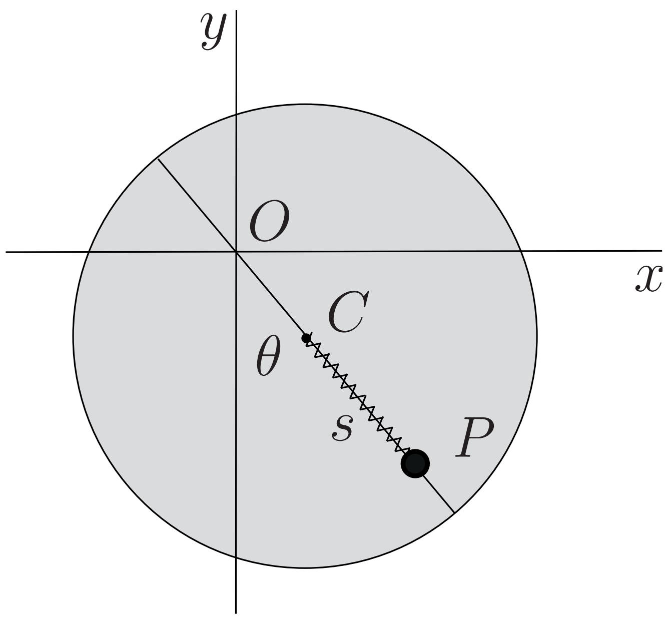
Corso di Laurea in Ingegneria Meccanica
Anno Accademico 2022/2023
Meccanica Razionale - Appello del 13/9/2023

Nome ....
N. Matricola .... Ancona, 13 settembre 2023

1. Un disco di centro C, raggio R e massa M è libero di ruotare attorno al suo punto interno O, sito a distanza $R/2$ dal centro. Il moto si svolge nel piano verticale $O(x, y)$. Un punto P di massa m scorre senza attrito lungo la scanalatura diametrale passante per O ed è collegato al centro C da una molla di costante elastica $k > 0$. Scrivere le equazioni di Lagrange per il sistema.



$$T = T_P + T_D$$

$$T_P = \frac{1}{2} m v_P^2 \qquad T_D = \frac{1}{2} I \dot{\theta}^2$$

$$I = \frac{1}{2} M R^2 + M \frac{R^2}{4} = \frac{3}{4} M R^2$$

$$P - O = \left(s + \frac{R}{2}\right)\left(\hat{\imath}\,\mathrm{sen}\,\vartheta - \hat{\jmath}\cos\vartheta\right)$$

$$\vec{v}_P = \dot{s}\left(\hat{\imath}\,\mathrm{sen}\,\vartheta - \hat{\jmath}\cos\vartheta\right) +$$

$$+ \left(s + \frac{R}{2}\right)\dot{\theta}\left(\hat{\imath}\cos\vartheta + \hat{\jmath}\,\mathrm{sen}\,\vartheta\right)$$

$$T_P = \frac{1}{2}m\left\{\dot{s}^2 + \left(s + \frac{R}{2}\right)^2\dot{\theta}^2\right\}$$

$$T = \frac{1}{2}\left\{\frac{3}{4}MR^2\dot{\theta}^2 + m\left[\dot{s}^2 + \left(s + \frac{R}{2}\right)^2\dot{\theta}^2\right]\right\}$$

$$V = -Mg\frac{R}{2}\cos\vartheta - mg\left(\frac{R}{2} + s\right)\cos\vartheta +$$

$$+ \frac{1}{2}ks^2 =$$

$$= -g\left[M\frac{R}{2} + m\left(\frac{R}{2} + s\right)\right]\cos\vartheta + \frac{1}{2}ks^2$$

Equil e stab

$$\frac{\partial V}{\partial s} = -mg\cos\vartheta + ks$$

$$\frac{\partial V}{\partial \vartheta} = g\left[\qquad\qquad\right]\sin\vartheta$$

$\sin\vartheta = 0 \qquad \vartheta = 0, \pi$

$$mg\frac{R}{2} + m\left(\frac{R}{2} + s\right)$$

$$\mathcal{L} = T - V$$

$$\frac{\partial \mathcal{L}}{\partial \dot{s}} = m\dot{s}$$

$$\frac{\partial \mathcal{L}}{\partial s} = m\left(s + \frac{R}{2}\right)\dot{\theta}^2 + mg\cos\theta - ks$$

$$\frac{\partial \mathcal{L}}{\partial \dot{\theta}} = \frac{3}{4}MR^2\dot{\theta} + m\left(s + \frac{R}{2}\right)^2\dot{\theta}$$

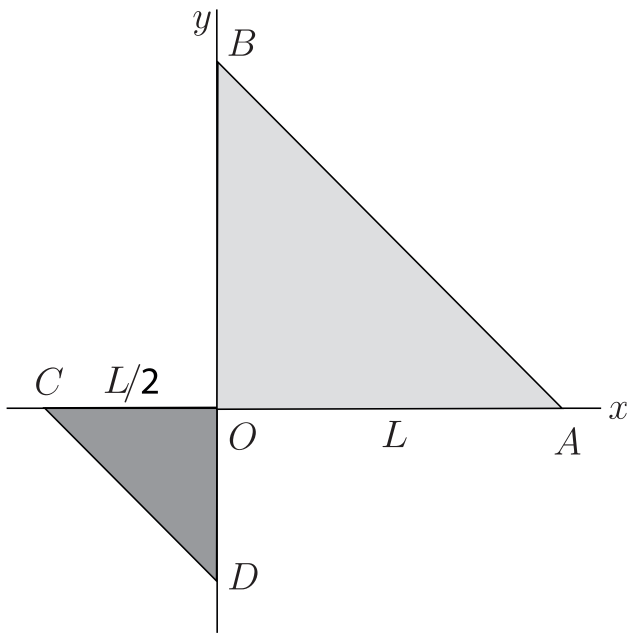
$$\frac{\partial \mathcal{L}}{\partial \theta} = -g\left[M\frac{R}{2} + m\left(\frac{R}{2} + s\right)\right]\sin\theta$$

Equazioni di Lagrange:

$$\boxed{m\ddot{s} - m\left(s+\frac{R}{2}\right)\dot{\theta}^2 - mg\cos\theta + ks = 0}$$

$$\boxed{\frac{3}{4}MR^2\ddot{\theta} + 2m\left(s+\frac{R}{2}\right)\dot{s}\dot{\theta} + m\left(s+\frac{R}{2}\right)^2\ddot{\theta} + g\left[M\frac{R}{2} + m\left(\frac{R}{2}+s\right)\right]\sin\theta = 0}$$

2. Nel sistema di riferimento $O(x, y, z)$ indicato in figura, calcolare la matrice d'inerzia della lamina $ABCD$ costituita da due triangoli rettangoli isosceli OAB, di massa $M/2$ e cateti L, e OCD, di massa M e cateti $L/2$, con i cateti lungo gli assi coordinati e disposti come in figura.



$$\underline{OAB}: \quad I_{11} = I_{22} = \sigma \int_0^L dy \int_0^{L-y} y^2\, dx$$

$$= \sigma \int_0^L dy\, y^2 (L-y) =$$

$$= \sigma \left\{ L \frac{y^3}{3}\Big|_0^L - \frac{y^4}{4}\Big|_0^L \right\} = \sigma \frac{L^4}{12} =$$

$$= \frac{M/2}{L^2/2} \frac{L^4}{12} = \frac{1}{12} M L^2$$

$$I_{12} = -\sigma \int_0^L dx \int_0^{L-x} dy \; x\, y =$$

$$= -\sigma \int_0^L dx \; x \; \frac{y^2}{2}\Big|_0^{L-x} =$$

$$= -\frac{\sigma}{2} \int_0^L dx \; x \,(L-x)^2 =$$

$$= -\frac{\sigma}{2} \int_0^L dx \; x^2 (L-x) =$$

$$= -\frac{\sigma}{2} \left\{ L \frac{x^3}{3}\Big|_0^L - \frac{x^4}{4}\Big|_0^L \right\} =$$

$$= -\frac{M/2}{L^2/2} \frac{1}{24} L^4 = -\frac{1}{24} M L^2$$

$\underline{OCD}$: $I_{11} = I_{22} =$

$$= \sigma \int_{-L/2}^{0} dy \int_{-y-L/2}^{0} y^2 \, dx =$$

$$= \sigma \int_{-L/2}^{0} dy \; y^2 \left(y + L/2 \right) =$$

$$= \frac{M}{L^2/8} \left\{ \frac{y^4}{4} \Big|_{-L/2}^{0} + \frac{L}{2} \; \frac{y^3}{3} \Big|_{-L/2}^{0} \right\} =$$

$$= \frac{8M}{L^2} \left\{ - \frac{L^4}{64} + \frac{L^4}{48} \right\} =$$

$$= ML^2 \left(\frac{1}{6} - \frac{1}{8} \right) = \frac{1}{24} ML^2$$

$$I_{12} = -\sigma \int_{-L/2}^{0} dx \int_{-L/2-x}^{0} dy \; xy =$$

$$= -\sigma \int_{-L/2}^{0} dx \; x \; \frac{y^2}{2} \Big|_{-L/2-x}^{0} =$$

$$= -\frac{\sigma}{2} \int_{-L/2}^{0} dx \, x \, (-1) \left(\frac{L}{2} + x \right)^2 =$$

$$= + \frac{\sigma}{2} \int_{-L/2}^{0} dx \; x^2 \left(x - \frac{L}{2} \right) =$$

$$= \frac{\sigma}{2} \left\{ \frac{x^4}{4} \Big|_{-L/2}^{0} - \frac{L}{2} \; \frac{x^3}{3} \Big|_{-L/2}^{0} \right\} =$$

$$= \frac{M}{L^2/4} \left\{ - \frac{L^4}{4 \cdot 16} - \frac{L}{2} \frac{1}{3} \frac{L^3}{8} \right\} =$$

$$= \frac{4M}{L^2} \left(\frac{L^4}{64} - \frac{L^4}{48} \right) =$$

$$= ML^2 \left(\frac{1}{16} - \frac{1}{12} \right) = - \frac{1}{48} ML^2$$

<u>Total</u>:

$$I_{11} = \left(\frac{1}{12} + \frac{1}{24} \right) ML^2 = \frac{1}{8} ML^2 = I_{22}$$

$$I_{12} = - \left(\frac{1}{24} + \frac{1}{48} \right) ML^2 = - \frac{1}{16} ML^2$$

$$I_{33} = \frac{1}{4} ML^2$$

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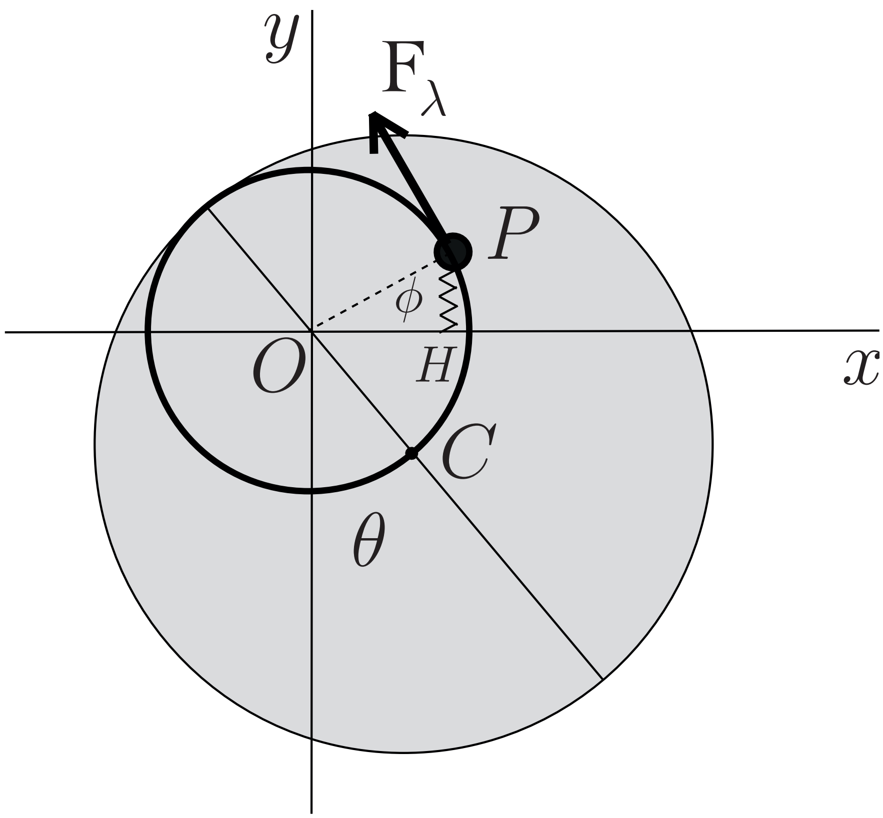


$T = T_p + T_D$

$$T_p = \frac{1}{2} m v_p^2 \qquad T_D = \frac{1}{2} I \dot{\theta}^2$$

$$I = \frac{3}{4} M R^2$$

$$P - O = \frac{R}{2} \left(\hat{\imath} \cos\varphi + \hat{\jmath} \sin\varphi \right)$$

$$\vec{v}_p = \frac{R}{2} \dot{\varphi} \left(-\hat{\imath} \operatorname{sen}\varphi + \hat{\jmath} \cos\varphi \right)$$

$$T = \frac{1}{2} m \frac{R^2}{4} \dot{\varphi}^2 + \frac{1}{2} \frac{3}{4} M R^2 \dot{\theta}^2 =$$

$$= \frac{1}{8} R^2 \left(m \dot{\varphi}^2 + 3 M \dot{\theta}^2 \right)$$

$$V = -Mg\frac{R}{2}\cos\theta + mg\frac{R}{2}\sin\varphi +$$

$$+\frac{1}{2}k\frac{R^2}{4}\sin^2\varphi$$

$$\vec{F}_\lambda = -\lambda\vec{v}_p = -\lambda\frac{R}{2}\dot{\varphi}\left(-\hat{\imath}\,\mathrm{sen}\,\varphi + \hat{\jmath}\cos\varphi\right)$$

$$Q_\varphi = -\lambda\frac{R}{2}\dot{\varphi}\left(-\hat{\imath}\,\mathrm{sen}\,\varphi + \hat{\jmath}\cos\varphi\right)\cdot$$

$$\frac{R}{2}\left(-\hat{\imath}\,\mathrm{sen}\,\varphi + \hat{\jmath}\cos\varphi\right) =$$

$$= -\lambda\frac{R^2}{4}\dot{\varphi}$$

$$\mathcal{L} = T - V$$

$$\frac{\partial \mathcal{L}}{\partial \dot{\theta}} = \frac{3}{4} M R^2 \dot{\theta}$$

$$\frac{\partial \mathcal{L}}{\partial \theta} = -Mg \frac{R}{2} \operatorname{sen} \theta$$

$$\frac{\partial \mathcal{L}}{\partial \dot{\varphi}} = m \frac{R^2}{4} \dot{\varphi}$$

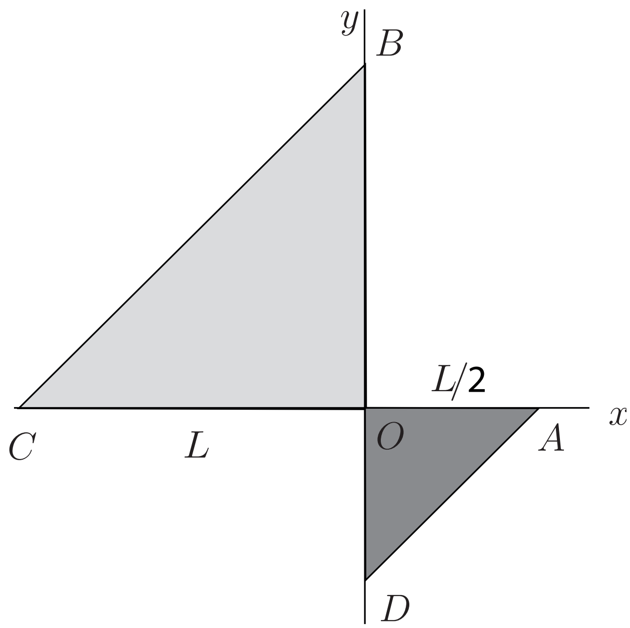
$$\frac{\partial \mathcal{L}}{\partial \varphi} = -mg \frac{R}{2} \cos \varphi + \frac{1}{4} k R^2 \cos \varphi \sin \varphi$$

Equazioni di Lagrange:

$$\boxed{\frac{3}{4} M R^2 \ddot{\theta} + M g \frac{R}{2} \operatorname{sen} \vartheta = 0}$$

$$\boxed{m \frac{R^2}{4} \ddot{\varphi} + m g \frac{R}{2} \cos \varphi - \frac{k R^2}{4} \cos \varphi \sin \varphi = -\lambda \frac{R^2}{4} \dot{\varphi}}$$

2. Nel sistema di riferimento $O(x, y, z)$ indicato in figura, calcolare la matrice d'inerzia della lamina costituita da due triangoli rettangoli isosceli OCB, di massa $M/2$ e cateti L, e OAD, di massa M e cateti $L/2$, con i cateti lungo gli assi coordinati e disposti come in figura.



OBC :

$$I_{11} = I_{22} = \sigma \int_{-L}^{0} dx \int_{0}^{x+L} dy \, x^2 =$$

$$= \sigma \int_{-L}^{0} dx \, x^2 (x+L) =$$

$$= \sigma \left\{ \frac{x^4}{4} \Big|_{-L}^{0} + L \frac{x^3}{3} \Big|_{-L}^{0} \right\} =$$

$$= \frac{M/2}{L^2/2} \left\{ -\frac{L^4}{4} + \frac{L^4}{3} \right\} = \frac{1}{12} M L^2$$

$$I_{33} = \frac{1}{6} M L^2$$

$$I_{12} = -\sigma \int_{-L}^{0} dx \int_{0}^{x+L} dy \; xy =$$

$$= -\sigma \int_{-L}^{0} dx \; x \left. \frac{y^2}{2} \right|_{0}^{x+L} =$$

$$= -\frac{M/2}{L^2/2} \int_{-L}^{0} dx \; x \frac{(x+L)^2}{2} =$$

$$= -\frac{M}{2L^2} \int_{0}^{L} dx \, (x-L) \, x^2 =$$

$$= -\frac{M}{2L^2} \left\{ \left. \frac{x^4}{4} \right|_{0}^{L} - L \left. \frac{x^3}{3} \right|_{0}^{L} \right\} =$$

$$= -\frac{M}{2L^2} \left\{ \frac{L^4}{4} - \frac{L^4}{3} \right\} =$$

$$= \frac{1}{24} ML^2$$

<u>OAD</u> $\quad I_{11} = I_{22} = \sigma \int_0^{L/2} dx \int_{x-L/2}^{0} dy \; x^2 =$

$$= \sigma \int_0^{L/2} dx \; x^2 \left(\frac{L}{2} - x \right) =$$

$$= \frac{M}{L^2/8} \left\{ \frac{L}{2} \frac{x^3}{3} \Big|_0^{L/2} - \frac{x^4}{4} \Big|_0^{L/2} \right\} =$$

$$= \frac{8M}{L^2} \left\{ \frac{L}{2} \frac{1}{3} \frac{L^3}{8} - \frac{1}{4} \frac{L^4}{16} \right\} =$$

$$= \frac{M}{L^2} \left(\frac{1}{6} - \frac{1}{8} \right) L^4 = \frac{1}{24} ML^2$$

$$I_{12} = -\sigma \int_0^{L/2} dx \int_{x-L/2}^{0} dy \; xy$$

$$= -\frac{8M}{L^2} \int_0^{L/2} dx \, x \left. \frac{y^2}{2} \right|_{x-L/2}^{0} =$$

$$= -\frac{8M}{2L^2} \int_0^{L/2} dx \, x \, (-) \left(x - L/2 \right)^2 =$$

$$= \frac{4M}{L^2} \int_{-L/2}^{0} dx \, (x + L/2) \, x^2 =$$

$$= \frac{4M}{L^2} \left\{ \left. \frac{x^4}{4} \right|_{-L/2}^{0} + \frac{L}{2} \left. \frac{x^3}{3} \right|_{-L/2}^{0} \right\} =$$

$$= \frac{4M}{L^2} \left\{ -\frac{L^4}{4 \cdot 16} + \frac{L^4}{6 \cdot 8} \right\} =$$

$$= ML^2\left(-\frac{1}{16} + \frac{1}{12}\right) = \frac{1}{48} ML^2$$

Totali:

$$I_{11} = \left(\frac{1}{12} + \frac{1}{24}\right) ML^2 = \frac{1}{8} ML^2 = I_{22}$$

$$I_{33} = \frac{1}{4} ML^2$$

$$I_{12} = \left(\frac{1}{24} + \frac{1}{48}\right) ML^2 = \frac{1}{16} ML^2$$