J. Differential Equations 246 (2009) 1354–1362

Contents lists available at ScienceDirect

Journal of Differential Equations

www.elsevier.com/locate/jde

On forced fast oscillations for delay differential equations on compact manifolds

Pierluigi Benevieri [a], Alessandro Calamai [b], Massimo Furi [a,*], Maria Patrizia Pera [a]

[a] Dipartimento di Matematica Applicata "Giovanni Sansone", Università degli Studi di Firenze, Via S. Marta 3, I-50139 Firenze, Italy
[b] Dipartimento di Scienze Matematiche, Università Politecnica delle Marche, Via Brecce Bianche, I-60131 Ancona, Italy

ARTICLE INFO

Article history:
Received 28 December 2006
Available online 13 December 2008

ABSTRACT

We prove an existence result for forced oscillations of delay differential equations on compact manifolds with nonzero Euler–Poincaré characteristic. When the period is smaller than the delay we need the asymptotic fixed point index theory for C^1 maps due to Eells and Fournier, and Nussbaum.

1. Introduction

In [1] we studied the delay differential equation

$$x'(t) = \lambda f\big(t, x(t), x(t-1)\big), \quad \lambda \geqslant 0, \tag{1.1}$$

where, given a smooth manifold $M \subseteq \mathbb{R}^k$ with boundary (∂-*manifold* for short), the map $f : \mathbb{R} \times M \times M \to \mathbb{R}^k$ is continuous, T-periodic in the first variable and tangent to M in the second one; that is

$$f(t+T, p, q) = f(t, p, q) \in T_pM, \quad \forall (t, p, q) \in \mathbb{R} \times M \times M$$

($T_pM \subseteq \mathbb{R}^k$ denotes the tangent space of M at p).

Call T-*periodic pair* of the above equation a pair (λ, x) such that $\lambda \geqslant 0$ and $x : \mathbb{R} \to M$ is a T-periodic solution of (1.1) corresponding to λ. The set of the T-periodic pairs will be regarded as a subset of $[0, +\infty) \times C_T(M)$, where $C_T(M)$ is the set of the continuous T-periodic maps from $\mathbb{R}$ to M with the metric induced by the Banach space $C_T(\mathbb{R}^k)$ of the continuous T-periodic $\mathbb{R}^k$-valued maps

* Corresponding author.
E-mail addresses: pierluigi.benevieri@unifi.it (P. Benevieri), calamai@dipmat.univpm.it (A. Calamai), massimo.furi@unifi.it (M. Furi), mpatrizia.pera@unifi.it (M.P. Pera).

0022-0396/$ – see front matter
doi:10.1016/j.jde.2008.11.005

(with the standard supremum norm). A T-periodic pair (λ, x) is called *trivial* when $\lambda = 0$. In this case x is a constant M-valued map and will be identified with a point of M.

Under the assumptions that M is compact with nonzero Euler–Poincaré characteristic, that $T \geqslant 1$, and that f satisfies a natural inward condition along ∂M (when nonempty), in [1] we proved the existence of an unbounded (with respect to λ) connected branch of nontrivial T-periodic pairs whose closure intersects the set of the trivial T-periodic pairs in the so-called *set of bifurcation points*. That result extends an analogous one of the last two authors for the undelayed case (see [4,5]).

The approach followed in [1] consists in applying to a Poincaré-type T-translation operator the fixed point index theory for locally compact maps on ANRs. To this purpose, the assumption $T \geqslant 1$ is crucial, since otherwise the compactness of the Poincaré operator fails.

In this work we continue the study of Eq. (1.1) tackling the case $0 < T < 1$, and we prove the same global bifurcation result as in [1] in the more restrictive assumption that M is boundaryless. The reason of this restriction is due to the fact that, because of the lack of compactness of the Poincaré operator, when $0 < T < 1$ we need the fixed point index theory of Eells and Fournier [3] and Nussbaum [7] instead of the classical one. This theory regards eventually condensing C^1 maps between C^1-ANRs and cannot be applied when M is a ∂-manifold with $\partial M \neq \emptyset$.

2. Preliminaries

Throughout the paper, M will be a compact, boundaryless, smooth manifold embedded in $\mathbb{R}^k$. Let $g : \mathbb{R} \times M \times M \to \mathbb{R}^k$ be a continuous map. We say that g is *tangent to M in the second variable* or, for short, that g is a *vector field on M* if $g(t, p, q) \in T_pM$ for all $(t, p, q) \in \mathbb{R} \times M \times M$, where T_pM denotes the tangent space of M at p.

Given a vector field $g : \mathbb{R} \times M \times M \to \mathbb{R}^k$ (on M), consider the following delay differential equation:

$$x'(t) = g\big(t, x(t), x(t-1)\big). \tag{2.1}$$

By a *solution* of (2.1) we mean a continuous function $x : J \to M$, defined on a (possibly unbounded) real interval with length greater than 1, which is of class C^1 on the subinterval $(\inf J + 1, \sup J)$ of J and verifies $x'(t) = g(t, x(t), x(t-1))$ for all $t \in J$ with $t > \inf J + 1$.

Given g as above and given a continuous map $\varphi : [-1, 0] \to M$, consider the following initial value problem:

$$\begin{cases} x'(t) = g\big(t, x(t), x(t-1)\big), \\ x(t) = \varphi(t), \quad t \in [-1, 0]. \end{cases} \tag{2.2}$$

A solution of this problem is a solution $x : J \to M$ of (2.1) such that $J \supseteq [-1, 0]$ and $x(t) = \varphi(t)$ for all $t \in [-1, 0]$.

The following technical lemma regards the existence and uniqueness of a persistent solution of problem (2.2). The proof is standard in the theory of ODEs and can be adapted to the delay case. Therefore, it will be omitted.

By a C^1 map defined on an arbitrary subset X of a Banach space E we mean the restriction of a C^1 map defined on an open subset of E containing X.

Lemma 2.1. *Let g be a vector field on M. Then problem* (2.2) *admits a solution defined on the whole half-line* $[-1, +\infty)$.

Assume moreover that g is of class C^1. Then problem (2.2) *has a unique solution which depends continuously on data. More precisely, if $\{g_n\}$ is a sequence of C^1 vector fields on M which converges uniformly to g and $\{\varphi_n\}$ is a sequence of continuous maps from* $[-1, 0]$ *to M which converges uniformly to φ, then the sequence of the solutions of the initial value problems*

$$\begin{cases} x'(t) = g_n\big(t, x(t), x(t-1)\big), & t > 0, \\ x(t) = \varphi_n(t), & t \in [-1, 0], \end{cases}$$

converges uniformly on compact subsets of $[-1, +\infty)$ *to the solution of* (2.2).

In the sequel we will need the fixed point index for eventually compact C^1 maps between C^1-ANRs. This index has been defined independently by Eells and Fournier [3] and Nussbaum [7] for the more general class of eventually condensing maps.

Recall that a metric space X is a C^s-ANR ($s \in \mathbb{N} \cup \{\infty\}$) if there exist an embedding of class C^s of X in a Banach space E, a neighborhood W of X in E and a retraction $r: W \to X$ of class C^s. A map $k: X \to X$ is said to be *eventually compact* if for some $n \in \mathbb{N}$ the nth iterate k^n is compact. Let I be a compact real interval, and let $H: I \times X \to X$ be given. Then, H is said to be an *eventually compact homotopy* if the map $\widehat{H}: I \times X \to I \times X$, defined as $\widehat{H}(\lambda, x) = (\lambda, H(\lambda, x))$, is eventually compact.

Let X be a C^1-ANR and consider an eventually compact map $k: X \to X$ of class C^1. Given an open subset U of X, if the set of fixed points of k in U is compact, the pair (k, U) is called *admissible*. Then (as proved in [3,7]) it is possible to associate to any admissible pair (k, U) an integer $\mathrm{ind}_X(k, U)$—the *fixed point index* of k in U—which satisfies all the classical properties of the fixed point index theory. Obviously, in this new theory, the continuity assumption of maps and homotopies is strengthened by assuming the C^1 regularity, and the local compactness is weakened by supposing the eventual compactness.

As far as we know, the problem whether or not the above theory holds for the merely C^0 case is still open.

3. Branches of periodic solutions

From now on we will adopt the following notation. By $C([-1, 0], M)$ we mean the complete metric space of the M-valued (continuous) functions defined on $[-1, 0]$ with the metric induced by the Banach space $C([-1, 0], \mathbb{R}^k)$. Given $T > 0$, by $C_T(\mathbb{R}^k)$ we denote the Banach space of the continuous T-periodic maps $x: \mathbb{R} \to \mathbb{R}^k$ (with the standard supremum norm) and by $C_T(M)$ we mean the metric subspace of $C_T(\mathbb{R}^k)$ of the M-valued maps.

Let $f: \mathbb{R} \times M \times M \to \mathbb{R}^k$ be a vector field on M which is T-periodic in the first variable. Consider the following delay differential equation depending on a parameter $\lambda \geqslant 0$:

$$x'(t) = \lambda f\big(t, x(t), x(t-1)\big). \tag{3.1}$$

We will say that $(\lambda, x) \in [0, +\infty) \times C_T(M)$ is a *T-periodic pair* (of (3.1)) if $x: \mathbb{R} \to M$ is a T-periodic solution of (3.1) corresponding to λ. A T-periodic pair of the type $(0, x)$ is said to be *trivial*. In this case the function x is constant and will be identified with a point of M, and vice versa.

A pair $(\lambda, \varphi) \in [0, +\infty) \times C([-1, 0], M)$ will be called a *T-starting pair* (of (3.1)) if there exists $x \in C_T(M)$ such that $x(t) = \varphi(t)$ for all $t \in [-1, 0]$ and (λ, x) is a T-periodic pair. A T-starting pair of the type $(0, \varphi)$ will be called *trivial*. Notice that in this case the map φ must be constant, being the restriction of a periodic eventually constant map defined on $\mathbb{R}$.

Clearly, the map $\rho: (\lambda, x) \mapsto (\lambda, \varphi)$ which associates to a T-periodic pair (λ, x) the corresponding T-starting pair (λ, φ) is continuous (φ being the restriction of x to the interval $[-1, 0]$). Moreover, if f is C^1, from Lemma 2.1 it follows that ρ is actually a homeomorphism between the set of T-periodic pairs and the set of T-starting pairs.

Given $p \in M$, it is convenient to regard the pair $(0, p)$ both as a trivial T-periodic pair and as a trivial T-starting pair. With this in mind, the restriction of the map ρ to $\{0\} \times M \subseteq [0, +\infty) \times C_T(M)$ as domain and to $\{0\} \times M \subseteq [0, +\infty) \times C([-1, 0], M)$ as codomain is the identity.

An element $p_0 \in M$ will be called a *bifurcation point* of Eq. (3.1) if every neighborhood of $(0, p_0)$ in $[0, +\infty) \times C_T(M)$ contains a nontrivial T-periodic pair (i.e. a T-periodic pair (λ, x) with $\lambda > 0$). The following result provides a necessary condition for a point $p_0 \in M$ to be a bifurcation point.

Theorem 3.1. *(See [1].) Assume that $p_0 \in M$ is a bifurcation point of Eq.* (3.1)*. Then the tangent vector field $w: M \to \mathbb{R}^k$ defined by*

$$w(p) = \frac{1}{T} \int_0^T f(t, p, p)\, dt$$

vanishes at p_0.

Our main result (Theorem 3.7 below) provides a sufficient condition for the existence of a bifurcation point in M. More precisely, under the assumption that the Euler–Poincaré characteristic of M is nonzero, we will prove the existence of a *global bifurcating branch* for Eq. (3.1); that is, an unbounded and connected set of nontrivial T-periodic pairs whose closure intersects the set $\{0\} \times M$ of the trivial T-periodic pairs. Observe that, $C_T(M)$ being bounded, a global bifurcating branch is necessarily unbounded with respect to λ. In particular, the existence of such a branch ensures the existence of a T-periodic solution of Eq. (3.1) for each $\lambda \geqslant 0$.

As already pointed out, $C([-1,0], M)$ is a closed subset of the Banach space $C([-1,0], \mathbb{R}^k)$ and, therefore, can be regarded as a metric space. Moreover, it is known that $C([-1,0], M)$ is a smooth infinite-dimensional manifold (see e.g. [2]), and it is not difficult to prove (see e.g. [3]) that it is a C^1-ANR as well. In fact, it is a C^1 retract of the open subset $C([-1,0], U)$ of $C([-1,0], \mathbb{R}^k)$, $U \subseteq \mathbb{R}^k$ being a tubular neighborhood of M. We stress that this argument fails if M is a ∂-manifold with $\partial M \neq \emptyset$ since, in that case, M cannot be a C^1 retract of an open subset of $\mathbb{R}^k$.

For simplicity, from now on, the metric space $C([-1,0], M)$ will be denoted by X.

Suppose, for the moment, that f is C^1 (this assumption will be removed in Theorem 3.7). Given $\lambda \geqslant 0$ and $\varphi \in X$, consider in M the following delay differential (initial value) problem:

$$\begin{cases} x'(t) = \lambda f\big(t, x(t), x(t-1)\big), & t > 0, \\ x(t) = \varphi(t), & t \in [-1,0]. \end{cases} \tag{3.2}$$

Denote by α_0 the unique solution of problem (3.2), ensured by Lemma 2.1. Given $\lambda \in [0, +\infty)$, consider the Poincaré-type operator

$$P_\lambda : X \to X$$

defined as $P_\lambda(\varphi)(s) = \alpha_0(s+T)$, $s \in [-1,0]$. The next lemmas regard some crucial properties of P_λ.

Lemma 3.2. *(See e.g. [1].) The fixed points of P_λ correspond to the T-periodic solutions of Eq.* (3.1) *in the following sense*: *φ is a fixed point of P_λ if and only if it is the restriction to* $[-1,0]$ *of a T-periodic solution.*

Lemma 3.3. *Assume that* $0 < T < 1$. *Then, the map $P : [0,+\infty) \times X \to X$, defined by $(\lambda, \varphi) \mapsto P_\lambda(\varphi)$, is of class C^1.*

Proof. Given $\lambda \geqslant 0$ and $\varphi \in X$, consider the following (undelayed) Cauchy problem:

$$\begin{cases} x'(t) = \lambda f\big(t, x(t), \varphi(t-1)\big), & t \in [0,1], \\ x(0) = \varphi(0). \end{cases} \tag{3.3}$$

Define the operator $\mathcal{S} : [0,+\infty) \times X \to C([0,1], M)$ by $\mathcal{S}(\lambda, \varphi) = x$, where x is the unique solution of problem (3.3). Then, it is not difficult to check that P can be defined in an equivalent way as

$$P(\lambda, \varphi)(s) = \begin{cases} \varphi(s+T), & -1 \leqslant s \leqslant -T, \\ \mathcal{S}(\lambda, \varphi)(s+T), & -T \leqslant s \leqslant 0, \end{cases}$$

where $(\lambda, \varphi) \in [0,+\infty) \times X$ and $s \in [-1,0]$.

To prove that P is of class C^1, we embed the metric space $[0,+\infty) \times X$ into the Banach space $\mathbb{R} \times C([-1,0], \mathbb{R}^k)$. It is enough to show that P is the restriction of a C^1 map $\widetilde{P} : \mathbb{R} \times C([-1,0], \mathbb{R}^k) \to C([-1,0], \mathbb{R}^k)$.

Since the vector field f is C^1, it admits a bounded global extension $\tilde{f} : \mathbb{R} \times \mathbb{R}^k \times \mathbb{R}^k \to \mathbb{R}^k$ of class C^1. To construct such a global extension notice first that, since f is C^1, it is the restriction of a C^1 map f_1 defined on an open set $U_1 \subseteq \mathbb{R} \times \mathbb{R}^k \times \mathbb{R}^k$ containing $\mathbb{R} \times M \times M$. Let now B be an open ball in $\mathbb{R}^k$ containing the compact set $f(\mathbb{R} \times M \times M)$—recall that f is T-periodic in the first variable—and set $U_2 = f_1^{-1}(B) \subseteq U_1$. Then, the restriction $f_2 = f_1|_{U_2}$ of f_1 to U_2 is a bounded C^1 extension of f. Now, to extend the domain of f_2 to the whole space, consider a C^1 map $g:\mathbb{R}^k \to \mathbb{R}^k$ which coincides with the identity on $f(\mathbb{R}\times M\times M)$ and vanishes outside a compact set $K\subseteq B$ containing $f(\mathbb{R}\times M\times M)$. Define $\tilde f:\mathbb{R}\times\mathbb{R}^k\times\mathbb{R}^k\to\mathbb{R}^k$ by

$$\tilde f(t,p,q)=\begin{cases} g(f_2(t,p,q)), & \text{if } (t,p,q)\in U_2,\\ 0, & \text{if } (t,p,q)\notin f_2^{-1}(K).\end{cases}$$

It is easy to see that this construction provides the desired bounded global C^1 extension of f.

Given $\lambda\in\mathbb{R}$ and $\varphi\in C([-1,0],\mathbb{R}^k)$, consider the following Cauchy problem:

$$\begin{cases} x'(t)=\lambda\tilde f\big(t,x(t),\varphi(t-1)\big), & t\in[0,1],\\ x(0)=\varphi(0).\end{cases} \tag{3.4}$$

Define the extension $\widetilde{\mathcal{S}}:\mathbb{R}\times C([-1,0],\mathbb{R}^k)\to C([0,1],\mathbb{R}^k)$ of $\mathcal{S}$ by $\widetilde{\mathcal{S}}(\lambda,\varphi)=\tilde x$, where $\tilde x$ is the unique solution of problem (3.4). We claim that $\widetilde{\mathcal{S}}$ is of class C^1. To see this, observe that $\tilde x$ solves the following integral equation:

$$x(t)=\varphi(0)+\lambda\int_0^t \tilde f\big(\tau,x(\tau),\varphi(\tau-1)\big)\,d\tau,\quad t\in[0,1].$$

Defining the operator $\mathcal{T}:C([0,1],\mathbb{R}^k)\times\mathbb{R}\times C([-1,0],\mathbb{R}^k)\to C([0,1],\mathbb{R}^k)$ as

$$\mathcal{T}(x,\lambda,\varphi)(t)=\varphi(0)+\lambda\int_0^t \tilde f\big(\tau,x(\tau),\varphi(\tau-1)\big)\,d\tau,$$

the above equation can be written equivalently as

$$x-\mathcal{T}(x,\lambda,\varphi)=0,\quad x\in C\big([0,1],\mathbb{R}^k\big). \tag{3.5}$$

Thus, $\widetilde{\mathcal{S}}(\lambda,\varphi)$ is the unique element in $C([0,1],\mathbb{R}^k)$ satisfying the condition

$$\widetilde{\mathcal{S}}(\lambda,\varphi)-\mathcal{T}\big(\widetilde{\mathcal{S}}(\lambda,\varphi),\lambda,\varphi\big)=0.$$

Now, by the Implicit Function Theorem, to prove that $\widetilde{\mathcal{S}}$ is C^1 it is sufficient to show that the Fréchet derivative of the map

$$x\mapsto x-\mathcal{T}(x,\lambda,\varphi)$$

is an isomorphism. That is, it is enough to prove that, given (x,λ,φ) satisfying (3.5), the linear map

$$I-\partial_1\mathcal{T}(x,\lambda,\varphi):C\big([0,1],\mathbb{R}^k\big)\to C\big([0,1],\mathbb{R}^k\big)$$

is an isomorphism (where I denotes the identity and ∂_1 the derivative w.r.t. the first variable). Consider now the equation

$$\big(I-\partial_1\mathcal{T}(x,\lambda,\varphi)\big)v=0,\quad v\in C\big([0,1],\mathbb{R}^k\big),$$

that is

$$v(t) = \lambda \int_0^t \partial_2 \tilde{f}\big(\tau, x(\tau), \varphi(\tau - 1)\big) v(\tau)\, d\tau, \quad t \in [0, 1].$$

This is equivalent to the linear Cauchy problem

$$\begin{cases} v'(t) = \lambda \partial_2 \tilde{f}\big(t, x(t), \varphi(t - 1)\big) v(t), & t \in [0, 1], \\ v(0) = 0 \end{cases}$$

which has only the trivial solution. Thus, the Fredholm alternative implies that the operator $I - \partial_1 \mathcal{T}(x, \lambda, \varphi)$, being injective, is surjective as well. Hence $\widetilde{\mathcal{S}}$ is C^1, as claimed.

Define now

$$\widetilde{P} : \mathbb{R} \times C\big([-1, 0], \mathbb{R}^k\big) \to C\big([-1, 0], \mathbb{R}^k\big)$$

as

$$\widetilde{P}(\lambda, \varphi)(s) = \begin{cases} \varphi(s + T), & -1 \leqslant s \leqslant -T, \\ \widetilde{\mathcal{S}}(\lambda, \varphi)(s + T), & -T \leqslant s \leqslant 0, \end{cases}$$

where $(\lambda, \varphi) \in \mathbb{R} \times C([-1, 0], \mathbb{R}^k)$ and $s \in [-1, 0]$. Observe that the restriction of $\widetilde{P}$ to $[0, +\infty) \times X$ coincides with P. Let us show that $\widetilde{P}$ is C^1. To this end, consider the linear (and hence C^∞) operators

$$\Theta_0 : C\big([-1, 0], \mathbb{R}^k\big) \to C\big([-1, -T], \mathbb{R}^k\big) \quad \text{defined by } \Theta_0(\psi)(s) = \psi(s + T)$$

and

$$\Theta_1 : C\big([0, 1], \mathbb{R}^k\big) \to C\big([-T, 0], \mathbb{R}^k\big) \quad \text{defined by } \Theta_1(\psi)(s) = \psi(s + T).$$

Moreover, define the (C^∞) gluing operator $G : C([-1, -T], \mathbb{R}^k) \times C([-T, 0], \mathbb{R}^k) \to C([-1, 0], \mathbb{R}^k)$ as

$$G(y_1, y_2)(s) = \begin{cases} y_1(s), & s \in [-1, -T], \\ y_2(s) + y_1(-T) - y_2(-T), & s \in [-T, 0]. \end{cases}$$

Then, $\widetilde{P}$ can be obtained by combining the operators defined above in the following way:

$$\widetilde{P}(\lambda, \varphi) = G\big(\Theta_0(\varphi), \Theta_1\big(\widetilde{\mathcal{S}}(\lambda, \varphi)\big)\big).$$

This shows that, since $\widetilde{\mathcal{S}}$ is C^1, so is $\widetilde{P}$. Hence, P is C^1 as well. This completes the proof. □

We observe that one could prove, with a more involved argument, that the operator P is still of class C^1 even in the case $T \geqslant 1$. However, this fact will not play any role in this paper.

Before stating the next lemma, we introduce the following notation. Given a continuous map $x : [-1, +\infty) \to M$, we will denote by $\hat{x}$ the restriction of x to the interval $[-1, 0]$.

Lemma 3.4. *If $nT \geqslant 1$, then the map $(\lambda, \varphi) \mapsto P_\lambda^n(\varphi)$ is locally compact, where P_λ^n denotes the nth iterate of P_λ. Thus, the map $(\lambda, \varphi) \mapsto (\lambda, P_\lambda(\varphi))$ is eventually locally compact.*

Proof. Fix $(\lambda, \varphi) \in [0, +\infty) \times X$ and denote by α_0 the unique solution of problem (3.2). For any integer $n \geqslant 1$, define

$$\alpha_n(t) = \alpha_{n-1}(t+T) = \alpha_0(t+nT), \quad t \in [-1, +\infty).$$

Let us show, by induction, that for any $n \in \mathbb{N}$ we have $P_\lambda^n(\varphi) = \hat{\alpha}_n$. Indeed, for $n = 0$ we have $P_\lambda^0(\varphi) = \varphi = \hat{\alpha}_0$. Assume that $P_\lambda^{n-1}(\varphi) = \hat{\alpha}_{n-1}$, and consider $P_\lambda^n(\varphi) = P_\lambda(P_\lambda^{n-1}(\varphi)) = P_\lambda(\hat{\alpha}_{n-1})$. Notice that if $t \in [-1, 0]$ we have $\alpha_{n-1}(t) = \hat{\alpha}_{n-1}(t)$ by definition and, if $t > 0$, we have

$$\begin{aligned}
\alpha'_{n-1}(t) &= \alpha'_0\big(t + (n-1)T\big) \\
&= \lambda f\big(t + (n-1)T, \alpha_0\big(t + (n-1)T\big), \alpha_0\big(t + (n-1)T - 1\big)\big) \\
&= \lambda f\big(t, \alpha_{n-1}(t), \alpha_{n-1}(t-1)\big).
\end{aligned}$$

That is, the function α_{n-1} is the (unique) solution of the initial value problem

$$\begin{cases} x'(t) = \lambda f\big(t, x(t), x(t-1)\big), & t > 0, \\ x(t) = \hat{\alpha}_{n-1}(t), & t \in [-1, 0]. \end{cases}$$

Consequently, $P_\lambda(\hat{\alpha}_{n-1})(s) = \alpha_{n-1}(s+T) = \alpha_n(s)$ for any $s \in [-1, 0]$. Hence, $P_\lambda(\hat{\alpha}_{n-1}) = \hat{\alpha}_n$ as claimed.

Choose now $n \in \mathbb{N}$ such that $nT \geqslant 1$. Then, $P_\lambda^n(\varphi)(t) = \hat{\alpha}_n(t) = \alpha_0(t+nT)$ for any $t \in [-1, 0]$. Denote $C = \max\{|f(t,p,q)|\colon (t,p,q) \in [0,T] \times M \times M\}$. Recalling that α_0 is the solution of problem (3.2), since $nT \geqslant 1$, we have $|\alpha'_0(s)| \leqslant \lambda C$ for any $0 \leqslant nT - 1 \leqslant s \leqslant nT$.

Hence, given $b > 0$ and $(\lambda, \varphi) \in [0, b] \times X$, the derivative of the function $P_\lambda^n(\varphi)$ is dominated by bC, and this implies our assertion by Ascoli's Theorem. □

The following Lemma 3.5 regards the existence of an unbounded connected branch of nontrivial T-starting pairs for Eq. (3.1) which emanates from the set of the trivial T-starting pairs.

Since we identify M with the subset of X of the constant maps, $\{0\} \times M$ will be regarded as a subset of $[0, +\infty) \times X$.

Lemma 3.5. *Let M be a compact boundaryless manifold with nonzero Euler–Poincaré characteristic, and let f be a C^1 vector field on M which is T-periodic in the first variable. Then, Eq.* (3.1) *admits a connected branch of nontrivial T-starting pairs whose closure in the set of the T-starting pairs is not compact and intersects $\{0\} \times M$.*

Proof. We distinguish two different cases.

($T \geqslant 1$) In this case the operator P is locally compact and the result can be found in [1, Lemma 4.5].

($0 < T < 1$) Let

$$\Sigma = \big\{(\lambda, \varphi) \in [0, +\infty) \times X\colon\ (\lambda, \varphi) \text{ is a } T\text{-starting pair of (3.1)}\big\}.$$

If we prove that Σ is locally compact, then the result follows with exactly the same proof as in [1, Lemma 4.5] applying the Eells–Fournier–Nussbaum fixed point index instead of the classical index. To this end, fix $n \in \mathbb{N}$ with $nT \geqslant 1$ and set

$$\Sigma_n = \big\{(\lambda, \varphi) \in [0, +\infty) \times X\colon\ P_\lambda^n(\varphi) = \varphi\big\}.$$

Observe that Σ_n is locally compact as a consequence of Lemma 3.4. Moreover, on the basis of Lemma 3.2, Σ coincides with the set

$$\{(\lambda, \varphi) \in [0, +\infty) \times X\colon\ P_\lambda(\varphi) = \varphi\}$$

which is a closed subset of Σ_n. Hence, the assertion follows. □

In Theorem 3.7 below, which deals with T-periodic pairs instead of T-starting pairs, the vector field f is assumed to be merely continuous. The result regards the existence of a global bifurcating branch of nontrivial T-periodic pairs, which, $C_T(M)$ being bounded, must be unbounded with respect to λ.

The proof is the same as in Theorem 4.6 of [1] and will be given for the reader's convenience.

The following topological lemma is needed.

Lemma 3.6. *(See [6].) Let K be a compact subset of a locally compact metric space Z. Assume that any compact subset of Z containing K has nonempty boundary. Then $Z \setminus K$ contains a connected set whose closure is not compact and intersects K.*

Theorem 3.7. *Let M be a compact boundaryless manifold with nonzero Euler–Poincaré characteristic, and let f be a vector field on M, T-periodic in the first variable. Then, Eq.* (3.1) *admits an unbounded connected set of nontrivial T-periodic pairs whose closure meets the set of the trivial T-periodic pairs.*

Proof. The proof will be divided into two steps. In the first one f is assumed to be C^1 (so that Lemma 3.5 applies) and in the second one f is merely continuous.

Step 1. Assume that f is of class C^1. Let $\Gamma \subseteq [0, +\infty) \times C_T(M)$ denote the set of the T-periodic pairs of (3.1) and $\Sigma \subseteq [0, +\infty) \times X$ the set of the T-starting pairs (of the same equation). Let $A \subseteq \Sigma$ be a connected branch of nontrivial T-starting pairs as in the assertion of Lemma 3.5. As already pointed out, the map $\rho : \Gamma \to \Sigma$, which associates to any T-periodic pair (λ, x) the corresponding T-starting pair (λ, φ), is a homeomorphism. Moreover, the restriction of ρ to $\{0\} \times M \subseteq \Gamma$ as domain and to $\{0\} \times M \subseteq \Sigma$ as codomain is the identity. Thus, the subset $\rho^{-1}(A)$ of Γ is connected, it is made up of nontrivial T-periodic pairs, and its closure in Γ is not compact and meets the set $\{0\} \times M$ of the trivial T-periodic pairs. One can easily check that Γ is closed in $[0, +\infty) \times C_T(M)$ and, because of Ascoli's Theorem, any bounded subset of Γ is relatively compact. Hence, $\rho^{-1}(A)$ must be unbounded and its closure in Γ is the same as in $[0, +\infty) \times C_T(M)$.

Step 2. Suppose now that f is continuous. We apply Lemma 3.6 with $\{0\} \times M$ in place of K and with Γ in place of Z. Here, as in the previous step, Γ denotes the set of the T-periodic pairs of (3.1). As already pointed out, Γ is a closed, locally compact subset of $[0, +\infty) \times C_T(M)$.

Assume, by contradiction, that there exists a compact set $\widehat{\Gamma} \subseteq \Gamma$ containing $\{0\} \times M$ and with empty boundary in the metric space Γ. Thus, $\widehat{\Gamma}$ is also an open subset of Γ and, consequently, both $\widehat{\Gamma}$ and $\Gamma \setminus \widehat{\Gamma}$ are closed in $[0, +\infty) \times C_T(M)$. Hence, there exists a bounded open subset W of $[0, +\infty) \times C_T(M)$ such that $\widehat{\Gamma} \subseteq W$ and $\partial W \cap \Gamma = \emptyset$.

Let now $\{f_n\}$ be a sequence of C^1 vector fields on M, T-periodic in the first variable, and such that $\{f_n(t, p, q)\}$ converges to $f(t, p, q)$ uniformly on $[0, T] \times M \times M$. Given any $n \in \mathbb{N}$, let Γ_n denote the set of the T-periodic pairs of the equation

$$x'(t) = \lambda f_n\bigl(t, x(t), x(t-1)\bigr).$$

Since W is bounded and contains $\{0\} \times M$, the previous step implies that for any $n \in \mathbb{N}$ there exists a pair $(\lambda_n, x_n) \in \Gamma_n \cap \partial W$. We may assume $\lambda_n \to \lambda_0$ and, by Ascoli's Theorem, $x_n(t) \to x_0(t)$ uniformly. Since $\{\lambda_n f_n(t, p, q)\}$ converges to $\lambda_0 f(t, p, q)$ uniformly on $[0, T] \times M \times M$, $x_0(t)$ is a T-periodic solution of the equation

$$x'(t) = \lambda_0 f\bigl(t, x(t), x(t-1)\bigr).$$

That is, (λ_0, x_0) is a T-periodic pair of (3.1) and, consequently, (λ_0, x_0) belongs to $\partial W \cap \Gamma$, which is a contradiction. Therefore, by Lemma 3.6 one can find a connected branch B of nontrivial T-periodic pairs of (3.1) whose closure in Γ (which is the same as in $[0, +\infty) \times C_T(M)$) intersects $\{0\} \times M$ and is not compact. Finally, B cannot be bounded since, otherwise, because of Ascoli's Theorem, its closure would be compact. This completes the proof. $\square$

We close this paper with an example illustrating how our main result applies. In this example M is a (two-dimensional) sphere in $\mathbb{R}^3$.

Example 3.8. Consider the following system of delay differential equations:

$$\begin{cases} x_1'(t) = -x_2(t)x_3(t-1), \\ x_2'(t) = x_1(t)x_3(t-1) - x_3(t)\sin(\omega t), \\ x_3'(t) = x_2(t)\sin(\omega t). \end{cases}$$

Let us show that for any $\omega > 0$ this system has a $\frac{2\pi}{\omega}$-periodic solution lying on the unit sphere S^2 of $\mathbb{R}^3$. Let $f : \mathbb{R} \times S^2 \times S^2 \to \mathbb{R}^3$ be defined by

$$f(t, p, q) = \bigl(-p_2 q_3,\, p_1 q_3 - p_3 \sin(\omega t),\, p_2 \sin(\omega t)\bigr),$$

where $p = (p_1, p_2, p_3)$ and $q = (q_1, q_2, q_3)$ belong to S^2. Clearly, f is a vector field on S^2 which is $\frac{2\pi}{\omega}$-periodic with respect to $t \in \mathbb{R}$. We need to prove that the equation

$$x'(t) = \lambda f\bigl(t, x(t), x(t-1)\bigr)$$

admits a $\frac{2\pi}{\omega}$-periodic solution (on S^2) for $\lambda = 1$. This is a consequence of Theorem 3.7, since $\chi(S^2) = 2$.

References

[1] P. Benevieri, A. Calamai, M. Furi, M.P. Pera, Global branches of periodic solutions for forced delay differential equations on compact manifolds, J. Differential Equations 233 (2007) 404–416.

[2] J. Eells, A setting for global analysis, Bull. Amer. Math. Soc. 72 (1966) 751–807.

[3] J. Eells, G. Fournier, La théorie des points fixes des applications à itérée condensante, Bull. Soc. Math. France 46 (1976) 91–120.

[4] M. Furi, M.P. Pera, Global branches of periodic solutions for forced differential equations on nonzero Euler characteristic manifolds, Boll. Unione Mat. Ital. 3-C (1984) 157–170.

[5] M. Furi, M.P. Pera, A continuation principle for forced oscillations on differentiable manifolds, Pacific J. Math. 121 (1986) 321–338.

[6] M. Furi, M.P. Pera, A continuation principle for periodic solutions of forced motion equations on manifolds and applications to bifurcation theory, Pacific J. Math. 160 (1993) 219–244.

[7] R.D. Nussbaum, Generalizing the fixed point index, Math. Ann. 228 (1977) 259–278.