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Nonlinear evolution of perturbations in marginally stable plasmas

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Abstract

We derive a general, self-consistent, reduced equation that describes the nonlinear evolution of electrostatic perturbations in marginally stable plasma equilibria. The equation is universal in the sense that it is independent of the equilibrium, and it contains as special cases the beam-plasma, the bump-on-tail, and the two-stream instability problems, among others. In particular, the present work offers a systematic justification of the O'Neil–Winfrey–Malmberg single-wave beam-plasma model. But more importantly, the analysis shows that the single-wave model has a wider range of applicability: it can be applied to localized perturbation in any marginally stable equilibrium. We discuss the linear theory, and construct families of exact nonlinear solutions. © 1998 Elsevier Science B.V.

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1. Introduction

A fundamental problem in plasma physics is the nonlinear evolution of perturbations of an equilibrium state. The objective of this Letter is to study this problem for a collisionless plasma. Previous works have considered the beam-plasma instability [1], and the bump-on-tail instability [2]. Recently, the problem has been addressed by constructing the amplitude equation for a weakly unstable mode [3]. Here we present a new approach: using a matched asymptotic expansion, we derive from the Vlasov–Poisson equation a general, self-consistent, reduced equation that describes the nonlinear evolution of perturbations in marginally stable plasma equilibria. This reduced equation is universal in the sense that it is independent of the equilibrium, and it contains as special cases the beam-plasma, the bump-on-tail, and the two-stream instability problems, among others. In particular, the present work offers a systematic derivation of the O'Neil–Winfrey–Malmberg [1] single-wave, beam-plasma model. But more importantly, the analysis shows that the single-wave model have a much wider range of applicability: it can be applied to study localized perturbations in any marginally stable equilibrium. Compared with the amplitude equation approach proposed in Ref. [3], the approach followed here has the advantage that it is not restricted to perturbations along the unstable manifold.

The starting point of our analysis is the dimensionless Vlasov–Poisson system for the electron distribution function in a uniform neutralizing ion background,

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$$\partial_T F + u\,\partial_X F + \partial_u (F_0 + F)\,\partial_X \Phi = 0\,, \tag{1}$$

$$G(k)\,\tilde{\Phi}(k,T) = -\int_{-\infty}^{\infty} \tilde{F}(k,u,T)\mathrm{d}u\,, \tag{2}$$

where the tilde denotes Fourier transform, $G(k) = k^2$, and $F(X,u,T)$ is the departure from the equilibrium $F_0(u)$.

Recently, it has been shown that the two-dimensional Euler equation describing the dynamics of localized vorticity perturbations in shear flow, can be reduced to the vorticity defect equation [4] which is the same as Eqs. (1) and (2) with $G(k) = 2k\coth k$, if one identifies (X,u) with the (x,y), f with the vorticity, and ϕ with the streamfunction. Because of this, plasma physics concepts like Landau damping, and BGK modes, and techniques like the Nyquist method, have an analogue in fluid dynamics [4]. In particular, the reduced equation derived here, describes the nonlinear evolution of vorticity perturbations in marginally stable shear flows. Also, the reduced equation bears similarities with models used to study globally coupled oscillators.

The key assumption we make is that the equilibrium is linearly stable and that it has a stationary inflection point at $u = c_0$, that is $F_0'(c_0) = F_0''(c_0) = 0$. From linear theory [5] it is known that $F_0'(c_0) = 0$ implies the existence of a neutral mode with wave number k_0 (which we will assume to be different from zero) given by the dispersion relation $D(k_0, c_0) = 0$, where

$$D(k,c) \equiv G(k) - \int_{-\infty}^{\infty} \frac{F_0'}{u-c}\,\mathrm{d}u\,. \tag{3}$$

We consider a domain periodic in X, and of size $L \approx 2\pi/k_0$. Using the Nyquist method, it can be shown that the condition $L \approx 2\pi/k_0$, together with the stationary inflection point condition,

$$F_0'(c_0) = F_0''(c_0) = 0,$$

implies that the system is marginally stable [6]. That is, that the equilibrium $F_0(u;\mu)$ is stable for $\mu = \mu_c$, but it is unstable for $\mu = \mu_c + \delta\mu$, where μ is a control parameter, and $\delta\mu \ll 1$. Some examples of marginally stable equilibria for which $k_0 \neq 0$, are shown in Fig. 1.

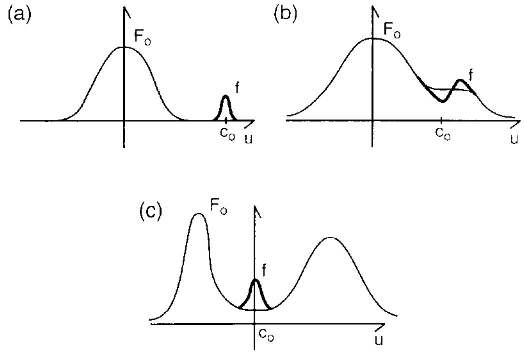

Fig. 1. Examples of marginally stable equilibria. Panels (a), (b), and (c) show the equilibria, F_0, used in the beam-plasma, the bump-on-tail, and the two-stream instability problems respectively. The reduced Vlasov–Poisson equation (18)–(20) describes the nonlinear evolution of a perturbation, f, localized around the stationary inflection point at c_0.

2. Derivation

Our goal is to derive from (1) and (2) a self-consistent equation for perturbations localized around the stationary inflection point, as those shown in Fig. 1. To do this we look for solutions of the form

$$F = \epsilon^2 f(x,u,t)\,, \quad \Phi = \epsilon^2 \phi(x,t)\,, \tag{4}$$

where $1 \gg \epsilon > 0$, $x \equiv (2\pi/L)(X - c_0 T)$, $t \equiv (2\pi/L)\,\epsilon T$. and $L \equiv 2\pi(1+\epsilon\Lambda)/k_0$. The construction in (4) represents a slowly varying, small perturbation, propagating on the background $F_0(u)$. The scaling in (4) corresponds to the trapping scaling [1–3] according to which $E \sim \gamma^2$, where γ is the growth rate of the instability, and E is the amplitude of the electric field after the linear instability has saturated. However, contrary to what is typically assumed, we consider the more general situation in which there is an $\mathrm{O}(\epsilon)$ detuning between the domain length, L, and the wavelength, $2\pi/k_0$, of the linear mode. The parameter Λ determines if L is larger ($\Lambda > 0$), smaller ($\Lambda < 0$), or equal ($\Lambda = 0$) to $2\pi/k_0$.

In terms of the variable x, the domain has period 2π, and we write $\phi(x,t) = \sum_n \tilde{\phi}(n,t)\mathrm{e}^{\mathrm{i}nx}$. Substituting (4) into (1) and (2) we get

$$\epsilon\,\partial_t f + (u - c_0)\,\partial_x f + \left(F_0' + \epsilon^2 \partial_u f\right)\,\partial_x \phi = 0\,, \tag{5}$$

$$G\left[nk_0\left(1+\epsilon\Lambda\right)^{-1}\right]\tilde{\phi}(n,t) = -\int_{-\infty}^{\infty}\tilde{f}(k,u,t)\mathrm{d}u\,. \tag{6}$$

Eqs. (5) and (6) are exact, and our goal is to simplify them using ϵ as a small parameter. The method we follow is inspired by the technique used in the study of critical layers in shear flows [7]. In fact, the scaling (4) is the critical layer scaling, and the critical layer singularity is mathematically similar to the singularity created by the resonant particles in a plasma.

We write $\phi(x,t) = \phi_0 + \epsilon\,\phi_1 + \ldots$, and following the method of matched asymptotic expansions [8], we divide the (x,u) space into two regions: an *inner* region where $u - c_0 = \mathrm{O}(\epsilon)$, and an *outer* region where $u - c_0 = \mathrm{O}(1)$. In each region we solve (5) by expanding f in powers of ϵ, and then we match the solutions in the intermediate region $\epsilon \ll u - c_0 \ll 1$. Once f is found, ϕ is determined self-consistently from (6). The end result of this is the reduced system (18)–(20) below.

2.1. Inner region

Consider first the inner region. Introducing the stretched coordinate $v \equiv (u-c_0)/\epsilon$, and substituting $f(x,v,t) = f_0^i + \epsilon f_1^i + \ldots$, into (5) we get at O(1),

$$\partial_t f_0^i + v\,\partial_x f_0^i + \partial_x\phi_0\,\partial_v f_0^i = 0\,, \tag{7}$$

and at $\mathrm{O}(\epsilon)$,

$$\partial_t f_1^i + v\,\partial_x f_1^i + \partial_x\phi_0\,\partial_v f_1^i + \partial_x\phi_1\,\partial_v f_0^i = v^2\partial_x\Gamma\,, \tag{8}$$

where we have used $F_0'(c_0) = F''(c_0) = 0$, and have defined $\Gamma \equiv -\phi_0 F_0'''(c_0)/2$.

2.2. Outer region

In the outer region, $u - c_0 = \mathrm{O}(1)$, we substitute $\tilde{f}(n,u,t) = \tilde{f}_0^o + \epsilon\,\tilde{f}_1^o + \ldots$ into (5), and get at O(1),

$$\tilde{f}_0^o = -\frac{F_0'}{u-c_0}\,\tilde{\phi}_0\,, \tag{9}$$

and at $\mathrm{O}(\epsilon)$,

$$\tilde{f}_1^o = -\frac{\mathrm{i}}{n}\,\frac{F_0'}{(u-c_0)^2}\,\partial_t\tilde{\phi}_0 - \frac{F_0'}{u-c_0}\,\tilde{\phi}_1\,. \tag{10}$$

Because $F_0'(c_0) = F_0''(c_0) = 0$, there are no singularities in (9) and (10): the stationary inflection point condition smooths the singularity created by the resonant particles at $\mathrm{O}(\epsilon)$, and moves the singularity to $\mathrm{O}(\epsilon^2)$ in the expansion.

2.3. Matching

We have constructed two asymptotic expansions for f: f^i in (7) and (8) which is valid in the inner region, $u - c_0 = \mathrm{O}(\epsilon)$; and f^o in (9) and (10) which is valid in the outer region $u - c_0 = \mathrm{O}(1)$. The approximation will be consistent provided the solutions match in the overlap, or intermediate, region $\epsilon \ll u - c_0 \ll 1$. To check this, it is convenient to introduce the intermediate variable $\eta \equiv (u-c_0)/\epsilon^{1/3} = \epsilon^{2/3}v$. To guarantee the matching, we impose the boundary condition $f_0^i \sim 1/v^2$ (or smaller) as $v \to \infty$ on Eq. (7), and get from (8) the following asymptotic expression for f^i is the intermediate region,

$$f^i(x,\eta,t) = \epsilon^{1/3}\,\eta\Gamma - \epsilon\int\partial_t\Gamma\,\mathrm{d}x + \mathrm{O}(\epsilon^{4/3})\,. \tag{11}$$

On the other hand, from (9) and (10), in the matching region,

$$\tilde{f}^o(n,\eta,t) = \epsilon^{1/3}\,\eta\tilde{\Gamma} + \epsilon\,\frac{\mathrm{i}}{n}\,\partial_t\tilde{\Gamma} + \mathrm{O}(\epsilon^{4/3})\,. \tag{12}$$

After taking the Fourier transform of (11) we conclude that at $\mathrm{O}(\epsilon)$ in the intermediate region $\tilde{f}^o = \tilde{f}^i$. Therefore, as required, the two solutions match.

2.4. Self-consistent potential

The next step is to determine the potential ϕ self-consistently from the Poisson equation (6). Expanding both sides of (6) in powers of ϵ, we get at O(1),

$$G(nk_0)\,\tilde{\phi}_0 = -\int_{-\infty}^{\infty}\tilde{f}_0^o\,\mathrm{d}u\,, \tag{13}$$

and at $\mathrm{O}(\epsilon)$,

$$G(nk_0)\,\tilde{\phi}_1 - nk_0\Lambda G'(nk_0)\,\tilde{\phi}_0 = -\int_{-\infty}^{\infty}\tilde{f}_1^o\,\mathrm{d}u - \int_{-\infty}^{\infty}\tilde{f}_0^i\,\mathrm{d}v\,. \tag{14}$$

Because $\int \tilde{f}^i du = \epsilon \int \tilde{f}^i dv$, the zeroth-order term in the inner field, f^i, contributes to the first-order term of ϕ. This is the reason why $\tilde{f}_0^i$ appears in (14) and not in (13). Note also that $F_0'(c_0) = F_0''(c_0) = 0$ guarantees that the integrals in (13) and (14) are not singular.

Substituting (9) into (13), and using $D(k_0, c_0) = 0$ in (3), we conclude $n = \pm 1$, and

$$\phi_0 = \tilde{\phi}_0(1,t)\, e^{ix} + \tilde{\phi}_0(-1,t)\, e^{-ix} . \tag{15}$$

That is, the single-wave spatial structure of the potential arises naturally from the leading-order balance in (6). To determine the time evolution of ϕ_0, we substitute (10) into (14) and get

$$\gamma \frac{d\tilde{\phi}_0}{dt} + i\lambda \tilde{\phi}_0 = i\langle e^{-ix} f_0^i \rangle , \tag{16}$$

where $\langle\ \rangle$ denotes average over x and v,

$$\langle \theta \rangle \equiv \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_0^{2\pi} \theta \, dx \, dv , \tag{17}$$

$\gamma \equiv \partial_c D(k_0, c_0)$, and $\lambda \equiv A k_0 \partial_k D(k_0, c_0)$, where $D(c,k)$ is the dispersion function (3).

2.5. Reduced Vlasov–Poisson equation

Eqs. (7) and (16) form a closed, self-consistent system of equations describing the nonlinearly saturated state in the vicinity of the stationary inflection point. To simplify the notation we define $\sigma \equiv \text{sign}(\gamma)$, $\ell \equiv \text{sign}(A)$, and use the convention $\text{sign}(0) = 0$. Also, we rescale $v \to |\lambda/\gamma|\, v$, $t \to |\gamma/\lambda|\, t$, define $f \equiv |\gamma|/\lambda^2 f_0^i$, $a \equiv (\gamma/\lambda)^2\, \tilde{\phi}_0$, and rewrite (7) and (16) as

$$\partial_t f + v\, \partial_x f + \partial_x \varphi\, \partial_v f = 0 , \tag{18}$$

$$\varphi = a(t)\, e^{ix} + a^*(t)\, e^{-ix} , \tag{19}$$

$$\sigma \frac{da}{dt} + i\ell\, a = i \langle e^{-ix} f \rangle . \tag{20}$$

The reduced system (18)–(20) is universal in the sense that it has no free parameters, and it is independent of the equilibrium F_0. The only requirements are that F_0 has a stationary inflection point at c_0, that $k_0 \neq 0$, and that the perturbation is localized around c_0. These requirements are satisfied for the beam-plasma, the bump-on-tail, and the two-stream instability problems, among others. In particular, discretizing the distribution function f as a finite number of point charges, and assuming $\ell = 0$, we recover as an special case the single-wave beam-plasma model [1]. Compared with the approach followed in Ref. [3], the reduced equation (18)–(20) has the advantage that it is not restricted to perturbations along the unstable manifold. Also, the amplitude equation in Ref. [3], contains an infinite number of terms which make difficult to draw conclusions on the nonlinear saturation of the perturbation.

The reduced equation (18)–(20) has the same structure as an equation derived by Churilov and Shukhman [9] for the nonlinear evolution of the critical layer in a marginally stable shear flow. The reason for this lays in the analogy between the Vlasov–Poisson equation, and the vorticity defect equation [4]. In fact, one can conceive the derivation of the critical layer equation in Ref. [9] as consisting of two steps: in the first step, one reduces the 2D Euler equation to the vorticity defect equation, as done in Ref. [4]; and in the second step, as it is shown here, one reduces the vorticity defect equation to the critical layer equation. Another example in which an advection equation of the form (18) is coupled to an amplitude equation of the form (20), is the system derived by Warn and Gauthier [10] in the study of marginally unstable baroclinic waves in a fluid.

Recently, there has been a great deal of interest in the study of the collective behavior of globally coupled oscillator systems, and some similarities between the Vlasov–Poisson system, and couple oscillator models have been noted [13]. It is interesting to observe that the reduced system (18)–(20) can be viewed as the kinetic equation for a distribution of identical, nonlinear Hamiltonian oscillators coupled through a mean field. What is important about this system is that the mean-field coupling of the oscillators comes from the self-consistent dynamics, and not from an ad-hoc model. The ideas discussed above, are to some extent independent of the specific form of the coupling, and therefore they can be applied to coupled oscillators Hamiltonian models in general.

We have neglected dissipative effects. However, the Krook collision model, $\partial F/\partial T = -\hat{\alpha} F$, can be incorporated into the analysis. Assuming that $\hat{\alpha} = \epsilon^2 \alpha$, it is easy to see that to $O(\epsilon)$, collisions are only important in the inner region, and that the right-hand side of (18) becomes $-\alpha f$.

The system (18)-(20) inherits all the conservation laws of the Vlasov-Poisson system (1)-(2); it conserves $\langle C(f)\rangle$ for any function $C(f)$, and it has the momentum-like, and the energy-like invariants

$$P = \langle vf\rangle + \sigma|a|^2,$$
$$E = \langle (v^2/2 - \varphi)\, f\rangle + \ell|a|^2. \qquad (21)$$

3. Linear theory and exact nonlinear solutions

The linear theory of equilibrium solutions of the form $f = f_0(v)$, $a = 0$, is straightforward. Substituting

$$f = f_0(v) + \sum_{n=-\infty}^{\infty} \chi_n(v)\, e^{in(x-ct)}, \quad a = \rho e^{-inct} \qquad (22)$$

into (18)-(20), and neglecting the nonlinear terms, we get the dispersion relation $d(c) = 0$, where

$$d(c) \equiv \sigma c - \ell - \int_{-\infty}^{\infty} \frac{f_0'}{v - c}\, dv. \qquad (23)$$

As expected, (23) is the leading-order part of the dispersion relation (3) for the equilibrium $F_0(u) + \epsilon^2 f_0((u - c_0)/\epsilon)$. In the same way as is done for the Vlasov-Poisson system, the linear initial value problem can be solved using Laplace transforms, and Landau damping, both linear and non-linear [11], can be studied in the context of the reduced system. Also, using the Nyquist method, necessary and sufficient conditions for the instability can be derived.

The reduced system admits a large class of exact nonlinear solutions which are the analogue of the BGK modes [12] for the Vlasov equation. These solutions are useful to model coherent structures, and to understand the nonlinear saturation of instabilities. To construct them, we substitute $f = f(\xi, v)$, with $\xi = x - ct$, and $a = \rho\, e^{i\delta - ict}$ into (18) and (20), and get

$$\frac{\partial(H, f)}{\partial(\xi, v)} = 0, \qquad (24)$$

$$\langle f\, e^{-i\xi}\rangle = (\ell - \sigma c)\,\rho, \qquad (25)$$

where $H = \frac{1}{2}(v - c)^2 - 2\rho\cos\xi$, and the integral in $\langle\cdot\rangle$ is taken over v and ξ. Any function of the form $f = f(H)$ is a solution of Eq. (24). However, $f = f(H)$ will be a self-consistent solution of (18) and (20), only if (25) is also satisfied. For example, $f(H) = \exp(-H)$ will be a solution provided $2\sqrt{\pi} I_1(2\rho) = \sqrt{2}\rho(\ell - \sigma c)$, where $I_1(z)$ is the modified Bessel function of order one. It is interesting to observe that in the small amplitude limit, $\rho \to 0$, Eq. (25), which determines the speed of the wave c in terms of its amplitude ρ, becomes the linear dispersion relation (23); and the nonlinear solution $f = f(H)$ becomes the neutral mode,

$$f \approx f((v - c)^2) - 2\rho \frac{f'}{v - c} \cos(x - ct). \qquad (26)$$

4. Concluding remarks

We conclude with some remarks on the problem of integrability and chaos in plasma and fluid systems. In an integrable Hamiltonian, particles move on KAM (Kolmogorov-Arnold-Moser) tori, and the problem of transition to chaos is to determine the fate of these tori when a perturbation is added. Since the pioneering work of Poincaré, important advances, including the well-known KAM theorem, have been done in the study of this problem in the case of finite degrees-of-freedom (d.o.f.) systems. However, the case of infinite d.o.f. systems has proved to be considerably harder to study. From the Lagrangian point of view, the exact solution (24) and (25) correspond to an infinite d.o.f. integrable Hamiltonian system in which the "particles" (i.e. electrons in the plasma physics context, or vorticity elements in fluid dynamics) move on the level sets of $f(H)$ which are in fact the KAM tori of the Hamiltonian $H = \frac{1}{2}(v - c)^2 - 2\rho\cos\xi$. Following the intuition of finite d.o.f. systems, one expects that, when a perturbation is added, some of the tori will be destroyed whereas others will persist in some sense. This picture is supported by recent numerical simulation of the beam plasma instability with a finite, but large (10^4), number of particles [14]. In these simulations, it is observed that a fraction of particles is trapped in what seems to be time-dependent invariant surfaces. The characterization of these KAM-like tori in the infinite d.o.f. limit, and the general problem of chaos and integrability in the reduced system are problems that we plan to address in a future publication. These problems are closely related to the problem of self-consistent chaotic vorticity mixing.

Another interesting problem is the relation between negative energy modes and instability. The reduced system applies to linearly stable equilibria with $F_0'(c_0) = F_0''(c_0) = 0$, and it is known [6] that such equilibria can support neutral modes with negative energy which can lead to instability via dissipation or nonlinearity.

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