The Invariance of Domain Theorem for compact perturbations of nonlinear Fredholm maps of index zero

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Abstract. We extend the Invariance of Domain Theorem to locally compact perturbations of nonlinear Fredholm maps of index zero between Banach spaces. We show that these maps are, up to the composition with a linear isomorphism, α-contractive perturbations of the identity, where α is Kuratowski's measure of noncompactness. Thus, we prove our result using Nussbaum's degree theory.

Keywords: *Domain invariance, Nonlinear Fredholm maps, Kuratowski's measure of noncompactness, α-contractive maps.*

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1 Introduction and preliminaries

The purpose of this paper is to generalize the Invariance of Domain Theorem to compact perturbations of nonlinear Fredholm maps of index zero between Banach spaces. In a recent work, Benevieri, Furi and Pera [1] showed that an injective nonlinear Fredholm map of index zero sends open sets into open sets, and they also generalized this result to nonlinear Fredholm maps of index zero with a perturbation of finite dimensional range. Moreover, they raised the question whether or not the same result is true if the perturbation is only locally compact. We give a positive answer to this problem showing that these maps are, up to the composition with a linear isomorphism, locally α-contractive perturbations of the identity (where α is the Kuratowski measure of noncompactness). Therefore, we can apply the following theorem, due to Nussbaum ([4], see also Deimling [2]).

Theorem 1.1. *Let E be a Banach space, U an open subset of E, and $C : U \to E$ a continuous locally α-contractive map such that $I - C$ is locally injective. Then $I - C$ is an open map.*

In order to do this, we have first to recall some concepts from nonlinear functional analysis, as the notions of Fredholm map and of Kuratowski measure of noncompactness. In the sequel E and F will denote infinite dimensional real Banach spaces, and we will assume that all the considered maps are continuous.

A bounded linear operator $L : E \to F$ between Banach spaces is said to be *Fredholm* if both $\operatorname{Ker} L$ and $\operatorname{CoKer} L = F/\operatorname{Im} L$ are finite dimensional. If L is a Fredholm operator, its *index* is defined by

$$\operatorname{ind} L = \dim \operatorname{Ker} L - \dim \operatorname{CoKer} L.$$

Let $f : U \to F$ be a map of class C^1 from an open subset U of a Banach space E into a Banach space F. We say that f is *Fredholm* if its Fréchet derivative $f'(x) : E \to F$ is a Fredholm operator for any fixed $x \in U$. A Fredholm map $f : U \to F$ is said to have index m if for every $x \in U$ the derivative $f'(x)$ is a Fredholm operator of index m. In this case, we denote the index of f by $\operatorname{ind} f$. In particular, since the index is locally constant, if U is connected and f is Fredholm, then $\operatorname{ind} f$ is the index of $f'(x)$ for any given $x \in U$.

Let A be a bounded subset of the Banach space E. The *Kuratowski measure of noncompactness* $\alpha(A)$ of A is defined as the infimum of the real numbers $d > 0$ such that A admits a finite covering by sets of diameter less than or equal to d. For the properties of the measure of noncompactness α, see e.g. Deimling [2].

A map $f : U \to F$ from an open subset U of a Banach space E into a Banach space F is said to be *α-Lipschitz* with constant $k \geq 0$ if $\alpha(f(A)) \leq k\alpha(A)$ for any bounded subset $A \subseteq U$. If the α-Lipschitz constant k is less than 1, then f is said to be *α-contractive.* We note that f is completely continuous if and only if it is α-Lipschitz with constant $k = 0$. Moreover, if f is Lipschitz continuous with constant k, then it is α-Lipschitz with the same constant k.

Let $f : U \to F$ be as above. We recall the definitions of $\alpha(f)$ and $\beta(f)$ given in [3]. Remind that the spaces E and F are infinite dimensional. First, we set

$$\alpha(f) = \sup \left\{ \frac{\alpha(f(A))}{\alpha(A)} \ : \ A \subseteq U \text{ bounded, } \alpha(A) > 0 \right\}.$$

This number is related to the property of compactness of the map f. In fact, $\alpha(f) = 0$ if and only if f is completely continuous. Below we recall some other properties of $\alpha(f)$ (see [3] for the proofs).

Proposition 1.1. *Let $f, g : U \to F$ be continuous. Then*

(1) $$\alpha(\lambda f) = |\lambda|\alpha(f), \quad \lambda \in \mathbb{R}.$$

(2) $$|\alpha(f) - \alpha(g)| \leq \alpha(f + g) \leq \alpha(f) + \alpha(g).$$

(3) $\alpha(f) = 0$ *if and only if f is completely continuous.*

Proposition 1.2. *Let E, F, G be Banach spaces, $U \subseteq E$ and $V \subseteq F$ open, $g : U \to V$ and $f : V \to G$ continuous. Then*

(4) $$\alpha(f \circ g) \leq \alpha(f)\alpha(g).$$

We recall the following definition:

$$\beta(f) = \inf \left\{ \frac{\alpha(f(A))}{\alpha(A)} \ : \ A \subseteq U \text{ bounded, } \alpha(A) > 0 \right\}.$$

The number $\beta(f)$ is related to the properness of the map. The following properties hold.

Proposition 1.3. *Let $f, g : U \to F$ be continuous. Then*

(5) $$\beta(\lambda f) = |\lambda|\beta(f), \quad \lambda \in \mathbb{R}.$$

(6) *If $\beta(f) > 0$, then f is proper on bounded closed sets.*

(7) $$\beta(f) \leq \alpha(f).$$

(8) $$\beta(f) - \alpha(g) \leq \beta(f + g) \leq \beta(f) + \alpha(g).$$

(9) $$|\beta(f) - \beta(g)| \leq \alpha(f - g).$$

(10) *If f is a homeomorphism and $\beta(f) > 0$, then $\alpha(f^{-1})\beta(f) = 1$.*

Proposition 1.4. *Let E, F, G be Banach spaces, $U \subseteq E$ and $V \subseteq F$ open, $g : U \to V$ and $f : V \to G$ continuous. Then*

(11) $$\beta(f)\beta(g) \leq \beta(f \circ g) \leq \alpha(f)\beta(g).$$

Moreover, in the case of a linear operator, we get the following properties.

Proposition 1.5. *Let $L : E \to F$ be a bounded linear operator. Then*

$$\alpha(L) \le ||L||. \tag{12}$$

$$\beta(L) > 0 \textit{ if and only if } \operatorname{Im} L \textit{ is closed and } \dim \operatorname{Ker} L < +\infty. \tag{13}$$

We observe that property (13) of β is closely linked to Fredholm operators. In fact, one can prove that L is Fredholm if and only if $\beta(L) > 0$ and $\beta(L^*) > 0$, where L^* is the adjoint of L.

Let $f : U \to F$ be, as before, a map from an open subset U of a Banach space E into a Banach space F, and let $p \in U$ be fixed. We introduce the new concepts of $\alpha_p(f)$ and $\beta_p(f)$.

Let $B_r(p)$ be the open ball in E centered at p with radius r. Suppose that $B_r(p) \subseteq U$ and consider $\alpha(f|_{B_r(p)})$, i.e.

$$\alpha(f|_{B_r(p)}) = \sup \left\{ \frac{\alpha(f(A))}{\alpha(A)} \, : \, A \subseteq B_r(p), \, \alpha(A) > 0 \right\}.$$

This is non-decreasing as a function of r, and clearly $\alpha(f|_{B_r(p)}) \le \alpha(f)$. Hence, the following definition makes sense:

$$\alpha_p(f) = \lim_{r \to 0} \alpha(f|_{B_r(p)}),$$

and in particular we have $\alpha_p(f) \le \alpha(f)$ for any p. In an analogous way, we define

$$\beta_p(f) = \lim_{r \to 0} \beta(f|_{B_r(p)}),$$

and we have $\beta_p(f) \ge \beta(f)$ for any p. It follows from these definitions that the above properties of α and β hold, with minor changes, for α_p and β_p as well. Therefore, only some statements in the next propositions will be proved.

Proposition 1.6. *Let $f, g : U \to F$ be continuous and $p \in U$. Then*

$$\alpha_p(\lambda f) = |\lambda| \alpha_p(f) \textit{ and } \beta_p(\lambda f) = |\lambda| \beta_p(f), \quad \lambda \in \mathbb{R}. \tag{14}$$

$$\beta_p(f) \le \alpha_p(f). \tag{15}$$

$$|\alpha_p(f) - \alpha_p(g)| \le \alpha_p(f + g) \le \alpha_p(f) + \alpha_p(g). \tag{16}$$

$$\beta_p(f) - \alpha_p(g) \le \beta_p(f + g) \le \beta_p(f) + \alpha_p(g). \tag{17}$$

$$|\beta_p(f) - \beta_p(g)| \le \alpha_p(f - g). \tag{18}$$

(19) *If f is locally compact, then $\alpha_p(f) = 0$.*

(20) *If $\beta_p(f) > 0$, then f is locally proper at p.*

(21) *If f is a local homeomorphism and $\beta_p(f) > 0$, then $\alpha_{f(p)}(f^{-1}) \beta_p(f) = 1$.*

Proposition 1.7. *Let E, F, G be Banach spaces, $U \subseteq E$ and $V \subseteq F$ open, $g : U \to V$ and $f : V \to G$ continuous. Fix $p \in U$ and let $q = g(p) \in V$. Then*

$$\alpha_p(f \circ g) \le \alpha_q(f) \alpha_p(g). \tag{22}$$

$$\beta_q(f) \beta_p(g) \le \beta_p(f \circ g) \le \alpha_q(f) \beta_p(g). \tag{23}$$

Proof. Let $\rho > 0$ be fixed. Since g is continuous, there exists $\bar{r} > 0$ such that $g(B_r(p)) \subseteq B_\rho(q)$ for $r < \bar{r}$. Recall that

$$\alpha(f \circ g|_{B_r(p)}) = \sup \left\{ \frac{\alpha(f(g(A)))}{\alpha(A)} \,:\, A \subseteq B_r(p),\, \alpha(A) > 0 \right\}.$$

If A is a subset of $B_r(p)$ such that $\alpha(g(A)) > 0$, then

$$\frac{\alpha(f(g(A)))}{\alpha(A)} = \frac{\alpha(f(g(A)))}{\alpha(g(A))} \, \frac{\alpha(g(A))}{\alpha(A)}.$$

We observe that $g(A) \subseteq B_\rho(q)$ if $r < \bar{r}$. Hence, taking the supremum with respect to the sets $A \subseteq B_r(p)$ such that $\alpha(A) > 0$, from the above we get

$$\alpha(f \circ g|_{B_r(p)}) \leq \alpha(f|_{B_\rho(q)})\alpha(g|_{B_r(p)}).$$

This is true for every $r < \bar{r}$, so we can pass to the limit for $r \to 0$ and we obtain $\alpha_p(f \circ g) \leq \alpha(f|_{B_\rho(q)})\alpha_p(g)$. The latter inequality holds for arbitrary $\rho > 0$. Consequently, we can take the limit for $\rho \to 0$ and we conclude that $\alpha_p(f \circ g) \leq \alpha_q(f)\alpha_p(g)$.

The proof of (23) is analogous. □

If L is a bounded linear operator, then the numbers $\alpha_p(L)$ and $\beta_p(L)$ do not depend on $p \in E$. We have in fact the following result.

Proposition 1.8. *Let $L : E \to F$ be a bounded linear operator. Then $\alpha_p(L) = \alpha(L)$ and $\beta_p(L) = \beta(L)$ for any $p \in E$.*

Proof. By the property of invariance of the Kuratowski measure of noncompactness α with respect to traslations, it is sufficient to show that $\alpha_0(L) = \alpha(L)$. Given $c > 0$, again by the properties of α we have $\alpha(cA) = c\alpha(A)$, and consequently $\alpha(L(cA)) = \alpha(cL(A)) = c\alpha(L(A))$. Now, recall that in the definition of $\alpha(L)$ the sets A must be bounded. Hence, it is clear from the above that in the linear case the supremum does not change if we consider only sets A included in a ball of a fixed radius $r > 0$. That is,

$$\alpha(L) = \sup \left\{ \frac{\alpha(L(A))}{\alpha(A)} \,:\, A \subseteq B_r(0),\, \alpha(A) > 0 \right\},$$

and consequently $\alpha_0(L) = \alpha(L)$.

The same argument holds for $\beta(L)$, and this completes the proof. □

In the case of a map f of class C^1, we get the following property.

Proposition 1.9. *Let $f : U \to F$ be of class C^1. Then, for any $p \in U$ we have $\alpha_p(f) = \alpha(f'(p))$ and $\beta_p(f) = \beta(f'(p))$.*

Proof. We show for instance that $\beta_p(f) = \beta(f'(p))$. Let $p \in U$ be given, and define

$$\phi(x) = f(x) - f'(p)(x - p).$$

The map ϕ is of class C^1 and its derivative at p vanishes. Thus, for any fixed $\varepsilon > 0$ there exists $r > 0$ such that for any x in $B_r(p)$ we have $\|\phi'(x)\| < \varepsilon$. It follows that in $B_r(p)$ the map ϕ is Lipschitz continuous with constant ε, and consequently it is α-Lipschitz with the same constant, that is, $\alpha(\phi|_{B_r(p)}) \leq \varepsilon$. Hence $\alpha_p(\phi) = 0$. Applying property (18) of β_p and the previous proposition, we get $\beta_p(f) = \beta_p(f'(p)) = \beta(f'(p))$.

The case of $\alpha_p(f)$ is analogous. $\square$

If the map f is of class C^1, Proposition 1.9 yields a simple method to evaluate $\alpha_p(f)$ and $\beta_p(f)$ by linearization. As an appication of this fact, we state the following result.

Proposition 1.10. *Let $g : U \to F$ and $\lambda : U \to \mathbb{R}$ be maps of class C^1, with $\lambda(x) \geq 0$. Consider the C^1 map $f : U \to F$ defined by $f(x) = \lambda(x)g(x)$. Then, for any $p \in U$ we have $\alpha_p(f) = \lambda(p)\alpha_p(g)$ and $\beta_p(f) = \lambda(p)\beta_p(g)$.*

Proof. Let $p \in U$ be fixed. By Proposition 1.9 we have $\alpha_p(f) = \alpha(f'(p))$. The linear operator $f'(p) : E \to F$ is defined by

$$f'(p)v = (\lambda'(p)v)g(p) + \lambda(p)g'(p)v.$$

By property (1) of α we get $\alpha(\lambda(p)g'(p)) = \lambda(p)\alpha(g'(p))$. Moreover, the operator $v \mapsto (\lambda'(p)v)g(p)$ is of finite dimensional range; hence, it is compact. By property (2) of α, this implies that $\alpha(f'(p)) = \lambda(p)\alpha(g'(p))$. Now the claim follows from the previous proposition.

The case of $\beta_p(f)$ is analogous. $\square$

Using Proposition 1.10, we can give examples of maps f with the property that $\alpha(f) = \infty$ but $\alpha_p(f) < \infty$ for any p, or that $\beta(f) = 0$ but $\beta_p(f) > 0$ for any p.

Example 1. Let $f = \lambda I$, where I is the identity, and $\lambda : E \to \mathbb{R}$ is a C^1 map such that

$$\lim_{||x|| \to +\infty} \lambda(x) = +\infty.$$

By Proposition 1.10, for any fixed $p \in E$ we have $\alpha_p(f) = \lambda(p)$. Moreover, as we already pointed out, we have $\alpha(f) \geq \alpha_p(f)$ for any p. Thus, $\alpha(f) = \infty$. In an analogous way, consider a C^1 map $\mu : E \to \mathbb{R}$ such that $\mu(x) > 0$ for any x and

$$\inf_{x \in E} \mu(x) = 0,$$

and define $f = \mu I$. By Proposition 1.10, for any fixed $p \in E$ we have $\beta_p(f) = \mu(p) > 0$. On the other hand, we have $\beta(f) \leq \beta_p(f)$ for any p. Hence, $\beta(f) = 0$. For instance, if E is a Hilbert space with norm $|| \cdot ||$, we can take $\lambda(x) = ||x||^2$ and $\mu(x) = \dfrac{1}{||x||^2 + 1}$.

We give now an example of a map f such that $\alpha(f) > 0$, even if $\alpha_p(f) = 0$ for any p.

Example 2. Let $E = F = \ell^2$. Fix an orthonormal basis $C = \{e_1, e_2, ...\}$ in ℓ^2, and let $\varepsilon > 0$ be such that the balls of the family $\{B_\varepsilon(e_i) : i = 1, 2, ...\}$ are pairwise disjoint. Define $U = \bigcup_i B_\varepsilon(e_i)$, and consider the map $f : U \to E$ defined by $f(x) = e_i$ if x belongs to $B_\varepsilon(e_i)$. The map f is locally constant, hence it is continuous and we have $\alpha_p(f) = 0$ for any $p \in U$. On the other hand, we want to show that $\alpha(f) > 0$. Since U is bounded, it suffices to prove that $\alpha(U) > 0$ and $\alpha(f(U))/\alpha(U) > 0$. We observe that $\alpha(C) = \sqrt{2}$ because $\|e_i - e_j\| = \sqrt{2}$ for $i \neq j$. Consequently, $\alpha(U) \geq \sqrt{2}$ since $U \supseteq C$. The fact that $f(U) = C$ implies $\alpha(f(U))/\alpha(U) > 0$, as claimed.

In the previous example the open set U is not connected. However, analogous examples with U connected could be given by considering, instead of the orthonormal basis C, any connected finite dimensional submanifold M of ℓ^2 with $\alpha(M) > 0$, and taking as U a tubolar neighborhood of M. In this case, if $f : U \to M$ is the associated retraction, we have $\alpha(f) > 0$ and $\alpha_p(f) = 0$ for any $p \in U$.

2 Invariance of Domain Theorem

We are now able to state and prove the main result of this paper, from which the generalization of the Invariance of Domain Theorem will follow.

Theorem 2.1. *Let E and F be Banach spaces, U an open subset of E and $p \in U$. Let $f : U \to F$ be locally injective, and suppose that f is the sum of two maps, g and h, where g is Fredholm of index zero and h is continuous with $\alpha_p(h) < \beta(g'(p))$. Then $f(p)$ belongs to the interior of the image $f(U)$ of the map f.*

Proof. By assumption, the derivative $g'(p)$ of g at p is a Fredholm operator of index zero. Thus, there exists a linear operator S of finite dimensional range such that $L := g'(p)+S$ is a linear isomorphism. Now, consider the composition $L^{-1} \circ f : U \to E$. It is sufficient to show that $L^{-1}f(p)$ is an interior point of the image of this map.

We claim that there exists a neighborhood V of p with the property that the restriction $L^{-1} \circ f|_V$ is an α-contractive perturbation of the identity or, equivalently, $\alpha((I - L^{-1} \circ f)|_V) < 1$. Clearly, it is enough to prove that $\alpha_p(I - L^{-1} \circ f) < 1$. To see this, write $f(x) = g(x) + Sx + h(x) - Sx$. By linearity we get

$$L^{-1}f(x) = L^{-1}(g(x) + Sx) + L^{-1}(h(x) - Sx).$$

First, consider the maps $T := L^{-1} \circ (g + S)$ and $\phi := I - T$. The latter is of class C^1 and its derivative at the point p vanishes, indeed by definition $\phi'(p) = I - L^{-1} \circ (g'(p) + S) = 0$. Hence, given $\delta > 0$, there exists $r > 0$ such that for any x in $B_r(p)$ we have $\|\phi'(x)\| < \delta$. This implies that in $B_r(p)$ the map ϕ is Lipschitz continuous with constant δ, and in particular $\alpha(\phi|_{B_r(p)}) \le \delta$. Thus, $\alpha_p(\phi) = 0$. On the other hand, since $L^{-1} \circ S$ is a compact linear operator, we have $\alpha(L^{-1} \circ S) = 0$ and consequently $\alpha_p(L^{-1} \circ S) = 0$. Hence, by property (16) of α_p, we only have to show that $\alpha_p(L^{-1} \circ h) < 1$. By property (22) of α_p, and since Proposition 1.8 implies that $\alpha_q(L^{-1}) = \alpha(L^{-1})$ for any $q \in F$, we have $\alpha_p(L^{-1} \circ h) \le \alpha(L^{-1})\alpha_p(h)$. As L is a linear isomorphism, by property (10) of β we get $\alpha(L^{-1}) = \beta(L)^{-1}$, and by definition of L, since S is a compact linear operator, applying property (8) of β we get $\beta(L) = \beta(g'(p) + S) = \beta(g'(p))$. Hence $\alpha(L^{-1}) = \beta(g'(p))^{-1}$ and we conclude that

$$\alpha_p(L^{-1} \circ h) \le \alpha(L^{-1})\alpha_p(h) = \beta(g'(p))^{-1}\alpha_p(h) < 1.$$

As we already pointed out, the latter inequality implies $\alpha_p(I - L^{-1} \circ f) < 1$. Consequently, there exists a neighborhood V of p such that $L^{-1} \circ f|_V$ is an α-contractive perturbation of the identity. Finally, applying Theorem 1.1 to $L^{-1} \circ f|_V$ we get that $L^{-1}f(V)$ is open, and in particular the point $L^{-1}f(p)$ belongs to the interior of the image of $L^{-1} \circ f$. □

We give a consequence of the previous theorem, in which only the known concepts of α and β are involved.

Corollary 2.1. *Let E and F be Banach spaces, and U an open subset of E. Let $f : U \to F$ be locally injective, and suppose that f is the sum of two maps, g and h, where g is Fredholm of index zero and h is continuous with $\alpha(h) < \beta(g)$. Then f is an open map.*

Proof. Let $p \in U$ be fixed. As we already pointed out, $\alpha_p(h) \le \alpha(h)$ and $\beta_p(g) \ge \beta(g)$. Moreover, by Proposition 1.9 we get $\beta_p(g) = \beta(g'(p))$. Hence, by assumption we have $\alpha_p(h) < \beta(g'(p))$, and the assertion follows from Theorem 2.1. □

The following result was our starting point, we state it as a straightforward consequence of Theorem 2.1.

Corollary 2.2. *Let E and F be Banach spaces, and U an open subset of E. Let $f : U \to F$ be locally injective, and suppose that f is the sum of two maps, g and h, where g is Fredholm of index zero and h is continuous and locally compact. Then f is an open map.*

Proof. Let $p \in U$ be fixed. Observe that by the local compactness of h we get $\alpha_p(h) = 0$, and by Proposition 1.5, since g is Fredholm, $\beta(g'(p)) > 0$. Thus, the claim follows from Theorem 2.1. □

Let us remark that, for our purposes, the right assumption for the Fredholm map g is to be of index zero. In some way this corresponds to the finite dimensional case in which the Invariance of Domain Theorem holds only for maps acting between spaces of equal dimension. The next proposition explains the situation.

Proposition 2.1. *If $f : U \to F$ is a locally injective Fredholm map of nonzero index, then f is not an open map.*

Proof. (i) If the index of f is negative, any element of its image is a critical value and we know, from Sard-Smale Theorem (see [5]), that the set of regular values is dense in F. Hence, the image of f has empty interior.

(ii) Suppose that the map f is locally injective with positive index, and let $\operatorname{ind}(f) = m > 0$. We want to show that this assumption leads to a contradiction. Define $\widetilde{f} : U \to F \times \mathbb{R}^m$ by $\widetilde{f}(x) = (f(x), 0)$. The map $\widetilde{f}$ is locally injective and Fredholm of index zero, as the composition of f with the inclusion $x \mapsto (x, 0)$ from F into $F \times \mathbb{R}^m$. Hence, we know that $\widetilde{f}$ is open (see [1]). On the other hand we have $\widetilde{f}(U) \subseteq F \times \{0\}$, a contradiction. □

We conclude this paper with a local result which, in some way, generalizes Corollary 2.2. Note in fact that a map h of class C^1 with $h'(p) = 0$ need not be locally compact at p.

Corollary 2.3. *Let E and F be Banach spaces, U an open subset of E and $p \in U$. Let $g : U \to F$ be Fredholm of index zero, $h_1 : U \to F$ locally compact, and $h_2 : U \to F$ of class C^1 and such that $h_2'(p) = 0$. Suppose that the map $f = g + h_1 + h_2$ is locally injective. Then $f(p)$ belongs to the interior of the image $f(U)$ of the map f.*

Proof. By Proposition 1.5, since g is Fredholm, $\beta(g'(p)) > 0$. Moreover, by the local compactness of h_1 we get $\alpha_p(h_1) = 0$, and by Proposition 1.9 we get $\alpha_p(h_2) = 0$. Applying Theorem 2.1 we have the claim. □

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