Scuola di Dottorato

THE WAVE EQUATION

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The vibrating string equation

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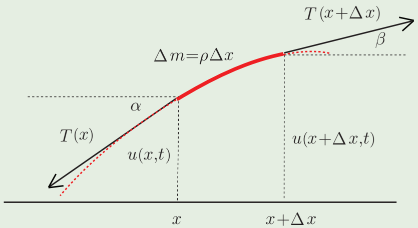

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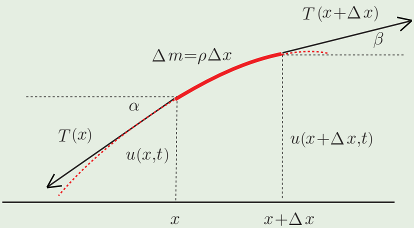

Newton's law for Δm along x and y gives

$$0 = T(x+\Delta x)\,\cos\beta - T(x)\,\cos\alpha$$

No horizontal displacement

$$\Rightarrow T(x+\Delta x)\,\cos\beta = T(x)\,\cos\alpha \equiv T_0$$

$$\Delta m \frac{\partial^2 u}{\partial t^2} = T(x+\Delta x)\,\sin\beta - T(x)\,\sin\alpha \qquad (1)$$

The vibrating string equation

With $\Delta m = \rho \, \Delta x$ and by substituting from (1):

$$\frac{\partial^2 u}{\partial t^2} = \frac{T_0}{\rho} \frac{\tan \beta - \tan \alpha}{\Delta x} = v^2 \, \frac{\frac{\partial u}{\partial x}(x + \Delta x, t) - \frac{\partial u}{\partial x}(x, t)}{\Delta x} \to v^2 \, \frac{\partial^2 u}{\partial x^2}$$

$$\frac{\partial^2 u}{\partial t^2} - v^2 \, \frac{\partial^2 u}{\partial x^2} = 0 \qquad (2)$$

which is called *wave equation* (in 1−D).

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Generalizing to 2 or 3 dimensions:

$$\frac{\partial^2 u}{\partial t^2} - v^2\,\Delta u = 0$$

The wave equation describes virtually all wave phenomena: sound waves, light waves, waves in fluids, gases and plasmas, waves in solids, etc.

Quasilinear second order PDEs

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General form in 2 variables

The wave equation is a particular case of a second-order PDE; the most general form is

$$a\,\frac{\partial^2 u}{\partial x^2} + 2\,b\,\frac{\partial^2 u}{\partial x \partial y} + c\,\frac{\partial^2 u}{\partial y^2} = d \tag{3}$$

where $a = a(x, y, u, \partial u/\partial x, \partial u/\partial y)$, and so for b, c and d (quasilinearity).

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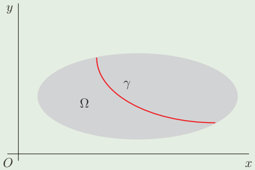
- Initial Value (Cauchy) Problem: when the auxiliary conditions are imposed on the function and its derivatives along a line belonging to the domain of the independent variables;
- Boundary Value Problem: the solution is sought for in a domain $\Omega \in \mathbb{R}^n$ and the auxiliary conditions are imposed on the function (or its derivatives) on the boundary $\partial\Omega$ of the domain.

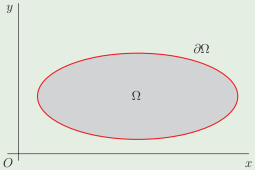
Quasilinear second order PDEs

Initial Value and Boundary Value Problems



Cauchy Problem:

$$u(x,y) = h(x,y)$$
$$\frac{\partial u}{\partial x}(x,y) = \phi(x,y) \qquad \text{On } \gamma$$
$$\frac{\partial u}{\partial y}(x,y) = \psi(x,y)$$



Boundary Value Problem:

$$u(x,y) = f(x,y)$$
or
$$\frac{\partial u}{\partial \boldsymbol{n}}(x,y) = \psi(x,y) \qquad \text{On } \partial\Omega$$

Quasilinear second order PDEs

Cauchy problem

Existence and uniqueness of the solution starting from assigned values of u, $\partial u/\partial x$ and $\partial u/\partial y$ on a smooth curve in the (x, y) plane. Let

$$\begin{cases} x = f(s) \\ y = g(s) \end{cases}$$

be a parametric representation of a curve γ in the (x, y) plane.

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Cauchy problem

We must prescribe

$$u(x(s), y(s)) = h(s) \tag{4}$$

$$\frac{\partial u}{\partial x}(x(s), y(s)) = \phi(s) \tag{5}$$

$$\frac{\partial u}{\partial y}(x(s), y(s)) = \psi(s) \tag{6}$$

and find the conditions for existence and uniqueness of the solution.

Quasilinear second order PDEs

Let us differentiate (5) and (6) w.r. to s; by adding these to the differential equation (3) we have the following system:

$$\frac{\partial^2 u}{\partial x^2}\, f'(s) + \frac{\partial^2 u}{\partial x\, \partial y}\, g'(s) = \phi'(s) \tag{7}$$

$$\frac{\partial^2 u}{\partial x\, \partial y}\, f'(s) + \frac{\partial^2 u}{\partial y^2}\, g'(s) = \psi'(s) \tag{8}$$

$$a\, \frac{\partial^2 u}{\partial x^2} + 2\, b\, \frac{\partial^2 u}{\partial x \partial y} + c\, \frac{\partial^2 u}{\partial y^2} = d, \tag{9}$$

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Unique solution if the determinant

$$\Delta = \begin{vmatrix} f'(s) & g'(s) & 0 \\ 0 & f'(s) & g'(s) \\ a & 2\, b & c \end{vmatrix} = a\, [g'(s)]^2 - 2\, b\, f'(s)\, g'(s) + c\, [f'(s)]^2 \neq 0$$

If $\Delta = 0$, γ is called *characteristic curve*.

Classification

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The condition $a\,[g'(s)]^2 - 2\,b\,f'(s)\,g'(s) + c\,[f'(s)]^2 = 0$ is equivalent to

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- If $b^2 - a\,c > 0$ we have

$$\frac{g'(s)}{f'(s)} = \frac{b \pm \sqrt{b^2 - a\,c}}{a} \tag{10}$$

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- If $b^2 - a\,c < 0$ we have no solutions for $g'(s)/f'(s)$ and the equation is called *elliptic*
- If $b^2 - a\,c = 0$ we have

$$\frac{g'(s)}{f'(s)} = \frac{b}{a}$$

  and the equation is called *parabolic*.

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- If a, b and c depend only on x and y (linear equation), equation (10) can be written as

$$\frac{dy}{dx} = \frac{b \pm \sqrt{b^2 - a\,c}}{a} \equiv \lambda_{\pm}(x, y) \tag{11}$$

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- The presence of two families of characteristics in hyperbolic equations corresponds to counter-propagating wave forms;

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Prototypical equations

Wave equation: $$\frac{\partial^2 u}{\partial t^2} - v^2 \frac{\partial^2 u}{\partial x^2} = 0$$ $b^2 - a\,c = v^2 > 0$, hyperbolic equation

Heat equation: $$\frac{\partial u}{\partial t} - \kappa \frac{\partial^2 u}{\partial x^2} = 0$$ $b^2 - a\,c = 0$, parabolic equation

Laplace equation: $$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$ $b^2 - a\,c = -1 < 0$, elliptic equation

Classification

Canonical form

- $$Lu = a\,\frac{\partial^2 u}{\partial x^2} + 2\,b\,\frac{\partial^2 u}{\partial x \partial y} + c\,\frac{\partial^2 u}{\partial y^2}$$

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- Polynomial form (a, b, c constants for simplicity)
  $p_L(\lambda) = a\lambda^2 + 2\,b\,\lambda + c = a(\lambda - \lambda_1)\,(\lambda - \lambda_2)$ and, by analogy,

$$Lu = a\left(\frac{\partial}{\partial x} - \lambda_1\,\frac{\partial}{\partial y}\right)\left(\frac{\partial}{\partial x} - \lambda_2\,\frac{\partial}{\partial y}\right)u$$

Classification

- Reduction to canonical form (after linear transformation to (ξ, η)):

$$(\lambda_1 \neq \lambda_2 \in \mathbb{R}) : Lu \sim \frac{\partial^2 u}{\partial \xi \partial \eta} \qquad \text{hyperbolic case}$$

or also $$Lu \sim \frac{\partial^2 u}{\partial \xi^2} - \frac{\partial^2 u}{\partial \eta^2}$$

$$(\lambda_1 = \lambda_2 \in \mathbb{R}) : Lu \sim \frac{\partial^2 u}{\partial \eta^2} \qquad \text{parabolic case}$$

$$(\lambda_1 = \lambda_2^* \in \mathbb{C}) : Lu \sim \frac{\partial^2 u}{\partial \xi^2} + \frac{\partial^2 u}{\partial \eta^2} \qquad \text{elliptic case}$$

The D'Alembert solution

Back to the wave equation

Characteristics

From equation (11) with
$a = 1$, $b = 0$, $c = -v^2$

$$\frac{dy}{dx} = \pm v$$

$$x + v\,t = \xi$$
$$x - v\,t = \eta$$

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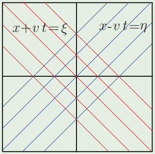

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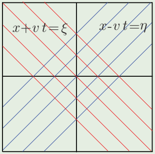

Canonical Form

The transformation $(x,t) \to (\xi,\eta)$ leads to:

$$\frac{\partial^2 u}{\partial t^2} - v^2\,\frac{\partial^2 u}{\partial x^2} = 0 \qquad \Longrightarrow \qquad \frac{\partial^2 U}{\partial \xi \partial \eta} = 0$$

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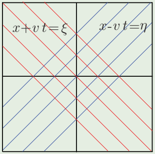

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General Solution

$$U(\xi, \eta) = F(\xi) + G(\eta) \tag{12}$$
$$u(x, t) = F(x + v\,t) + G(x - v\,t) \tag{13}$$

D'Alembert's form of the solution

Initial value problem

- We prescribe the initial data

$$u(x,0) = h(x) \tag{14}$$

$$\frac{\partial u}{\partial t}(x,0) = \psi(x) \tag{15}$$

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$$F(x) + G(x) = h(x) \tag{16}$$

$$v\,[F'(x) - G'(x)] = \psi(x) \tag{17}$$

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- and, by differentiating (16):

$$F'(x) = \frac{v\,h'(x) + \psi(x)}{2\,v} \tag{18}$$

$$G'(x) = \frac{v\,h'(x) - \psi(x)}{2\,v} \tag{19}$$

D'Alembert's form of the solution

- After integration

$$F(x) = \frac{h(x)}{2} + \frac{1}{2\,v}\int_0^x \psi(\lambda)d\lambda \qquad (20)$$

$$G(x) = \frac{h(x)}{2} - \frac{1}{2\,v}\int_0^x \psi(\lambda)d\lambda \qquad (21)$$

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- We finally obtain the d'Alembert solution of the Initial Value Problem:

$$u(x,t) = F(x + v\,t) + G(x - v\,t) =$$
$$= \frac{1}{2}[h(x + v\,t) + h(x - v\,t)] + \frac{1}{2\,v}\int_{x-v\,t}^{x+v\,t} \psi(\lambda) d\lambda \qquad (22)$$

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- Works well on infinite domains; on a finite domain, we must use it piecewise and take reflections into account.

Dependence domain and Influence cone

Domain of dependence

- From the d'Alembert form (22), we see that $u(x,t)$ depends upon the values of the functions h and ψ in the interval $[x - v\,t, x + v\,t]$ on the $x-$axis. This is called *domain of dependence.*

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Cone of influence

We also see that the initial condition at $x = x_0$ influences the solution at later times within the cone (a triangle in a plane) delimited by the surfaces $x_0 - v\,t = 0$ and $x_0 + v\,t = 0$. This is called *cone of influence.*

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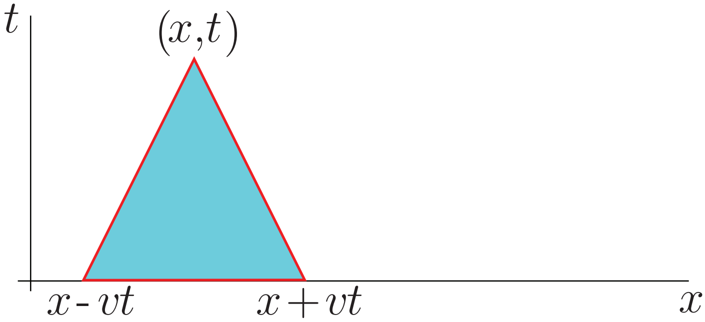
These concepts are consistent with the experience of a finite propagation speed.

Dependence domain and Influence cone



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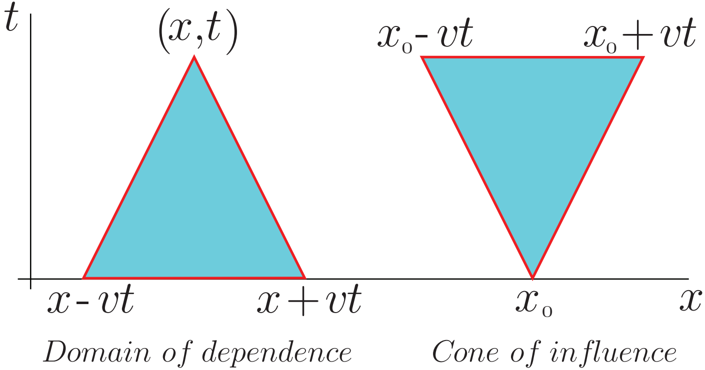

The Klein-Gordon and the telegrapher's equations

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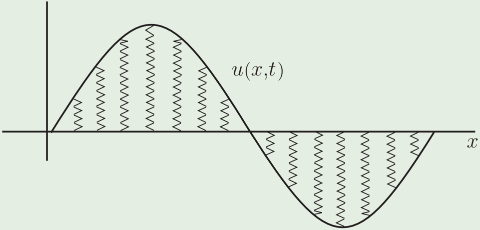
The Klein-Gordon equation

If the vibrating string is subjected to a uniformly distributed elastic force, we obtain the Klein-Gordon equation:

$$\frac{\partial^2 u}{\partial t^2} - v^2 \frac{\partial^2 u}{\partial x^2} + \gamma u = 0 \tag{23}$$

where γ is a constant (e.g., proportional to the elastic constant).



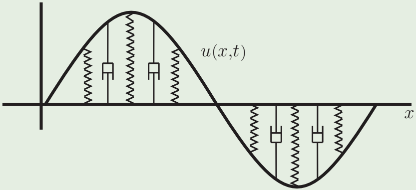
The Klein-Gordon and the telegrapher's equations

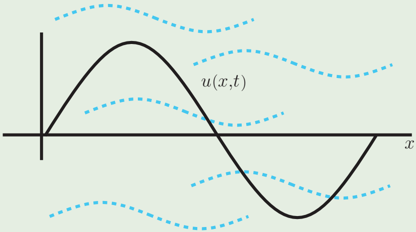
The telegrapher's equation

Propagation of waves rarely happens without dissipation; when taken into account, we obtain the telegrapher's equation (or transmission line equation):

$$\frac{\partial^2 u}{\partial t^2} - v^2 \frac{\partial^2 u}{\partial x^2} + 2\,\lambda \frac{\partial u}{\partial t} + \gamma u = 0 \tag{24}$$

where λ is a damping constant.





Solutions on the real line (Fourier Transforms)

Solutions by Fourier Transforms

The wave equation

Assume that $u(x,t)$ can be written as a Fourier integral (which means that $|u(x,t)| \to 0$ fast enough as $x \to \pm\infty$)

$$u(x,t) = \int_{-\infty}^{\infty} \widehat{u}(k,t)\, e^{i\,k\,x}\, dk \qquad \widehat{u}(k,t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} u(x,t)\, e^{-i\,k\,x} dx$$

and substitute in the wave equation. One obtains

$$\frac{\partial^2 \widehat{u}}{\partial t^2} + \omega(k)^2\, \widehat{u} = 0$$

with $\omega(k) = k\,v$. The general solution is given by

$$\widehat{u}(k,t) = A(k)\, e^{i\omega t} + B(k) e^{-i\omega t}$$

Inversion of the Fourier transform gives back the d'Alembert solution.

Solutions by Fourier Transforms

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Inversion of the Fourier transform gives back the d'Alembert solution.

However, the Fourier form says something very important: the solution of the wave equation can be written as a superposition of plave waves with constant group velocity $v_g = d\omega/dk = v$, so there is no dispersion.

Solutions by Fourier Transforms

The Klein-Gordon Equation

Again, let

$$u(x,t) = \int_{-\infty}^{\infty} \widehat{u}(k,t)\, e^{i\,k\,x}\, dk \qquad \widehat{u}(k,t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} u(x,t)\, e^{-i\,k\,x} dx$$

and substitute in the Klein-Gordon equation. One obtains

$$\frac{\partial^2 \widehat{u}}{\partial t^2} + \omega(k)^2\, \widehat{u} = 0$$

with $\omega(k) = \sqrt{k^2\, v^2 + \gamma}$. The general solution is again given by

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Solutions by Fourier Transforms

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Now the group velocity $v_g = d\omega/dk$ is not constant and there is dispersion: waves with different k's propagate at different speeds and the shape (signal) acquires a distorsion in time.

Solutions by Fourier Transforms

The telegrapher's equation

Again, with

$$u(x,t) = \int_{-\infty}^{\infty} \widehat{u}(k,t)\, e^{i\,k\,x}\, dk \qquad \widehat{u}(k,t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} u(x,t)\, e^{-i\,k\,x} dx$$

and $\gamma = 0$, one obtains

$$\frac{\partial^2 \widehat{u}}{\partial t^2} + 2\lambda \frac{\partial \widehat{u}}{\partial t} + \omega(k)^2\, \widehat{u} = 0$$

with $\omega(k) = k\,v$. The general solution is again given by

$$\begin{aligned} \widehat{u}(k,t) &= e^{-\lambda t} \left[A_1(k)\, e^{\Omega t} + A_2(k)\, e^{-\Omega t} \right] \quad \text{for} \quad |k| \leq \frac{\lambda}{v} \\ &= e^{-\lambda t} \left[B_1(k)\, e^{i\nu t} + B_2(k) e^{-i\nu t} \right] \quad \text{for} \quad |k| > \frac{\lambda}{v} \end{aligned}$$

where $\Omega = \sqrt{\lambda^2 - k^2\, v^2}$ and $\nu = \sqrt{k^2\, v^2 - \lambda^2}$.

Solutions by Fourier Transforms

The telegrapher's equation

Again, with

$$u(x,t) = \int_{-\infty}^{\infty} \widehat{u}(k,t)\, e^{i\,k\,x}\, dk \qquad \widehat{u}(k,t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} u(x,t)\, e^{-i\,k\,x} dx$$

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where $\Omega = \sqrt{\lambda^2 - k^2\, v^2}$ and $\nu = \sqrt{k^2\, v^2 - \lambda^2}$.

The solution shows both dispersion and diffusion

An example

Initial condition

We compare solutions of the wave equation, the Klein-Gordon equation and the telegrapher's equation with initial condition

$$u(x,0) = e^{-x^2/2} \qquad \frac{\partial u}{\partial t}(x,0) = 0$$

(black line: wave eq., red line: Klein-Gordon, blue line: telegr.)

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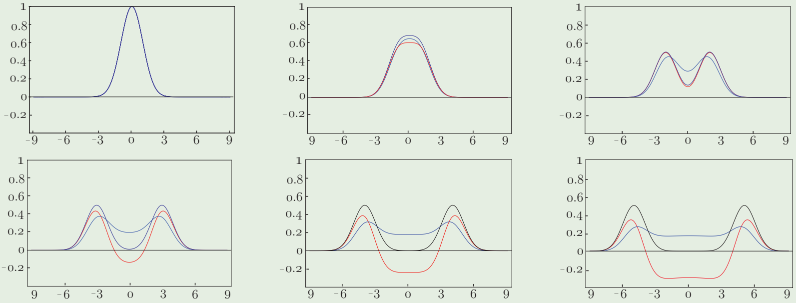

The wave equation on a finite domain

The wave equation on a finite domain

Vector space with scalar product

Consider now the wave equation in the interval $0 \le x \le l$. On a finite domain, we must impose two boundary conditions (the equation is of 2^{nd} order in x). Let

$$u(0,t) = u(l,t) = 0 \tag{25}$$

be the homogeneous Dirichlet boundary conditions and consider the linear operator

$$L\,u = -\frac{\partial^2 u}{\partial x^2}$$

defined on all twice differentiable functions on $[0, l]$ with boundary conditions (25). Equipped with the scalar product

$$(u, v) = \int_0^l u(x)\, v(x)\, dx$$

it becomes a vector space with scalar product (pre-Hilbert space) and it is easily seen that L is a self-adjoint operator w.r. to this scalar product.

The wave equation on a finite domain

Eigenfunctions

Then, L has a set of real eigenvalues, $k_n^2 = (n\pi/l)^2$, for $n = 1, 2, ...$ and real orthogonal eigenfunctions $\phi_n(x)$, given by

$$\phi_n(x) = \sqrt{\frac{2}{l}}\, \sin k_n\, x \tag{26}$$

which form an orthonormal basis for the vector space, that is

$$(\phi_n, \phi_m) = \int_0^l \phi_n(x)\, \phi_m(x)\, dx = \delta_{mn} \tag{27}$$

The wave equation on a finite domain

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Eigenfunction expansion

Then, any function of this vector space can be expressed as

$$u(x,t) = \sum_{n=1}^{\infty} c_n(t)\,\phi_n(x) = \sum_{n=1}^{\infty} c_n(t)\,\sqrt{\frac{2}{l}}\,\sin k_n\, x \tag{28}$$

The wave equation on a finite domain

Solution of the wave equation

By substituting (28) into the wave equation (2) we obtain

$$\sum_{n=1}^{\infty} \left\{ \ddot{c}_n(t)\, \phi_n(x) - v^2\, c_n(t)\, \phi_n''(x) \right\} = 0$$

$$\sum_{n=1}^{\infty} \left\{ \ddot{c}_n(t)\, \phi_n(x) + v^2\, c_n(t)\, L\, \phi_n(x) \right\} = 0$$

$$\sum_{n=1}^{\infty} \left\{ \ddot{c}_n(t)\, \phi_n(x) + v^2\, c_n(t)\, k_n^2\, \phi_n(x) \right\} = 0$$

$$\sum_{n=1}^{\infty} \left\{ \ddot{c}_n(t) + \omega_n^2\, c_n(t) \right\}\, \phi_n(x) = 0 \quad \text{with} \quad \omega_n = v\, k_n$$

By taking the scalar product with ϕ_m we obtain

$$\ddot{c}_m(t) + \omega_m^2\, c_m(t) = 0$$

whose general solution is

$$c_m(t) = A_m\, \cos \omega_m t + B_m\, \sin \omega_m t$$

The wave equation on a finite domain

Solution of the wave equation

The general solution of the wave equation (2) can then be written as a superposition of all eigenfunctions as

$$u(x,t) = \sum_{n=1}^{\infty} \{A_n \cos \omega_n t + B_n \sin \omega_n t\} \, \phi_n(x) \tag{29}$$

Each of the terms in the sum is called *mode of vibration* or “free vibrations”. The general solution is thus a linear superposition of the vibration modes.

The wave equation on a finite domain

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Initial conditions

The wave equation must be accompanied by the initial conditions (14)-(15):

$$u(x,0) = h(x) \qquad \frac{\partial u}{\partial t}(x,0) = \psi(x)$$

from which the coefficents A_n and B_n can be determined:

$$A_n = \int_0^l h(x)\,\phi_n(x)\,dx \qquad B_n = \frac{1}{\omega_n}\int_0^l \psi(x)\,\phi_n(x)\,dx$$

The wave equation on a finite domain

The Klein-Gordon equation and the telegrapher's equation

By following similar steps, we may write the general solutions of the Klein-Gordon equation (23) and the telegrapher's equation (24):

$$u(x,t) = \sum_{n=1}^{\infty} \{A_n \cos\omega_n t + B_n \sin\omega_n t\}\, \phi_n(x) \qquad \text{(K.-G.)} \tag{30}$$

$$u(x,t) = e^{-\lambda t} \sum_{n=1}^{\infty} \{A_n \cos\nu_n t + B_n \sin\nu_n t\}\, \phi_n(x) \qquad \text{(telegr.)} \tag{31}$$

where the frequencies now are $\omega_n = \sqrt{v^2\, k_n^2 + \gamma}$ and $\nu_n = \sqrt{v^2\, k_n^2 - \lambda^2}$. We again observe dispersion in the Klein-Gordon equation and both dispersion and diffusion in the telegrapher's equation.

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Numerical simulations

FiniteDomain.W, FiniteDomain.KG, FiniteDomain.tele,

Non homogeneous problems

Non homogeneous problems

Generalities

Non homogeneous terms arise whenever nonhomogeneous boundary conditions or external forces are present. As customary in these cases, the solution is a sum of the general solution of the homogeneous problem and a particular solution of the complete equation.

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Nonhomogeneous boundary conditions

$$\frac{\partial^2 u}{\partial t^2} - v^2 \frac{\partial^2 u}{\partial x^2} = 0$$
$$u(0,t) = 0 \qquad u(l,t) = A$$

General solution: $u(x,t) = u_p(x) + \tilde{u}(x,t)$ with $u_p(x)$ and $\tilde{u}$ such that

$$-v^2\, u_p'' = 0 \quad \text{stationary solution}$$
$$u_p(0) = 0 \qquad u_p(l) = A$$

$$\frac{\partial^2 \tilde{u}}{\partial t^2} - v^2 \frac{\partial^2 \tilde{u}}{\partial x^2} = 0$$
$$\tilde{u}(0,t) = \tilde{u}(l,t) = 0$$

which gives $u_p(x) = A\, x/l$.

Non homogeneous problems

The solution then is:

$$u(x,t) = A\,\frac{x}{l} + \sum_{n=1}^{\infty} \{A_n\,\cos\omega_n t + B_n\,\sin\omega_n t\}\;\phi_n(x)$$

where now the coefficients A_n and B_n are given by

$$A_n = \int_0^l \left[h(x) - A\,\frac{x}{l}\right]\,\phi_n(x)\,dx \qquad B_n = \frac{1}{\omega_n}\int_0^l \psi(x)\,\phi_n(x)\,dx$$

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Numerical simulations

FiniteDomain.WNH

Non homogeneous problems

External loads, e.g. gravity

$$\frac{\partial^2 u}{\partial t^2} - v^2 \frac{\partial^2 u}{\partial x^2} + k = 0$$
$$u(0,t) = 0 \qquad u(l,t) = A$$

General solution: $u(x,t) = u_p(x) + \tilde{u}(x,t)$ with $u_p(x)$ and $\tilde{u}$ such that

$$-v^2 u_p'' + k = 0$$ stationary solution

$$u_p(0) = 0 \qquad u_p(l) = 0$$

which gives

$$u_p(x) = \frac{k\,x}{2\,v^2}\,(x-l)$$

$$\frac{\partial^2 \tilde{u}}{\partial t^2} - v^2 \frac{\partial^2 \tilde{u}}{\partial x^2} = 0$$
$$\tilde{u}(0,t) = \tilde{u}(l,t) = 0$$

Non homogeneous problems

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$$u(x,t) = \frac{k\,x}{2\,v^2}\,(x-l) + \sum_{n=1}^{\infty} \{A_n \cos\omega_n t + B_n \sin\omega_n t\}\,\phi_n(x)$$

where now the coefficients A_n and B_n are given by

$$A_n = \int_0^l \left[h(x) - \frac{k\,x}{2\,v^2}\,(x-l)\right]\phi_n(x)\,dx \qquad B_n = \frac{1}{\omega_n}\int_0^l \psi(x)\,\phi_n(x)\,dx$$

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Numerical simulations

FiniteDomain.WNH

Non homogeneous problems

Exteral excitations and resonances

Resonance phenomena occur when harmonic external sources (forcing terms) or harmonic boundary conditions are present:

Sources:

$$\frac{\partial^2 u}{\partial t^2} - v^2 \frac{\partial^2 u}{\partial x^2} = f(x,t)$$
$$u(0,t) = 0 \qquad u(l,t) = 0$$

Non hom. b.c.:

$$\frac{\partial^2 u}{\partial t^2} - v^2 \frac{\partial^2 u}{\partial x^2} = 0$$
$$u(0,t) = 0 \qquad u(l,t) = f(t)$$

Consider the case of harmonic boundary conditions with $f(t) = A_0 \sin \mu t$. The unknown function can be written as

$$u(x,t) = A_0 \frac{x}{l} \sin \mu t + \tilde{u}(x,t)$$

Then, $\tilde{u}$ obeys the equation (looks like a problem with source)

$$\frac{\partial^2 \tilde{u}}{\partial t^2} - v^2 \frac{\partial^2 \tilde{u}}{\partial x^2} = A_0 \frac{x}{l} \mu^2 \sin \mu t \qquad (32)$$
$$\tilde{u}(0,t) = \tilde{u}(l,t) = 0$$

Non homogeneous problems

External excitation

We expand $\tilde{u}$ in the usual eigenfunctions:

$$\tilde{u}(x,t) = \sum_{n=1}^{\infty} c_n(t)\,\phi_n(x).$$

After substituting into equation (32):

$$\sum_{n=1}^{\infty} \left\{ \ddot{c}_n(t) + \omega_n^2\, c_n(t) \right\}\, \phi_n(x) = A_0\, \frac{x}{l}\, \mu^2\, \sin \mu t \quad \text{with} \quad \omega_n = v\, k_n$$

and, by taking scalar products with the ϕ_m's,

$$\ddot{c}_m(t) + \omega_m^2\, c_m(t) = A_0\, \frac{\gamma_m}{l}\, \mu^2\, \sin \mu t \quad \text{with} \quad \gamma_m = (x, \phi_m(x))$$

whose general solution is

$$c_m(t) = A_m\, \cos \omega_m t + B_m\, \sin \omega_m t + \frac{A_0}{l}\, \frac{\gamma_m\, \mu^2}{\omega_m^2 - \mu^2}\, \sin \mu t$$

Non homogeneous problems

Solution of the wave equation

The general solution of the wave equation (2) can then be written as a superposition of all eigenfunctions as

$$u(x,t) = A_0 \frac{x}{l} \sin \mu t + \sum_{n=1}^{\infty} \left\{ A_n \cos \omega_n t + B_n \sin \omega_n t + \frac{A_0}{l} \frac{\gamma_n \mu^2}{\omega_n^2 - \mu^2} \sin \mu t \right\} \phi_n(x)$$

With the initial conditions

$$u(x,0) = h(x) \qquad \frac{\partial u}{\partial t}(x,0) = \psi(x)$$

we have for the coefficients A_n and B_n:

$$A_n = \int_0^l h(x) - \phi_n(x)\, dx$$

$$B_n = \frac{1}{\omega_n} \left\{ \int_0^l \left[\psi(x) - A_0 \frac{\mu x}{l} \right] \phi_n(x)\, dx - \frac{A_0 \mu \gamma_n}{l} \left(1 + \frac{\mu^2}{\omega_n^2 - \mu^2} \right) \right\}$$

Numerical simulations: FiniteDomain.WNH