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A numerical solution to the Vlasov equation

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Abstract

A fast, accurate and robust method to obtain a numerical solution to the Vlasov equation is presented. The method is based on time splitting to separate the initial equation into a set of simple transport type equations, and the solution of the resulting split equations by a fluid flux balance method. The goal of the present paper is to present this method for constant coefficients, and to show its extension to second order in phase space variable for equations such as the relativistic or the gyrokynetic Vlasov equation. Numerical results of simulations in 1 and 2 dimensions are presented. © 1999 Elsevier Science B.V.

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1. Introduction

Since the introduction of the time splitting method to solve numerically the Vlasov equation (Eq. (1)) many techniques have been used to solve the equations obtained after the splitting is done: Finite Elements [8], Finite Differences [9], Spline Interpolation [3].

The earliest of these techniques is the use of Fast Fourier Transforms (FFT) [4]. The great difficulty of the method comes from the FFT principle: as the FFT is a Fourier series development it is valid only on a periodic space, and as the velocity (or momentum) space is never periodic, we must surround the function values by an equal number of zeros to avoid aliasing. The present obligation is very expensive in computer memory and as a corollary in computer time [10]. The method presented in [11] reduces the need for storage, but is still expensive in computer time, every transform being evaluated twice.

The technique of formal integration and calculation of the function values and shift of the calculated values at the computational grid knots by cubic splines interpolation (or MOS, Method Of Splines) [3] is much faster than the previous one, but still slower than the method we present in the actual work (by a factor 3.5 on a scalar computer, and up to a factor 10 on a fast vector computer).

The present paper deals with the "Flux Balance" (FB) method, a technique based on application of the ENO [7] schemes method, and a way how to apply that basic idea to solve the equations following from Vlasov equation splitting.

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2. The Flux Balance method

2.1. The time splitting

The aim of this paper is to present a numerical solution to the Vlasov equation,

$$\frac{\partial f}{\partial t} + \boldsymbol{p} \cdot \nabla_x f + G(\boldsymbol{x}, t) \nabla_p f(\boldsymbol{x}, \boldsymbol{p}, t) = 0 , \tag{1}$$

coupled to Maxwell equations in the most general case, and to the Poisson equation

$$\nabla_x \cdot \boldsymbol{E}(\boldsymbol{x}, t) = \int_{-\infty}^{+\infty} f(\boldsymbol{x}, \boldsymbol{p}, t) \, d\boldsymbol{p} - N_0 \tag{2}$$

in the electrostatic case.

In the equations presented in this paper, time is normalised to the inverse of the plasma frequency: ω_p, velocity to thermal speed (and so $\boldsymbol{p}$ to mv_{thermal}), and space to Debye length. $G(\boldsymbol{x}, t)$ is a generalised force term, $\boldsymbol{E}$ in the electrostatic case, $(\boldsymbol{E} + \boldsymbol{v} \times \boldsymbol{B})$ in the presence of a magnetic field $\boldsymbol{B}$, and $(\boldsymbol{E} + \boldsymbol{p} \times \boldsymbol{B}/\gamma)/\gamma$ in the relativistic case $(v = p/\gamma)$.

To simplify the presentation of the method we present the 1D electrostatic case, the generalisation to higher dimensions being straightforward. In this case the Vlasov equation simplifies to

$$\frac{\partial f}{\partial t} + v \frac{\partial f}{\partial x} + E(x, t) \frac{\partial f}{\partial v} = 0 , \tag{3}$$

with initial condition $f(x, v, t = 0) = f_0(x, v)$.

Eq. (3) can be written using a differential operator form,

$$\frac{\partial f}{\partial t} + Lf = 0 , \qquad L = L_1 + L_2 , \tag{4}$$

with

$$L_1 = v \frac{\partial}{\partial x} , \tag{5}$$

$$L_2 = E(x, t) \frac{\partial}{\partial v} . \tag{6}$$

Following the splitting method idea, Eq. (3) is now replaced by the couple of equations (7) and (8),

$$\frac{\partial f^*}{\partial t} + v \frac{\partial f^*}{\partial x} = 0 ; \qquad f^*(x, v, t = n\Delta t) = f^{**}(x, v, t = (n-1)\Delta t) , \tag{7}$$

$$\frac{\partial f^{**}}{\partial t} + E(x, t = n\Delta t) \frac{\partial f^{**}}{\partial v} = 0 ; \qquad f^{**}(x, v, t = n\Delta t) = f^*(x, v, t = n\Delta t) , \tag{8}$$

which are solved iteratively, the result of one of the equations being used as the initial condition for the other.

In spite of the fact that $E(x, t)$ is time dependent, since a change in v-space does not change the density in the Poisson equation (2), the electric field remains constant when solving Eq. (8).

In practice, since we look for a scheme of second order in time, we take the splitting symmetrical in time, obtaining a leap-frog type scheme.

2.2. The basic idea

In order to obtain a numerical solution to the Vlasov equation, we have to solve Eqs. (7) and (8), which have the form of a one-dimensional fluid equation, and can then be written in the form

$$\frac{\partial g(x,t)}{\partial t} + \frac{\partial}{\partial x}\left(a g(x,t)\right) = 0. \tag{9}$$

Now we shall follow [7] to obtain a non-oscillatory scheme. Our goal is the solution of equations of Eq. (9) type on a grid, assuming the function to be smooth in each elementary cell. We replace $g(x,t)$ by its smoothed approximation

$$\overline{g(x,t)} = \frac{1}{\Delta x} \int_{-\Delta x/2}^{\Delta x/2} g(x+h,t)\, dh, \tag{10}$$

and we introduce Eq. (10) in (9),

$$\frac{\partial g(x,t)}{\partial t} + \frac{1}{\Delta x}\left[a g(x+\Delta x/2,t) - a g(x-\Delta x/2,t)\right] = 0. \tag{11}$$

The equation so obtained is of the same degree of smoothness as Eq. (9). If "a" is a function of x and/or t as in relativistic or the gyrokinetic Vlasov equations, Eq. (11) becomes

$$\frac{\partial g(x,t)}{\partial t} + \frac{1}{\Delta x}\left[a(x+\Delta x/2,t) g(x+\Delta x/2,t) - a(x-\Delta x/2,t) g(x-\Delta x/2,t)\right] = 0. \tag{12}$$

Integrating Eq. (11) for a time step gives

$$g(x,t+\Delta t) = g(x,t) - \frac{\Delta t}{\Delta x} \int_{t}^{t+\Delta t} \left[a g(x+\Delta x/2,t) - a g(x-\Delta x/2,t)\right] dt. \tag{13}$$

2.3. The flux balance

The integral term in Eq. (13) is the quantity of fluid gained or lost by the cell at its right and left boundaries. To find that quantity formally we introduce an evolutionary function $X(t_0,t,x)$ (with t_0 the initial time). That function had to obey to the following rules:

$$g(x,t) = g(X(t_0,t,x))\, \frac{\partial X}{\partial x} \tag{14}$$

is the solution of the PDE

$$\frac{\partial X}{\partial t} + a \frac{\partial X}{\partial x} = 0 \tag{15}$$

with initial condition $X(t_0,t_0,x) = x$ and boundary condition $\frac{\partial X}{\partial t}(t_0,t,x) = -a(x,t)$.

After introduction of (14) into Eq. (13), and using (15), we obtain finally, for every cell x_i,

$$g(x_i,t+\Delta t) = g(x_i,t) - \frac{1}{\Delta x}\left[\int_{X(t,t+\Delta t,\, x_i+\Delta x/2)}^{x_i+\Delta x/2} g(h-x_i,t)\, dh - \int_{X(t,t+\Delta t,x_{i-1}+\frac{\Delta x}{2})}^{x_{i-1}+\Delta x/2} g(h-x_{i-1},t)\, dh\right] \tag{16}$$

and $X(t, t+\Delta t, x_i)$ taking the value $x_i + \Delta x/2 - a\Delta t$ in the case of a = constant, or as a first order solution of (15) for $a = a(x,t)$. For the later case a second order solution to Eq. (15) is obvious,

$$X(t, t+\Delta t, x_i) = x_i + \Delta x/2 a(x_i + \Delta x/2 - a(x_i + \Delta x/2, t)\Delta t/2, t)\Delta t\,. \tag{17}$$

Many options are available to solve Eq. (16). The simplest, and fastest model is to represent $g(x,t)$ in every cell x_i at time t by a linear second order function,

$$g(x) = g(x_i) + (g(x_{i+1}) - g(x_{i-1}))\frac{x}{2\Delta x}\,, \qquad x_i - \Delta x/2 \le x \le x_i + \Delta x/2\,. \tag{18}$$

Going back now to Eq. (16), we note that the first integral term is the decrease of $g(x_i,t)$ due to loss of fluid to the $(i+1)$th cell (for positive "a", to $i-1$ for "a" negative) whereas the second integral term is the gain from the $(i-1)$th cell and is equal to the fluid the $(i-1)$th cell looses.

We shall now calculate the loss of fluid for the x_i cell, $Dg(x_i)$, introducing (18) into the integral term of (16),

$$\begin{aligned}
Dg(x_i) &= +\frac{1}{\Delta x}\int\limits_{x_i+\Delta x/2-a\Delta t}^{x_i+\Delta x/2} g(h - x_i)\, dh \\
&= +\frac{1}{\Delta x}\int\limits_{\Delta x/2-a\Delta t}^{\Delta x/2} g(h)\, dh \\
&= +\frac{1}{\Delta x}\int\limits_{\Delta x/2-a\Delta t}^{\Delta x/2} \left[g(x_i) + (g(x_{i+1}) - g(x_{i-1}))\frac{h}{2\Delta x}\right] dh \\
&= +g(x_i)a\frac{\Delta t}{\Delta x} + (g(x_{i+1}) - g(x_{i-1}))\left(\frac{a\Delta t}{4\Delta x}\left(1 - \frac{a\Delta t}{\Delta x}\right)\right).
\end{aligned} \tag{19}$$

The new value of the function $g(x_i, t+\Delta t)$ is the old value, plus the gain from the left, minus the loss to the right,

$$g(x_i, t+\Delta t) = g(x_i, t) + Dg(x_{i-1}) - Dg(x_i)\,, \tag{20}$$

for $a(x,t)$ negative the equivalent to Eq. (19) is

$$Dg(x_i) = g(x_i)|a|\frac{\Delta t}{\Delta x} - (g(x_{i+1}) - g(x_{i-1}))\left(\frac{|a|\Delta t}{4\Delta x}\left(1 - |a|\frac{\Delta t}{\Delta x}\right)\right), \tag{21}$$

and the function $g(x_i, t+\Delta t)$ takes now the new value

$$g(x_{i,t+\Delta t}) = g(x_i, t) + Dg(x_{i+1}) - Dg(x_i)\,. \tag{22}$$

3. Application of the FB method to the Vlasov equation

To solve the Vlasov equation in 1D, we have to solve Eqs. (7) and (8) iteratively. Using the same recipe as for Eqs. (19) and (21), we get for advancing equation (7),

$$D_x f(x_i, v_j) = v_j\frac{\Delta t}{\Delta x}\left[f(x_i, v_j, t) + \{f(x_{i+1}, v_j, t) - f(x_{i-1}, v_j, t)\}\left(1 - v_j\frac{\Delta t}{\Delta x}\right)/4\right] \tag{23}$$

for $v_j > 0$. For $v_j < 0$, Eq. (23) transform to

$$D_x f(x_i, v_j) = |v_j| \frac{\Delta t}{\Delta x} \left[f(x_i, v_j, t) - \{ f(x_{i+1}, v_j, t) - f(x_{i-1}, v_j, t) \} \left(1 - |v_j| \frac{\Delta t}{\Delta x} \right) /4 \right] \tag{24}$$

and the new value of the distribution function is

$$f(x_i, v_j, t^*) = f(x_i, v_j, t) + D_x f(x_{i1}, v_j) - D_x f(x_i, v_j) , \tag{25}$$

the minus sign is for $v_j > 0$ and the plus sign is for $v_j < 0$.

The same process is used to solve Eq. (8), the value of the distribution function $f(x_i, v_j, t^*)$ obtained in Eq. (25) being used as the initial condition of the equation to be solved,

$$D_v f(x_i, v_j) = |E_i| \frac{\Delta t}{\Delta v} \left[f(x_i, v_j, t^*) \pm \{ f(x_i, v_{j+1}, t^*) - f(x_i, v_{j-1}, t^*) \} \left(1 - |E_i| \frac{\Delta t}{\Delta x} \right) /4 \right] . \tag{26}$$

The $\pm$ sign is plus for $E_i > 0$ and minus for $E_i < 0$.

The final result for $f(x_i, v_j, t)$ after a full time step is

$$f(x_i, v_j, t + \Delta t) = f(x_i, v_j, t^*) + D_v f(x_i, v_{j\pm 1}) - D_v f(x_i, v_j) , \tag{27}$$

again the plus sign is for $E_i < 0$ and the minus sign for $E_i > 0$.

4. Vlasov equation in 2D

To extend the method to two dimensions, we start again with Eq. (1), with the differential forms (4) extended to two dimensions.

Rewriting the 2D Vlasov equation with a magnetic field B gives

$$\frac{\partial f}{\partial t}(x, y, v_x, v_y, t) + v_x \frac{\partial f}{\partial x} + v_y \frac{\partial f}{\partial y} (E_x + B_z v_y) \frac{\partial f}{\partial v_x} + (E_y - v_x B_z) \frac{\partial f}{\partial v_y} = 0 , \tag{28}$$

so that the differential operator L is now

$$\begin{aligned} L_1 &= v_x \frac{\partial}{\partial x} , \\ L_2 &= v_y \frac{\partial}{\partial y} , \qquad L = L_1 + L_2 + L_3 + L_4 , \\ L_3 &= (E_x + v_y B_z) \frac{\partial}{\partial v_x} , \\ L_4 &= (E_y - v_x B_z) \frac{\partial}{\partial v_y} . \end{aligned} \tag{29}$$

We have now to apply the FB method four times, and not two as in the 1D case. Four fluid losses (or gains) have now to be calculated using (19)–(22).

For the first, a is replaced by v_x, x remains x,

$$\begin{aligned} D_x f(x_i) = {} & f(x_{i_x}, y_{i_y}, v_{x_{j_x}}, v_{y_{j_y}}, t) v_x \frac{\Delta t}{\Delta x} + \left(\frac{v_x \Delta t}{4\Delta x} \left(1 - v_x \frac{\Delta t}{\Delta x} \right) \right) \\ & \times (f(x_{i_x+1}, y_{i_y}, v_{x_{j_x}}, v_{y_{j_y}}, t) - f(x_{i_x-1}, y_{i_y}, v_{x_{j_x}}, v_{y_{j_y}}, t)) . \end{aligned} \tag{30}$$

The second is of the same form $a \Rightarrow v_y$, $x \Rightarrow y$ and the shifted indexes are the i_y's.

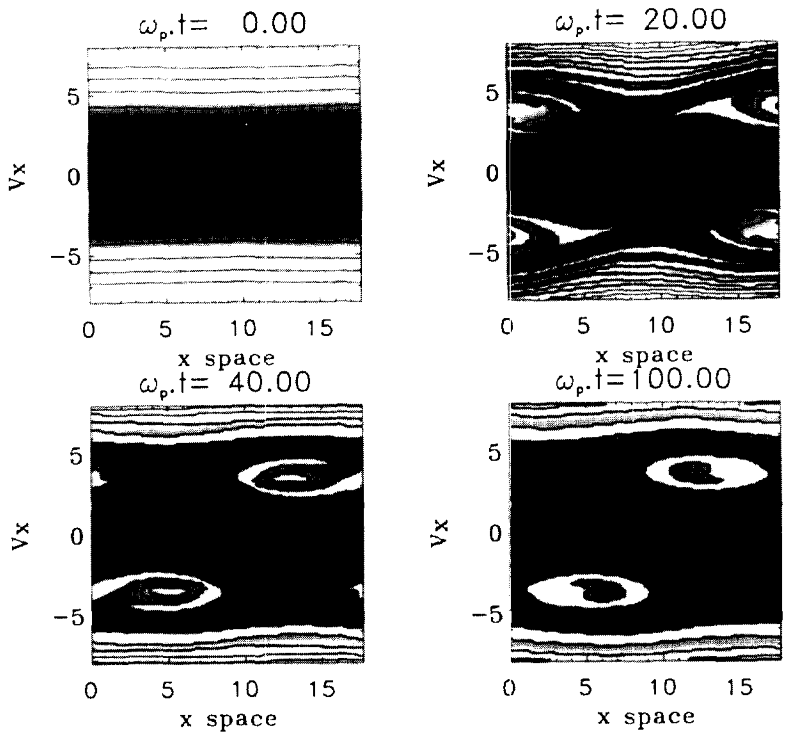

Fig. 1. Phase space.

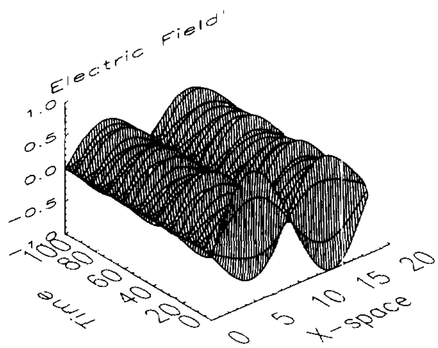

Fig. 2. The electric field in space and time.

For the third loss term, $a \Rightarrow E_x + v_y B_z$, $x \Rightarrow v_x$, $i \Rightarrow j_x$. The equation so obtained is now

$$
\begin{aligned}
D_{v_x} f(j_x) = {} & f(x_{i_x}, y_{i_y}, v_{x_{j_x}}, v_{y_{j_y}}, t)(E_x + v_y B_z)\,\frac{\Delta t}{\Delta v_x} \\
& + \left[(E_x + v_y B_z \frac{\Delta t}{4\Delta v_x}\left(1 - (E_x + v_y B_z)\,\frac{\Delta t}{\Delta v_x}\right)\right] \qquad (31) \\
& \times [\, f(x_{i_x}, y_{i_y}, v_{x_{j_x}+1}, v_{y_{j_y}}, t) - f(x_{i_x}, y_{i_y}, v_{x_{j_x}-1}, v_{y_{j_y}}, t)\,]\,. \qquad (32)
\end{aligned}
$$

The last loss term is of the same global form as (31) with now $a \Rightarrow E_y - v_x B_z$, $x \to v_y$, $i \Rightarrow j_y$.

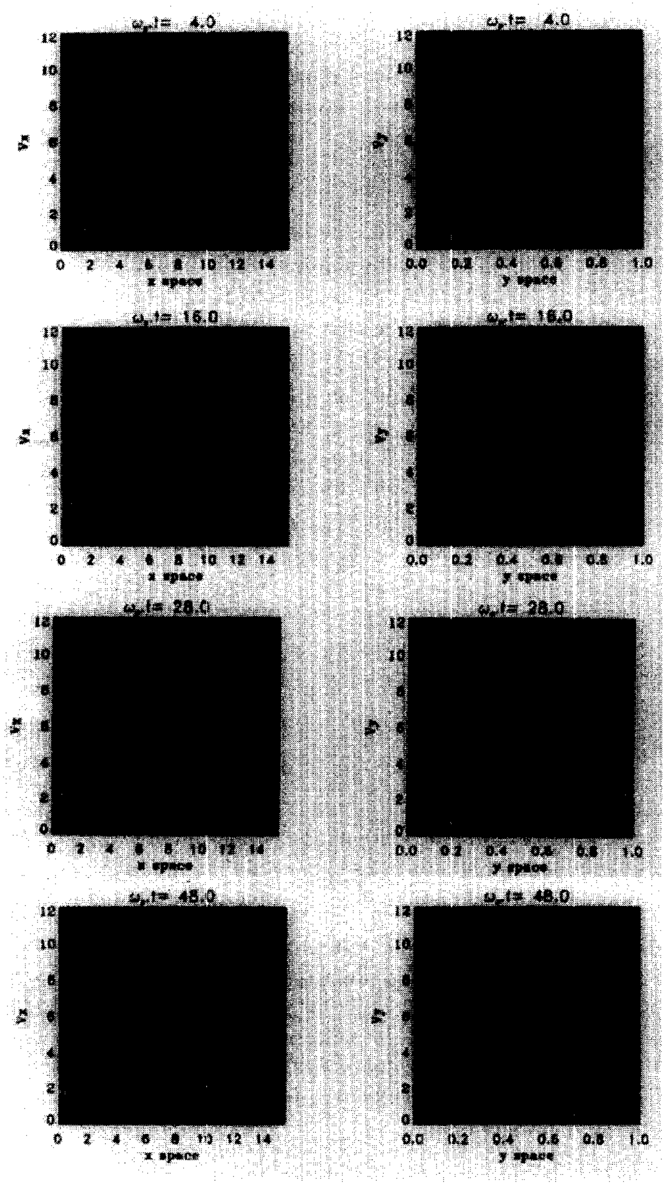

Fig. 3. The two-dimensional phase space projections.

5. Numerical results

To show the capability of the computer codes based on the F.B. method, we present in this section some results in one and two dimensions.

In 1D a full output of the phase space function is possible, and from the knowledge of that function the whole global behaviour can be obtained. Fig. 1 present a phase space representation of a 1D problem starting

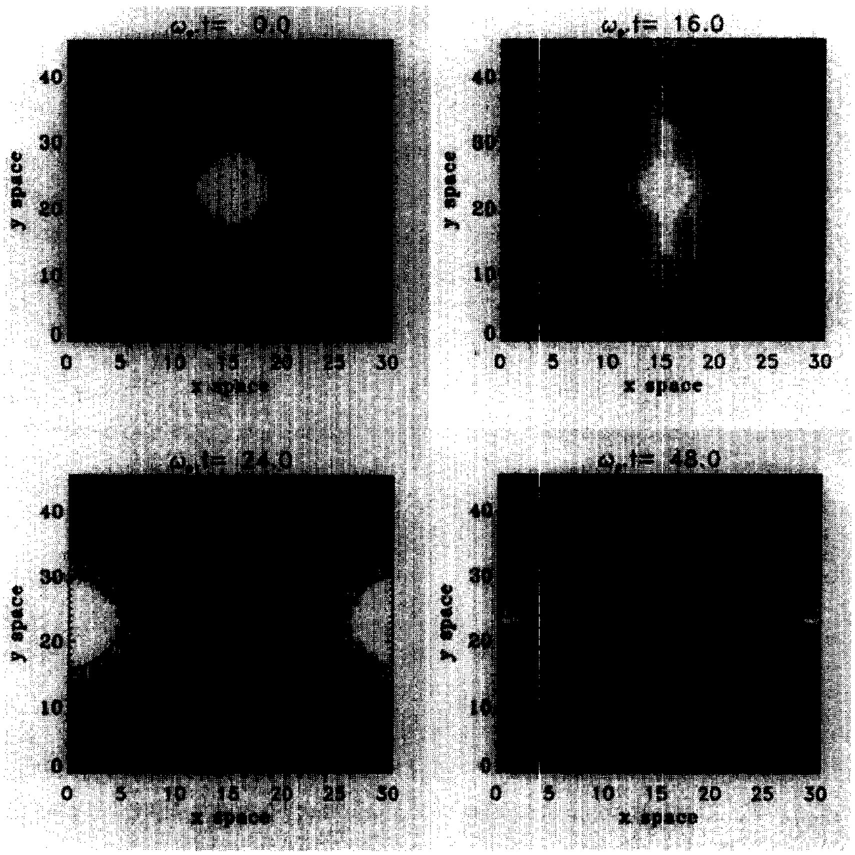

Fig. 4. The two-dimensional density.

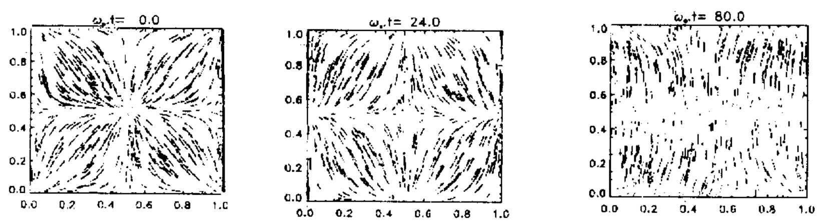

Fig. 5. Two-dimensional electric field in space, at time 0, 24, 80.

with a Maxwellian in v, homogeneous in x, with a perturbation in x of the form

$$\text{pert} = 0.3 \cos k_0 x \,, \tag{33}$$

so that the initial condition is

$$f(x, v_1 t = 0) = e^{-v^2/2}(1 + \text{pert})/\sqrt{2\pi}\,. \tag{34}$$

In Fig. 1, we plot the phase space for $\omega_p t = 0, 20, 40100$. The phase space holes, the plasma response due to nonlinear Landau damping are well defined. Fig. 2 present the electric field behaviour in space and time.

In 2D the output of results is more difficult, since it is impossible to plot a 4-dimensional function. The huge quantity of results generated by the code does not allow the full time evolution of the distribution to be stored. We have to decide before running the code what data we wish to store and display. Fig. 3 present phase-spaces projections (34), (35) at $\omega_p t = 4, 16, 28, 48$,

$$f(x, v_x) = \sum_{j_y} \sum_{i_y} f(i_x, i_y, j_x, j_y)/m_y n_y \,, \tag{35}$$

$$f(y, v_y) = \sum_{j_x} \sum_{i_x} f(i_x, i_y, j_x, j_y)/m_x n_x \,. \tag{36}$$

The problem is again a nonlinear Landau damping with $k_{ox} = .3$, $k_{oy} = .4$. The density is given in Fig. 4, at the time values 0, 16, 24, 48. The electric field behaviour is given for $\omega_p t = 0, 24, 80$ in Fig. 5. Other data can be stored, for example the velocity space, the energy, etc.

6. Conclusion

We have presented the flux balance method, and its application to solve the Vlasov equation in one and two dimensions. We also show how the equations of the flow gain and loss for a system known on a rectangular grid are solved. From the calculated gains and losses the time evolution of the Vlasov distribution function is obtained.

The last part of this paper shows the phase space behaviour in one and 2D, electric field and density evolution with time, as they are computed by codes written according to the flux balance method.

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