Self-consistent Lagrangian study of nonlinear Landau damping

Francesco Valentini, Vincenzo Carbone, and Pierluigi Veltri
Dipartimento di Fisica and Istituto Nazionale di Fisica della Materia, Unità di Cosenza, Università della Calabria, I-87030 Arcavacata di Rende, Italy

André Mangeney
Section de Meudon 5, LESIA–Observatoire de Paris, place Jules Janssen, 92195 Meudon Cedex, France
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The electric field computed by numerically solving the one-dimensional Vlasov-Poisson system is used to calculate Lagrangian trajectories of particles in the wave-particle resonance region. The analysis of these trajectories shows that, when the initial amplitude of the electric field is above some threshold, two populations of particles are present: a first one located near the separatrix, which performs flights in the phase space and whose trajectories become ergodic and chaotic, and a second population of trapped particles, which displays a nonergodic dynamics. The complex, nonlinear interaction between these populations determines the oscillating long-time behavior of solutions.

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The Landau conjecture [1] that wave-particle interactions in a plasma can give rise to wave damping also when collisions are absent represents a milestone in physics not only by its impact on laboratory and space plasma, but also as a paradigm for processes that occur in different systems [2]. In unmagnetized plasmas, Landau damping is described, in the framework of the kinetic theory, by the one-dimensional nonlinear Vlasov-Poisson system of equations

$$\frac{\partial f}{\partial t} + v\frac{\partial f}{\partial x} - \frac{eE}{m}\frac{\partial f}{\partial v} = 0, \tag{1}$$

$$\frac{\partial E}{\partial x} = 4\pi e\left(n_0 - \int_{-\infty}^{+\infty} f\, dv\right), \tag{2}$$

where f is the electron distribution function (the ions are considered as a motionless back-ground of neutralizing positive charge with density n_0) and E is the self-consistent longitudinal electric field.

In the linear theory [1], the damping is produced by the interaction between a wave with phase velocity v_ϕ and particles with velocity $v \simeq v_\phi$. The physical content of the linear interaction is conceptually quite simple: particles whose velocity is just below the wave phase velocity in their *tail-on collision* with the wave gain some energy, while particles whose velocity is just above lose it (*head-on collisions*). When the former particles are more numerous than the latter the wave exhibits exponential damping.

The nonlinear regime of the plasma oscillations was first studied by O'Neil [3], who found that after a time $\tau_p \simeq \sqrt{m/(eEk)}$ (k is the wave vector), particles have time to make both tail-on and head-on collisions with the wave so that the net energy exchange between wave and particles when averaged in time is null. O'Neil [3] predicted then a damping rate that, after oscillating with a period of the order of τ_p, becomes asymptotically zero through a phase mixing process, thus stopping the wave dissipation.

In 1996 Isichenko [4] reconsidered the long-time evolution of generic initial perturbations in a Vlasov plasma and suggested that an algebraic asymptotic damping for one-dimensional (1D) plasma should occur, in spite of the nonlinear interaction effects. These conclusions are based on the idea that the motion of the resonant particles is not simply oscillatory, but there are a significant number of them that escape from the potential well; so the energy balance between wave and particles is not kept. The Isichenko theory requires then particles which, after *colliding* with the wave, perform long flights in the space.

At variance with Isichenko's theory, numerical simulations [5] show that when starting with a sufficiently large initial wave amplitude, in the final asymptotic state the wave energy displays an oscillatory behavior and wave damping is stopped. The results of numerical simulations have also been substantiated by Lancellotti and Dorning [6], who showed that there exists a critical threshold value of the initial electric field amplitude above which the Landau damping is asymptotically stopped, and by Danielson *et al.* [7], who experimentally observed asymptotic oscillation in the electric field amplitude.

We have faced the Landau damping problem from a different point of view: we have followed and analyzed Lagrangian trajectories of resonant particles in a self-consistent way. Our analysis is based (i) on the integration of the equations of motion in the phase space

$$\frac{dx}{dt} = v, \quad \frac{dv}{dt} = -\frac{e}{m}E(x,t), \tag{3}$$

where E is the self-consistent electric field, calculated by solving numerically the Vlasov-Poisson system (1) and (2), and (ii) on the calculation of the Lyapunov exponents associated with the phase space trajectories for a large number (about 4000) of initial conditions corresponding to particles which, in the wave reference, are trapped in the wave potential well. The numerical integration of the Vlasov equation

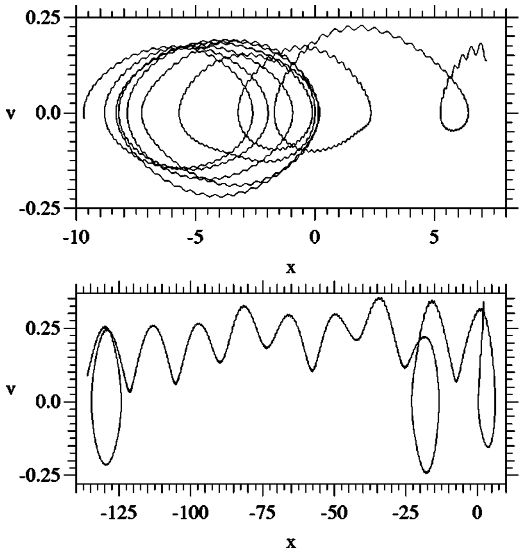

FIG. 1. Trapped particle phase space trajectory (at the top); escaped particle phase space trajectory (at the bottom).

has been performed using the splitting method in the electrostatic approximation [8], coupled with a finite difference upwind scheme [9]. In the physical space we have imposed periodic boundary conditions. In solving Eqs. (1) and (2), time is normalized to the inverse of the electron plasma frequency $\omega_{\rm pe}$ and velocity to the initial equilibrium thermal velocity $v_{\rm th}$; consequently, E is normalized to $m\omega_{\rm pe}v_{\rm th}/e$ where e is the electron charge. Finally, the distribution function f is normalized to the equilibrium particle density n_0.

The initial distribution function is a Maxwellian in the velocity space, over which a modulation in the physical space with amplitude A and wave vector k is superposed:

$$f(x,v,t=0)=\frac{1}{\sqrt{2\pi}}e^{-v^2/2}[1+A\cos(kx)]. \qquad (4)$$

The simulation domain in the phase space is given by $D=[0,L_x]\times[-v_{\rm max},v_{\rm max}]$, where $L_x=2\pi/k$ and $v_{\rm max}=6$. Outside the velocity simulation interval the distribution function is put equal to zero. Typically a simulation is performed using $N_x=512$ grid points in the physical space and $N_v=1600$ grid points in the velocity space. The time step $\Delta t \simeq 0.005-0.001$ has been chosen in such a way that the Courant-Friedrichs-Lewy condition (see, for example, Ref. [12]) is satisfied. An energy conservation equation has been used to control numerical accuracy. The total energy variations remain always 10^{-2} times smaller with respect to typical electric and kinetic energy fluctuations, throughout the simulation.

We have performed numerical simulations, with a wave number $k\simeq 0.4$ ($L_x=\lambda=15.5$), which corresponds to a phase velocity of the wave $v_\phi\simeq 3.162$, and the asymptotic evolution of the resonant particle trajectories has been investigated for a set of initial perturbation amplitudes larger than the threshold value A^* predicted in [6]. The time evolution of the electric field has been followed up to $t\simeq 1200$ and the previously observed [5] phenomenology has been correctly reproduced. In the following the results are presented for $A=0.05$, but the described physical behavior remains the same.

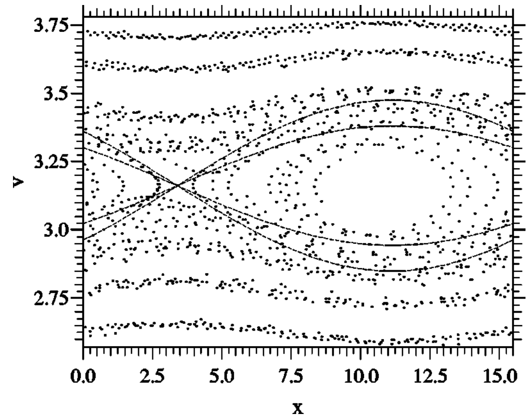

FIG. 2. Poincaré sections in the phase space.

Solving (3) coupled to (1) and (2), two different kinds of motion have been observed as shown in Fig. 1. The trajectories are described in the wave reference frame, in which each sign change of the velocity represents a wave-particle interaction. At the top in the figure, a trapped trajectory is shown: the characteristic length of the particle oscillation is of the order of the wavelength of the initial sinusoidal perturbation ($\lambda=15.5$). In the trajectory at the bottom the resonant particle, initially trapped in the potential well, performs a long flight in the phase space, before being retrapped by the wave potential well. The length of this flight is larger than 6–7 wavelengths.

In Fig. 2, we show Poincaré sections that have been obtained by following a small number of initial conditions (ten particles), uniformly distributed in a domain somewhat larger than the resonant region. Points on the x-v phase space have been plotted separated by a time interval equal to $2\pi/\omega$, where ω is the oscillation frequency of the wave, which has been accurately determined by performing a time Fourier spectral analysis on the electric field signal. Only points corresponding to times larger than 450 have been plotted.

In the asymptotic regime, i.e., for $t\geqslant 450$, the electric field envelope displays more or less regular oscillations in time, which we have reported in Fig. 3 by previously separating the contribution due to the two counterpropagating waves, that are present in our simulation. As a consequence, the separatrix in phase space, defined in terms of the single par-

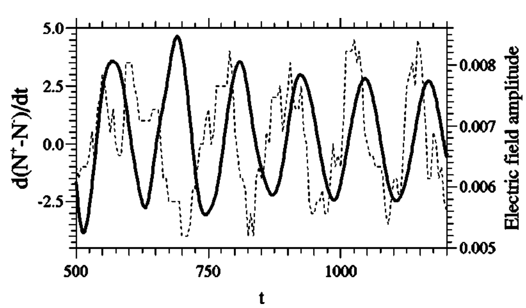

FIG. 3. Time evolution of the electric field amplitude (solid line) and of the difference between head-on and tail-on collisions for unit time (dashed line).

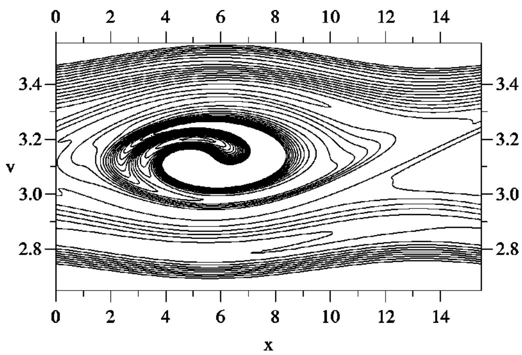

FIG. 4. Contour plot of the electron distribution function in the phase space for $t=1200$.

ticle Hamiltonian, also oscillates in time. In Fig. 2, we then reported the position of the separatrix corresponding to the maximum and the minimum values of the electric field envelope. It is worth noting that the behavior of the particles remains nonergodic outside the exterior dashed lines and inside the interior ones, but, in the region in between the trajectories diffuse, displaying an ergodic behavior. The main differences between the domain delimited by the dashed lines and the region where trajectories display an ergodic behavior are localized around the x point of the separatrix. Actually, in a time dependent Hamiltonian particle energy is not conserved, so that the boundary between trapped and untrapped trajectories is only approximately represented by the separatrix. Looking at the electron distribution function (Fig. 4), it can be seen that in this figure the x point is split into two branches which appear superposed. The form of the ergodicity region near the x point seems then to reproduce the form of the electron distribution function level curves.

In Fig. 5 we have represented in the x-v phase space the contour plot of the Lyapunov exponent distribution, which is a measure of the chaoticity of trajectories. The maximum values of the Lyapunov exponents appear in a critical zone around the separatrix between the trapping hole and the free motion region [10]. As for the Poincaré sections, the position of the separatrix corresponding to the maximum and the minimum values of the electric field envelope are indicated in the figure by dashed lines. Once again these lines delimit rather well the zone where we find the maximum values of the Lyapunov exponents. In conclusion, particles moving in the phase space between the maximum and the minimum

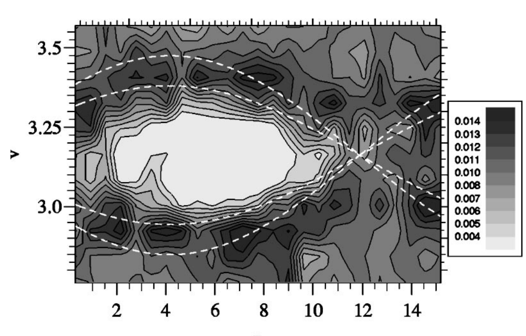

FIG. 5. Asymptotic Lyapunov exponent distribution in the phase space.

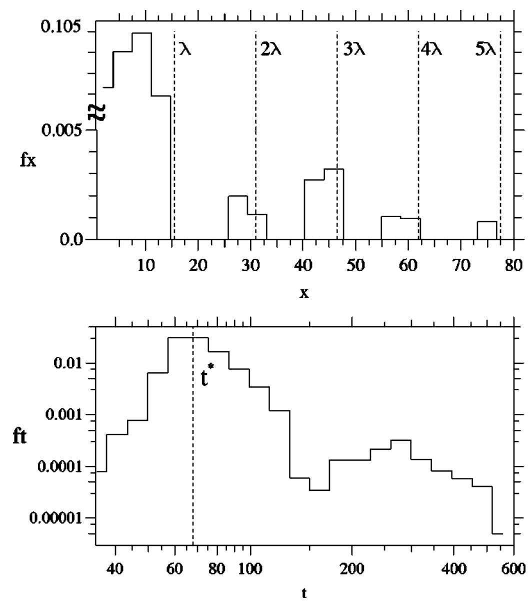

FIG. 6. x-flights histogram (at the top) and t-flights histogram (at the bottom).

sizes of the resonant region, display an ergodic and chaotic dynamics, the contrary occurring for particles which remain trapped all the time.

We then studied the statistics of phase space flights (a flight is defined as the portion of trajectory between two successive sign changes of the velocity in the wave reference frame). In Fig. 6, the probability distributions of flights as a function of length in space and in time are shown. In both histograms two clearly separated populations can be identified: the first one that performs flights smaller than one wavelength ($\lambda=L_x=15.5$) in space and with time duration between 30 and 150 and a second one whose flights are larger than a wavelength and last more than 150 in time.

The first population is formed by particles which remains always trapped in the potential well, in the second one we find those particles which, after performing a tail-on collision, are able to escape from the potential well but are almost all retrapped, the most probable flight being 3λ large, while very few flights are larger than 5λ.

Let us compare the O'Neil scenario with simulation results: the oscillating energy exchange between wave and particles is able to stop the Landau damping effect, as predicted by O'Neil, but according to the O'Neil point of view, the characteristic time of the wave-particle energy exchange is of the order of the trapping time τ_p, which for the initial value of the wave amplitude $A=0.05$ is $\tau_p \simeq 4.5$. Moreover in the O'Neil scenario the assumed ergodic behavior of resonant particles produces a phase mixing, stopping the wave energy dissipation but also stopping oscillations in the growth rate. From our simulation, the electric field envelope in the asymptotic limit oscillates with a period of the order of 120, which is strongly different with respect to τ_p, but whose half value $t^* \simeq 60$ (see Fig. 6) represents the average (most probable) value of the flight duration for particles trapped inside the potential well. It is also worth noting that both the oscillation period and the average value of the flight duration scale with the oscillation amplitude in the same way: i.e., as $A^{-1/2}$. This means that the energy exchange mechanism in the asymptotic state is totally different from that predicted by Isichenko, but is also somewhat more complicated with respect to the O'Neil view.

The difference from Isichenko's view is related to the fact that in [4] a pure ballistic motion is assumed for detrapped particles, these particles being subject only to integrable fields. The possibility of Lagrangian chaos and subsequent diffusion-induced retrapping was ruled out. On the contrary, the set of self-consistent Lagrangian equations (3) is in general *nonintegrable* [11], and, as we have shown, displays a chaotic behavior in the region around the separatrix. In this region particles are subject to Lagrangian chaos and cannot follow simply ballistic trajectories at constant velocity. Diffusion in the phase space leads to a retrapping of those particles escaped from the potential well. Correlations present in the electric field might induce also very long flights in the diffusive motion before the particle is retrapped, but infinite flights are not observed (they would require a physically unrealistic electric field, able to produce an infinite variance in particle velocity fluctuations), so the damping must necessarily saturate.

The main difference from the O'Neil view, is represented by the presence of a population of trapped particles whose behavior remains all the time nonergodic and which are then subject to more or less regular oscillations in the potential well (their flights correspond to one-half oscillation). Since the oscillation period in a sinusoidal well depends on the oscillation amplitude, even if at some time the phases of those particles were uniformly distributed, later on their distribution could display a larger number of particles at some particular phase. This situation can switch on, for example, a decreasing in the potential barrier if the largest number of particles occurs in correspondence with tail-on collisions. The decreasing level of the barrier then causes the escape of other particles in the chaotic zone which in turn produces a further reduction of the energy barrier, and so on. The situation can be inverted when the peak in the distribution of the phase of trapped particles in the wave potential well arrives in correspondence to head-on collisions. This phenomenon is clearly visible in Fig. 3 where the electric field envelope amplitude grows when the number of head-on collisions per unit time is larger than that of tail-on collisions and decreases in the opposite situation. The characteristic time of this phenomenon is clearly related to the average oscillation period of particles trapped inside the potential well, which, as seen in Fig. 6, is of the order of $2t^*$, i.e., the oscillation time for the electric field envelope.

The mechanism outlined above furnishes a physical explanation for asymptotic solutions obtained in numerical simulations. Clearly it does not rule out the possibility that, for times longer than those numerically investigated, all the resonant region becomes chaotic and ergodic, giving rise to the phase mixing predicted by O'Neil and thus stopping the oscillation in the wave amplitude.

[1] L. D. Landau, J. Phys. (Moscow) **10**, 25 (1946).

[2] S. H. Strogatz, R. E. Mirollo, and P. C. Matthews, Phys. Rev. Lett. **68**, 2730 (1992); P. Stubbe and A. I. Sukhorukov, Phys. Plasmas **6**, 2976 (1999).

[3] T. O'Neil, Phys. Fluids **8**, 2255 (1965).

[4] M. B. Isichenko, Phys. Rev. Lett. **78**, 2369 (1996).

[5] G. Manfredi, Phys. Rev. Lett. **79**, 2815 (1997); M. Brunetti, F. Califano, and F. Pegoraro, Phys. Rev. E **62**, 4109 (2000).

[6] C. Lancellotti and J. J. Dorning, Phys. Rev. Lett. **81**, 5137 (1998).

[7] J. R. Danielson, F. Anderegg, and C. F. Driscoll, Phys. Rev. Lett. **92**, 245003 (2004).

[8] C. Z. Cheng and Georg Knorr, J. Comput. Phys. **22**, 330 (1976).

[9] A. Mangeney, F. Califano, C. Cavazzoni, and P. Travnicek, J. Comput. Phys. **179**, 495 (2002).

[10] M. Colonna and A. Bonasera, Phys. Rev. E **60**, 444 (1999); M.-C. Firpo, F. Doveil, Y. Elskens, P. Bertrand, M. Poleni, and D. Guyomarc'h, *ibid.* **64**, 026407 (2001); M.-C. Firpo and F. Doveil, *ibid.* **65**, 016411 (2001).

[11] E. Ott, *Chaos in Dynamical Systems* (Cambridge University Press, Cambridge, U.K., 1997).

[12] R. Peyret and T. D. Taylor, *Computational Methods for Fluid Flow* (Springer-Verlag, New York, 1986).