ON THE EXISTENCE OF FORCED OSCILLATIONS FOR THE SPHERICAL PENDULUM ACTED ON BY A RETARDED PERIODIC FORCE

PIERLUIGI BENEVIERI, ALESSANDRO CALAMAI, MASSIMO FURI, AND MARIA PATRIZIA PERA

ABSTRACT. We show that, under mild conditions, T-periodic retarded functional motion equations on even-dimensional spheres admit forced oscillations. In this way we extend analogous results for the undelayed case due to the last two authors. A crucial role in our argument is played by a quite general continuation result, obtained in a recent paper, for forced oscillations of retarded functional motion equations on compact topologically nontrivial boundaryless manifolds.

Dedicated to Professor Russell A. Johnson for his outstanding contributions
in the theory of dynamical systems and ordinary differential equations

1. INTRODUCTION

Let $M \subseteq \mathbb{R}^k$ be a smooth (i.e. C^∞) compact boundaryless manifold, and let

$$F : \mathbb{R} \times C((-\infty, 0], M) \to \mathbb{R}^k$$

be a continuous map such that

$$F(t, \varphi) \in T_{\varphi(0)}M\,, \quad \forall\, (t, \varphi) \in \mathbb{R} \times C((-\infty, 0], M),$$

where, given $q \in M$, $T_qM \subseteq \mathbb{R}^k$ denotes the tangent space of M at q. Assume that F is T-periodic in the first variable and consider the *retarded functional motion equation* on M

$$x''_\pi(t) = F(t, x_t), \tag{1.1}$$

where $x''_\pi(t)$ stands for the tangential component of the acceleration $x''(t) \in \mathbb{R}^k$ at the point $x(t) \in M$, and $x_t \in C((-\infty, 0], M)$ is the function $s \mapsto x(t + s)$. We are interested in the problem of the existence of *forced oscillations of* (1.1), namely, solutions of (1.1) which are globally defined on $\mathbb{R}$ and periodic of the same period T as the forcing term F.

As a particular case of (1.1) we recover the undelayed differential equation

$$x''_\pi(t) = f(t, x(t)), \tag{1.2}$$

which corresponds to the choice $F(t, \varphi) = f(t, \varphi(0))$ in (1.1). In a series of papers (see e.g. [2, 3, 4]), the last two authors conjectured that equation (1.2) admits forced oscillations if the Euler–Poincaré characteristic $\chi(M)$ of M is different from zero. The conjecture is suggested by the fact that when f is autonomous the well-known Poincaré–Hopf Theorem (see e.g. [8]) implies that f vanishes at some point $q_0 \in M$. So that q_0 is an equilibrium point of (1.2) and is clearly a periodic solution of any period.

In the special case of the spherical pendulum (i.e. $M = S^2$) an affirmative answer to the above conjecture has been given in [3] and extended in [4] to the case $M = S^{2n}$. A crucial argument in [3] is

2000 *Mathematics Subject Classification.* 34K13, 55M25, 34C40, 70K42.
Key words and phrases. Retarded functional differential equations, periodic solutions, winding number, motion problems on manifolds.

the use of the classical concept of winding number in order to assign in a continuous way an integer to any T-periodic solution of (1.2) with sufficiently high energy. This integer, called *rotation index*, counts the number of rotations that a curve on S^2 makes in the subset of the sphere obtained by removing a pair of antipodal points (depending only on the chosen curve).

The purpose of this paper is to extend the above results to the case of retarded functional motion equations. Our main result, Theorem 3.1 below, asserts that, when $M = S^2$, equation (1.1) admits forced oscillations provided that F is bounded and satisfies a suitable Lipschitz-type assumption. Our proof is based on a quite general continuation result for forced oscillations of parametrized retarded functional motion equations on compact topologically nontrivial boundaryless manifolds, Theorem 2.2 below, that we have previously obtained in [1]. The strategy to prove the main result is similar to that followed by the last two authors in the undelayed case. In particular, we adapt to the retarded case some technical lemmas from [3, 4] strictly related to the geometry of the sphere. Finally, in Theorem 3.5 below we extend the existence result to the case $M = S^{2n}$.

Among the wide bibliography on retarded functional differential equations in Euclidean spaces we refer to the works of Gaines and Mawhin [5], Nussbaum [9, 10], and Mallet-Paret, Nussbaum and Paraskevopoulos [7]. For equations on manifolds we cite the papers of Oliva [11, 12]. For general reference we suggest the monograph by Hale and Verduyn Lunel [6].

2. Preliminaries: continuation results for motion equations

Let $M \subseteq \mathbb{R}^k$ be a smooth boundaryless manifold. Given $q \in M$, by $T_qM \subseteq \mathbb{R}^k$ and $(T_qM)^\perp \subseteq \mathbb{R}^k$ we denote the tangent and normal space of M at q, respectively. Since $\mathbb{R}^k = T_qM \oplus (T_qM)^\perp$, any vector $u \in \mathbb{R}^k$ can be uniquely decomposed into the sum of the *parallel* (or *tangential*) *component* $u_\pi \in T_qM$ *of* u *at* q and the *normal component* $u_\nu \in (T_qM)^\perp$ *of* u *at* q. By

$$TM = \{(q, v) \in \mathbb{R}^k \times \mathbb{R}^k : \ q \in M, \ v \in T_qM\}$$

we denote the *tangent bundle of* M, which is a smooth manifold containing a natural copy of M via the embedding $q \mapsto (q, 0)$. The natural projection of TM onto M is just the restriction (to TM as domain and to M as codomain) of the projection of $\mathbb{R}^k \times \mathbb{R}^k$ onto the first factor.

As is well known, there exists a smooth map $R : TM \to \mathbb{R}^k$, called *reactive force* (or *inertial reaction*), with the following properties:

(a) $R(q, v) \in (T_qM)^\perp$ for any $(q, v) \in TM$;
(b) R is quadratic in the second variable;
(c) given $(q, v) \in TM$, $R(q, v)$ is the unique vector such that $(v, R(q, v))$ belongs to the tangent space $T_{(q,v)}(TM)$ of TM at (q, v);
(d) given any C^2 curve $\gamma : (a, b) \to M$, the normal component $\gamma''_\nu(t)$ of $\gamma''(t)$ at $\gamma(t)$ equals $R(\gamma(t), \gamma'(t))$.

By $C((-\infty, 0], M)$ we mean the metrizable space of the M-valued continuous functions defined on $(-\infty, 0]$ with the topology of the uniform convergence on compact subintervals of $(-\infty, 0]$.

Given a continuous function $x : J \to M$ defined on a real interval J with $\inf J = -\infty$, and given $t \in J$, we adopt the standard notation $x_t : (-\infty, 0] \to M$ for the function $x_t(s) = x(t + s)$.

Let $F : \mathbb{R} \times C((-\infty, 0], M) \to \mathbb{R}^k$ be a continuous map. We say that F is a *functional field on* M if $F(t, \varphi) \in T_{\varphi(0)}M$ for all $(t, \varphi) \in \mathbb{R} \times C((-\infty, 0], M)$.

Consider the retarded functional motion equation on M

$$x''_\pi(t) = F(t, x_t), \tag{2.1}$$

where $x''_\pi(t)$ stands for the parallel component of the acceleration $x''(t) \in \mathbb{R}^k$ at the point $x(t)$. By properties (a) and (d) above, equation (2.1) can be equivalently written as

$$x''(t) = R(x(t), x'(t)) + F(t, x_t). \tag{2.2}$$

By a *solution* of (2.2) or, equivalently, of (2.1) we mean a continuous function $x : J \to M$, defined on a real interval J with $\inf J = -\infty$, which verifies eventually the equality $x''(t) = R(x(t), x'(t)) + F(t, x_t)$. This means that there exists $\bar{t}$, with $-\infty \le \bar{t} < \sup J$, such that x is C^2 on the subinterval $(\bar{t}, \sup J)$ of J and verifies $x''(t) = R(x(t), x'(t)) + F(t, x_t)$ for all $t \in (\bar{t}, \sup J)$.

Notice that, in the case when F is identically zero, equation (2.2) reduces to the so-called *inertial equation*

$$x''(t) = R(x(t), x'(t)),$$

and one obtains the geodesics of M as solutions.

Equation (2.2) is equivalent to the retarded functional differential equation on TM

$$\begin{cases} x'(t) = y(t), \\ y'(t) = R(x(t), y(t)) + F(t, x_t), \end{cases} \tag{2.3}$$

in the following sense: a function $x : J \to M$ is a solution of (2.2) if and only if the pair (x, x') is a solution of (2.3). For more details see [1].

Following [1], we say that a subset Q of $C((-\infty, 0], M)$ is a *brush* if there exists $\sigma \le 0$ such that

$$\varphi(s) = \psi(s),$$

for all $s \le \sigma$ and $\varphi, \psi \in Q$. We will make the following assumption:

(H) Given $\bar{t} > 0$ and any compact brush Q of $C((-\infty, 0], M)$, there exists $L \ge 0$ such that

$$\|F(t, \varphi) - F(t, \psi)\| \le L \sup_{s \le 0} \|\varphi(s) - \psi(s)\|$$

for all $t \in [0, \bar{t}]$ and $\varphi, \psi \in Q$.

Remark 2.1. Assumption (H) extends an analogous Lipschitz condition in [6], where the authors study equations of the type

$$x'(t) = h(t, x_t),$$

with $h : \mathbb{R} \times C([-r, 0], \mathbb{R}^k) \to \mathbb{R}^k$ Lipschitz in the second variable in any compact subset of $\mathbb{R} \times C([-r, 0], \mathbb{R}^k)$. In fact, if this condition is satisfied, the functional field F on $\mathbb{R}^k$ defined by

$$F(t, \varphi) = h(t, \varphi|_{[-r,0]})$$

verifies (H).

As pointed out in [1], if F is bounded and verifies (H), then one gets existence and uniqueness results of solutions for initial value problems associated to the retarded functional differential equation (2.3). These assumptions are crucial in the proof of Theorem 2.2 (given in [1]) and, therefore, from now on we will suppose that they are always satisfied.

Assume in addition that F is T-periodic in the first variable. By a *T-periodic solution*, or *forced oscillation*, of equation (2.1) we mean a solution which is globally defined on $\mathbb{R}$ and T-periodic. Let us point out that it will be useful, for or purposes, to consider the forced oscillations of (2.1) as elements of $C^1_T(M)$, the metric subspace of the Banach space $C^1_T(\mathbb{R}^k)$ of the T-periodic C^1 maps $x:\mathbb{R}\to M$.

In order to study the set of forced oscillations of (2.1), it is convenient to embed equation (2.1) into the following family of parametrized equations, depending on $\lambda \geq 0$:

$$x''_\pi(t) = \lambda F(t, x_t). \tag{2.4}$$

We will say that $(\lambda, x) \in [0,+\infty) \times C^1_T(M)$ is a *T-forced pair* of (2.4) if $x:\mathbb{R}\to M$ is a forced oscillation of (2.4) corresponding to λ. Observe that the subset of $[0,+\infty)\times C^1_T(M)$ of all the T-forced pairs of (2.4) is closed and, because of Ascoli's Theorem, locally compact. Given $q \in M$, we denote by $\bar{q} \in C^1_T(M)$ the constant map $t \mapsto q$, $t \in \mathbb{R}$. Among the T-forced pairs we shall distinguish those of the type $(0,\bar{q})$, $q \in M$, that will be considered *trivial*. Notice that there may exist nontrivial T-forced pairs $(0,x)$, provided that $x:\mathbb{R}\to M$ is a non-constant T-periodic geodesic in M.

The following continuation result has been proved in [1].

Theorem 2.2. *Let $M \subseteq \mathbb{R}^k$ be a smooth compact boundaryless manifold whose Euler–Poincaré characteristic $\chi(M)$ is different from zero, and $F:\mathbb{R}\times C((-\infty,0],M)\to\mathbb{R}^k$ a functional field which is T-periodic in the first variable. Suppose that F is bounded and verifies (H). Then, the equation* (2.4) *admits an unbounded connected set of nontrivial T-forced pairs whose closure meets the set*

$$\big\{(0,\bar{q}) \in [0,+\infty)\times C^1_T(M) : q \in M\big\}$$

of the trivial T-forced pairs.

3. Applications to forced oscillations

From now on we will adopt the following notation. The inner product of two vectors v and w in $\mathbb{R}^3$ will be denoted by $\langle v, w\rangle$, the vector product by $v\times w$, and $|v|$ will stand for the Euclidean norm of v (i.e. $|v| = \langle v, v\rangle^{1/2}$).

Let

$$S = \{q \in \mathbb{R}^3 : |q| = r\}$$

be the two-dimensional sphere centered at the origin with radius $r>0$, and let $F:\mathbb{R}\times C((-\infty,0],S)\to\mathbb{R}^3$ be a functional field on S which is T-periodic in the first variable. Regarding F as a force acting on a point of mass m constrained on S, consider the retarded functional motion equation on S

$$m x''_\pi(t) = F(t, x_t). \tag{3.1}$$

It is well known that in this case the reactive force (or force of constraint) at $q \in S$ corresponding to the velocity $v \in T_qS$ is given by $R(q,v) = m(|v^2|/r^2)q$ and, consequently, (3.1) can be equivalently written as

$$m x''(t) = m(|x'(t)|^2/r^2)x(t) + F(t, x_t). \tag{3.2}$$

Let $C^1_T(S)$ denote the metric subspace of the Banach space $C^1_T(\mathbb{R}^3)$ of the T-periodic C^1 maps $x:\mathbb{R}\to S$, endowed with the usual C^1 norm $\|x\|^1_\infty = \|x'\|_\infty + \|x\|_\infty$, where, given a continuous T-periodic function $y:\mathbb{R}\to\mathbb{R}^3$, $\|y\|_\infty$ stands for $\max\{|y(t)| : t \in \mathbb{R}\}$. Our result is the following.

Theorem 3.1. *Let $F:\mathbb{R}\times C((-\infty,0],S)\to\mathbb{R}^3$ be a functional field on S which is T-periodic in the first variable. Suppose that F is bounded and verifies (H). Then, the equation* (3.1) *admits a forced oscillation.*

In order to prove Theorem 3.1 we need some preliminary results. Namely, Lemmas 3.2, 3.3, 3.4 below. First, let us briefly recall the notion of *winding number*.

Let $\gamma : \mathbb{R} \to \mathbb{R}^2 \setminus \{0\}$ be a T-periodic C^1 curve. Regarding $\mathbb{R}^2$ as the complex plane, let $\theta(t)$ be the argument of $\gamma(t)$. Notice that, while $\theta(t)$ is defined up to integer multiples of 2π, the rate of change $\theta'(t)$ is a well defined function. The integer

$$w(\gamma) = \frac{1}{2\pi}\int_0^T \theta'(t)\,dt,$$

is called the *winding number* of the curve γ with respect to the origin. Roughly speaking, $w(\gamma)$ represents the number of counterclockwise rotations of γ around the origin in an interval of length T. It is well known that the winding number depends continuously on γ. More precisely, the function $w : C^1_T(\mathbb{R}^2\setminus\{0\}) \to \mathbb{Z}$ is locally constant, where $C^1_T(\mathbb{R}^2\setminus\{0\})$ denotes the metric space of the T-periodic C^1 maps $\gamma : \mathbb{R} \to \mathbb{R}^2\setminus\{0\}$.

Let now $x \in C^1_T(S)$. We say that x is *admissible* if for any $\tau, t \in \mathbb{R}$ one has $x'(\tau) \neq 0$ and $\rho^\tau(t) > 0$, where $\rho^\tau(t)$ denotes the distance of $x(t)$ from the axis α^τ through the origin spanned by the vector product $x(\tau) \times x'(\tau)$. Clearly, the set of all the admissible curves is an open subset of $C^1_T(S)$. In the sequel we will show that forced oscillations of (3.1) with sufficiently high energy are admissible.

Let $x \in C^1_T(S)$ be admissible. For any $\tau \in \mathbb{R}$, define $v_1(\tau) = x(\tau)/|x(\tau)|$, $v_2(\tau) = x'(\tau)/|x'(\tau)|$, and $v_3(\tau) = v_1(\tau) \times v_2(\tau)$. Observe that $\{v_1(\tau), v_2(\tau), v_3(\tau)\}$ is an orthonormal basis in $\mathbb{R}^3$ and for all $\tau, t \in \mathbb{R}$ we have $\rho^\tau(t)^2 = \langle v_1(\tau), x(t)\rangle^2 + \langle v_2(\tau), x(t)\rangle^2 > 0$.

Now, for $\tau, t \in \mathbb{R}$ define

$$x^\tau(t) = (\langle v_1(\tau), x(t)\rangle, \langle v_2(\tau), x(t)\rangle) \in \mathbb{R}^2.$$

Observe that x^τ is a T-periodic C^1 curve with values in $\mathbb{R}^2 \setminus \{0\}$. Thus, the winding number $w(x^\tau)$ is well defined. Moreover, since $w(x^\tau)$ depends continuously on τ and is integer valued, it is independent of τ. Hence, it makes sense to define the integer $\mathfrak{i}(x) = w(x^\tau)$, that will be called the *rotation index* of the admissible curve x. This integer is clearly a locally constant function defined on the open subset of $C^1_T(S)$ of all the admissible curves.

The following two lemmas provide some inequalities directly involving the mechanics of the considered motion and will be crucial to prove our result. The proofs of Lemmas 3.2 and 3.3 are quite similar to the undelayed case (see [3]), and will be given for the sake of completeness.

It is convenient to embed equation (3.2) into the following family of parametrized equations, depending on $\lambda \geq 0$:

$$mx''(t) = m(|x'(t)|^2/r^2)x(t) + \lambda\, F(t, x_t). \tag{3.3}$$

Lemma 3.2. *Let $x : \mathbb{R} \to S$ be a T-periodic solution of (3.3) corresponding to a given $\lambda \geq 0$. Let $K = \sup\{|F(t,\varphi)| : t \in \mathbb{R}, \varphi \in C((-\infty, 0], S)\}$. Then the norm of the momentum vector $p(t) = mx'(t)$ is a Lipschitz function with constant λK. So, in particular, for any $t_1, t_2 \in \mathbb{R}$, one has*

$$m|u(t_2) - u(t_1)| \leq \lambda K T, \tag{3.4}$$

where $u(t) = |x'(t)|$.

Proof. If for a given $t \in \mathbb{R}$ one has $u(t) \neq 0$, then

$$mu'(t) = \frac{m\langle x'(t), x''(t)\rangle}{u(t)} = \frac{\langle x'(t), \lambda F(t, x_t)\rangle}{u(t)}.$$

Therefore, $|mu'(t)| \leq \lambda K$ for all $t \in \mathbb{R}$ such that $u(t) \neq 0$.

Consider now $t_1, t_2 \in \mathbb{R}$, with $t_1 < t_2$. If $u(t) \neq 0$ in the interval (t_1, t_2), then the inequality (3.4) follows from the above argument. Otherwise, without loss generality, we may assume $u(t_1) \leq u(t_2)$ and $u(t_2) > 0$. Let $\hat{t} = \max\{t \in [t_1, t_2] : u(t) = 0\}$. Then, since the function u is nonnegative, one obtains

$$m|u(t_2) - u(t_1)| \leq mu(t_2) = m(u(t_2) - u(\hat{t})) \leq \lambda K|t_2 - \hat{t}| \leq \lambda K|t_2 - t_1|.$$

Finally, since u is T-periodic we may assume $|t_2 - t_1| \leq T$, and inequality (3.4) follows. □

Lemma 3.3. *Let x, u, F, K be as in Lemma 3.2. Assume that $mu(t) > \lambda KT$ for each $t \in \mathbb{R}$. Take any $\tau \in \mathbb{R}$ and let α^τ be the straight line through the origin spanned and oriented by the vector product $x(\tau) \times x'(\tau)$. Denote by $\rho^\tau(t)$ the distance of $x(t)$ from α^τ. Then, for any $t \in \mathbb{R}$, the angular momentum $M^\tau(t)$ with respect to the axis α^τ is such that*

$$(mu(t) - \lambda KT)r \leq M^\tau(t) \leq (mu(t) + \lambda KT)r. \tag{3.5}$$

Moreover, $\rho^\tau(t)$ satisfies the inequality

$$\rho^\tau(t) \geq \frac{mu(\tau) - \lambda KT}{mu(\tau) + \lambda KT} r.$$

So, in particular, $x(\cdot)$ lies in $S \setminus \alpha^\tau$.

Proof. Since $M^\tau(t)$ is the orthogonal projection of the angular momentum $x(t) \times mx'(t)$ onto the axis α^τ, one has

$$M^\tau(t) \leq m\rho^\tau(t)u(t), \quad \text{for all } t \in \mathbb{R}.$$

In particular,

$$M^\tau(\tau) = mru(\tau).$$

In addition,

$$|M'^\tau(t)| \leq \lambda\rho^\tau(t)|F(t, x_t)| \leq \lambda Kr,$$

and thus, for any $t \in \mathbb{R}$,

$$|M^\tau(t) - M^\tau(\tau)| = \left| \int_\tau^t M'^\tau(s)ds \right| \leq \lambda KrT,$$

so that

$$(mu(\tau) - \lambda KT)r \leq M^\tau(t) \leq (mu(\tau) + \lambda KT)r.$$

Finally, by applying the inequality (3.4) to t and τ, it follows

$$(mu(\tau) - \lambda KT)r \leq M^\tau(t) \leq m\rho^\tau(t)u(t) \leq (mu(\tau) + \lambda KT)r.$$

This implies

$$\rho^\tau(t) \geq \frac{mu(\tau) - \lambda KT}{mu(\tau) + \lambda KT} r,$$

and the proof is complete. □

In the following lemma (see [4] for the analogous result in the undelayed case) we show that a T-periodic solution of (3.3) with high speed is admissible and makes a large numbers of turns around the origin in each period.

Lemma 3.4. *Let x, u, F, K be as in Lemma 3.2. Assume that $\|x'\|_\infty > 2\lambda KT/m$. Then, x is admissible and its rotation index $\mathrm{i}(x)$ satisfies the inequality*

$$\mathrm{i}(x) \geq \frac{T}{2\pi} \frac{(m\|x'\|_\infty - KT)}{mr}. \tag{3.6}$$

Proof. Let $\tau \in \mathbb{R}$ be such that $|x'(\tau)| = \|x'\|_\infty$. By assumption

$$mu(\tau) - \lambda KT > \lambda KT$$

and, by Lemma 3.2,

$$mu(t) \geq mu(\tau) - \lambda KT$$

for any $t \in \mathbb{R}$. Therefore, as a consequence of Lemma 3.3, we get that x is admissible.

To prove inequality (3.6) observe that, given $\tau \in \mathbb{R}$, the rate of change of the angle $\theta(t) = \arg(x^\tau(t))$ is given, with the notation of Lemma 3.3, by $\theta'(t) = M^\tau(t)/m\rho^\tau(t)^2$. Hence, from (3.5) one obtains

$$\theta'(t) \geq \frac{(m|x'(\tau)| - \lambda KT)r}{m\rho^\tau(t)^2} \geq \frac{m|x'(\tau)| - KT}{mr} = \frac{m\|x'\|_\infty - KT}{mr},$$

so that

$$\mathrm{i}(x) \geq \frac{T}{2\pi}\,\frac{(m\|x'\|_\infty - KT)}{mr},$$

as claimed. □

We are now in a position to give the

Proof of Theorem 3.1. As we already pointed out, equation (3.1) is equivalent to (3.2). Let us associate to (3.2) the parametrized equation (3.3). Since the Euler–Poincaré characteristic of S is $\chi(S) = 2 \neq 0$, Theorem 2.2 implies that the equation (3.3) admits an unbounded connected set $\Sigma \subseteq [0,+\infty) \times C^1_T(S)$ of nontrivial T-forced pairs whose closure intersects the set of the trivial T-forced pairs. We will prove Theorem 3.1 by showing that Σ must contain a T-forced pair of the type $(1,x)$. Suppose not. Thus Σ is contained in $[0,1) \times C^1_T(S)$. So, necessarily, its projection onto $C^1_T(S)$ is unbounded. Let us prove that this leads to a contradiction.

Since S is bounded, Lemma 3.4 implies the existence of a constant $C > 0$ such that any T-periodic solution x of (3.3), corresponding to some $\lambda \in [0,1)$, is admissible provided that $\|x\|^1_\infty \geq C$. Consider the closure $\overline{\Sigma}$ of Σ in $[0,+\infty) \times C^1_T(S)$. Observe that this is a set of (possibly trivial) T-forced pairs. Let now

$$Y = \{(\lambda,x) \in \overline{\Sigma} : \|x\|^1_\infty \geq C\}$$

and consider the continuous function $\eta : Y \to \mathbb{Z}$ defined by $\eta(\lambda,x) = \mathrm{i}(x)$. Since Y is a closed subset of the metric space $\overline{\Sigma}$, the Tietze Extension Theorem implies the existence of a continuous extension $\hat{\eta} : \overline{\Sigma} \to \mathbb{R}$ of η. The inequality (3.6) of Lemma 3.4 shows that the image of $\hat{\eta}$ is unbounded. In addition, since $\overline{\Sigma}$ is connected, this image must actually be an unbounded interval. This is impossible because $\hat{\eta}$ takes integer values outside of the set

$$\Sigma_0 = \{(\lambda,x) \in \overline{\Sigma} : \|x\|^1_\infty \leq C\},$$

which, due to Ascoli's Theorem, is compact. This contradiction shows that there exists $x_0 \in C^1_T(S)$ such that $(1,x_0) \in \Sigma$, as claimed. □

Arguing as in [4] and making use of some ideas and definitions contained therein, one can prove the following generalization of Theorem 3.1 to retarded functional motion equations on even dimensional spheres.

Theorem 3.5. *Let $S^n = \{q \in \mathbb{R}^{n+1} : |q| = 1\}$ be the n-dimensional sphere, and $F : \mathbb{R}\times C((-\infty, 0], S^n) \to \mathbb{R}^{n+1}$ a functional field on S^n which is T-periodic in the first variable. Suppose that F is bounded and verifies (H). Then, the equation*

$$x''_\pi(t) = F(t, x_t)$$

admits a forced oscillation provided that the dimension of S^n is even.

References

[1] P. Benevieri, A. Calamai, M. Furi, and M.P. Pera, *A continuation result for forced oscillations of constrained motion problems with infinite delay*, preprint (2009).

[2] M. Furi and M.P. Pera, *On the existence of forced oscillations for the spherical pendulum*, Boll. Un. Mat. Ital. (7) **4-B** (1990), 381–390.

[3] M. Furi and M.P. Pera, *The forced spherical pendulum does have forced oscillations.* Delay differential equations and dynamical systems (Claremont, CA, 1990), 176–182, Lecture Notes in Math., 1475, Springer, Berlin, 1991.

[4] M. Furi and M.P. Pera, *On the notion of winding number for closed curves and applications to forced oscillations on even-dimensional spheres*, Boll. Un. Mat. Ital. (7), **7-A** (1993), 397–407.

[5] R. Gaines and J. Mawhin, *Coincidence degree and nonlinear differential equations*, Lecture Notes in Math., **568**, Springer, Berlin, 1977.

[6] J.K. Hale and S.M. Verduyn Lunel, *Introduction to Functional Differential Equations*, Springer Verlag, New York, 1993.

[7] J. Mallet-Paret, R.D. Nussbaum and P. Paraskevopoulos, *Periodic solutions for functional-differential equations with multiple state-dependent time lags*, Topol. Methods Nonlinear Anal. **3** (1994), 101–162.

[8] J.M. Milnor, *Topology from the differentiable viewpoint*, Univ. Press of Virginia, Charlottesville, 1965.

[9] R.D. Nussbaum, *Periodic solutions of some nonlinear autonomous functional differential equations*, Ann. Mat. Pura Appl. **101** (1974), 263–306.

[10] R.D. Nussbaum, *The fixed point index and fixed point theorems*, Topological methods for ordinary differential equations (Montecatini Terme, 1991), 143–205, Lecture Notes in Math., **1537**, Springer, Berlin, 1993.

[11] W.M. Oliva, *Functional differential equations on compact manifolds and an approximation theorem*, J. Differential Equations **5** (1969), 483–496.

[12] W.M. Oliva, Functional differential equations–generic theory. Dynamical systems (Proc. Internat. Sympos., Brown Univ., Providence, R.I., 1974), Vol. I, pp. 195–209. Academic Press, New York, 1976.

Pierluigi Benevieri, Massimo Furi, and Maria Patrizia Pera,
Dipartimento di Matematica Applicata ''Giovanni Sansone''
Università degli Studi di Firenze
Via S. Marta 3
I-50139 Firenze, Italy

Alessandro Calamai
Dipartimento di Scienze Matematiche
Università Politecnica delle Marche
Via Brecce Bianche
I-60131 Ancona, Italy.

e-mail addresses:
pierluigi.benevieri@unifi.it
calamai@dipmat.univpm.it
massimo.furi@unifi.it
mpatrizia.pera@unifi.it