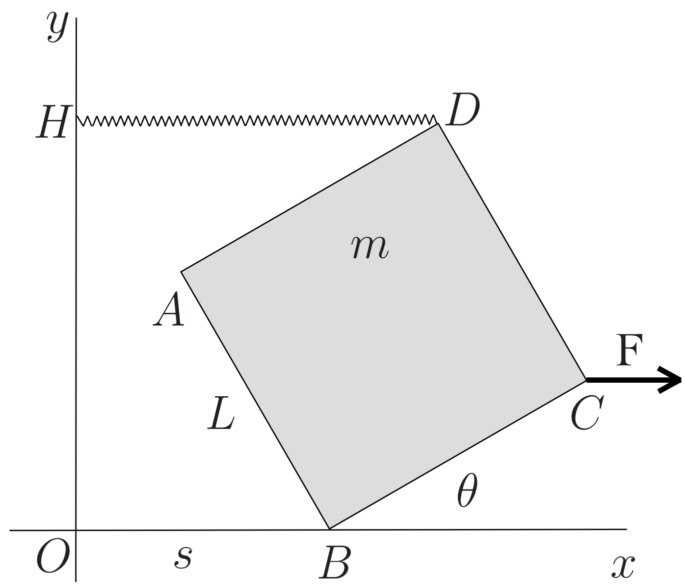
Corso di Laurea in Ingegneria Meccanica
Anno Accademico 2022/2023
Meccanica Razionale - Appello del 19/4/2023

Nome ....
N. Matricola .... Ancona, 19 aprile 2023

1. Un quadrato $ABCD$ di lato L e massa m si muove nel piano verticale $O(x, y)$, libero di ruotare attorno al vertice vertice B che scorre senza attrito sull'asse x. Una molla di costante elastica $k > 0$ collega il vertice D con il punto H, proiezione ortogonale di D sull'asse y. Una forza costante $\mathbf{F} = m\,g\,\hat{\mathbf{i}}$ agisce sul vertice C. Utilizzando le coordinate lagrangiane s (ascissa di B) e θ (angolo di BC con l'asse x) si chiede di determinare le configurazioni di equilibrio e discuterne la stabilità.



$\ell = 2 \qquad q_1 = s \qquad q_2 = \vartheta$

$$P_0 - O = \left[s + \frac{L\sqrt{2}}{2}\cos\left(\vartheta + \frac{\pi}{4}\right)\right]\hat{\imath} +$$

$$+ \frac{L\sqrt{2}}{2}\sin\left(\vartheta + \frac{\pi}{4}\right)\hat{\jmath}$$

$$D - O = \left[S + L\sqrt{2}\cos\left(\vartheta + \frac{\pi}{4}\right)\right]\hat{i}$$

$$+ \; L\sqrt{2}\sin\left(\vartheta + \frac{\pi}{4}\right)\hat{j}$$

$$x_c = S + L\cos\vartheta$$

$$V = V_g + V_k + V_F = mg\,\frac{L\sqrt{2}}{2}\sin\left(\vartheta + \frac{\pi}{4}\right) +$$

$$+ \; \frac{1}{2}k\left[S + L\sqrt{2}\cos\left(\vartheta + \frac{\pi}{4}\right)\right]^2 - F\left(S + L\cos\vartheta\right)$$

$$(\text{con } F = mg)$$

$$\frac{\partial V}{\partial s} = k\left[s + L\sqrt{2}\cos\left(\vartheta + \frac{\pi}{4}\right)\right] - mg$$

$$\frac{\partial V}{\partial \vartheta} = mg\,\frac{L\sqrt{2}}{2}\cos\left(\vartheta + \frac{\pi}{4}\right) -$$

$$- k\left[s + L\sqrt{2}\cos\left(\vartheta + \frac{\pi}{4}\right)\right] L\sqrt{2}\,\mathrm{sen}\left(\vartheta + \frac{\pi}{4}\right) +$$

$$+ mgL\,\mathrm{sen}\,\vartheta$$

$$\frac{\partial V}{\partial s} = 0 \quad ; \quad \frac{\partial V}{\partial \vartheta} = 0$$

$$\Downarrow$$

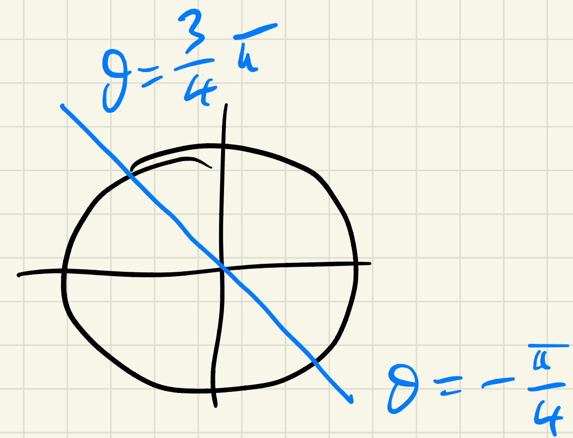
$$\begin{cases} k\left[s + L\sqrt{2}\cos\left(\vartheta + \frac{\pi}{4}\right)\right] = mg \\ mg\,\frac{L\sqrt{2}}{2}\cos\left(\vartheta + \frac{\pi}{4}\right) - mg\,L\sqrt{2}\,\mathrm{sen}\left(\vartheta + \frac{\pi}{4}\right) + \\ \quad + mg\,L\,\mathrm{sen}\,\vartheta = 0 \end{cases}$$

$$
\begin{cases}
k\left[s + L\sqrt{2}\cos\left(\vartheta + \frac{\pi}{4}\right)\right] = mg \\
mg\,\frac{L\sqrt{2}}{2}\,\frac{\sqrt{2}}{2}\left(\cos\vartheta - \sin\vartheta\right) - mgL\,\sqrt{2}\,\frac{\sqrt{2}}{2}\left(\cos\vartheta + \sin\vartheta\right) + \\
\quad + mgL \operatorname{sen}\vartheta \qquad = 0
\end{cases}
$$

$$
\begin{cases}
k\left[s + L\sqrt{2}\cos\left(\vartheta + \frac{\pi}{4}\right)\right] = mg \\
\frac{mgL}{2}\left(\cos\vartheta - \sin\vartheta\right) - mgL\left(\cos\vartheta + \cancel{\sin\vartheta}\right) + \cancel{mgL \operatorname{sen}\vartheta} = 0
\end{cases}
$$

$$
\begin{cases}
k\left[s + L\sqrt{2}\cos\left(\vartheta + \frac{\pi}{4}\right)\right] = mg \\
-\frac{mgL}{2}\cos\vartheta - \frac{mgL}{2}\operatorname{sen}\vartheta = 0
\end{cases}
$$

$$
\begin{cases}
k\left[s + L\sqrt{2}\cos\left(\vartheta + \frac{\pi}{4}\right)\right] = mg \\
\cos\vartheta + \sin\vartheta = 0
\end{cases}
\Longrightarrow
$$



$$\theta = \frac{3}{4}\pi \quad : \quad s + L\sqrt{2}\cos\pi = mg/k$$

$$s - L\sqrt{2} = mg/k \qquad s = \frac{mg}{k} + L\sqrt{2}$$

$$\theta = -\frac{\pi}{4} \qquad s = \frac{mg}{k} - L\sqrt{2}$$

$$\frac{\partial^2 V}{\partial s^2} = k \quad ; \quad \frac{\partial^2 V}{\partial s \partial \theta} = -k L\sqrt{2}\,\mathrm{sen}\left(\theta + \frac{\pi}{4}\right)$$

$$\frac{\partial^2 V}{\partial \theta^2} = -mg\frac{L\sqrt{2}}{2}\sin\left(\theta + \frac{\pi}{4}\right) + mgL\cos\theta +$$

$$+k\left[-L\sqrt{2}\,\mathrm{sen}\left(\theta + \frac{\pi}{4}\right)\right]L\sqrt{2}\,\mathrm{sen}\left(\theta + \frac{\pi}{4}\right) -$$

$$-k\left[s + L\sqrt{2}\cos\left(\theta + \frac{\pi}{4}\right)\right]L\sqrt{2}\cos\left(\theta + \frac{\pi}{4}\right) =$$

$$= mgL\cos\vartheta - mg\frac{L\sqrt{2}}{2}\sin\left(\vartheta + \frac{\pi}{4}\right) -$$

$$- 2kL^2 - kLs\sqrt{2}\cos\left(\vartheta + \frac{\pi}{4}\right)$$

Matrice Hessiana per $\theta = \frac{3}{4}\pi$:

$$V_{ss} = k > 0 \qquad V_{s\theta} = 0$$

$$V_{\theta\theta} = - mgL\frac{\sqrt{2}}{2} \cancel{-2kL^2} + kL\sqrt{2}\left(\frac{mg}{k} \cancel{+ L\sqrt{2}}\right)$$

$$= - mgL\frac{\sqrt{2}}{2} + mgL\sqrt{2} = mgL\frac{\sqrt{2}}{2} > 0$$

STABILE

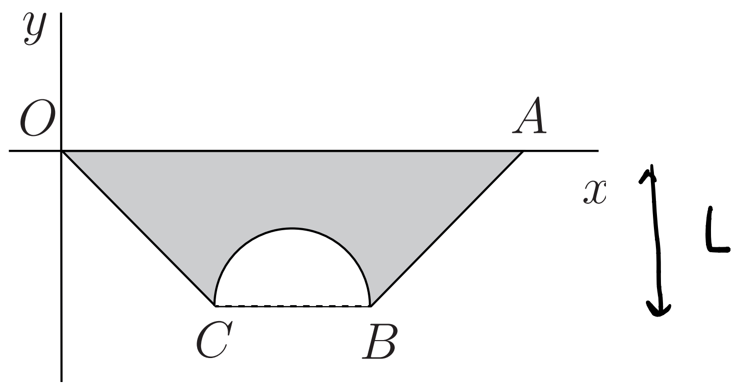
Matrice Hessiana per $\theta = -\frac{\pi}{4}$

$V_{ss} = k > 0$ $V_{s\theta} = 0$

$V_{\theta\theta} = \dots = -mgL\frac{\sqrt{2}}{2} < 0$

INSTABILE

2. Nel sistema di riferimento $O(x, y, z)$ indicato in figura, calcolare la matrice d'inerzia della lamina di massa m, costituita dal trapezio isoscele $OABC$, di base maggiore $OA = 3\,L$, base minore $BC = L$ e lati obliqui OC e AB inclinati di $\pi/4$ rispetto alla base, privato del semicerchio di diametro BC. Il vertice A sta sull'asse x mentre i vertici B e C appartengono al IV quadrante.



Non si possono usare le formule notevoli dei momenti d'inerzia.

m_T = massa Trapezio

m_C = massa semic.

$$A_T = 4L \cdot L \, \frac{1}{2} = 2L^2$$

$$A_C = \frac{\pi R^2}{2} = \frac{\pi L^2}{8}$$

$$\begin{cases} m_T - m_c = m \\ \dfrac{m_T}{m_c} = 4L \cdot L \, \dfrac{1}{2} \, \dfrac{8}{\pi L^2} = \dfrac{16}{\pi} \end{cases}$$

$$m_T = \frac{16}{\pi} m_c$$

$$m_c \left(\frac{16}{\pi} - 1 \right) = m$$

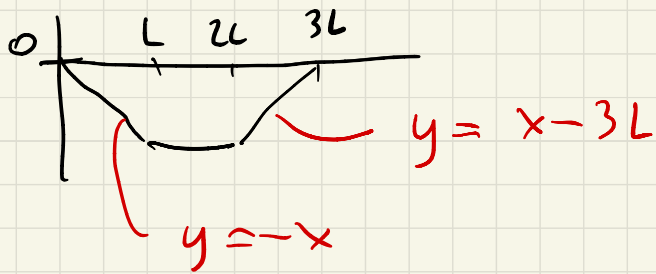
$$m_c \, \frac{16 - \pi}{\pi} = m$$

$$m_c = \frac{\pi}{16 - \pi} m$$

$$m_T = \frac{16}{16 - \pi} m$$

$$\underline{\underline{I}} = \underline{\underline{I}}_T - \underline{\underline{I}}_C$$

Trapezio



$$I_{11} = \sigma \int_{-L}^{0} dy \int_{-y}^{y+3L} dx \; y^2 = \sigma \int_{-L}^{0} dy \, y^2 (2y + 3L) =$$

$$= \sigma \left\{ 2 \left. \frac{y^4}{4} \right|_{-L}^{0} + 3L \left. \frac{y^3}{3} \right|_{-L}^{0} \right\} = \sigma \left\{ -\frac{L^4}{2} + L^4 \right\} = \sigma \frac{L^4}{2} =$$

$$= m_T \frac{L^2}{4}$$

$$I_{22} = \sigma \int_{-L}^{0} dy \int_{-y}^{y+3L} x^2 dx = \sigma \int_{-L}^{0} dy \, \frac{(y+3L)^3 + y^3}{3} =$$

$$= \frac{\sigma}{3} \int_{-L}^{0} \left(2y^3 + 27L^3 + 9Ly^2 + 27L^2 y \right) dy =$$

$$= \frac{\sigma}{3} \left\{ 2 \frac{y^4}{4} \Big|_{-L}^{0} + 27L^3 \cdot L + 9L \frac{y^3}{3} \Big|_{-L}^{0} + 27L^2 \frac{y^2}{2} \Big|_{-L}^{0} \right\} =$$

$$= \frac{\sigma}{3} \left\{ -\frac{L^4}{2} + 27L^4 + 3L^4 - \frac{27}{2} L^4 \right\} = \frac{\sigma}{3} \cdot 16 L^4 =$$

$$= \frac{8}{3} m_T L^2 \qquad I_{33} = \left(\frac{8}{3} + \frac{1}{4} \right) m_T L^2 = \frac{35}{12} m_T L^2$$

$$I_{12} = -\sigma \int_{-L}^{0} dy\, y \int_{-y}^{y+3L} dx\, x = -\sigma \int_{-L}^{0} dy\, y \left. \frac{x^2}{2} \right|_{-y}^{y+3L} =$$

$$= -\frac{\sigma}{2} \int_{-L}^{0} dy\, y \left[(y+3L)^2 - y^2 \right] =$$

$$= -\frac{\sigma}{2} \int_{-L}^{0} dy\, y\, (3L)(2y+3L) =$$

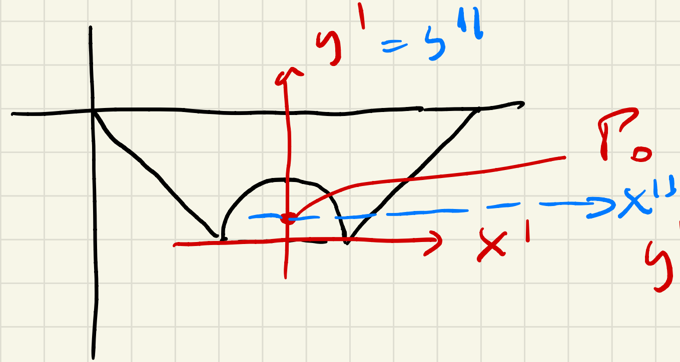
$$= -\frac{\sigma}{2} \cdot 3L \int_{-L}^{0} dy \left(2y^2 + 3Ly \right) =$$

$$= -\frac{3}{2}\sigma L\left(2\left.\frac{y^3}{3}\right|_{-L}^{0} + 3L\left.\frac{y^2}{2}\right|_{-L}^{0}\right) =$$

$$= -\frac{3}{2}\sigma L\left(\frac{2}{3}L^3 - \frac{3}{2}L^3\right) = -\frac{3}{2}\sigma L^4\,\frac{-5}{6} =$$

$$= \frac{5}{4}\sigma L^4 = \frac{5}{8}m_T L^2$$

Semicerchio



$$y'(P_0) = \frac{4R}{3\pi} = \frac{2L}{3\pi}$$

In x', y':

$$I_{11} = \sigma \int_0^R dr\, r \int_0^\pi d\vartheta\, r^2 \sin^2\vartheta =$$

$$= \sigma \int_0^R dr\, r^3\, \frac{\pi}{2} = \sigma \frac{\pi}{2} \frac{R^4}{4} = \frac{1}{4} m_c R^2 = \frac{1}{16} m_c L^2$$

$$= I_{22}$$

$$I_{33} = I_{11} + I_{22} = \frac{1}{8} m_c L^2 \qquad I_{12} = 0$$

In x'' y'' :

$$I_{22} = \frac{1}{16} m_c L^2 \qquad I_{12} = 0$$

$$I_{11} = \frac{1}{16} m_c L^2 - m_c \left(\frac{2L}{3\pi}\right)^2$$

In $O(x,y)$:

$$I_{11} = \frac{1}{16} m_c L^2 - m_c \left(\frac{2L}{3\pi}\right)^2 + m_c \left(L - \frac{2L}{3\pi}\right)^2$$

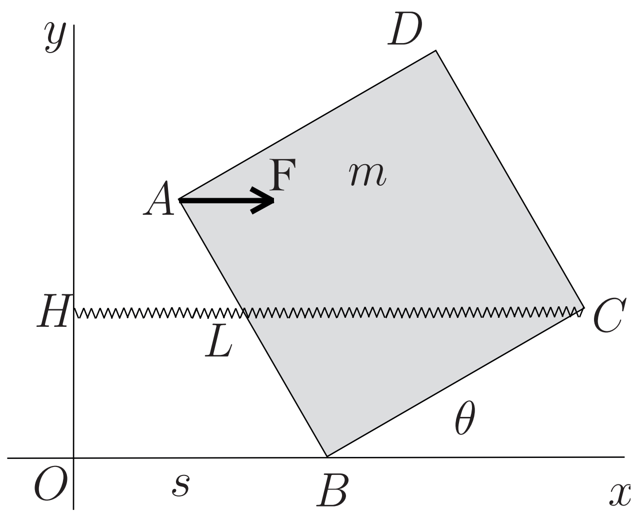
$$I_{22} = \frac{1}{16} m_c L^2 + \frac{9}{4} m_c L^2 \qquad I_{33} = I_{11} + I_{22}$$

$$I_{12} = 0 + m_c \frac{3}{2} L \left(L - \frac{2L}{3\pi}\right)$$

Corso di Laurea in Ingegneria Meccanica
Anno Accademico 2022/2023
Meccanica Razionale - Appello del 19/4/2023

Nome ....
N. Matricola .... Ancona, 19 aprile 2023

1. Un quadrato $ABCD$ di lato L e massa m si muove nel piano verticale $O(x, y)$, libero di ruotare attorno al vertice vertice B che scorre senza attrito sull'asse x. Una molla di costante elastica $k > 0$ collega il vertice C con il punto H, proiezione ortogonale di C sull'asse y. Una forza costante $\mathbf{F} = m\, g\, \widehat{\mathbf{i}}$ agisce sul vertice A. Utilizzando le coordinate lagrangiane s (ascissa di B) e θ (angolo di BC con l'asse x) si chiede di determinare le configurazioni di equilibrio e discuterne la stabilità.



$l = 2$ $\qquad q_1 = s$ $\qquad q_2 = \theta$

$$P_0 - O = \left[s + \frac{L\sqrt{2}}{2}\cos\left(\theta + \frac{\pi}{4}\right)\right]\hat{\imath} + $$
$$+ \frac{L\sqrt{2}}{2}\sin\left(\theta + \frac{\pi}{4}\right)\hat{\jmath}$$

$$x_C = S + L \cos\vartheta$$

$$x_A = S - L \operatorname{sen}\vartheta$$

$$V = V_g + V_k + V_F = mg \frac{L\sqrt{2}}{2} \sin\left(\vartheta + \frac{\pi}{4}\right) +$$

$$+ \frac{1}{2} k \left(S + L\cos\vartheta\right)^2 - F\left(S - L\sin\vartheta\right)$$

(con $F = mg$)

$$\frac{\partial V}{\partial s} = k\left(s + L\cos\vartheta\right) - mg$$

$$\frac{\partial V}{\partial \vartheta} = mg\,\frac{L\sqrt{2}}{2}\cos\left(\vartheta + \frac{\pi}{4}\right) - kL\left(s + L\cos\vartheta\right)\sin\vartheta -$$

$$+ mgL\cos\vartheta$$

$$\frac{\partial V}{\partial s} = 0 \quad ; \quad \frac{\partial V}{\partial \vartheta} = 0$$

$$\Downarrow$$

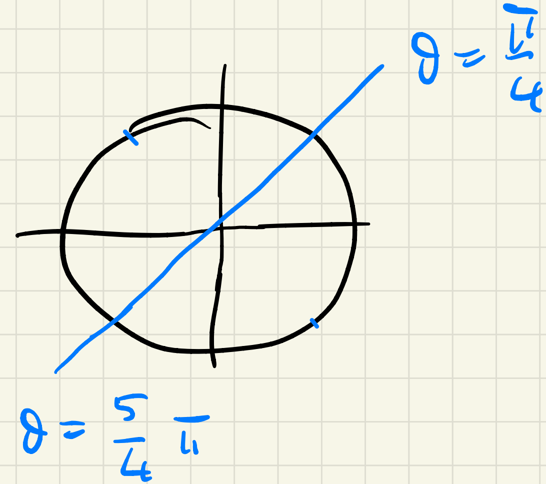
$$\begin{cases} k\,(s + L\cos\vartheta) = mg \\ mgL\frac{\sqrt{2}}{2}\frac{\sqrt{2}}{2}(\cos\vartheta - \sin\vartheta) - kL(s+L\cos\vartheta)\sin\vartheta - \end{cases}$$

$$+ mgL\cos\vartheta = 0$$

$$\begin{cases} k\left(s+L\cos\vartheta\right) = mg \\ mg\,\frac{L\sqrt{2}}{2}\,\frac{\sqrt{2}}{2}\left(\cos\vartheta - \sin\vartheta\right) - mgL\sin\vartheta + mgL\cos\vartheta = 0 \\ \end{cases}$$

$$\begin{cases} k\left(s+L\cos\vartheta\right) = mg \\ \frac{3}{2}mgL\cos\vartheta - \frac{3}{2}mgL\sin\vartheta = 0 \end{cases}$$

$$\begin{cases} k(s+L\cos\vartheta) = mg \\ \cos\vartheta - \sin\vartheta = 0 \end{cases} \Longrightarrow$$



$$\theta = \frac{\pi}{4} \quad : \quad s + \frac{L}{\sqrt{2}} = mg/k$$

$$s = \frac{mg}{k} - \frac{L}{\sqrt{2}}$$

$$\theta = \frac{5}{4}\pi \qquad s = \frac{mg}{k} + \frac{L}{\sqrt{2}}$$

$$\frac{\partial^2 V}{\partial s^2} = k \quad ; \quad \frac{\partial^2 V}{\partial s \partial \theta} = -k L \operatorname{sen}\theta$$

$$\frac{\partial^2 V}{\partial \theta^2} = -mg \frac{L\sqrt{2}}{2} \sin\left(\theta + \frac{\pi}{4}\right) -$$

$$- kLs\cos\theta - kL^2(\cos^2\theta - \sin^2\theta) - mgL\sin\theta$$

$$= -\frac{mgL}{2}(\sin\theta + \cos\theta) - kLs\cos\theta - mgL\sin\theta -$$

$$- kL^2(\cos^2\theta - \sin^2\theta)$$

Matrice Hessiana per $\theta = \frac{\pi}{4}$:

$$V_{ss} = k > 0 \qquad V_{s\theta} = -kL\frac{\sqrt{2}}{2}$$

$$V_{\theta\theta} = -\frac{mgL}{\sqrt{2}} - kL\left(\frac{mg}{k} - \frac{L}{\sqrt{2}}\right)\frac{\sqrt{2}}{2} - mgL\frac{\sqrt{2}}{2} =$$

$$= -mg\frac{L}{\sqrt{2}} - \frac{mgL}{\sqrt{2}} + \frac{kL^2}{2} - mgL\frac{\sqrt{2}}{2} =$$

$$= \frac{1}{2}kL^2 - 3\frac{mgL}{\sqrt{2}} = \frac{1}{2}kL^2\left(1 - 3\sqrt{2}\,\frac{mg}{kL}\right)$$

$$\mathrm{Det}(H) = \frac{1}{2} k^2 L^2 \left(\cancel{1} - 3\sqrt{2}\,\frac{mg}{kL} \right) - \cancel{\frac{k^2 L^2}{2}} =$$

$$= -\frac{3}{\sqrt{2}}\, mg k L < 0$$

INSTABILE

Matrice Hessiana per $\theta = \frac{5}{4}\pi$

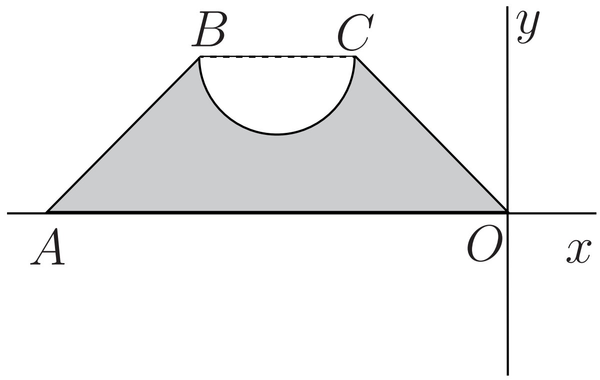
$V_{ss} = k > 0$ $\qquad$ $V_{s\theta} = kL\frac{\sqrt{2}}{2}$

$V_{\theta\theta} = \; --- \; = \frac{1}{2}kL^2\left(1 + 3\sqrt{2}\,\frac{mg}{kL}\right)$

$Det(H) = \frac{3}{\sqrt{2}}\,mgkL > 0$

STABILE perché $V_{ss} > 0$

2. Nel sistema di riferimento $O(x, y, z)$ indicato in figura, calcolare la matrice d'inerzia della lamina di massa m, costituita dal trapezio isoscele $OABC$, di base maggiore $OA = 3\,L$, base minore $BC = L$ e lati obliqui OC e AB inclinati di $\pi/4$ rispetto alla base, privato del semicerchio di diametro BC. Il vertice A sta sull'asse x mentre i vertici B e C appartengono al II quadrante.



Non si possono usare le formule notevoli dei momenti d'inerzia.

m_T = massa Trapezio

m_C = massa semic.

$$A_T = 4L \cdot L \frac{1}{2} = 2L^2$$

$$A_C = \frac{\pi R^2}{2} = \frac{\pi L^2}{8}$$

$$\begin{cases} m_T - m_C = m \\ \dfrac{m_T}{m_C} = 4L \cdot L \dfrac{1}{2} \dfrac{8}{\pi L^2} = \dfrac{16}{\pi} \end{cases}$$

$$m_T = \frac{16}{\pi} m_C$$

$$m_C \left(\frac{16}{\pi} - 1 \right) = m$$

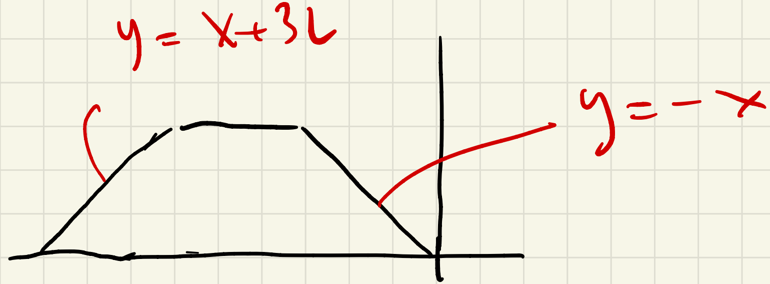
$$m_C \frac{16 - \pi}{\pi} = m$$

$$m_C = \frac{\pi}{16 - \pi} m$$

$$m_T = \frac{16}{16 - \pi} m$$

$$\underline{\underline{I}} = \underline{\underline{I}}_T - \underline{\underline{I}}_C$$

Trapezio

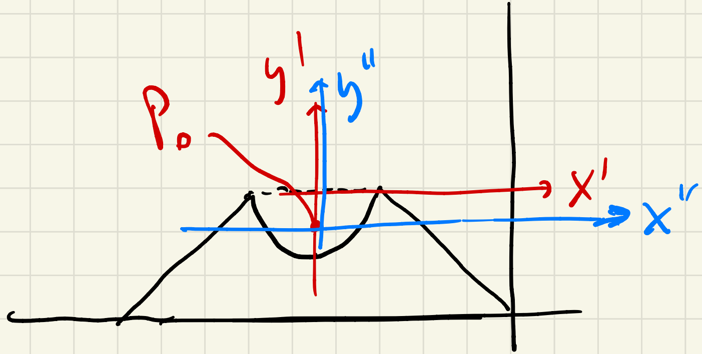


$$I_{11} = \sigma \int_0^L dy \int_{y-3L}^{-y} dx \; y^2 = \sigma \int_0^L dy \, y^2 (3L - 2y) = m_T \frac{L^2}{4}$$

$$I_{22} = \sigma \int_0^L dy \int_{y-3L}^{-y} x^2 \, dx = \frac{8}{3} m_T L^2$$

$$I_{12} = -\sigma \int_0^L dy\, y \int_{y-3L}^{-y} dx\, x = \frac{5}{8} m_T L^2$$

Semicerchio



Gm

$$y'(P_0) = -\frac{4R}{3\pi} = -\frac{2L}{3\pi}$$

In x', y' :

$$I_{11} = \sigma \int_0^R dr\, r \int_{\pi}^{2\pi} d\vartheta\, r^2 \sin^2\vartheta = \frac{1}{16} m_c L^2 = I_{22}$$

$$I_{33} = I_{11} + I_{22} = \frac{1}{8} m_c L^2 \qquad I_{12} = 0$$

In x^u y^u :

$$I_{22} = \frac{1}{16} m_c L^2 \qquad I_{12} = 0$$

$$I_{11} = \frac{1}{16} m_c L^2 - m_c \left(\frac{2L}{3\pi}\right)^2$$

In $O(x, y)$:

$$I_{11} = \frac{1}{16} m_c L^2 - m_c \left(\frac{2L}{3\pi}\right)^2 + m_c \left(L - \frac{2L}{3\pi}\right)^2$$

$$I_{22} = \frac{1}{16} m_c L^2 + \frac{9}{4} m_c L^2 \qquad I_{33} = I_{11} + I_{22}$$

$$I_{12} = 0 + m_c \frac{3}{2} L \left(L - \frac{2L}{3\pi}\right)$$