Branches of harmonic solutions for a class of periodic differential-algebraic equations

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Abstract

We study a class of T-periodic parametrized differential-algebraic equations, which are equivalent to suitable ordinary differential equations on manifolds. By combining a recent result on the degree of tangent vector fields, due to Spadini, with an argument on periodic solutions of ODEs on manifolds, we get a global continuation result for T-periodic solutions of our equations.

1 Introduction and preliminaries

In a recent paper [9], Spadini investigated the set of periodic solutions of a particular class of periodic differential-algebraic equations (DAEs) by means of topological methods. Namely, given $T > 0$, [9] considers the T-periodic solutions of

$$\begin{cases} \dot{x} = h(x, y) + \lambda f(t, x, y), \\ g(x, y) = 0, \end{cases} \tag{1.1}$$

where λ is a nonnegative real parameter, $g : U \to \mathbb{R}^s$ and $h : U \to \mathbb{R}^k$ are continuous maps defined on an open connected set $U \subseteq \mathbb{R}^k \times \mathbb{R}^s$, and $f : \mathbb{R} \times U \to \mathbb{R}^k$ is a continuous map which is T-periodic in the first variable. The map g is assumed to be of class C^∞ with $\partial_2 g(p, q)$, the partial derivative of g with respect to the second variable, invertible for each $(p, q) \in U$. Equation (1.1) is a T-periodic perturbation of the autonomous equation

$$\begin{cases} \dot{x} = h(x, y), \\ g(x, y) = 0. \end{cases} \tag{1.2}$$

To study the T-periodic solutions of (1.1) it is natural to adopt a bifurcation approach.

Since $\partial_2 g(p, q)$ is invertible for all $(p, q) \in U$, it follows that $0 \in \mathbb{R}^s$ is a regular value of g. Thus $M := g^{-1}(0)$ is a C^∞ submanifold of $\mathbb{R}^k \times \mathbb{R}^s$ and equations (1.1) and (1.2) give rise to ordinary differential equations on M (see also [7]). Moreover the manifold M can be locally represented as a graph of some map from an open subset of $\mathbb{R}^k$ to $\mathbb{R}^s$. Thus equations (1.1) and (1.2) can be in principle locally decoupled. However, this might not be true globally. Observe also that even when M is a graph of some map η, it might happen that the expression of η is complicated (or even impossible to determine analytically), so that the decoupled version of (1.1) or (1.2) may be impractical.

The crucial idea in [9] is to take advantage of the equivalence between the given DAEs and suitable ODEs on M, without requiring an explicit knowledge of the manifold M itself. In this way, by means of topological methods, based on the fixed point index, as well as results about periodic solutions of ordinary differential equations on differentiable manifolds, in [9] a global continuation result for T-periodic solutions of (1.1) is proved.

Here we tackle a related problem. Namely, we consider the differential-algebraic equation, depending on $\lambda \geq 0$,

$$\begin{cases} \dot{x} = \lambda f(t, x, y), \\ g(x, y) = 0, \end{cases} \tag{1.3}$$

where the maps f and g satisfy the same assumptions as above, and we investigate the structure of the set of T-periodic solutions of (1.3). In our main result (Theorem 2.2 below) we will give conditions ensuring the existence of a connected component of elements $(\lambda; x, y)$, with $\lambda > 0$ and (x, y) a T-periodic solution to (1.3), that emanates from the set of constants functions, and is not compact. This kind of results may be regarded as a useful tool to study the existence of T-periodic solutions of the equation

$$\begin{cases} \dot{x} = f(t, x, y), \\ g(x, y) = 0 \end{cases}$$

which corresponds to the choice $\lambda = 1$ in (1.3) (see Corollary 2.3 below).

We stress that it is not possible to deduce our results from those in [9], although the equation we consider here is a particular case of (1.1), i.e., corresponding to h identically zero. In fact it is not difficult to see that the degree-theoretical assumption needed in the main result of [9] cannot be satisfied when $h \equiv 0$. Therefore, equation (1.3) requires a slightly different approach. However, this paper follows essentially the same scheme as [9].

Our first observation will be that, as $\partial_2 g(p, q)$ is invertible for all $(p, q) \in U$, equation (1.3) induces the T-periodic tangent vector field $(t, p, q) \mapsto \Psi(t, p, q)$ on $M = g^{-1}(0)$ given by

$$\Psi(t, p, q) = \big(f(t, p, q), -[\partial_2 g(p, q)]^{-1} \partial_1 g(p, q) f(t, p, q)\big) ,$$

and leads to an ordinary differential equation on M. In Theorem 2.2 we will get information on the set of T-periodic solutions of (1.3) by means of an argument of Furi and Pera (see [2]) about periodic solutions of ordinary differential equations on a differentiable manifold. We recall the result of Furi and Pera in Theorem 1.1. However, to apply Theorem 1.1 one has to know the degree of the following *mean value* tangent vector field on M:

$$\Phi(p, q) = \frac{1}{T} \int_0^T \Psi(t, p, q) dt.$$

In this context it will be crucial to deduce a formula (Theorem 2.1 below) relating the degree of the vector field Φ to that of $F : U \to \mathbb{R}^k \times \mathbb{R}^s$, given by

$$(p, q) \mapsto \left(\frac{1}{T} \int_0^T f(t, p, q) dt \, , g(p, q) \right) .$$

In this way, combining Theorems 1.1 and 2.1, we shall give conditions only on the degree of F.

We point out that, since in Euclidean spaces vector fields can be regarded as maps and vice versa, the degree of the vector field F is essentially the well known Brouwer degree, with respect to 0, of F (seen as a map). Thus the degree of F has a simpler nature than that of Φ and, as a consequence, it is also easier to compute. In addition, the manifold M is known only implicitly and the form of Φ may not be simple.

We conclude the paper with an example of application of Theorem 2.2 below to a bifurcation problem for a second order DAE, depending on a nonnegative parameter.

1.1 Associated vector fields

Our first step, following [7, 9], is to associate to (1.3) an ordinary differential equation on the manifold $M = g^{-1}(0)$.

Given $\lambda \geq 0$, a solution of equation (1.3) is a pair of functions, $x : I \to \mathbb{R}^k$ of class C^1 and $y : I \to \mathbb{R}^s$ continuous, both of them defined on some interval I, with the property that

$$\begin{cases} \dot{x}(t) = \lambda f\big(t, x(t), y(t)\big), \\ g\big(x(t), y(t)\big) = 0, \end{cases}$$

for each $t \in I$. Notice that, as a consequence of the regularity assumptions on g, the Implicit Function Theorem implies that y is actually a C^1 function. In fact, in what follows, it will be convenient to consider a solution of (1.3) as a C^1 function $\zeta := (x, y)$ defined on I with values in $\mathbb{R}^k \times \mathbb{R}^s$.

Let (x, y) be a solution of (1.3) for a given $\lambda \geq 0$, defined on $I \subseteq \mathbb{R}$. Differentiating the identity $g\big(x(t), y(t)\big) = 0$ we get

$$\partial_1 g\big(x(t), y(t)\big)\dot{x}(t) + \partial_2 g\big(x(t), y(t)\big)\dot{y}(t) = 0,$$

which yields

$$\dot{y}(t) = -\lambda \left[\partial_2 g\big(x(t), y(t)\big)\right]^{-1} \partial_1 g\big(x(t), y(t)\big)\, f\big(t, x(t), y(t)\big) \tag{1.4}$$

for all $t \in I$.

Now, consider the map $\Psi : \mathbb{R} \times M \to \mathbb{R}^k \times \mathbb{R}^s$ given by

$$\Psi(t, p, q) = \big(f(t, p, q), -[\partial_2 g(p, q)]^{-1}\partial_1 g(p, q) f(t, p, q)\big)\,. \tag{1.5}$$

It is not hard to prove that Ψ is tangent to M in the sense that, for any $(t, p, q) \in \mathbb{R} \times M$, $\Psi(t, p, q)$ belongs to the tangent space $T_{(p,q)}M$ to M at (p, q). Taking (1.4) into account, one can see that (1.3) is equivalent to the following ODE on M:

$$\dot{\zeta} = \lambda\, \Psi(t, \zeta), \qquad \lambda \geq 0, \tag{1.6}$$

where $\zeta = (x, y)$.

Observe that, when f is of class C^1, so is the vector field Ψ. Thus, the local resuts on existence, uniqueness and continuous dependence of local solutions of the initial value problems translate to (1.3) from the theory of ordinary differential equations on manifolds by virtue of the equivalence of (1.3) with (1.6).

In order to investigate the set of T-periodic solutions of (1.3) we will study the T-periodic solutions of the equivalent equation (1.6). For this purpose it will be crucial to determine a formula for the computation of the degree (sometimes called characteristic or rotation) of the mean value tangent vector field $\Phi : M \to \mathbb{R}^k \times \mathbb{R}^s$ given by formula (2.2) below. Before doing that, however, we will recall some basic facts about the notion of the degree of a tangent vector field.

1.2 The degree of a tangent vector field

Let $M \subseteq \mathbb{R}^n$ be a manifold. Given any $p \in M$, by $T_pM \subseteq \mathbb{R}^n$ we denote the tangent space of M at p. Let w be a tangent vector field on M, and let V be an open subset of M. We say that the pair (w, V) is *admissible* (or, equivalently, that w is admissible on V) if $w^{-1}(0) \cap V$ is compact. In this case one can associate to the pair (w, V) an integer, $\deg(w, V)$, called the *degree (or characteristic)* of the tangent vector field w on V which, roughly speaking, counts algebraically the number of zeros of w in V (for general references see e.g. [5, 6, 8]).

We recall that, when w is a C^1 tangent vector field on M, a zero $p \in M$ of w is said to be *nondegenerate* if $w'(p) : T_pM \to \mathbb{R}^n$ is one-to-one. Since the condition $w(p) = 0$ implies that $w'(p)$ maps T_pM into itself (see e.g. [8]), then $w'(p)$ is actually an isomorphism of T_pM. Thus, the determinant $\det(w'(p))$ is nonzero and its sign is called the *index* of w at p. In the particular case when an admissible pair (w, V) is *regular* (i.e. w is smooth with only nondegenerate zeros), one can show that $\deg(w, V)$ coincides with the sum of the indices at the zeros of w in V. This makes sense, since $w^{-1}(0) \cap V$ is compact (w being admissible in V) and discrete; therefore, the sum is finite.

When $M = \mathbb{R}^n$, that is, V is an open subset of $\mathbb{R}^n$, $\deg(w, V)$ is just the classical Brouwer degree, $\deg(w, V, 0)$, of the map w on V with respect to zero.

All the standard properties of the Brouwer degree for continuous maps on open subsets of Euclidean spaces, such as homotopy invariance, excision, additivity, existence, still hold in the more general context of differentiable manifolds (see e.g. [3]).

1.3 A continuation result of Furi and Pera

We conclude the preliminary part of the paper with a continuation result for ordinary differential equations on manifolds due to Furi and Pera (see [2]).

Let $M \subseteq \mathbb{R}^n$ be a boundaryless smooth manifold and $\psi : \mathbb{R} \times M \to \mathbb{R}^n$ a T-periodic continuous tangent vector field on M. Consider the parametrized equation

$$\dot{\zeta} = \lambda\, \psi(t, \zeta), \qquad \lambda \geq 0. \tag{1.7}$$

By $C_T(M)$ we denote the metric subspace of the Banach space $C_T(\mathbb{R}^n)$ of all the continuous T-periodic functions with values in M. We say that $(\lambda, \zeta) \in [0, \infty) \times C_T(M)$ is a *solution pair of* (1.7) if ζ is a T-periodic solution of (1.7) corresponding to λ. Given any $s \in M$, it is convenient to denote by $\hat{s}$ the map in $C_T(M)$ that is constantly equal to s. A solution pair of the form $(0, \hat{s})$ is called *trivial*.

Given an open subset $\mathcal{O}$ of $[0, \infty) \times C_T(M)$, with the symbol $V_{\mathcal{O}}$ we will denote the open subset of M given by

$$V_{\mathcal{O}} = \{s \in M : (0, \hat{s}) \in \mathcal{O}\}.$$

The following result is an immediate consequence of Theorem 2.2 of [2].

Theorem 1.1. *Let $M \subseteq \mathbb{R}^n$ be a boundaryless smooth manifold, $\psi : \mathbb{R} \times M \to \mathbb{R}^n$ a T-periodic continuous tangent vector field on M, and $\phi : M \to \mathbb{R}^n$ the mean value autonomous vector field given by*

$$\phi(s) = \frac{1}{T} \int_0^T \psi(t, s) dt.$$

Let $\mathcal{O}$ be an open subset of $[0, \infty) \times C_T(M)$, and assume that $\deg(\phi, V_{\mathcal{O}})$ is defined and nonzero. Then, the equation (1.7) *admits in $\mathcal{O}$ a connected set Γ of nontrivial solution pairs whose closure in $\mathcal{O}$ meets the set $\big\{(0, \hat{s}) \in \mathcal{O} : \phi(s) = 0\big\}$ and is not contained in any compact subset of $\mathcal{O}$.*

2 Main results

This section is devoted to the study of the set of T-periodic solutions of equation (1.3). Recall that $U \subseteq \mathbb{R}^k \times \mathbb{R}^s$ is open and connected, $f : \mathbb{R} \times U \to \mathbb{R}^k$ and $g : U \to \mathbb{R}^s$ are continuous maps, and we assume that f is T-periodic in the first variable for a given $T > 0$. We also assume that g is C^∞ and such that $\det \partial_2 g(p, q) \neq 0$ for all $(p, q) \in U$. Define $F : U \to \mathbb{R}^k \times \mathbb{R}^s$ by

$$F(p, q) = \left(\frac{1}{T} \int_0^T f(t, p, q) dt\, , g(p, q) \right). \tag{2.1}$$

Let $M = g^{-1}(0)$ and define the tangent vector field $\Psi : \mathbb{R} \times M \to \mathbb{R}^k \times \mathbb{R}^s$ as in (1.5). Moreover, define $\Phi : M \to \mathbb{R}^k \times \mathbb{R}^s$ by

$$\Phi(p, q) = \frac{1}{T} \int_0^T \Psi(t, p, q) dt = \left(\frac{1}{T} \int_0^T f(t, p, q) dt, -\frac{1}{T} [\partial_2 g(p, q)]^{-1} \partial_1 g(p, q) \int_0^T f(t, p, q) dt \right). \tag{2.2}$$

Theorem 2.1 below gives a simple formula for the computation of the degree of Φ that does not require the explicit expression of the manifold M.

We will omit the proof since it is an immediate consequence of Theorem 4.1 of [9].

Theorem 2.1. *Let $U \subseteq \mathbb{R}^k \times \mathbb{R}^s$ be open and connected, and let $g : U \to \mathbb{R}^s$ and $f : \mathbb{R} \times U \to \mathbb{R}^k$ be such that f is continuous and g is C^∞ with $\partial_2 g(p, q)$ invertible for all $(p, q) \in U$. Let $M = g^{-1}(0)$. Let also $F : U \to \mathbb{R}^k \times \mathbb{R}^s$ and $\Phi : M \to \mathbb{R}^k \times \mathbb{R}^s$ be given by* (2.1) *and* (2.2)*, respectively. Given $W \subseteq U$ open, if either $\deg(\Phi, M \cap W)$ or $\deg(F, W)$ is well defined, so is the other, and*

$$\big| \deg(\Phi, M \cap W) \big| = \big| \deg(F, W) \big|.$$

As mentioned in the Introduction, the main result of this section, Theorem 2.2 below, follows from a combination of Theorems 1.1 and 2.1 above.

We have to introduce some further notation. By $C_T(U)$ we mean the metric subspace of $C_T(\mathbb{R}^k \times \mathbb{R}^s)$ of all the continuous T-periodic functions with values in U. We say that $(\sigma; x, y) \in [0, \infty) \times C_T(U)$

is a *solution pair of* (1.3) if (x, y) satisfies (1.3) for $\lambda = \sigma$; here the pair (x, y) is thought of as a single element of $C_T(U)$. It is convenient, given any $(p, q) \in \mathbb{R}^k \times \mathbb{R}^s$, to denote by $(\hat{p}, \hat{q})$ the element of $C_T(\mathbb{R}^k \times \mathbb{R}^s)$ that is constantly equal to (p, q). A solution pair of the form $(0; \hat{p}, \hat{q})$ is called *trivial*. Observe that $(\hat{p}, \hat{q}) \in C_T(U)$ is a constant solution of (1.3) corresponding to $\lambda = 0$ if and only if $g(p, q) = 0$.

Given an open subset Ω of $[0, \infty) \times C_T(U)$, with the symbol W_Ω we will denote the open subset of U given by

$$W_\Omega = \{(p, q) \in U : (0; \hat{p}, \hat{q}) \in \Omega\}.$$

We are now ready to state and prove our main result concerning the T-periodic solutions of (1.3).

Theorem 2.2. *Let $U \subseteq \mathbb{R}^k \times \mathbb{R}^s$ be open and connected, and let $f : \mathbb{R} \times U \to \mathbb{R}^k$ be continuous and T-periodic in the first variable. Assume that $g : U \to \mathbb{R}^s$ is C^∞ with $\partial_2 g(p, q)$ invertible for all $(p, q) \in U$. Define $F : U \to \mathbb{R}^k \times \mathbb{R}^s$ by* (2.1)*. Given $\Omega \subseteq [0, \infty) \times C_T(U)$ open, suppose that* $\deg(F, W_\Omega)$ *is well-defined and nonzero. Then, there exists a connected set of nontrivial solution pairs of* (1.3) *whose closure in Ω meets the set*

$$\{(0; \hat{p}, \hat{q}) \in \Omega : F(p, q) = (0, 0)\}$$

and is not contained in any compact subset of Ω.

Proof. Let Ψ be as in (1.5). Then, as we already pointed out, equation (1.3) is equivalent to (1.6) on $M = g^{-1}(0)$. Moreover, define Φ as in (2.2). From Theorem 2.1 above we get that $|\deg(\Phi, M \cap W_\Omega)| = |\deg(F, W_\Omega)|$. Denote by $\mathcal{O}$ the open subset of $[0, \infty) \times C_T(M)$ given by

$$\mathcal{O} = \Omega \cap ([0, \infty) \times C_T(M)).$$

Recalling the notation introduced in Subsection 1.3, we have $V_\mathcal{O} = M \cap W_\Omega$. Since, by assumption, $\deg(F, W_\Omega)$ is well defined and nonzero, we get that $\deg(\Phi, V_\mathcal{O}) = \deg(\Phi, M \cap W_\Omega) \neq 0$. Hence, Theorem 1.1 implies the existence of a connected subset Γ of $\mathcal{O}$ of nontrivial solution pairs of (1.6), whose closure $\overline{\Gamma}$ in $\mathcal{O}$ is noncompact and meets the set

$$\big\{(0; \hat{p}, \hat{q}) \in \mathcal{O} : \Phi(p, q) = (0, 0)\big\}.$$

Now, since $[0, \infty) \times C_T(M)$ is contained in $[0, \infty) \times C_T(U)$, the equivalence of equations (1.3) and (1.6) on M implies that each pair $(\lambda; x, y) \in \Gamma$ can be also thought as a nontrivial solution pair of (1.3). Moreover, since M is closed in U, it is not difficult to prove that $[0, \infty) \times C_T(M)$ is closed in $[0, \infty) \times C_T(U)$. Hence, $\overline{\Gamma}$ coincides with the closure of Γ in Ω, and thus $\overline{\Gamma}$ cannot be contained in any compact subset of Ω as well. Finally, observe that

$$\big\{(p, q) \in M : \Phi(p, q) = (0, 0)\big\} = \{(p, q) \in U : F(p, q) = (0, 0)\}$$

and, consequently, the set $\big\{(0; \hat{p}, \hat{q}) \in \mathcal{O} : \Phi(p, q) = (0, 0)\big\}$ coincides with $\{(0; \hat{p}, \hat{q}) \in \Omega : F(p, q) = (0, 0)\}$. Therefore, Γ satisfies the assertion. □

As an immediate consequence of our main result we get the following *Continuation Principle*.

In Corollary 2.3 below the manifold $M = g^{-1}(0)$ is assumed to be a closed subset of $\mathbb{R}^k \times \mathbb{R}^s$ so that, in this case, $C_T(M)$ is a complete metric space.

Corollary 2.3. *Let U, f, g, F and Ω be as in Theorem 2.2. Assume also that $M = g^{-1}(0)$ is closed in $\mathbb{R}^k \times \mathbb{R}^s$. Let $\deg(F, W_\Omega)$ be nonzero. Then there exists a connected component of the set of solution pairs of* (1.3) *that meets $\{(0; \hat{p}, \hat{q}) \in \Omega : F(p, q) = (0, 0)\}$ and cannot be both bounded and contained in Ω.*

If, in particular, Ω is of the form $[0, \infty) \times A$, with $A \subseteq C_T(U)$ open and bounded, and such that there are no T-periodic solutions of (1.3) *on the boundary* Fr A *of A for $\lambda \in [0, 1]$, then equation*

$$\begin{cases} \dot{x} = f(t, x, y), \\ g(x, y) = 0. \end{cases} \tag{2.3}$$

admits a T-periodic solution in A.

Proof. From Theorem 2.2 it follows that there exists a connected set Γ of nontrivial solution pairs of (1.3) whose closure $\overline{\Gamma}$ in $[0,\infty) \times C_T(U)$ meets the set $\{(0;\hat{p},\hat{q}) \in \Omega : F(p,q) = (0,0)\}$ and is not contained in any compact subset of Ω.

Denote by Σ be the connected component of the set of all solution pairs that contains $\overline{\Gamma}$. Since, by assumption, M is a closed subset of $\mathbb{R}^k \times \mathbb{R}^s$, the metric space $[0,\infty) \times C_T(M)$ is complete. Moreover, the Ascoli-Arzelà Theorem implies that any bounded set of T-periodic solutions of (1.6) is totally bounded. Thus, if Σ is bounded, then it is also compact. If, in addition, Σ is contained in Ω then so is $\overline{\Gamma} \subseteq \Sigma$, which is impossible. This contradiction proves that Σ cannot be both bounded and contained in Ω.

To prove the last part of the assertion, let $A \subseteq C_T(U)$ be open and bounded and let $\Omega = [0,\infty) \times A$. Consider the subset $[0,1] \times A$ of Ω. Since Σ is connected, contains some nontrivial pair (as $\Gamma \subseteq \Sigma$) and cannot be contained in $[0,1] \times A$, then Σ necessarily meets the boundary of $(0,1] \times A$. Since there are no solution pairs of (1.1) in $[0,1] \times \mathrm{Fr}\, A$, the set Σ intersects $\{1\} \times A$. This completes the proof. $\square$

Remark 2.1. *One can deduce an existence result from Corollary 2.3 as follows. Let V be a relatively compact open subset of U, and assume that the following properties hold:*

- *the set $F^{-1}(0,0) \cap V$ is compact and $\deg(F,V) \neq 0$;*
- *there are no T-periodic solutions of (1.3) whose image intersects the boundary ∂V of V for $\lambda \in (0,1]$.*

In this situation, taking $A = C_T(V) \subseteq C_T(U)$ and $\Omega = [0,\infty) \times A$, we have $W_\Omega = V$, and thus $\deg(F,W_\Omega) = \deg(F,V) \neq 0$. Hence, Corollary 2.3 yields a T-periodic solution of (2.3).

2.1 Some examples

We conclude the paper with some examples of applications of our results to some special class of parametrized second order DAEs.

Consider the following second order equation, depending on a parameter $\mu \geq 0$:

$$\begin{cases} \ddot{x} = \mu\,\varphi(t,x,y), \\ g(x,y) = 0, \end{cases} \tag{2.4}$$

where $\varphi : \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^s \to \mathbb{R}^n$ is continuous and T-periodic in the first variable, and $g : \mathbb{R}^n \times \mathbb{R}^s \to \mathbb{R}^s$ is of class C^∞ and such that the partial derivative of g with respect to the second variable, $\partial_2 g(p,q)$, is invertible for each $(p,q) \in \mathbb{R}^n \times \mathbb{R}^s$.

Given $\mu \geq 0$, a solution of equation (2.4) is a pair of functions $x \in C^2(I,\mathbb{R}^n)$ and $y \in C(I,\mathbb{R}^s)$, defined on some interval I, with the property that

$$\begin{cases} \ddot{x}(t) = \mu\,\varphi\big(t,x(t),y(t)\big), \\ g\big(x(t),y(t)\big) = 0, \end{cases}$$

for each $t \in I$. Notice that, as in Section 1, the regularity assumptions on g together with the Implicit Function Theorem imply that y is actually a C^2 function.

Since $\mu \geq 0$, equation (2.4) can be equivalently written as a first order system as follows:

$$\begin{cases} \dot{x} = \sqrt{\mu} z, \\ \dot{z} = \sqrt{\mu}\varphi(t,x,y), \\ g(x,y) = 0, \end{cases} \tag{2.5}$$

Further, system (2.6) can be written in the form (1.3) setting $\sqrt{\mu} = \lambda$, $k = 2n$ and

$$f : \mathbb{R} \times \mathbb{R}^{2n} \times \mathbb{R}^s \to \mathbb{R}^{2n}, \qquad f(t,p,r,q) = (r,\varphi(t,p,q))$$

(for convenience, here and in what follows we identify $\mathbb{R}^{2n} \times \mathbb{R}^s$ with $\mathbb{R}^{2n+s}$).

Now, define $F : \mathbb{R}^n \times \mathbb{R}^s \to \mathbb{R}^n \times \mathbb{R}^s$ by

$$F(p,q) = \left(\frac{1}{T} \int_0^T \varphi(t,p,q)dt\,, g(p,q) \right),$$

and $G : \mathbb{R}^{2n} \times \mathbb{R}^s \to \mathbb{R}^{2n} \times \mathbb{R}^s$ by

$$G(p,r,q) = \left(r, \frac{1}{T} \int_0^T \varphi(t,p,q)dt\,, g(p,q) \right).$$

Clearly, denoting by $F_i(p,q)$, $i = 1,2$, the components of the map F, one has

$$G(p,r,q) = (r, F_1(p,q), F_2(p,q)).$$

Lemma 2.1. *We have*

$$\deg(G, \mathbb{R}^{2n} \times \mathbb{R}^s) = (-1)^n \deg(F, \mathbb{R}^n \times \mathbb{R}^s).$$

Proof. Observe first that, by a standard approximation argument (see e.g. [1]) it is not restrictive to assume that the map F (and consequently G) is smooth with only nondegenerate zeros. Let $F^{-1}(0,0) = \{(p_i,q_i)\}_{i=1,\dots,N}$ be the zeros of F. Hence, $G^{-1}(0,0,0) = \{(p_i,0,q_i)\}_{i=1,\dots,N}$. Denote by $\mathrm{Id}_{\mathbb{R}^n}$ the identity on $\mathbb{R}^n$. Then, the differential of F at the point (p_i,q_i) is given by

$$d_{(p_i,q_i)}F = \begin{pmatrix} \partial_1 F_1(p_i,q_i) & \partial_2 F_1(p_i,q_i) \\ \partial_1 F_2(p_i,q_i) & \partial_2 F_2(p_i,q_i) \end{pmatrix}$$

and analogously

$$d_{(p_i,0,q_i)}G = \begin{pmatrix} 0 & \mathrm{Id}_{\mathbb{R}^n} & 0 \\ \partial_1 F_1(p_i,q_i) & 0 & \partial_2 F_1(p_i,q_i) \\ \partial_1 F_2(p_i,q_i) & 0 & \partial_2 F_2(p_i,q_i) \end{pmatrix}.$$

Thus, a straightforward computation yields

$$\deg(G, \mathbb{R}^{2n} \times \mathbb{R}^s) = \sum_{i=1}^N \operatorname{sign}\det d_{(p_i,0,q_i)}G = (-1)^n \sum_{i=1}^N \operatorname{sign}\det d_{(p_i,q_i)}F = (-1)^n \deg(F, \mathbb{R}^n \times \mathbb{R}^s),$$

and the assertion follows. □

The following is a consequence of Theorem 2.2 and Lemma 2.1.

Corollary 2.4. *Let $g : \mathbb{R}^n \times \mathbb{R}^s \to \mathbb{R}^s$, $\varphi : \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^s \to \mathbb{R}^n$ and $T > 0$ be such that φ is continuous and T-periodic in the first variable, and g is C^∞ with $\partial_2 g(p,q)$ invertible for all $(p,q) \in \mathbb{R}^n \times \mathbb{R}^s$. Define $F : \mathbb{R}^n \times \mathbb{R}^s \to \mathbb{R}^n \times \mathbb{R}^s$ by*

$$F(p,q) = \left(\frac{1}{T} \int_0^T \varphi(t,p,q)dt\,, g(p,q) \right),$$

and $G : \mathbb{R}^{2n} \times \mathbb{R}^s \to \mathbb{R}^{2n} \times \mathbb{R}^s$ by

$$G(p,r,q) = \left(r, \frac{1}{T} \int_0^T \varphi(t,p,q)dt\,, g(p,q) \right).$$

Assume that $\deg(F, \mathbb{R}^n \times \mathbb{R}^s)$ is well-defined and nonzero. Then, there exists an unbounded connected set Γ of nontrivial solution pairs of the system

$$\begin{cases} \dot{x} = \lambda z, \\ \dot{z} = \lambda \varphi(t,x,y), \\ g(x,y) = 0, \end{cases} \tag{2.6}$$

whose closure in $[0,\infty) \times C_T(\mathbb{R}^{2n} \times \mathbb{R}^s)$ meets the set $\{(0; \hat{p}, \hat{r}, \hat{q}) \in C_T(\mathbb{R}^{2n} \times \mathbb{R}^s) : G(p,r,q) = (0,0,0)\}$.

Let us now deduce some consequence of Corollary 2.4 on the structure of the set of solutions to (2.4). For this purpose, we introduce the following further notation.

We denote by $C^1_T(\mathbb{R}^n\times\mathbb{R}^s)$ the Banach space of the C^1 T-periodic functions with values in $\mathbb{R}^n\times\mathbb{R}^s$. We say that $(\sigma;x,y)\in[0,\infty)\times C^1_T(\mathbb{R}^n\times\mathbb{R}^s)$ is a *T-pair of* (2.4) if (x,y) satisfies (2.4) for $\mu=\sigma$; here the pair (x,y) is thought of as a single element of $C^1_T(\mathbb{R}^n\times\mathbb{R}^s)$. Given any $(p,q)\in\mathbb{R}^n\times\mathbb{R}^s$, it is convenient to denote by $(\hat p,\hat q)$ the map in $C^1_T(\mathbb{R}^n\times\mathbb{R}^s)$ that is constantly equal to (p,q). A T-pair of the form $(0;\hat p,\hat q)$ is called *trivial*.

Now, suppose that $\deg(F,\mathbb{R}^n\times\mathbb{R}^s)\neq 0$, and let $\Gamma\subseteq[0,\infty)\times C_T(\mathbb{R}^{2n}\times\mathbb{R}^s)$ be an unbounded branch of nontrivial solution pairs of (2.6) as in the assertion of Corollary 2.4. Consider the map

$$\Theta:[0,+\infty)\times C_T(\mathbb{R}^{2n}\times\mathbb{R}^s)\to[0,+\infty)\times C^1_T(\mathbb{R}^n\times\mathbb{R}^s)$$

defined by

$$\Theta(\lambda;x,z,y)=(\lambda^2;x,y).$$

Observe that:

– Θ sends solution pairs of (2.6) into T-pairs of (2.4);
– Θ sends trivial solution pairs into trivial T-pairs;
– the restriction of Θ to $(0,+\infty)\times C_T(\mathbb{R}^{2n}\times\mathbb{R}^s)$ is continuous with inverse given by

$$(\mu;x,y)\mapsto\left(\sqrt{\mu};x,\frac{\dot x}{\sqrt{\mu}},y\right).$$

From Corollary 2.4 we get that the set $\Upsilon=\Theta(\Gamma)\subseteq[0,\infty)\times C^1_T(\mathbb{R}^n\times\mathbb{R}^s)$ is connected, made up of nontrivial T-pairs of (2.4), moreover its closure in $[0,\infty)\times C^1_T(\mathbb{R}^n\times\mathbb{R}^s)$ meets the set of the trivial T-pairs. However, we cannot deduce that Υ is unbounded. In fact, it may happen that the elements of Υ are bounded in $[0,\infty)\times C^1_T(\mathbb{R}^n\times\mathbb{R}^s)$, while the norm of $\dot x/\sqrt{\mu}$ "blows up" as $\mu\to 0^+$. In other words, as a consequence of Corollary 2.4 we have the following partial result.

Corollary 2.5. *Assume that* $\deg(F,\mathbb{R}^n\times\mathbb{R}^s)$ *is well-defined and nonzero. Then, there exists a connected set* Υ *of nontrivial T-pairs of* (2.4) *whose closure in* $[0,\infty)\times C^1_T(\mathbb{R}^n\times\mathbb{R}^s)$ *meets the set* $\{(0;\hat p,\hat q):F(p,q)=(0,0)\}$ *and is either unbounded or contains a sequence* $\{(\mu_j;x_j,y_j)\}$ *which is bounded in* $[0,\infty)\times C^1_T(\mathbb{R}^n\times\mathbb{R}^s)$ *and such that* $\mu_j\to 0^+$ *and* $\dfrac{\|\dot x_j\|_\infty}{\sqrt{\mu_j}}\to+\infty$ *as* $j\to+\infty$.

As a final remark, we observe that to obtain the existence of an unbounded branch of nontrivial T-pairs of (2.4), we need suitable *a priori* bounds on the quotients $\frac{\|\dot x\|_\infty}{\sqrt{\mu}}$ as $\mu\to 0^+$, x being any T-periodic solution of (2.4) corresponding to μ. It is possible to prove such a priori bounds, for instance, in the special case when (2.4) is equivalent to a linear second order ODE.

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