On existence and uniqueness of solutions for ordinary differential equations with nonlinear boundary conditions

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Sunto. – *Si prova un teorema di esistenza e unicità per un "problema ai limiti funzionale non-lineare", ossia un'equazione differenziale ordinaria con condizione al bordo non-lineare. La dimostrazione di questo risultato si basa su un teorema di inversione globale di Ambrosetti e Prodi: tale teorema viene applicato all'operatore al bordo ristretto alla varietà delle soluzioni globali dell'equazione differenziale ordinaria data. Questo risultato generalizza un analogo teorema di G. Vidossich. Inoltre vengono forniti esempi che mostrano come tale generalizzazione sia effettiva.*

Abstract. – We prove an existence and uniqueness theorem for a nonlinear functional boundary value problem, that is, an ordinary differential equation with a nonlinear boundary condition. The proof is based on a Global Inversion Theorem of Ambrosetti and Prodi, which is applied to the boundary operator restricted to the manifold of the global solutions to the equation. Our result is a generalization of an analogous existence and uniqueness theorem of G. Vidossich, as it is shown with some examples.

1. – Introduction and preliminaries

We consider a functional boundary value problem (in short, BVP) of the form

$$x' = f(t, x), \qquad L(x) = r, \tag{1}$$

where $f : [a, b] \times \mathbb{R}^n \to \mathbb{R}^n$ is a continuous map with continuous partial derivative with respect to the second variable, and L is a nonlinear map from the space $C([a, b], \mathbb{R}^n)$ into $\mathbb{R}^n$ of class C^1, i.e. Fréchet-differentiable with continuous derivative

$$L' : C([a, b], \mathbb{R}^n) \to \mathcal{L}(C([a, b], \mathbb{R}^n), \mathbb{R}^n).$$

Our aim is to prove an existence and uniqueness theorem for the functional BVP (1), generalizing an analogous result of Vidossich [4, Theorem 3].

For this purpose, we will use the following Global Inversion Theorem of Ambrosetti and Prodi [1, Theorem 1.8, p. 47]. We recall that if X and Y are metric spaces and $F : X \to Y$ is a continuous map, F is said to be *proper* if $F^{-1}(K)$ is compact for any compact set $K \subset Y$; furthermore, F is said to be *locally invertible* at a point $x \in X$ if there exist neighborhoods U of x in X and V of $y = F(x)$ in Y

such that F is a homeomorphism of U onto V, and F is said to be locally invertible on X if it is locally invertible at every point of X.

THEOREM **1.1 (Global Inversion)** – *Let X and Y be metric spaces, and let $F : X \to Y$ be proper. Suppose that the map F is locally invertible on X, the space X is arcwise connected, and the space Y is simply connected. Then F is a homeomorphism of X onto Y.*

Let us briefly set the notations which will be used in this paper. As usual, $\mathbb{R}^n$ denotes the euclidean n-dimensional space, and if $x = (x_1, \dots, x_n)$ is in $\mathbb{R}^n$ we set $|x| = \sum_{i=1}^n |x_i|$. By $M_n(\mathbb{R})$ we will denote the space of $n \times n$ real-valued matrices, and if $A = (a_{ij})$ belongs to $M_n(\mathbb{R})$ we set $|A| = \sum_{i,j=1}^n |a_{ij}|$. Moreover, $C([a,b], \mathbb{R}^n)$ is the Banach space of the continuous maps from the closed and bounded real interval $[a,b]$ into $\mathbb{R}^n$, with the supremum norm $|| \cdot ||_\infty$. For brevity, except in the statements, we will often write C^0 instead of $C([a,b], \mathbb{R}^n)$.

Let us recall that, if $f : [a,b] \times \mathbb{R}^n \to \mathbb{R}^n$ is a continuous map, a solution x to the ordinary differential equation

$$x' = f(t,x)$$

is said to be *maximal* if it is not a proper restriction of another solution, and it is said to be *global* if it is defined on the whole interval $[a,b]$.

We will make use of some properties of Fredholm maps between Banach spaces (see e.g. Smale [3]). We recall that, if $T : X \to Y$ is a linear and continuous operator between Banach spaces, T is said to be a *Fredholm operator* if both $\operatorname{Ker} T$ and $\operatorname{CoKer} T = Y / \operatorname{Im} T$ are of finite dimension. If T is a Fredholm operator, its *index* is defined by

$$\operatorname{ind} T = \dim \operatorname{Ker} T - \dim \operatorname{CoKer} T.$$

Let $F : X \to Y$ be a map of class C^1 between Banach spaces. We say that F is a *Fredholm map* if its Fréchet-derivative $F'(x) : X \to Y$ is a Fredholm operator for any fixed $x \in X$. Moreover, a Fredholm map $F : X \to Y$ is said to have index m if for every $x \in X$ the derivative $F'(x)$ is a Fredholm operator of index m. In particular, since the index is locally constant, if the space X is connected and F is a Fredholm map we may define the index of F (and still denote it by $\operatorname{ind} F$) as the index of $F'(x)$ for any given $x \in X$.

2. – Existence and uniqueness theorem

The following is the main result of this paper.

THEOREM **2.1** – *Let $f : [a,b] \times \mathbb{R}^n \to \mathbb{R}^n$ be continuous with continuous partial derivative*

$$D_2 f : [a,b] \times \mathbb{R}^n \to M_n(\mathbb{R}),$$

and let $L : C([a,b], \mathbb{R}^n) \to \mathbb{R}^n$ be of class C^1. Consider the ordinary differential equation in $\mathbb{R}^n$

$$x' = f(t,x), \tag{2}$$

and assume that the following hypotheses hold:

($H1$) *For any global solution x to (2), the linearized functional BVP*

$$y' = D_2 f(t, x(t)) y, \qquad L'(x) y = 0$$

has only the trivial solution.

$(H2)$ *For every $M > 0$, the solution set of the problem*

$$x' = f(t, x), \qquad |L(x)| \leq M$$

is bounded in $C([a, b], \mathbb{R}^n)$.

Assume that the set of the initial points of the global solutions to (2) is connected. Then, for every $r \in \mathbb{R}^n$ the functional BVP

$$x' = f(t, x), \qquad L(x) = r \tag{3}$$

has a unique solution.

Proof. – Let

$$S = \{x \in C^0 : x' - f(t, x) = 0\}$$

be the set of the global solutions to the equation (2), and let $A \subseteq \mathbb{R}^n$ be the open set consisting of the initial points of the global solutions to (2). We observe that, since the map f is locally Lipschitz with respect to x, the solutions to (2) enjoy local uniqueness. Thus, by the continuous dependence of solutions with respect to initial values, the set A is homeomorphic to S. Moreover, as we will show later (see Lemma 3.1), the set S is closed in the space C^0 and it is also a differentiable manifold of class C^1 and dimension n. Note that, consequently, S is an unbounded subset of C^0; indeed, if we suppose S bounded, then it must be compact because of Ascoli's Theorem, but this is impossible since S is homeomorphic to an open subset of $\mathbb{R}^n$.

Now, consider the restriction $L|_S : S \to \mathbb{R}^n$ of the map L to S. We want to show that $L|_S$ is locally invertible on S and proper.

(i) $L|_S$ is locally invertible on S.
By assumption, the map L is of class C^1. Moreover, by Lemma 3.1 below, S is a C^1-manifold of dimension n. Hence, by the (local) Inverse Function Theorem, it is enough to show that the Fréchet-derivative

$$L'(x)|_{T_xS} : T_xS \to \mathbb{R}^n,$$

where $T_xS \subset C^0$ is the tangent space to the manifold S at x, is a linear isomorphism for every $x \in S$. Let $x \in S$ be given, and let $v \in C^0$ be such that $v \in T_xS$ and $L'(x)v = 0$. These two conditions imply that v is a solution to the linearized problem

$$v' = D_2 f(t, x(t))v, \qquad L'(x)v = 0.$$

By hypothesis $(H1)$, this problem has only the trivial solution. Therefore $v = 0$ and $T_xS \cap \operatorname{Ker} L'(x) = \{0\}$ for every $x \in S$. Since $L'(x)|_{T_xS}$ is a linear map between two linear spaces of dimension n, then, being injective, it is surjective as well. Hence we conclude that such an operator is an isomorphism for every $x \in S$. Consequently, $L|_S$ is locally invertible on S.

(ii) $L|_S$ is proper.
Let $K \subset \mathbb{R}^n$ be compact; in particular K is bounded, i.e. there is a constant $M > 0$ such that for any $r \in K$ we have $|r| \leq M$. Let x_0 be an element of $(L|_S)^{-1}(K)$, i.e. $x_0 \in S$ and $L(x_0) = r_0$ with $r_0 \in K$ (we may assume that such a set is not empty). These two conditions imply that x_0 is a solution to the problem

$$x' = f(t, x), \qquad |L(x)| \leq M.$$

By hypothesis $(H2)$, there is a constant $\delta > 0$, independent of x_0, such that $||x_0||_\infty \leq \delta$. Consequently the set $(L|_S)^{-1}(K)$ is bounded in C^0. We claim that this set is equicontinuous as well. Indeed, if we denote by D_δ the closed ball centered at the origin with radius δ in $\mathbb{R}^n$, we have

$$(t, x_0(t)) \in [a,b] \times D_\delta$$

for any $t \in [a,b]$ and $x_0 \in (L|_S)^{-1}(K)$. By the compactness of $[a,b] \times D_\delta$ and the continuity of f, there exists a constant $\gamma > 0$ such that

$$|x_0'(t)| = |f(t, x_0(t))| \leq \gamma,$$

for any $t \in [a,b]$ and $x_0 \in (L|_S)^{-1}(K)$. Hence, the elements of $(L|_S)^{-1}(K)$ are equi-Lipschitz and in particular the set $(L|_S)^{-1}(K)$ is equicontinuous. By Ascoli's Theorem, this set is relatively compact in C^0. Now we observe that, by Lemma 3.1 below, S is closed; consequently the set $(L|_S)^{-1}(K)$, being closed in S by continuity, is closed in the space C^0 as well. Thus, being closed and relatively compact, $(L|_S)^{-1}(K)$ is compact. Since this fact holds for any $K \subset \mathbb{R}^n$ compact, we conclude that $L|_S$ is proper.

As we already pointed out, S is homeomorphic to the set A; in particular, since by hypothesis A is connected, so is S. Hence the manifold S is arcwise connected, being connected and locally arcwise connected. All the assumptions of Global Inversion Theorem 1.1 are verified: consequently $L|_S$ is a bijection, and this fact implies that the functional BVP (3) has a unique solution for every $r \in \mathbb{R}^n$, thus proving Theorem 2.1. ∎

3. – A geometrical lemma

As we observed in the proof of the previous theorem, we now have to show that the set S of the global solutions to

$$x' = f(t,x)$$

is an n-dimensional differentiable manifold of class C^1, which is closed in the space C^0. To this end, we will apply the following known result about the regularity of the set of solutions (see e.g. Smale [3]).

THEOREM **3.1** – *Let $\Phi : X \to Y$ be a Fredholm map between Banach spaces. If $y \in Y$ is a regular value for Φ, then $S = \Phi^{-1}(y)$ is a differentiable manifold of class C^1 and dimension equal to the index of Φ.*

We will also denote by C^0_0 the linear space consisting of the continuous maps $x : [a,b] \to \mathbb{R}^n$ such that $x(a) = 0$. We note that C^0_0 is a subspace of C^0 of codimension n, since it is the kernel of the linear operator from C^0 onto $\mathbb{R}^n$ defined by $x \mapsto x(a)$.

LEMMA **3.1** – *Let $f : [a,b] \times \mathbb{R}^n \to \mathbb{R}^n$ be continuous with continuous partial derivative with respect to the second variable. Then, the set*

$$S = \{x \in C^0 : x' - f(t,x) = 0\}$$

of the global solutions to the equation

$$x' = f(t,x)$$

is closed in the space $C([a,b], \mathbb{R}^n)$ and it is also a differentiable manifold of class C^1 and dimension n.

Proof. – Define $F : C^0 \to C^0_0$ by

$$F(x)(t) = x(t) - x(a) - \int_a^t f(\tau, x(\tau))d\tau, \qquad t \in [a,b].$$

Clearly F is continuous and $S = F^{-1}(0)$; consequently, S is closed.

Let us show that S is also a C^1-manifold of dimension n. To apply the above Theorem 3.1, we have to prove that F is a Fredholm map. The Fréchet-derivative $F'(x) : C^0 \to C^0_0$ of F at $x \in C^0$ is given by

$$[F'(x)h](t) = h(t) - h(a) - \int_a^t D_2 f(\tau, x(\tau))h(\tau)d\tau, \qquad t \in [a,b].$$

In particular, $F' : C^0 \to \mathcal{L}(C^0, C^0_0)$ is continuous. Now let $x \in C^0$ be fixed; we claim that $F'(x)$ is surjective and that $\operatorname{Ker} F'(x)$ is isomorphic to $\mathbb{R}^n$.

(i) $F'(x)$ is surjective.

We need to show that the equation

$$h(t) - h(a) - \int_a^t D_2 f(\tau, x(\tau))h(\tau)d\tau = g(t), \qquad t \in [a,b]$$

has a solution in C^0 for any given $g \in C^0_0$. Searching for a particular solution h to this equation, which verifies the additional condition $h(a) = 0$, we get the following Volterra equation:

$$h(t) - \int_a^t D_2 f(\tau, x(\tau))h(\tau)d\tau = g(t), \qquad t \in [a,b].$$

It is well-known that this equation has a unique solution in C^0 for any $g \in C^0_0$; consequently, the operator $F'(x)$ is surjective.

(ii) $\operatorname{Ker} F'(x)$ is isomorphic to $\mathbb{R}^n$.

Let $h \in \operatorname{Ker} F'(x)$; this means that $h \in C^0$ is a solution to the linear integral equation

$$h(t) - h(a) - \int_a^t D_2 f(\tau, x(\tau))h(\tau)d\tau = 0, \qquad t \in [a,b].$$

In particular h is a C^1-map and verifies

$$h' = D_2 f(t, x(t))h.$$

Since the solutions to a homogeneous linear differential equation are in bijective correspondence with their initial values, the kernel of $F'(x)$ is isomorphic to $\mathbb{R}^n$.

From properties (i) and (ii) it follows that $F'(x)$ is a Fredholm operator of index n for any given $x \in C^0$. Hence, F is a Fredholm map of index n. Finally, since $F'(x)$ is surjective for any $x \in C^0$, there are neither critical points nor critical values. We then apply the above Theorem 3.1, and we conclude that $S = F^{-1}(0)$ is a C^1-manifold of dimension $n = \operatorname{ind} F$. ■

4. – Existence and uniqueness theorem of Vidossich

In this section we state the following existence and uniqueness theorem of Vidossich [4, Theorem 3], and then we show that Theorem 2.1 is a generalization of this result.

Theorem **4.1** – *Let $f : [a,b] \times \mathbb{R}^n \to \mathbb{R}^n$ be continuous with continuous partial derivative*

$$D_2 f : [a,b] \times \mathbb{R}^n \to M_n(\mathbb{R}),$$

and let $L : C([a,b], \mathbb{R}^n) \to \mathbb{R}^n$ be of class C^1. Consider the ordinary differential equation in $\mathbb{R}^n$

$$x' = f(t,x), \tag{4}$$

and assume that the following hypotheses hold:

$(I1)$ *Every maximal solution to (4) exists on $[a,b]$.*

$(H1)$ *For any global solution x to (4), the linearized functional BVP*

$$y' = D_2 f(t, x(t))y, \qquad L'(x)y = 0$$

has only the trivial solution.

$(I2)$ *The solutions z to the linearized functional BVPs*

$$z' = D_2 f(t, x(t))z, \qquad L'(x)z = c,$$

where c belong to the sphere S^{n-1} and x is any global solution to (4), are uniformly bounded with respect to the solutions x and $c \in S^{n-1}$.

Then, for every $r \in \mathbb{R}^n$ the functional BVP

$$x' = f(t,x), \qquad L(x) = r$$

has a unique solution.

We recall that the proof of this result is based on the following Hadamard's Global Inversion Theorem (see Schwartz [2, Theorem 1.22]).

Theorem **4.2 (Hadamard)** – *Let X and Y be Banach spaces and let $F : X \to Y$ be of class C^1. Suppose that the linear operator $F'(x)$ is invertible for any $x \in X$, and that there is a constant $\alpha > 0$ such that*

$$\|(F'(x))^{-1}\| \leq \alpha < \infty, \qquad \forall x \in X.$$

Then, F is a homeomorphism of X onto Y.

We also observe that, by the Local Inverse Function Theorem, F is actually a C^1-diffeomorphism of X onto Y.

Let us show that Theorem 4.1 is a particular case of the above Theorem 2.1. We consider the functional BVP

$$\begin{cases} x' = f(t,x) \\ L(x) = r. \end{cases} \tag{5}$$

Assume that all the hypotheses of Theorem 4.1 hold; then, we want to prove that problem (5) satisfies all the assumptions of Theorem 2.1 as well. In particular, we have to show that $(I1)$ and $(I2)$ imply $(H2)$ of Theorem 2.1.

Let $u \in \mathbb{R}^n$ be given, and consider the Cauchy problem

$$x' = f(t,x), \qquad x(a) = u.$$

By $(I1)$, and by the fact that f is locally Lipschitz with respect to x, this problem has a unique solution existing on $[a, b]$, which we will denote by $\varphi(\cdot, u)$. Now, define $T : \mathbb{R}^n \to \mathbb{R}^n$ by

$$T(u) = L(\varphi(\cdot, u)).$$

If all the assumptions of Theorem 4.1 hold, one can prove (see [4]) that the map T verifies the hypotheses of Global Inversion Theorem 4.2. Consequently T is globally invertible and its inverse $T^{-1} : \mathbb{R}^n \to \mathbb{R}^n$ is of class C^1. For any $r \in \mathbb{R}^n$, if we set $u = T^{-1}(r)$, we get $(T^{-1})'(r) = (T'(u))^{-1}$. Thus, by the properties of T, even the derivative of its inverse is uniformly bounded with respect to r. Hence T^{-1} is Lipschitz, and in particular it sends bounded sets into bounded sets.

Now, fix $M > 0$ and consider the open ball U centered at the origin with radius M in $\mathbb{R}^n$. We have seen that $T^{-1}(U)$ is a bounded subset of $\mathbb{R}^n$. By definition, the boundedness of $T^{-1}(U)$ is equivalent to the fact that the initial values $u = x(a)$ of the solutions x to the problem

$$x' = f(t, x), \qquad |L(x)| \leq M \tag{6}$$

are contained in a compact subset of $\mathbb{R}^n$. By assumption $(I1)$ of global existence, and by continuity with respect to initial values, even the solutions to the problem (6) are contained in a compact subset of C^0; in particular they are bounded. Hence, we have shown that $(I1)$ and $(I2)$ imply the boundedness of the solutions to (6).

Finally, from hypothesis $(I1)$ of global existence it follows that the open set of the initial points of the global solutions to the equation (4) coincides with $\mathbb{R}^n$. In particular such a set is connected, and consequently all the assumptions of Theorem 2.1 hold. This shows that Theorem 4.1 of Vidossich is a particular case of Theorem 2.1.

5. – Remarks and examples

In this section we will make some considerations about the above existence and uniqueness results. In particular, we will see two examples showing that Theorem 2.1 is actually more general than Theorem 4.1.

First, we observe that in Theorem 2.1 the assumption of connectedness of the set A, consisting of the initial points of the global solutions, could be removed. In fact, with only minor changes in the proof, one can prove that if k denotes the number of connected components of A, the functional BVP

$$x' = f(t, x), \qquad L(x) = r$$

has exactly k solutions for every $r \in \mathbb{R}^n$. The reason why we preferred to state our result assuming A connected, is that we were not able to find an example of a problem which satisfies the assumptions $(H1)$, $(H2)$ and such that $k > 1$.

In fact, the following example shows that there are equations for which the open set A may have an arbitrary (even infinite) number of connected components.

Example **1** – Consider the following system:

$$\begin{cases} x' = \dfrac{1 + x^2}{1 + y^2} \\ y' = 0. \end{cases} \tag{7}$$

First, we study the behaviour of the solutions to (7) on the half-line $[0, +\infty)$. Setting the initial condition

$$\begin{cases} x(0) = x_0 \\ y(0) = y_0, \end{cases}$$

we find that the unique solution to (7) is given by

$$\begin{cases} x(t) = \tan\left(\dfrac{t}{1+y_0^2} + \arctan x_0\right) \\ y(t) = y_0. \end{cases}$$

These solutions have a maximal interval of existence depending on the initial value, i.e. of the form $[0, \omega(x_0, y_0))$ where

$$\omega(x_0, y_0) = \left(\frac{\pi}{2} - \arctan x_0\right)(1 + y_0^2).$$

Restricting the variable t to an interval $[0, h]$, that is, considering the system (7) in the strip $[0, h] \times \mathbb{R}^2$, the set A is given by the pairs (x_0, y_0) such that $\omega(x_0, y_0) > h$. Setting for example $h = 4$, we get

$$A = \left\{(x, y) : \left(\frac{\pi}{2} - \arctan x\right)(1 + y^2) > 4\right\},$$

a set which has two connected components (to see that A is not connected, observe that the x-axis does not intersect it). More generally, considering a system of the form

$$\begin{cases} x' = \dfrac{1 + x^2}{g(y)} \\ y' = 0, \end{cases}$$

we get

$$A = \left\{(x, y) : \left(\frac{\pi}{2} - \arctan x\right) g(y) > 4\right\},$$

and with a suitable choice of the function $g(y)$, one can make the set A have an arbitrary (even infinite) number of connected components. ■

We conclude this paper with two examples showing that Theorem 2.1 actually extends Theorem 4.1. First, in the following Example 2 we see a problem which does not satisfy hypothesis $(I1)$ of global existence, while $(H1)$ and $(H2)$ of Theorem 2.1 still hold.

Example **2** – Consider the following equation in the strip $[0, 1] \times \mathbb{R}$:

$$x' = x^2. \tag{8}$$

There exist maximal solutions to (8) which are not global, that is, they are not defined on $[0, 1]$. Indeed, if we set the initial condition $x(0) = \alpha \in \mathbb{R}$, the solutions to (8) are given by

$$x(t) = \frac{\alpha}{1 - \alpha t}, \tag{9}$$

and these functions are defined on $[0, 1]$ if and only if $x(0) = \alpha < 1$; in other words, we have $A = (-\infty, 1)$. Hence, hypothesis $(I1)$ of Theorem 4.1 does not hold.

Now, consider the linear integral operator L defined by

$$Lx = \int_0^1 x(\tau)d\tau,$$

and the corresponding problem

$$\begin{cases} x' = x^2 \qquad \text{on } [0, 1] \\ Lx = r, \end{cases} \tag{10}$$

where $r \in \mathbb{R}$. Let us show that this problem verifies all the assumptions of Theorem 2.1.

First, we have to prove that the linearized problems

$$\begin{cases} y' = 2x(t)y & \text{on } [0,1] \\ \displaystyle\int_0^1 y(\tau)d\tau = 0, \end{cases}$$

where x is any solution to (8) on $[0,1]$, have only the trivial solution. The solutions to the linear equation $y' = 2x(t)y$ are of the form

$$y(t) = ce^{2\int_0^t x(\tau)d\tau},$$

and the further condition $\int_0^1 y(\tau)d\tau = 0$ implies $c = 0$, thus $y \equiv 0$ and hypothesis $(H1)$ holds.

Moreover, we have to consider the solution set of the problem

$$\text{(11)} \qquad \begin{cases} x' = x^2 & \text{on } [0,1] \\ \displaystyle\left|\int_0^1 x(\tau)d\tau\right| \le M, \end{cases}$$

where $M > 0$ is fixed. By replacing the global solutions x to (8) with their explicit expression (9), depending on the initial value $\alpha < 1$, one can easily check that the condition of boundedness of the integral implies that the solutions to (11) are bounded in the supremum norm on $[0,1]$. Hence the functional BVP (10) also verifies hypothesis $(H2)$, as it was to prove. ■

Finally, in Example 3 below we see a problem such that hypothesis $(I2)$ of Theorem 4.1 does not hold, but again $(H1)$ and $(H2)$ are satisfied. Hence, Examples 2 and 3 show that we have weakened both assumptions $(I1)$ and $(I2)$ of Theorem 4.1.

EXAMPLE **3** – Consider the nonlinear integral operator L from $C([0,1],\mathbb{R})$ into $\mathbb{R}$ defined by

$$L(x) = \int_0^1 \log\left(x(\tau) + \sqrt{1 + x^2(\tau)}\right) d\tau,$$

and the following problem:

$$\text{(12)} \qquad \begin{cases} x' = 0 & \text{on } [0,1] \\ L(x) = r, \end{cases}$$

where $r \in \mathbb{R}$. We claim that this problem verifies all the assumptions of Theorem 2.1. Indeed, for any x in $C([0,1],\mathbb{R})$ we have

$$L'(x)y = \int_0^1 \frac{y(\tau)}{\sqrt{1 + x^2(\tau)}} d\tau.$$

Since the solutions to $x' = 0$ are the constants $x \equiv x_0$, the linearized problems are given by

$$\begin{cases} y' = 0 & \text{on } [0,1] \\ \displaystyle\int_0^1 \frac{y(\tau)}{\sqrt{1 + x_0^2}} d\tau = 0, \end{cases}$$

and clearly these problems have only the trivial solution. Hence, hypothesis $(H1)$ holds.

Moreover, for any $M > 0$ the solutions to

$$\begin{cases} x' = 0 & \text{on } [0,1] \\ \left| \int_0^1 \log\left(x(\tau) + \sqrt{1 + x^2(\tau)} \right) d\tau \right| \leq M \end{cases}$$

are given by the constants $x_0 \in \mathbb{R}$ such that

$$\left| \log\left(x_0 + \sqrt{1 + x_0^2} \right) \right| \leq M.$$

These points are contained in a bounded interval, thus $(H2)$ is verified as well.

Finally, let us show that problem (12) does not satisfy assumption $(I2)$ of Theorem 4.1. We have to study the behaviour of the solutions z to the linearized problems

$$\begin{cases} z' = 0 & \text{on } [0,1] \\ \displaystyle\int_0^1 \frac{z(\tau)}{\sqrt{1 + x_0^2}} d\tau = c, \end{cases}$$

where $c \in \mathbb{R}$ with $|c| = 1$. These solutions are the constants $z \equiv z_0$ such that

$$z_0 = \pm\sqrt{1 + x_0^2}.$$

Since these solutions are not uniformly bounded with respect to $x_0 \in \mathbb{R}$, hypothesis $(I2)$ of Theorem 4.1 does not hold as it was to prove. ■

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