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Instability of the filtering method for Vlasov's equation

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1 Introduction

The earliest numerical methods introduced to solve the Vlasov-Poisson system were polynomial expansions [1]. In these methods, the position dependence is usually expanded in Fourier modes and the velocity dependence is treated either through Fourier modes [2, 3, 4, 5, 6] or Hermite polynomials [7, 8, 9, 10, 11]. Then splitting schemes appeared. In those schemes the initial Vlasov equation is splitted in two partial derivative equations, one in x, t the other in v, t. These equations must be solved alternatively [1]. A simple way to solve the splitted equations is to use Fourier transform both for x and v subspaces [12, 13] . The tendency of the distribution function $f(x,v,t)$ to develop steep gradients in phase space ("the filamentation") inhibits the numerical solution to Vlasov-Poisson system [13]. In order to ward of this problem Klimas has introduced a smoothed Fourier-Fourier method [14] . This method consists in convolving the original distribution function with a Gaussian distribution function, and, next, in solving the new system with a transformed splitting algorithm. Unfortunately, a second-order term appears in the new equation. In this work, we study how this term affects the numerical equation. In particular we prove that instability occurs in the linear version of the Vlasov equation obtained by considering only free non-interacting particles. We prove also that the use of Fourier-Fourier transform is a fundamental requirement to solve this new equation. We point out an important property, which is not completely clarified in [14], concerning the filtered distribution function in the transformed space. The paper is organized as follows. In the second section we define the mathematical model, in Section 3 we prove the instability of the smoothed equation. Section 4 is devoted to the need of using Fourier-Fourier transforms to obtain a stable splitting scheme. Our conclusions are exposed in Section 5.

2 The Mathematical Model

The evolution of a one-dimensional electron plasma in a periodic box can be described by the normalized Vlasov-Poisson system.

$$\frac{\partial f}{\partial t} + v\frac{\partial f}{\partial x} + E(x,t)\frac{\partial f}{\partial v} = 0, \tag{1}$$

$$\frac{\partial E}{\partial x} = \int f(x,v,t)dv - 1, \qquad \frac{1}{L}\int\int f(x,v,t)dxdv = 1, \tag{2}$$

where $f(x,v,t)$ denotes the electron distribution function, $E(x,t)$ the electric field and L is the length of the periodic spatial box. In this units t is normalized to the inverse of plasma frequency ω_p, v to thermal velocity v_{th} and x to Debye length λ_D. The idea to use a splitting algorithm in time to integrate the Vlasov equation (1) was introduced first in [2]. As it is difficult to distinguish between the mathematical filamentation and the numerical noise, the method of filtering was introduced in [14]. Its philosophy consists in a convolution of the distribution function f by a

Gaussian filter in the variable v to obtain the smoothed function $\bar{f}$,

$$\bar{f}(x,v,t) = \int F(v-u)f(x,u,t)du, \tag{3}$$

where

$$F(v) = \frac{1}{\sqrt{2\pi}v_0} e^{-\frac{1}{2}(\frac{v}{v_0})^2}, \tag{4}$$

and v_0 is a constant parameter giving the width of the Gaussian filter in thermal velocity units. The function $\bar{f}$ solves

$$\frac{\partial \bar{f}}{\partial t} + v\frac{\partial \bar{f}}{\partial x} + \bar{E}(x,t)\frac{\partial \bar{f}}{\partial v} = -v_0^2 \frac{\partial^2 \bar{f}}{\partial x \partial v}, \tag{5}$$

$$\frac{\partial \bar{E}}{\partial x} = \int \bar{f}(x,v,t)dv - 1. \tag{6}$$

In (5) and (6), we have $\bar{E} = E$. The aim of the present paper is to compare the stability properties of the solutions to equations (1) and (5). The conclusion we got is that the solutions to (5) can be obtained only by the use of Fourier Transforms, and so are very sensitive to perturbations. Consequently we have to be extremely careful when using such a method for numerical computation, in the general case of initial conditions.

Since the filamentation process is associated to the free streaming term $v\frac{\partial f}{\partial x}$, it is sufficient to consider the free streaming problem, dropping in (5) the term $\bar{E}(x,t)\frac{\partial \bar{f}}{\partial v}$. Thus let us consider the equation

$$\begin{cases} \dfrac{\partial g}{\partial t} + v\dfrac{\partial g}{\partial x} = -v_0^2 \dfrac{\partial^2 g}{\partial x \partial v}, \\ g(x,v,0) = g_0(x,v). \end{cases} \tag{7}$$

In order to describe the equation (7), let us define the Fourier-Transform $\tilde{g}$ of g by

$$\tilde{g}(m,\nu,t) = \frac{1}{L}\int_{x=0}^{L}\int_{v\in \mathbb{R}} e^{-i(m\frac{2\pi}{L}x+\nu v)} g(x,v,t)dxdv, \tag{8}$$

Introducing (8) in (7), we obtain

$$\frac{\partial \tilde{g}}{\partial t} - k_0 m \frac{\partial \tilde{g}}{\partial \nu} = v_0^2 k_0 m \nu \tilde{g}, \tag{9}$$

where k_0 is the fundamental wave number $k_0 = \frac{2\pi}{L}$.

Now let us study the Cauchy problem, which consists in solving equation (9) with initial condition

$$\tilde{g}(m,\nu,0) = \tilde{g}_0(m,\nu). \tag{10}$$

The solution of the system (9)–(10) is given by

$$\tilde{g}(m,\nu,t) = \tilde{g}_0(m,\nu + mk_0 t)\mathrm{e}^{v_0^2 m k_0 \nu t}\mathrm{e}^{\frac{1}{2}v_0^2 m^2 k_0^2 t^2}. \tag{11}$$

Then, in order to obtain a solution to (7), we need to find a function g such that its Fourier transform is $\tilde{g}$ defined in (11). If $\tilde{g}_0(m,\nu)$ is an arbitrary function, we observe that asymptotically, if ν and m have the same sign then the term $\mathrm{e}^{v_0^2 m k_0 \nu t}\mathrm{e}^{\frac{1}{2}v_0^2 m^2 k_0^2 t^2}$ in (11) tends exponentially to infinity, and therefore there is no function having $\tilde{g}$ as Fourier-Transform. Consequently there is no solution to (7). On the contrary, let the initial distribution function g_0 takes the form

$$g_0(x,v) = f_0(x,v) * F(v) \tag{12}$$

then

$$\tilde{g}_0(m,\nu) = \tilde{f}_0(m,\nu)\mathrm{e}^{-\frac{1}{2}v_0^2\nu^2}. \tag{13}$$

Hence, we get from (11)

$$\tilde{g}(m,\nu,t) = \tilde{f}_0(m,\nu + mk_0 t)\mathrm{e}^{-\frac{1}{2}v_0^2\nu^2}, \tag{14}$$

or equivalentely

$$\tilde{f}(m,\nu,t) = \tilde{f}_0(m,\nu + mk_0 t) = \tilde{g}(m,\nu,t)\mathrm{e}^{\frac{1}{2}v_0^2\nu^2}, \tag{15}$$

which is the solution to the Vlasov equation. By these formulas we see that the fact that g_0 has the form (12) is crucial, and, as we shall see in section 3, we have to keep this property for all times in approximate numerical schemes.

3 Stability

It might be interesting to investigate the stability of (7). For that purpose, we compare the exact solution of (7), which can be written as

$$g(t) = S(t)g(0), \tag{16}$$

with $S(t)$ the resolution operator, which can be expressed by (11), and an approximate solution h_n computed by

$$h_{n+1} = A(\Delta t)h_n, \tag{17}$$

with $A(\Delta t)$ an approximate resolution operator, and $h_0 = g(0)$. At a fixed time $T = n\Delta t$, we assume that there is a slight difference between h_n and g as

$$h_n = g(T) + \delta g. \tag{18}$$

Then, in the next step, since the operator S is linear, the difference between h_{n+1} and g takes the following form

$$h_{n+1} - g(t) = (A(\Delta t) - S(\Delta t))h_n + S(\Delta t)\delta g. \tag{19}$$

For this difference to be small, we need both terms in the right-hand side of (19) to be small. The first one depends on the way A approaches S, but for the second one, as we discussed before, $\tilde{\delta g}$ needs to be small with respect to $e^{-v_0^2\nu^2/2}$. Therefore, a necessary condition for the approximate method to be stable is that the operator $A(\Delta t)$ preserves the exponential decrease at infinity of the Fourier-Transform. Generally, this property is very difficult to obtain unless $A(\Delta t)$ is defined itself by Fourier-Transform. Moreover, it might be lost when taking into account the acceleration term due to the electric field.

4 Need to use Fourier-Fourier transform

In the following, we show that the solution to equation (5) can be obtained only by the use of Fourier transform (without the term $E\dfrac{\partial f}{\partial v}$). It will be proved that the direct solution to equation (7) leads to the solution of an unstable heat equation. Equation (7) is a second order linear partial differential equation. It can be solved by Fourier-Fourier transform, but let us try a splitting method as follow

$$\frac{\partial g}{\partial t} + v\frac{\partial g}{\partial x} = 0, \tag{20}$$

$$\frac{\partial g}{\partial t} + v_0^2\frac{\partial^2 g}{\partial x \partial v} = 0. \tag{21}$$

The system so obtained represents a linear transport equation (20) and a second-order parabolic equation in a non canonical form.

The solution of the transport equation (20) is given by

$$g(x, v, t) = g(x - vt, v, 0). \tag{22}$$

In order to solve equation (21) we introduce the change of variables

$$\begin{cases} x = x_1 - y_1, \\ \\ v = x_1 + y_1. \end{cases} \tag{23}$$

Introducing this last relation into equation (21), gives

$$\frac{\partial g}{\partial t} - v_0^2\frac{\partial^2 g}{\partial y_1^2} + v_0^2\frac{\partial^2 g}{\partial x_1^2} = 0. \tag{24}$$

We have obtained a canonical linear partial differential equation, which can be solved *a priori* by a splitting method as follows

$$\frac{\partial g}{\partial t} - v_0^2 \frac{\partial^2 g}{\partial y_1^2} = 0, \tag{25}$$

$$\frac{\partial g}{\partial t} + v_0^2 \frac{\partial^2 g}{\partial x_1^2} = 0. \tag{26}$$

The difference between the last equations reside in their stability. It is well known that the first type of equation is stable but the last one, called the retrograde temperature equation is unstable. Consequently a solution to equation (7) by the splitting method (20)-(21) is unstable.

Now we Remark that the stability of partial differential equations depends mainly on their highest-order terms. Therefore, since we have seen above that equation (21) is unstable, the solution to equation (7) is merely unstable also. Hence we must not separate the terms $v\frac{\partial g}{\partial x}$ and $v_0^2\frac{\partial^2 g}{\partial x \partial v}$, and we must be very careful in the treatment of these two terms. As shown in Sections 2-3, this can be achieved only by the use of Fourier-Fourier transform. In this case the filtering of the initial distribution function becomes a simple multiplication which consists to damp high wave lengths as we have seen in section 2. That operation hides but does not remove the filamentation.

5 Conclusion

The numerical integration of the Vlasov equation has been studied intensely during the recent years, since a knowledge of its non-linear evolution is indispensable in the understanding of plasmas. A major problem encountered in these studies is the phase space filamentation of the distribution function. The filtering method introduced by Klimas is reminescent of the Fokker-Planck term introduced in [7, 8] in the Fourier-Hermite method. But the comparison is fallacious. The finite number of Hermite polynomials introduced a bouncing of the information and triggers instability. The Fokker-Planck term damps the high order Hermite coefficients supressing the instability but at the price of a modification of the physics of the problem. The method of Klimas seems to remove filamentation. But, it is important to point out that filamentation is a physical property, and that the splitting method does not trigger any numerical instability. We have proved that the only way to accomplish smoothing and keep stability is by Fourier-Fourier transfom, as outlined by Klimas himself. But, in this case, the velocity wavenumbers are simply multiplied by $e^{-\frac{1}{2}v_0^2\nu^2}$. Their smallness at the border is just an artefact and tends to hide the reality of the approximation.

The only advantage of the Klimas method is to erase parasites from figures, allowing a better understanding of the phase structures.

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