{pmatrix} 0 & 0 & I_m \\ 0 & I_m & -I_m \\ I_m & 0 & 0 \end{pmatrix}.$$

This follows from (A.4) and (A.8) by a straightforward but somewhat lengthy calculation. The factorization (A.9) is motivated by the equivalence notion for matrix polynomials (see, e.g., [16, Chap. S1.6]).

Recall a well-known result on Fredholm properties of first order operators from [30].

PROPOSITION A.1. *Consider a first order system*

$$(\partial_\xi - Q(\xi))V = R \in L^2(\mathbb{R}, \mathbb{C}^N), \tag{A.10}$$

where the matrix-valued function $Q : \mathbb{R} \to \mathbb{C}^{N,N}$ *is continuous and has limits*

$$Q_\pm = \lim_{\xi \to \pm\infty} Q(\xi). \tag{A.11}$$

Further assume that $Q_\pm$ *have no eigenvalues on the imaginary axis. Then the operator*

$$\mathcal{Q} = \partial_\xi - Q(\cdot) : H^1(\mathbb{R}, \mathbb{C}^N) \to L^2(\mathbb{R}, \mathbb{C}^N)$$

is Fredholm of index $\dim E_+^s - \dim E_-^s$*, where* $E_\pm^s \subseteq \mathbb{C}^N$ *is the stable subspace of* $Q_\pm$ *(i.e., the maximal invariant subspace associated with eigenvalues of negative real part).*

A consequence of this result for parametrized systems is the following.

PROPOSITION A.2. *Consider a first order system*

$$\mathcal{Q}(\lambda)V = (\partial_\xi - Q(\xi, \lambda))V = R \in L^2(\mathbb{R}, \mathbb{C}^l), \tag{A.12}$$

with a matrix polynomial $Q(\xi, \lambda) = \sum_{j=0}^q Q_j(\xi)\lambda^j$, $Q_j \in C(\mathbb{R}, \mathbb{C}^{l,l})$*. Assume that the limits* $\lim_{\xi\to\pm\infty} Q_j(\xi) = Q_j^\pm$ *exist and let* $Q^\pm(\lambda) = \sum_{j=0}^q Q_j^\pm \lambda^j$*. Then the dispersion set*

$$\sigma_{\text{disp}}(\mathcal{Q}) = \{\lambda \in \mathbb{C} : \det(i\omega I - Q^\pm(\lambda)) = 0 \text{ for some } \omega \in \mathbb{R} \text{ and some sign } \pm\} \tag{A.13}$$

is contained in the essential spectrum $\sigma_{\text{ess}}(\mathcal{Q})$*. For* $\lambda \notin \sigma_{\text{disp}}(\mathcal{Q})$*, the operator* $\mathcal{Q}(\lambda) : H^1(\mathbb{R}, \mathbb{C}^l) \to L^2(\mathbb{R}, \mathbb{C}^l)$ *is Fredholm of index* $\dim E_+^s(\lambda) - \dim E_-^s(\lambda)$*, where* $E_\pm^s(\lambda)$ *denotes the stable subspace of* $Q^\pm(\lambda)$*.*

This result may be found in [23, Thm. 3.1.13] (note that the dispersion set is called the Fredholm border there).

If we replace ∂_ξ by $i\omega$ and let $\xi \to \pm\infty$ in (A.9), then the left and right factors in (A.9) are λ-dependent matrices with a constant determinant (see the equivalence notion of matrix polynomials in [16, Chap. S1.6]). Hence the dispersion set of the first order operator $\mathcal{P}_{1\mathrm{st}}(\lambda)$ is completely determined by the dispersion set (3.8) of the second order operator $\tilde{\mathcal{P}}(\lambda)$ and the first order operator $\mathcal{P}_-(\lambda)+cI_m$. Since $N+\mu_\star I_m$ has nonzero real eigenvalues $\lambda_j+\mu_\star$, $j=1,\ldots,m$, by (4.5) we find from Propositions A.1 and A.2

$$\sigma_{\mathrm{disp}}(\mathcal{P}_- + cI_m) = \{-c + (\lambda_j+\mu_\star)i\omega : \omega\in\mathbb{R}, j=1,\ldots,m\} = -c + i\mathbb{R}.$$

This yields the following result.

Proposition A.3. *The dispersion sets satisfy*

$$\sigma_{\mathrm{disp}}(\mathcal{P}_{1\mathrm{st}}) = \sigma_{\mathrm{disp}}(\mathcal{P}) \cup (-c+i\mathbb{R}). \tag{A.14}$$

This proposition leads to a proper choice of the shift parameter c. Taking $c>\delta$, condition (vi) immediately follows from Assumption 4.6. The following proposition relates the point spectra of the second order operator $\mathcal{P}$ and the first order operator $\mathcal{P}_{1\mathrm{st}}$ to each other.

Proposition A.4. *The following assertions hold:*

(a) *There exists a $\lambda_\star > -c$ such that $\sigma_{\mathrm{disp}}(\mathcal{P}_{1\mathrm{st}}) \cap [\lambda_\star,\infty) = \emptyset$.*

(b) *Let ρ_+ be the connected component of $\{\lambda\in\mathbb{C} : \mathrm{Re}\,\lambda > -c\} \setminus \sigma_{\mathrm{disp}}(\mathcal{P}_{1\mathrm{st}})$ containing $[\lambda_\star,\infty)$. Then the operator $\mathcal{P}_{1\mathrm{st}}(\lambda) : H^1(\mathbb{R},\mathbb{C}^{2m}) \to L^2(\mathbb{R},\mathbb{C}^{2m})$ is Fredholm of index 0 for all $\lambda\in\rho_+$.*

(c) *The point spectra of $\mathcal{P}_{1\mathrm{st}}$ and $\mathcal{P}(\lambda) : H^2(\mathbb{R},\mathbb{C}^m) \to L^2(\mathbb{R},\mathbb{C}^m)$ in ρ_+ coincide,*

$$\sigma_{\mathrm{point}}(\mathcal{P}) \cap \rho_+ = \sigma_{\mathrm{point}}(\mathcal{P}_{1\mathrm{st}}) \cap \rho_+. \tag{A.15}$$

Eigenvalues in these sets have the same geometric and maximum partial multiplicity.

Let us first note that this proposition implies condition (vii). For the choice $c>\delta$ the set ρ_+ contains $\{\mathrm{Re}\,\lambda > -\delta\}$ by Assumption 4.6 and Proposition A.3. Condition (vii) is then a consequence of Assumption 4.7 and assertion (c) of Proposition A.4.

Proof. Using Assumption 4.5 we can rewrite the operator from (A.6) as follows:

$$\mathcal{P}_{1\mathrm{st}}(\lambda) = -(E+\mu_\star I_{3m})(\partial_\xi - (E+\mu_\star I_{3m})^{-1}(\lambda I_{3m} - DF(V_\star))).$$

The matrix $(E+\mu_\star I_{3m})^{-1}$ is hyperbolic by Assumption 4.5 and this property persists for the matrix $(E+\mu_\star I_{3m})^{-1}(\lambda I_{3m} - DF(V_\pm))$ for $\lambda\geq\lambda_\star$ sufficiently large, independently of the sign $\pm$ and with the same number of stable and unstable eigenvalues. Therefore $\mathcal{P}_{1\mathrm{st}}(\lambda)$ is Fredholm of index 0 by Proposition A.2 for $\lambda\in[\lambda_\star,\infty)$. Since the Fredholm index is continuous in ρ_+ and can only change at $\sigma_{\mathrm{disp}}(\mathcal{P}_{1\mathrm{st}})$ or at $-c+i\mathbb{R}$, assertion (b) also follows.

Consider an eigenvalue $\lambda_0\in\sigma_{\mathrm{point}}(\mathcal{P}_{1\mathrm{st}})\cap\rho_+$ with nonzero eigenfunction $V = (V_1,V_2,V_3)^\top$ in $H^1(\mathbb{R},\mathbb{C}^{3m})$. The first block equation reads $(\mathcal{P}_-(\lambda_0)+cI_m)V_1 = V_3 \in H^1$ from which we infer $V_1\in H^2(\mathbb{R},\mathbb{C}^m)$. In the following let us write the factorization (A.9) in the short form

$$T_1\mathcal{P}_{1\mathrm{st}}(\lambda) = R(\lambda)T_2(\lambda) \tag{A.16}$$

and apply it to V. Then $W(\lambda_0) := T_2(\lambda_0)V$ satisfies $R(\lambda_0)W(\lambda_0)=0$, and from the triangular structure of R and the invertibility of $\mathcal{P}_-(\lambda_0)+cI_m$ we obtain $W_3=0$,

$W_2 = 0$ as well as $\tilde{\mathcal{P}}(\lambda_0)V_1 = \tilde{\mathcal{P}}(\lambda_0)W_1 = 0$. If $V_1 = 0$, then $V_2 = 0, V_3 = 0$ follows from $W_2 = 0, W_3 = 0$, hence $V_1 \neq 0$. In a similar manner, if $\tilde{\mathcal{P}}(\lambda_0)W_1 = 0$ for some $W_1 \in H^2(\mathbb{R}, \mathbb{C}^m), W_1 \neq 0$, then $\mathcal{P}_{\text{1st}}(\lambda_0)V = 0$ and $V \neq 0$ for $V = T_2(\lambda_0)^{-1}\begin{pmatrix} W_1 & 0 & 0\end{pmatrix}^\top$. By the same argument the null spaces $\mathcal{N}(\mathcal{P}_{\text{1st}}(\lambda_0))$ and $\mathcal{N}(\tilde{\mathcal{P}}(\lambda_0))$ have equal dimension.

Finally, consider a root polynomial $V(\lambda) = \sum_{j=0}^{n} V_{[j]}(\lambda - \lambda_0)^j$, $V_{[j]} \in H^1(\mathbb{R}, \mathbb{C}^{3m})$ satisfying

$$V(\lambda_0) = V_{[0]} \neq 0, \quad (\mathcal{P}_{\text{1st}}V)^{(\nu)}(\lambda_0) = 0, \nu = 0, \dots, n-1.$$

As above we find $V_{[0],1} \in H^2(\mathbb{R}, \mathbb{C}^m)$, $V_{[0],1} \neq 0$, and then by induction $V_{[j],1} \in H^2(\mathbb{R}, \mathbb{C}^m), j = 1, \dots, n$, from the equations

$$\nu!\ \mathcal{P}_{\text{1st}}(\lambda_0)V_{[\nu]} = -\sum_{\ell=1}^{\nu} \binom{\nu}{\ell} \mathcal{P}_{\text{1st}}^{(\ell)}(\lambda_0)V^{(\nu-\ell)}(\lambda_0).$$

Note that the right-hand side is in $H^1(\mathbb{R}, \mathbb{C}^{3m})$ since the λ-derivative of $\mathcal{P}_{\text{1st}}$ is I_{3m}. Setting $W(\lambda) = T_2(\lambda)V(\lambda)$ then leads via (A.16) to

$$(RW)^{(\nu)}(\lambda_0) = 0, \nu = 0, \dots, n-1.$$

Working backward through the components of this equation gives $W_k^{(\nu)}(\lambda_0) = 0, \nu = 0, \dots, n-1$, for $k = 3, 2$, and therefore,

$$0 = (\tilde{\mathcal{P}}W_1)^{(\nu)}(\lambda_0), \nu = 0, \dots, n-1,$$

with $W_1(\lambda_0) = V_1(\lambda_0) \neq 0$.

Conversely, let $W_1(\lambda) = \sum_{j=0}^{n-1}(\lambda - \lambda_0)^j W_{[j],1}$ be a root polynomial of $\tilde{\mathcal{P}}$ in $H^2(\mathbb{R}, \mathbb{C}^m)$ with $W_{[0],1} \neq 0$. Then we set $W(\lambda) = \begin{pmatrix} W_1(\lambda) & 0 & 0\end{pmatrix}^\top$ and find that

$$V(\lambda) = T_2(\lambda)^{-1}W(\lambda) = \begin{pmatrix} W_1(\lambda) & \mathcal{P}_+(\lambda)W_1(\lambda) & (-\mathcal{P}_-(\lambda) + cI_m)W_1(\lambda)\end{pmatrix}^\top$$

lies in $H^1(\mathbb{R}, \mathbb{C}^{3m})$ and satisfies $V(\lambda_0) \neq 0$ as well as

$$T_1(\mathcal{P}_{\text{1st}}V)^{(\nu)}(\lambda_0) = (RW)^{(\nu)}(\lambda_0) = 0, \nu = 0, \dots, n-1. \qquad \square$$

A.3. Stability for the second order system. In the following we consider the Cauchy problem (4.4) and recall the function spaces (4.6). We need two auxiliary results. The first one is regularity of solutions with respect to source terms taken from the theory of linear first order systems (see [31, Cor. 2.2.2]).

LEMMA A.5. *Consider a first order system*

$$u_t = A_1 u_x + B_1 u + r, \quad u(x, 0) = u_0(x), \quad x \in \mathbb{R}, t \geq 0, \tag{A.17}$$

where $A_1 \in \mathbb{R}^{l,l}$ is real diagonalizable and $B_1 \in \mathbb{R}^{l,l}$. If $u_0 \in H^k(\mathbb{R}, \mathbb{R}^l)$ for some $k \geq 1$ and $r \in \mathcal{CH}^{k-1}([0,\infty); \mathbb{R}^l)$, then the system (A.17) *has a unique solution in $u \in \mathcal{CH}^k([0,\infty); \mathbb{R}^l)$.*

The second one concerns commuting weak and strong derivatives with respect to space and time.

LEMMA A.6. *For $u \in C^1([0,\infty); H^1(\mathbb{R},\mathbb{R}^l))$ let $\frac{d}{dt}u \in C^0([0,\infty); H^1(\mathbb{R},\mathbb{R}^l))$ be its time derivative and let $\partial_x u(\cdot,t)$ be its weak space derivative pointwise in $t \in [0,\infty)$. Then $\partial_x u \in C^1([0,\infty); L^2(\mathbb{R},\mathbb{R}^l))$ and its time derivative agrees with the weak spatial derivative of $\frac{d}{dt}u$ evaluated pointwise in $t \in [0,\infty)$, i.e.,*

$$\frac{d}{dt}(\partial_x u) = \partial_x \left(\frac{d}{dt}u\right). \tag{A.18}$$

Proof. Let $t, t+h \in [0,\infty)$ with $h \neq 0$ and note that

$$\begin{aligned}&\left\|\frac{1}{h}(\partial_x u(\cdot,t+h) - \partial_x u(\cdot,t)) - \partial_x\left(\frac{d}{dt}u(\cdot,t)\right)\right\|_{L^2}\\ &\qquad \le \left\|\frac{1}{h}(u(\cdot,t+h) - u(\cdot,t)) - \frac{d}{dt}u(\cdot,t)\right\|_{H^1},\end{aligned}$$

where the right-hand side converges to zero as $h \to 0$ by assumption. Therefore, the derivative $\frac{d}{dt}(\partial_x u)$ exists in $L^2(\mathbb{R},\mathbb{R}^l)$ for all $t \in [0,\infty)$ and coincides with $\partial_x(\frac{d}{dt}u) \in C^0([0,\infty); L^2(\mathbb{R},\mathbb{R}^l))$. ∎

Remark A.7. Broadly speaking, we may write (A.18) as commuting partial derivatives $u_{xt} = u_{tx}$. However, this equality has to be interpreted with care since time and space derivatives are taken with respect to different norms.

We proceed with the proof of Theorem 4.8 by using the stability statements from (A.2), (A.3). Due to (4.7) and (4.16c), the initial difference $V_0 - V_\star$ equals

$$(u_0 - v_\star, v_0 + \mu_\star v_{\star,\xi} + N(u_{0,\xi} - v_{\star,\xi}), v_0 + \mu_\star v_{\star,\xi} - N(u_{0,\xi} - v_{\star,\xi}) + c(u_0 - v_\star))^\top. \tag{A.19}$$

Therefore, we have a constant $C_\star = C_\star(c, \|N\|)$ with

$$\|V_0 - V_\star\|_{H^2} \le C_\star(\|u_0 - v_\star\|_{H^3} + \|v_0 + \mu_\star v_{\star,\xi}\|_{H^2}) \le C_\star \rho, \tag{A.20}$$

and we take ρ such that $C_\star\rho \le \rho_0$. Let $V \in V_\star + \mathcal{CH}^1([0,\infty);\mathbb{R}^{3m})$ be the unique solution of (4.4) for $\|V_0 - V_\star\|_{H^2} \le \rho_0$. The first component V_1 satisfies

$$V_{1,t} = NV_{1,x} - cV_1 + V_3, \quad V_1(\cdot,0) = u_0, \tag{A.21}$$

so that $\tilde{V}_1 = V_1 - v_\star$ solves the Cauchy problem

$$\tilde{V}_{1,t} = N\tilde{V}_{1,x} - c\tilde{V}_1 + V_3 - V_{\star,3} - \mu_\star v_{\star,x}, \quad \tilde{V}_1(\cdot,0) = u_0 - v_\star.$$

Then Lemma A.5 applies with $k = 2, A_1 = N, B_1 = -cI_m,\ r = V_3 - V_{\star,3} - \mu_\star v_{\star,x}$ and yields $\tilde{V}_1 = V_1 - v_\star \in \mathcal{CH}^2([0,\infty);\mathbb{R}^m)$. By Lemma A.6 we obtain $\tilde{V}_{1,x} \in C^1([0,\infty); L^2(\mathbb{R},\mathbb{R}^m))$ as well as $\tilde{V}_{1,tx} = \tilde{V}_{1,xt} \in C^0([0,\infty); L^2(\mathbb{R},\mathbb{R}^m))$. Since $v_\star$ does not depend on t we also have $V_{1,tx} = \tilde{V}_{1,tx} = \tilde{V}_{1,xt} = V_{1,xt}$. For the same reason $\tilde{V}_{1,tt} = V_{1,tt} \in C^0([0,\infty); L^2(\mathbb{R},\mathbb{R}^m))$, and $\tilde{V}_{1,xx} = (V_1 - v_\star)_{xx} \in C^0([0,\infty); L^2(\mathbb{R},\mathbb{R}^m))$ implies $V_{1,xx} \in C^0([0,\infty); L^2(\mathbb{R},\mathbb{R}^m))$ since $v_{\star,xx} \in H^2(\mathbb{R},\mathbb{R}^m)$ by Assumption 4.4. Thus we can take space and time derivative of (A.21) and obtain from the third row of (4.4)

$$\begin{aligned}\tilde{f}(V) &= V_{3,t} + NV_{3,x} - cV_2\\ &= V_{1,tt} - N^2V_{1,xx} - NV_{1,xt} + cV_{1,t} + NV_{1,tx} + cNV_{1,x} - cV_2\\ &= V_{1,tt} - N^2V_{1,xx} - c(V_2 - V_{1,t} - NV_{1,x}).\end{aligned} \tag{A.22}$$

Next introduce the functions

$$W_2 = V_2 - V_{1,t} - NV_{1,x}, \quad W_3 = V_3 - V_{1,t} + NV_{1,x} - cV_1. \tag{A.23}$$

Using (A.22), the last two rows of (4.4), and Lemma A.6 again, these functions solve the hyperbolic system

(A.24)
$$\begin{aligned} W_{2,t} - NW_{2,x} &= V_{2,t} - V_{1,tt} - NV_{1,xt} - NV_{2,x} + NV_{1,tx} + N^2V_{1,xx} = -cW_2, \\ W_{3,t} + NW_{3,x} &= V_{3,t} + NV_{3,x} - V_{1,tt} - NV_{1,tx} + NV_{1,xt} + N^2V_{1,xx} - cV_{1,t} - cNV_{1,x} \\ &= \tilde{f}(V) + cV_2 - (\tilde{f}(V) + cW_2) - c(V_{1,t} + NV_{1,x}) = 0. \end{aligned}$$

Using from (4.4) the initial data and the differential equation at $t = 0$ one finds that $W_2(\cdot, 0) = 0$, $W_3(\cdot, 0) = 0$. Since (A.24) with homogeneous initial data has only the trivial solution we conclude $W_2 \equiv 0, W_3 \equiv 0$. Therefore, by setting $u = V_1$, (A.22) and (A.23) finally lead to

$$\begin{aligned} u_{tt} - N^2u_{xx} &= \tilde{f}(V) = M^{-1}f\left(V_1, \frac{1}{2}N^{-1}(V_2 - V_3 + cV_1), \frac{1}{2}(V_2 + V_3 - cV_1)\right) \\ &= M^{-1}f(u, u_x, u_t). \end{aligned}$$

Using (A.2) and (A.20) we obtain for the asymptotic phase φ_∞ the estimate

$$|\varphi_\infty| \le C\|V_0 - V_\star\|_{H^2} \le CC_\star(\|u_0 - v_\star\|_{H^3} + \|v_0 + \mu_\star v_{\star,x}\|_{H^2}).$$

Further, we have for $t \ge 0$ the stability estimate

$$\|V(\cdot, t) - V_\star(\cdot - \mu_\star t - \varphi_\infty)\|_{H^1} \le CC_\star(\|u_0 - v_\star\|_{H^3} + \|v_0 + \mu_\star v_{\star,x}\|_{H^2})e^{-\eta t},$$

where C depends only on η, ρ. From this we retrieve the estimate (4.13) for the original variables by taking the H^1-norm of the equation

(A.25)
$$V(\cdot, t) - V_\star(\cdot - \mu_\star t - \varphi_\infty) = \begin{pmatrix} I_m & 0 & 0 \\ 0 & N & I_m \\ cI_m & -N & I_m \end{pmatrix} \begin{pmatrix} u(\cdot, t) - v_\star(\cdot - \mu_\star t - \varphi_\infty) \\ u_x(\cdot, t) - v_{\star,x}(\cdot - \mu_\star t - \varphi_\infty) \\ u_t(\cdot, t) + \mu_\star v_{\star,x}(\cdot - \mu_\star t - \varphi_\infty) \end{pmatrix}$$

and using that the left factor of the right-hand side is invertible.

A.4. Lyapunov stability of the freezing method. Let us first recall from [33, Thm. 2.7] the stability theorem for the freezing method associated with the first order formulation (A.1),

$$W_t = \Lambda_E W_x + G(W) + \mu W_x, \quad x \in \mathbb{R}, t \ge 0, W(x, t) \in \mathbb{R}^l, \tag{A.26a}$$
$$W(\cdot, 0) = W_0, \tag{A.26b}$$
$$\Psi(W - \hat{W}) = 0. \tag{A.26c}$$

Here, $\Psi : L^2(\mathbb{R}, \mathbb{R}^l) \to \mathbb{R}$ is a linear functional and $\hat{W} : \mathbb{R} \to \mathbb{R}^l$ is a template function for which we assume

(viii) $\Psi(W_{\star,\xi}) \neq 0$, Ψ is bounded,

(ix) $\hat{W} \in W_\star + H^1(\mathbb{R}, \mathbb{R}^l)$ and $\Psi(W_\star - \hat{W}) = 0$.

Under the combined assumptions of (i)–(vii) and (viii), (ix) the result is the following. For every $0 < \eta < \delta$ there exists $\rho_0 > 0$ such that for all initial data $W_0 \in W_\star + H^2(\mathbb{R},\mathbb{R}^l)$ with $\|W_0 - W_\star\|_{H^2} \le \rho_0$ the system (A.26) has a unique solution (W,μ) in $(W_\star + \mathcal{CH}^1([0,\infty);\mathbb{R}^{3m})) \times C([0,\infty),\mathbb{R})$. Moreover, there is a constant $C = C(\eta)$ such that the solution satisfies

$$\|W(t) - W_\star\|_{H^1} + |\mu(t) - \mu_\star| \le C(\eta)\|W_0 - W_\star\|_{H^2} e^{-\eta t}, \quad t \ge 0. \tag{A.27}$$

We apply this to the frozen version of (4.5) with the functional Ψ and the function $\hat{W}$ defined by

$$\begin{aligned} \hat{V} &= \begin{pmatrix} \hat{v} & 0 & 0 \end{pmatrix}^\top, \hat{W} = T^{-1}\hat{V}, \\ \Psi(W - \hat{W}) &= \langle T(W - \hat{W}), T\hat{W}_\xi \rangle_{L^2}. \end{aligned} \tag{A.28}$$

While conditions (i)–(vii) have already been verified, the conditions (viii) and (ix) easily follow from Assumption 4.9 and the settings $W_\star = T^{-1}V_\star$ and (4.7). Thus the above result applies. By (W,μ) we denote the unique solution of (A.26) for $\|W_0 - W_\star\|_{H^2} \le \rho_0$, and we let $(V = TW, \mu)$ be the unique solution in the space $(V_\star + \mathcal{CH}^1([0,\infty);\mathbb{R}^{3m})) \times C([0,\infty),\mathbb{R})$ of the transformed equation

$$V_t = EV_\xi + F(V) + \mu V_\xi, \quad \xi \in \mathbb{R}, t \ge 0, \tag{A.29a}$$

$$V(\cdot,0) = V_0, \tag{A.29b}$$

$$\langle V_1 - \hat{v}, \hat{v}_\xi \rangle_{L^2} = 0. \tag{A.29c}$$

We impose two conditions on the radius ρ appearing in (4.20). The first one is $C_\star \rho \le \rho_0$ as in the argument following (A.20). The second one is to ensure for some constant $\underline{C} > 0$

$$|\langle V_{1,\xi}(\cdot,t), \hat{v}_\xi \rangle_{L^2}| \ge \underline{C} \quad \forall t \ge 0 \tag{A.30}$$

for all solutions satisfying (4.20). In fact, from (A.27), (A.20), and Assumption 4.9 we obtain

$$\begin{aligned} |\langle V_{1,\xi}(\cdot,t), \hat{v}_\xi \rangle_{L^2}| &\ge |\langle v_{\star,\xi}, \hat{v}_\xi \rangle_{L^2}| - \|V_1(\cdot,t) - v_\star\|_{H^1}\|\hat{v}_\xi\|_{L^2} \\ &\ge |\langle v_{\star,\xi}, \hat{v}_\xi \rangle_{L^2}| - C(\eta)e^{-\eta t}\|T\|\|W_0 - W_\star\|_{H^2}\|\hat{v}_\xi\|_{L^2} \\ &\ge |\langle v_{\star,\xi}, \hat{v}_\xi \rangle_{L^2}| - \rho C(\eta)\|T\|\|T^{-1}\|C_\star\|\hat{v}_\xi\|_{L^2}. \end{aligned}$$

The next step is to prove regularity of the solution in the sense that

$$V_1 \in v_\star + \mathcal{CH}^2([0,\infty);\mathbb{R}^m), \quad \mu \in C^1([0,\infty),\mathbb{R}). \tag{A.31}$$

For this we define $\gamma \in C^1([0,\infty),\mathbb{R})$ by $\gamma(t) = \int_0^t \mu(s)ds$ and return to the original variables via $U(x,t) := V(x - \gamma(t), t)$ for $x \in \mathbb{R}, t \ge 0$. Then we have that U is in $V_\star + \mathcal{CH}^1([0,\infty);\mathbb{R}^{3m})$ and solves the first order system (4.4). Hence the regularity $U_1 \in v_\star + \mathcal{CH}^2([0,\infty);\mathbb{R}^m)$ is obtained via Lemma A.5 by the same arguments as those following (A.21). In particular, $U_{1,x} \in \mathcal{CH}^1([0,\infty);\mathbb{R}^m)$ and thus $V_{1,\xi} \in \mathcal{CH}^1([0,\infty);\mathbb{R}^m)$ since $V_{1,\xi}(\cdot,t) = U_{1,x}(\cdot + \gamma(t), t)$ and $\gamma \in C^1([0,\infty),\mathbb{R})$. For the smoothness of μ we differentiate the phase condition (A.29c) with respect to t and use (A.29a)

$$0 = \langle V_{1,t}, \hat{v}_\xi \rangle_{L^2} = \langle NV_{1,\xi} - cV_1 + V_3, \hat{v}_\xi \rangle_{L^2} + \mu \langle V_{1,\xi}, \hat{v}_\xi \rangle_{L^2}.$$

By (A.30) this can be solved for μ and yields $\mu \in C^1([0,\infty),\mathbb{R})$ since the other terms are known to be C^1-smooth. Thus we have $\gamma \in C^2([0,\infty),\mathbb{R})$ and then finally $V_1 \in v_\star + \mathcal{CH}^2([0,\infty);\mathbb{R}^m)$ from $V(\xi,t) = U(\xi+\gamma(t),t)$ and $U_1 \in v_\star + \mathcal{CH}^2([0,\infty);\mathbb{R}^m)$.

Retrieving the frozen second order equation (4.17) now uses the same arguments as in the nonfrozen case. Therefore we only indicate the revised equations and leave out computations. Equation (A.22) is replaced by

(A.32)
$$\tilde{f}(V) = V_{1,tt} - N^2 V_{1,\xi\xi} + \mu^2 V_{1,\xi\xi} - 2\mu V_{1,t\xi} - \mu_t V_{1,\xi} + c(V_{1,t} - V_2 + (N - \mu I_m)V_{1,\xi}).$$

In view of (4.18) the analogous functions of (A.23) are defined as follows:

$$\text{(A.33)} \quad W_2 = V_2 - V_{1,t} - (N - \mu I_m)V_{1,\xi}, \quad W_3 = V_3 - V_{1,t} + (N + \mu I_m)V_{1,\xi} - cV_1.$$

They solve the hyperbolic homogeneous Cauchy problem

$$\begin{aligned} W_{2,t} - (N + \mu I_m)W_{2,\xi} &= -cW_2, \quad W_2(\cdot,0) = 0, \\ W_{3,t} + (N - \mu I_m)W_{3,\xi} &= 0, \quad W_3(\cdot,0) = 0, \end{aligned}$$

hence vanish identically. Inserting this into (A.32) shows that $v = V_1$ lies in $v_\star + \mathcal{CH}^2([0,\infty);\mathbb{R}^m)$ and, together with μ, solves the frozen second order system (4.17).

Concerning the estimate (4.21), we note the following relation, which replaces (A.25):

$$\text{(A.34)} \qquad V(\cdot,t) - V_\star = \begin{pmatrix} I_m & 0 & 0 \\ 0 & N & I_m \\ cI_m & -N & I_m \end{pmatrix} \begin{pmatrix} v(\cdot,t) - v_\star \\ v_\xi(\cdot,t) - v_{\star,\xi} \\ v_t(\cdot,t) + \mu_\star v_{\star,\xi} - \mu(t)v_\xi(\cdot,t) \end{pmatrix}.$$

Taking the H^1-norm of this equation and using the estimate (A.27) with $V, V_\star, V_0$ instead of $W, W_\star, W_0$ then gives the exponential estimate in (4.21) for $\|v(\cdot,t) - v_\star\|_{H^2}$, $|\mu - \mu_\star|$, and $\|v_t(\cdot,t) + \mu_\star v_{\star,\xi} - \mu(t)v_\xi(\cdot,t)\|_{H^1}$. Using the estimates for the first two terms and the triangle inequality on the last term then yields an exponential estimate for $\|v_t(\cdot,t)\|_{H^1}$. This finishes the proof.

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