Yet Another Spectrum for Nonlinear Operators in Banach Spaces

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Abstract. In this paper we study a nonlinear spectrum which is related to a similar spectrum introduced recently by Furi, Vignoli, and the second author. We compare this spectrum with other nonlinear spectra, study its analytical and topological properties, and briefly indicate a possible application.

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In view of the eminent importance of spectral theory for linear operators in both mathematics and physics, it is not surprising that various attempts have been made to define and study some kind of spectrum also for nonlinear operators. Meanwhile there is a large variety of spectra for different classes of nonlinear operators, for a self-contained description of the state of the art of nonlinear spectral theory we refer to the book [3] or the recent survey article [2].

We point out, however, that not all of these spectra are suitable from the viewpoint of applications to nonlinear problems. A pleasant exception is provided by the *asymptotic spectrum* defined by Furi, Martelli and Vignoli [11] in 1978, the *global spectrum* given by Feng [9] in 1997, and the *local spectrum* introduced by Väth [21] in 2001. These three spectra, although being quite different in nature, admit surprisingly strong applications, coincide for homogeneous operators and, as one should expect, reduce to the familiar spectrum in the linear case. Let us mention, in particular, that Väth actually defines two similar, but different spectra; in the sequel we will call them the *large Väth spectrum* $\sigma_V(f)$ and the *small Väth spectrum* $\sigma_v(f)$, respectively.

Quite recently, a new notion called *spectrum of a nonlinear operator at some point* has been defined by Calamai, Furi, and Vignoli [4]. The idea of "localizing" the spectrum seems to be reasonable, since many concepts in nonlinear analysis are in fact of local nature. (It suffices to mention the derivative of a map at some point as a typical example.) This spectrum, which we will call the *CFV-spectrum* $\sigma_{CFV}(f)$ in what follows, has some natural properties and in some cases may be calculated explicitly.

A scrutiny of the large Väth spectrum $\sigma_V(f)$ and the Calamai-Furi-Vignoli spectrum $\sigma_{CFV}(f)$ reveales that they have not only many interesting features in common, but that the latter may considered as a *localization* of the former. So it seems to be a tempting idea to define yet another spectrum which may be viewed as a localization of the small Väth spectrum $\sigma_v(f)$. This is the purpose of the present paper in which we introduce and study a spectrum called *small Calamai-Furi-Vignoli spectrum* and denoted by $\sigma_{cfv}(f)$.

This paper is organized as follows. In the first section we introduce some unavoidable notation. Afterwards we recall basic definitions and facts from nonlinear spectral theory; in particular, we discuss Väth's spectra in some detail and give some modifications which seem to be of independent interest. The third section is concerned with the large Calamai-Furi-Vignoli spectrum $\sigma_{CFV}(f)$ and its new analogue $\sigma_{cfv}(f)$. In the fourth section we discuss some illustrative examples in more detail, with a particular emphasis on homogeneous operators) of any positive degree). Finally, in the last section we briefly indicate an application to the eigenvalue problem for the p-Laplace operator which arises in many fields of mechanics, physics, and engineering.

1. Prerequisites and notation. Throughout this paper, we denote by $B_r(p) := \{x \in X : \|x - p\| < r\}$ the open ball of radius $r > 0$ centered at $p \in X$, where X is a real or complex Banach space, by $\overline{B}_r(p)$ the corresponding closed ball, and by $S_r(p) = \partial B_r(p)$ the corresponding sphere. In case $p = 0$ we will use the shortcut $B_r := B_r(0)$, $\overline{B}_r := \overline{B}_r(0)$, and $S_r := S_r(0)$.

Given a set $M \subseteq X$ with nonempty interior, we denote by $\mathfrak{OBC}(M)$ the family of all open, bounded, connected subsets of M containing 0. Moreover, for a bounded set $M \subset X$ we write $\alpha(M)$ for the Hausdorff measure of noncompactness of M, i.e., the infimum of all numbers $\varepsilon > 0$ such that M has a finite ε-net in X. All maps considered in this paper are assumed to be continuous. Following the notation of [4], we will use in the sequel the following characteristics for a map $f : X \to X$ and $p \in X$:

$$\alpha(f) := \inf\{\lambda : \alpha(f(M)) \le \lambda\alpha(M)\}, \quad \omega(f) := \sup\{\lambda : \alpha(f(M)) \ge \lambda\alpha(M)\}, \tag{1.1}$$

$$\alpha_p(f) := \lim_{r\to 0} \alpha(f|_{B_r(p)}), \quad \omega_p(f) := \lim_{r\to 0} \omega(f|_{B_r(p)}), \tag{1.2}$$

$$|f| := \limsup_{\|x\|\to\infty} \frac{\|f(x)\|}{\|x\|}, \quad d(f) := \liminf_{\|x\|\to\infty} \frac{\|f(x)\|}{\|x\|}, \tag{1.3}$$

$$|f|_p := \limsup_{\|x\|\to 0} \frac{\|f(p+x) - f(p)\|}{\|x\|}, \quad d_p(f) := \liminf_{\|x\|\to 0} \frac{\|f(p+x) - f(p)\|}{\|x\|}. \tag{1.4}$$

In what follows, we will restrict ourselves to the special case $p = 0$; the general case may be recovered by passing from f to the map f_p defined by $f_p(x) = f(p + x) - f(p)$. In fact, $\alpha_p(f) = \alpha_0(f_p)$, $\omega_p(f) = \omega_0(f_p)$, $|f|_p = |f_p|_0$, and $d_p(f) = d_0(f_p)$. The characteristic (1.1) plays a crucial role in Darbo's fixed point principle [5]: every operator f which leaves a convex closed bounded subset of a normed space invariant and satisfies $\alpha(f) < 1$ has a fixed point in this set. This contains Schauder's fixed point principle as a special case, since $\alpha(f) = 0$ if and only if f is compact.

In the following definition we collect some notions for maps which are defined on the closure of an open, bounded, connected subset of X (typically, a ball); these notions have been introduced in [12], [15], and [20], see also Chapter 7 in [3].

Definition 1.1. Let $\Omega \in \mathfrak{OBC}(X)$. A map $f : \overline{\Omega} \to X$ is called *epi* on $\overline{\Omega}$ if $f(x) \neq 0$ for $x \in \partial\Omega$ and the coincidence equation

$$f(x) = h(x) \tag{1.5}$$

has a solution in Ω for any compact map $h : \overline{\Omega} \to X$ satisfying $h(x) \equiv 0$ for $x \in \partial\Omega$. More generally, $f : \overline{\Omega} \to X$ is called *k-epi* ($k \ge 0$) on $\overline{\Omega}$ if $f(x) \neq 0$ for $x \in \partial\Omega$ and the coincidence equation (1.5) has a solution in Ω for any map $h : \overline{\Omega} \to X$ satisfying $\alpha(h) \le k$ and $h(x) \equiv 0$ for $x \in \partial\Omega$. Moreover, f is called *strictly epi* on $\overline{\Omega}$ if

$$\inf_{x\in\partial\Omega} \|f(x)\| > 0 \tag{1.6}$$

and f is k-epi for some $k > 0$. Finally, $f : \overline{\Omega} \to X$ is called *properly epi* on $\overline{\Omega}$ if f is epi and $\omega(f|_\Omega) > 0$.

The following characteristic which was introduced in [20] gives a precise measure for "how noncompact" the map g in (1.5) may be chosen. For $\Omega \in \mathfrak{OBC}(X)$ and $f : \overline{\Omega} \to X$ we put

$$\nu(f;\Omega) := \inf\{k > 0 : f \text{ is not } k\text{-epi on } \overline{\Omega}\}. \tag{1.7}$$

The problem of finding a map f which is epi but not strictly epi (i.e., is not k-epi for any $k > 0$) was open for a long time. It was solved by Furi [10] who showed that the map $f : X \to X$ defined by

$$f(x) = \|x\|x \tag{1.8}$$

has this property in any infinite dimensional Banach space X. Such a map must necessarily satisfy $\omega(f) = 0$ (see (1.1)); this follows from the following quite remarkable theorem which was given by Väth in [23] and may be found with a slightly different proof in [4].

Theorem 1.2. *Let $\Omega \in \mathfrak{OBC}(X)$ and $f : \overline{\Omega} \to X$ be properly epi on $\overline{\Omega}$. Then $\nu(f;\Omega) \geq \omega(f|_\Omega)$, and so f is strictly epi on $\overline{\Omega}$.*

The map (1.8) has other interesting properties. It is a *homeomorphism* with continuous inverse

$$f^{-1}(y) = \begin{cases} \|y\|^{-1/2}y & \text{if } y \neq 0, \\ 0 & \text{if } y = 0, \end{cases} \tag{1.9}$$

and so behaves quite well from the analytical viewpoint. Nevertheless, since f is a bijection between the sphere S_r and the sphere S_{r^2}, it follows that

$$\alpha_0(f) = \omega_0(f) = |f|_0 = d_0(f) = 0, \tag{1.10}$$

which shows that the topological characteristics (1.2) and (1.4) may be zero even for very nice maps.

For further reference, we still introduce the notation

$$N(f;r) := \{x \in \overline{B}_r : f(x) = 0\} \qquad (r > 0) \tag{1.11}$$

for the set of zeros of a map $f : X \to X$ in the closed ball $\overline{B}_r$. Of course, in case $f(0) = 0$ and $d_0(f) > 0$ we have $N(f;r) = \{0\}$ for all sufficiently small $r > 0$, and trivial examples show that the converse need not be true.

2. The large and small Väth spectra. Following [18,22], we call a map $f : X \to X$ *V-regular* if there exists some $\Omega \in \mathfrak{OBC}(X)$ such that f is properly epi on $\overline{\Omega}$. A certain modification of this is given in the following

Definition 2.1. We call a map $f : X \to X$ *locally V-regular* (at zero) if for all $r > 0$ there exists some $\Omega \in \mathfrak{OBC}(B_r)$ such that f is properly epi on $\overline{\Omega}$.

Clearly, local V-regularity implies V-regularity, but the converse is not true, as the following elementary example shows.

Example 2.2. Let $f : \mathbb{R} \to \mathbb{R}$ be defined by $f(x) := x+1$ for $x < -1$, $f(x) := 0$ for $-1 \leq x \leq 1$, and $f(x) := x - 1$ for $x > 1$. Choosing $\Omega = (-r, r)$ with $r > 1$, from the classical intermediate value theorem we immediately conclude that f is V-regular. However, f cannot be locally V-regular, as may be seen by choosing $r < 1$. □

In [23] (see also Chapter 8 of [3]) the *large Väth spectrum* of $f : X \to X$ is defined by

$$\sigma_V(f) := \{\lambda \in \mathbb{K} : \lambda I - f \text{ is not } V\text{-regular}\}. \tag{2.1}$$

It is also shown there that V-regularity is stable under small perturbations (in a suitable metric), which has the pleasant consequence that the spectrum $\sigma_V(f)$ is *closed*. However, for local V-regularity in the sense of Definition 2.1 this is not true, as the following example from [19] shows.

Example 2.3. Consider the space $X = \ell_1(\mathbb{R})$ of all absolutely summable real sequences $x = (\xi_n)_n$ with the usual norm, and denote by e_n the canonical basis element in X and by P_n the corresponding projection, i.e., $P_n(\xi_1, \xi_2, \xi_3, \ldots) = \xi_n$ $(n \in \mathbb{N})$. Let $f : X \to X$ be defined by

$$f(\xi_1, \xi_2, \xi_3, \ldots) = (\phi_1(\xi_1), \phi_2(\xi_2), \phi_3(\xi_3), \ldots),$$

where

$$\phi_n(s) := \begin{cases} s + 2^{-n} - 4^{-n} & \text{if } \; s \leq -2^{-n}, \\ 2^{-n}s & \text{if } \; -2^{-n} < s < 2^{-n}, \\ s - 2^{-n} + 4^{-n} & \text{if } \; s \geq 2^{-n}. \end{cases} \tag{2.2}$$

It is not hard to see that $\phi_n : \mathbb{R} \to \mathbb{R}$ is a continuous bijection with inverse

$$\psi_n(t) := \begin{cases} t + 4^{-n} - 2^{-n} & \text{if } \; t \leq -4^{-n}, \\ 2^{n}t & \text{if } \; -4^{-n} < t < 4^{-n}, \\ t - 4^{-n} + 2^{-n} & \text{if } \; t \geq 4^{-n}. \end{cases} \tag{2.3}$$

Consequently, f is a homeomorphism on X with inverse

$$f^{-1}(\eta_1, \eta_2, \eta_3, \ldots) = (\psi_1(\eta_1), \psi_2(\eta_2), \psi_3(\eta_3), \ldots).$$

Below we will use the estimates

$$\|f(x) - f(\tilde{x})\| = \sum_{k=1}^{\infty} |\phi_k(\xi_k) - \phi_k(\tilde{\xi}_k)| \leq \|x - \tilde{x}\| \qquad (x = (\xi_k)_k, \tilde{x} = (\tilde{\xi}_k)_k \in X) \tag{2.4}$$

and

$$\sum_{k=n}^{\infty} |\psi_k(\eta_k) - \psi_k(\tilde{\eta}_k)| \leq \|y - \tilde{y}\| + \frac{1}{2^{n-2}} \qquad (y = (\eta_k)_k, \tilde{y} = (\tilde{\eta}_k)_k \in X) \tag{2.5}$$

which follow from the definition of ϕ_n and ψ_n. Choosing $n = 1$ in (2.5) we see that $\|f^{-1}(y) - f^{-1}(\tilde{y})\| \leq \|y - \tilde{y}\| + 2$, and so both f and f^{-1} are bounded operators on X.
We claim that the operator f is locally V-regular, but $2^{-n}I - f$ is not locally V-regular for any $n \in \mathbb{N}$, and so this property is not stable w.r.t. small perturbations.
First of all, let us show that f is properly epi on each ball B_r. If $h : \overline{B}_r \to X$ is any compact map satisfying $h(x) \equiv 0$ on S_r, then $f^{-1} \circ h$ is a compact operator mapping S_r into $\overline{B}_r$ (actually, into zero), and so there is some $\hat{x} \in B_r$ with $(f^{-1} \circ h)(\hat{x}) = \hat{x}$, i.e., a solution of (1.5).

Observe that, in contrast to f, the operator $f^{-1}: X \to X$ cannot satisfy a Lipschitz condition near zero, because $\|4^{-n}e_n\| = 4^{-n}$, but $\|f^{-1}(4^{-n}e_n)\| = 2^{-n}$. However, f^{-1} is α-nonexpansive on the whole space X, i.e., $\alpha(f^{-1}) \leq 1$, which may be seen as follows. Fix $\hat{y} = (\hat{\eta}_n)_n \in X$ and $\delta > 0$, and choose $N \in \mathbb{N}$ so large that $2^{2-N} \leq \delta$. Consider the set

$$\hat{H} := \{(\xi_n)_n \in X : \xi_n = \psi_n(\hat{\eta}_n) \text{ if } n \geq N+1\}.$$

Clearly, this set lies in a finite dimensional subspace of X, and so the set $\hat{H}_r := \hat{H} \cap f^{-1}(B_r(\hat{y}))$ is precompact, since f^{-1} is a bounded operator. Consequently, there exists a finite $\frac{\delta}{2}$-net $\{z_1, \ldots, z_m\}$ for $\hat{H}_r$ which means that for $y \in B_r(\hat{y})$ we may choose $j \in \{1, 2, \ldots, m\}$ with

$$\sum_{k=1}^{N} |P_k f^{-1}(y) - P_k z_j| \leq \frac{\delta}{2}.$$

Since $z_j \in \hat{H}$, hence $P_n z_j = \psi_n(\hat{\eta}_n)$ for $n \geq N+1$, this implies that

$$\begin{aligned}
\|f^{-1}(y) - z_j\| &= \sum_{k=1}^{N} |P_k f^{-1}(y) - P_k z_j| + \sum_{k=N+1}^{\infty} |P_k f^{-1}(y) - P_k z_j| \\
&\leq \frac{\delta}{2} + \sum_{k=N+1}^{\infty} |P_k f^{-1}(y) - P_k z_j| = \frac{\delta}{2} + \sum_{k=N+1}^{\infty} |\psi_k(\eta_k) - \psi_k(\hat{\eta}_k)| \\
&\leq \frac{\delta}{2} + \|y - \hat{y}\| + \frac{1}{2^{N-1}} \leq \|y - \hat{y}\| + \delta \leq r + \delta,
\end{aligned}$$

where we have used (2.5). Since $y \in B_r(\hat{y})$ was arbitrary, we conclude that

$$f^{-1}(B_r(\hat{y})) \subseteq \bigcup_{j=1}^{m} B_{r+\delta}(z_j).$$

Now, if $M \subset X$ is any bounded set with $\alpha(M) = r > 0$, we may find a finite $(r+\delta)$-net $\{\hat{y}_1, \ldots, \hat{y}_k\}$ for M in X. Using the same construction as above for each of the points $\hat{y}_1, \ldots, \hat{y}_k$, we find then a finite $(r+2\delta)$-net for $f^{-1}(M)$. Since $\delta > 0$ was arbitrary, we see that $\alpha(f^{-1}) \leq 1$, hence $\omega(f) = \alpha(f^{-1})^{-1} \geq 1$, and so we have proved that f is indeed properly epi.

On the other hand, to see that $2^{-n}I - f$ is not locally V-regular for any $n \in \mathbb{N}$ is easy. In fact, given $n \in \mathbb{N}$, $r > 0$ with $r < 2^{-n}$, and $\Omega \in \mathfrak{OBC}(B_r)$, we may find $t \in (0, 2^{-n}]$ such that $te_n \in \partial\Omega$. For this element we have $(2^{-n}I - f)(te_n) = 0$, and so $2^{-n}I - f$ is not locally V-regular. □

Example 2.3 shows that defining a "local" spectrum $\sigma_V^{loc}(f)$ consisting of all $\lambda \in \mathbb{K}$ such that $\lambda I - f$ is not locally V-regular would not be a good idea, because such a spectrum would not be closed. Observe, however, that the map $2^{-n}I - f$ in Example 2.3, although not being locally V-regular, is V-regular. More generally, one can even show by means of a simple homotopy argument that $\lambda I - f$ is V-regular for any $|\lambda| < 1$ in this example.

We pass now to the definition of the small Väth spectrum which was also given in [23], see again Chapter 8 of [3]. A map $f : X \to X$ is called *v-regular* if there exists some $\Omega \in \mathfrak{OBC}(X)$ such that f is strictly epi on $\overline{\Omega}$. Again, the following is a local variant of this.

Definition 2.4. We call a map $f : X \to X$ *locally v-regular* (at zero) if for all $r > 0$ there exists some $\Omega \in \mathfrak{OBC}(B_r)$ such that f is strictly epi on $\overline{\Omega}$.

Similarly as before, local v-regularity implies v-regularity, but the converse is not true, as Example 2.2 shows. Moreover, in contrast to v-regularity, local v-regularity is again not stable under small perturbations, as Example 2.3 shows.

Now, the *small Väth spectrum* of $f : X \to X$ is defined in [23] (see also Chapter 8 of [3]) by

$$\sigma_v(f) := \{\lambda \in \mathbb{K} : \lambda I - f \text{ is not } v\text{-regular}\}. \tag{2.6}$$

It is also shown there that the small spectrum is always closed. We point out that both Väth spectra $\sigma_V(f)$ and $\sigma_v(f)$ are bounded (hence compact) if $\alpha(f) < \infty$, see (1.1).

The following proposition which we state for further reference, is an immediate consequence of Väth's Theorem 1.2.

Proposition 2.5. *Every V-regular map is v-regular, and every locally V-regular map is locally v-regular. In particular, the inclusion*

$$\sigma_v(f) \subseteq \sigma_V(f) \tag{2.7}$$

holds true.

3. The large and small Calamai-Furi-Vignoli spectra. The following notion was introduced under a different name in [4]. We call a map $f : X \to X$ *CFV-regular* (at zero) if $d_0(f) > 0$, $\omega_0(f) > 0$, and f is epi on $\overline{B}_r$ for all sufficiently small $r > 0$ (i.e., for $r \in (0, r_0]$ with some suitable $r_0 > 0$). For example, a sufficient condition for a scalar map $f : \mathbb{R} \to \mathbb{R}$ to be CFV-regular is given in terms of the upper and lower Dini derivatives of f in [4, Remark 4.10]. More illuminating examples will be given in Section 4 below.

The *large Calamai-Furi-Vignoli spectrum* of a map $f : X \to X$ is defined in [4] by

$$\sigma_{CFV}(f) := \{\lambda \in \mathbb{K} : \lambda I - f \text{ is not } CFV\text{-regular}\}. \tag{3.1}$$

It is also shown in [4] that the spectrum $\sigma_{CFV}(f)$ is always closed; in addition, in case $\alpha_0(f) < \infty$ and $|f|_0 < \infty$ it is also bounded, hence compact.

Suppose that $f : X \to X$ is CFV-regular. Then we may find $r_0 > 0$ such that $\omega(f|_{B_{r_0}}) > 0$ and f is epi on $\overline{B}_r$ for $r \leq r_0$. This implies that f is properly epi on $\overline{B}_r$ for $r \leq r_0$, and so also V-regular. In this way, we have proved the following

Proposition 3.1. *Every CFV-regular map is locally V-regular. In particular, the inclusion*

$$\sigma_V(f) \subseteq \sigma_{CFV}(f) \tag{3.2}$$

holds true.

Now we are going to introduce a new regularity notion and a corresponding new spectrum. A detailed exposition of the following material is contained, together with some more examples, in the thesis [19].

Definition 3.2. Let $f : X \to X$ be a map satisfying $N(f; r) = \{0\}$ for all sufficiently small $r > 0$, where $N(f; r)$ is given by (1.11). We define

$$\nu_0(f) := \lim_{r \to 0} \nu(f; B_r), \tag{3.3}$$

with $\nu(f; B_r)$ as in (1.7), and call (3.3) the *local measure of solvability of* f (at zero). Moreover, we call a map $f : X \to X$ *cfv-regular* (at zero) if both $d_0(f) > 0$ and $\nu_0(f) > 0$.

Since the map $r \mapsto \nu(f; B_r)$ is decreasing in the setting of Definition 3.2, the equality $\nu_0(f) = 0$ means that $\nu(f; B_r) = 0$ for sufficiently small $r > 0$, and so f cannot be strictly epi. However, f may be very well epi, as the map (1.8) shows which obviously satisfies $\nu_0(f) = 0$.

Definition 3.3. For $f : X \to X$ we put

$$\sigma_{cfv}(f) := \{\lambda \in \mathbb{K} : \lambda I - f \text{ is not } cfv\text{-regular}\}, \tag{3.4}$$

and call the set (3.4) the *small Calamai-Furi-Vignoli spectrum* of f.

The following proposition is parallel to Proposition 2.5 and Proposition 3.1.

Proposition 3.4. *Every CFV-regular map is cfv-regular, and every cfv-regular map is locally v-regular. In particular, the inclusions*

$$\sigma_v(f) \subseteq \sigma_{cfv}(f) \subseteq \sigma_{CFV}(f) \tag{3.5}$$

hold true.

Proof: Let f be CFV-regular, i.e., $d_0(f) > 0$, $\omega_0(f) > 0$, and there exists $r_1 > 0$ such that f is epi on $\overline{B}_r$ for each $r \leq r_1$. From $\omega_0(f) > 0$ it follows that there exists $r_2 > 0$ such that $\omega(f|_{B_r}) > 0$ for each $r \leq r_2$. So Theorem 1.2 implies that $\nu(f|_{B_r}) \geq \omega(f|_{B_r})$ for $r \leq \min\{r_1, r_2\}$. Passing to the limit $r \to 0$ we obtain $\nu_0(f) \geq \omega_0(f) > 0$ which means that f is cfv-regular.

Now suppose that f is cfv-regular, hence $d_0(f) > 0$ and $\nu_0(f) > 0$. Choose $r_0 > 0$ such that $\|f(x)\| \geq \frac{1}{2} d_0(f)\|x\|$ for $\|x\| \leq r_0$, and f is $\frac{1}{2}\nu_0(f)$-epi on B_r for all $r \leq r_0$. Then

$$\inf_{\|x\|=r} \|f(x)\| \geq r\frac{d_0(f)}{2} > 0 \qquad (0 < r \leq r_0),$$

which means that f is strictly epi on B_r for $r \leq r_0$, and hence locally v-regular. □

The following Theorem shows that the spectrum (3.4) shares some natural properties with the spectra introduced before.

Theorem 3.5. *The spectrum* $\sigma_{cfv}(f)$ *is always closed; in case* $\alpha_0(f) < \infty$ *and* $|f|_0 < \infty$ *it is also bounded, hence compact.*

We will prove Theorem 3.5 as a consequence of the following Rouché type perturbation result for cfv-regular operators.

Lemma 3.6. *Suppose that* $f_1 : X \to X$ *is cfv-regular and* $f_2 : X \to X$ *satisfies*

$$|f_2|_0 < d_0(f_1), \quad \alpha_0(f_2) < \nu_0(f_1). \tag{3.6}$$

Then the map $f_1 + f_2$ *is also cfv-regular.*

Proof: Without loss of generality we may assume that $f_1(0) = f_2(0) = 0$. In [4, Prop. 2.6 (5)] it was shown that $d_0(f_1 + f_2) \geq d_0(f_1) - |f_2|_0$, and so $d_0(f_1 + f_2) > 0$. It remains to prove that $\nu_0(f_1 + f_2) > 0$ as well.

Fix k with $\alpha_0(f_2) < k < \nu_0(f_1)$. Since the map $r \mapsto \nu(f|_{B_r})$ is monotonically decreasing, we find some $r_1 > 0$ such that $\nu(f_1|_{B_r}) > k$ for $r \leq r_1$. Likewise, since the map $r \mapsto \alpha(f|_{B_r})$ is monotonically increasing, we find some $r_2 > 0$ such that $\alpha(f_2|_{B_r}) < k$ for $r \leq r_2$. Finally, the inequality $|f_2|_0 < d_0(f_1)$ implies that

$$\sup_{\|x\|=r} \|f_2(x)\| < \inf_{\|x\|=r} \|f_1(x)\|$$

for sufficiently small $r > 0$, i.e., for $0 < r \leq r_0$ and some suitable $r_0 > 0$. So for $0 < r \leq \min\{r_0, r_1, r_2\}$ we have

$$\nu((f_1 + f_2)|_{B_r}) \geq k - \alpha(f_2|_{B_r}) \geq k - \alpha(f_2|_{B_{r_2}}),$$

by Lemma 7.4 in [3]. But this implies that also $\nu_0(f_1 + f_2) \geq k - \alpha(f_2|_{B_{r_2}}) > 0$, and so $f_1 + f_2$ is in fact cfv-regular. □

Proof of Theorem 3.5. Given $\lambda \in \mathbb{K}\backslash\sigma_{cfv}(f)$, we know that $d_0(\lambda I - f) > 0$ and $\nu_0(\lambda I - f) > 0$. Choose $\mu \in \mathbb{K}$ with $|\lambda - \mu| < \min\{d_0(\lambda I - f), \nu_0(\lambda I - f)\}$. Then all hypotheses of Lemma 3.6 are fulfilled for $f_1 := \lambda I - f$ and $f_2 := (\mu - \lambda)I$, and so $f_1 + f_2 = \mu I - f$ is cfv-regular. Consequently, $\mu \in \mathbb{K} \setminus \sigma_{cfv}(f)$ which implies that $\mathbb{K} \setminus \sigma_{cfv}(f)$ is open and $\sigma_{cfv}(f)$ is closed.

The fact that $\sigma_{cfv}(f)$ is bounded, hence compact, in case $\alpha_0(f) < \infty$ and $|f|_0 < \infty$ follows from the fact that the spectrum $\sigma_{CFV}(f)$ is bounded under these conditions, and the second inclusion in (3.5). □

We remark that Lemma 3.6 has a consequence which seems to be of independent interest. Suppose that $f, g : X \to X$ are two maps satisfying $|f - g|_0 = 0$ and $\alpha_0(f - g) = 0$; then

$$\sigma_{cfv}(f) = \sigma_{cfv}(g). \tag{3.7}$$

In fact, to see this it suffices to apply Lemma 3.6 to $f_1 := \lambda I - f$ and $f_2 := f - g$ with $\lambda \notin \sigma_{cfv}(f)$. The equality (3.7) in turn implies that the small Calamai-Furi-Vignoli spectrum shares an important property with the large Calamai-Furi-Vignoli spectrum (cf. [4, Remark 4.10]).

Theorem 3.7. *If f is Fréchet differentiable at 0, then the equality $\sigma_{cfv}(f) = \sigma(f'(0))$ holds, where $\sigma(f'(0))$ is the familiar spectrum of the linear operator $f'(0)$.*

The proof of Theorem 3.7 follows immediately from (3.7) choosing $g = f'(0)$. Of course, an analogous result holds for the small Calamai-Furi-Vignoli spectrum at an arbitrary point $p \in X$: if f is Fréchet differentiable at p, then this spectrum coincides with $\sigma(f'(p))$.

Now we give two examples, the first one being elementary, the second one (taken from the thesis [19]) quite sophisticated, which show that the converse of Proposition 3.4 is not true.

Example 3.8. Let $f : \mathbb{R} \to \mathbb{R}$ be defined by $f(x) := x \sin \frac{1}{x^2}$. Then f is locally v-regular, but not cfv-regular, since $d_0(f) = 0$.

Before stating our main example, we formulate and prove an auxiliary result about the continuity and compactness behaviour of a special class of operators in the space $C[0, 1]$.

Lemma 3.9. *Let $X = C[0, 1]$ be equipped with the usual maximum norm, let $K \subset X$ be an arbitrary precompact set consisting of continuous functions $\phi : [0, 1] \to [0, 1]$, and let $\Phi : X \to K$ be a map which associates to each $x \in X$ a function $\phi_x \in K$. Suppose that the operator $f : X \to X$ defined by $f(x)(s) = x(\phi_x(s))$, is continuous. Then f is α-nonexpansive, i.e., $\alpha(f(M)) \leq \alpha(M)$ for each bounded subset M of X.*

Proof: Observe first that the continuity of Φ at some point $x_0 \in X$ implies the continuity of f at this point. Given $\varepsilon > 0$, choose $\delta > 0$ such that $|s-t| \le \delta$ implies $|x_0(s) - x_0(t)| \le \varepsilon$, which is possible by the uniform continuity of x_0 on $[0,1]$. Since K is precompact, by assumption, we find $\phi_1, \ldots, \phi_m \in K$ such that $K \subset \overline{B}_\delta(\phi_1) \cup \ldots \cup \overline{B}_\delta(\phi_m)$. Fix $r > 0$ and choose any $x \in \overline{B}_r(x_0)$; since $\Phi(X) \subseteq K$ we have $\Phi(x) = \phi_x \in \overline{B}_\delta(\phi_j)$ for some $j \in \{1, 2, \ldots, m\}$. Using the inequality

$$|x(\phi_x(t)) - x_0(\phi_j(t))| \le |x(\phi_x(t)) - x_0(\phi_x(t))| + |x_0(\phi_x(t)) - x_0(\phi_j(t))|$$

and taking norms we arrive at

$$\|f(x) - x_0 \circ \phi_j\| \le \|x \circ \phi_x - x_0 \circ \phi_x\| + \|x_0 \circ \phi_x - x_0 \circ \phi_j\| \le \|x - x_0\| + \varepsilon \le r + \varepsilon,$$

where we have used the fact that ϕ_x maps the interval $[0,1]$ into itself and that $\|\phi_x - \phi_j\| \le \delta$. Since $x \in \overline{B}_r(x_0)$ was arbitrary, we have proved that

$$f(\overline{B}_r(x_0)) \subseteq \bigcup_{j=1}^{m} \overline{B}_{r+\varepsilon}(x_0 \circ \phi_j),$$

which shows that f is indeed α-nonexpansive as claimed. □

Example 3.10. Let $X = C[0,1]$ be equipped with the usual maximum norm. For any $x \in X \setminus \{0\}$ we put $\gamma_x := \|Px\|/\|x\|$, where P denotes the projection

$$Px(t) := \begin{cases} x(\frac{1}{2}) & \text{if} \quad 0 \le t \le \frac{1}{2}, \\ x(t) & \text{if} \quad \frac{1}{2} \le t \le 1. \end{cases}$$

To apply Lemma 3.9 we take as K the set of all Lipschitz continuous functions $\phi : [0,1] \to [0,1]$ with Lipschitz constant ≤ 1 and define $\Phi : X \setminus \{0\} \to K$ by

$$\Phi(x)(s) = \phi_x(s) := \begin{cases} s & \text{if } s \le \frac{1}{2}, \\ \frac{1}{2}s + \frac{1}{4} & \text{if } s \ge \frac{1}{2} \text{ and } \gamma_x \le \frac{1}{2}, \\ \gamma_x s + \frac{1-\gamma_x}{2} & \text{if } \gamma_x \ge \frac{1}{2} \end{cases} \tag{3.8}$$

and $\Phi(0) = 0$. A straightforward calculation shows that ϕ_x admits a continuous left inverse $\psi_x : [0,1] \to [0,1]$ which is given by

$$\psi_x(t) := \begin{cases} t & \text{if } t \le \frac{1}{2}, \\ 2t - \frac{1}{2} & \text{if } \frac{1}{2} \le t \le \frac{3}{4} \text{ and } \gamma_x \le \frac{1}{2}, \\ \frac{1}{\gamma_x}t - \frac{1}{2\gamma_x} + \frac{1}{2} & \text{if } \frac{1}{2} \le t \le \frac{\gamma_x+1}{2} \text{ and } \gamma_x \ge \frac{1}{2}, \\ 1 & \text{otherwise.} \end{cases} \tag{3.9}$$

Now we define an operator $f : X \to X$ as in Lemma 3.9, i.e.,

$$f(x)(s) := \begin{cases} x(\phi_x(s)) & \text{if} \quad x \ne 0, \\ 0 & \text{if} \quad x = 0. \end{cases}$$

Since $\gamma_{\tau x} \equiv \gamma_x$, hence $\phi_{\tau x} = \phi_x$, we conclude that

$$f(\tau x) = \tau f(x) \qquad (\tau > 0), \tag{3.10}$$

which means that f is *homogeneous* (of degree 1). Moreover, f is bounded, continuous, and even a weak isometry in the sense that $\|f(x)\| = \|x\|$ for all $x \in X$, which implies that $d_0(f) = 1$. Finally, from Lemma 3.9 we conclude that $\alpha(f) \leq 1$.

However, f is not proper on closed bounded sets. To see this, observe that the (noncompact) set M of all functions $x \in X$ which satisfy $x(s) := 1 - 2s$ for $0 \leq s \leq \frac{3}{4}$, but take arbitrary values $x(s) \in [-\frac{1}{2}, \frac{1}{2}]$ for $\frac{3}{4} \leq s \leq 1$, are all mapped by f into the single function $y = f(x)$ given by $y(t) = 1 - 2t$ for $0 \leq t \leq \frac{1}{2}$ and $y(t) = \frac{1}{2} - t$ for $\frac{1}{2} \leq t \leq 1$. Consequently, we have $\omega_0(f) = 0$ in this example; in particular, f cannot be CFV-regular.

To show that f is cfv-regular, i.e., $\nu_0(f) > 0$, requires a more careful scrutiny. We claim that f is surjective. In fact, given $y \in X$ we define $x \in X$ by $x(t) := y(\psi_y(t))$. Since ψ_y leaves the intervals $[0, \frac{1}{2}]$ and $[0, 1]$ invariant, we have

$$\gamma_x = \frac{\max\left\{|y(\psi_y(t))| : 0 \leq t \leq \frac{1}{2}\right\}}{\max\left\{|y(\psi_y(t))| : 0 \leq t \leq 1\right\}} = \frac{\max\left\{|y(s)| : 0 \leq s \leq \frac{1}{2}\right\}}{\max\left\{|y(s)| : 0 \leq s \leq 1\right\}} = \gamma_y,$$

and so also $\psi_x = \psi_y$. Consequently, for $0 \leq s \leq 1$ we obtain

$$f(x)(s) = x(\phi_x(s)) = y(\psi_y(\phi_x(s))) = y(\psi_x(\phi_x(s))) = y(s)$$

as claimed. Being surjective, the map f admits a right inverse g, which means that $f(g(y)) = y$ for all $y \in X$; moreover, $\|g(y)\| = \|y\|$. Observe that the map g may be explicitly calculated, in fact, $g(y) = y \circ \psi_y$, with ψ_x given by (3.9). To see this, we use the fact that $\phi_{g(y)} = \phi_y$ and obtain

$$f(g(y)) = f(y \circ \psi_y) = y \circ \psi_y \circ \phi_{g(y)} = y \circ \psi_y \circ \phi_y = y,$$

since ψ_y is left inverse to ϕ_y. From the definition (3.9) it follows that the set $\{\psi_x : x \in X\}$ consists entirely of functions with Lipschitz constant ≤ 2, and so this set is precompact, by the classical Arzelà-Ascoli criterion. From Lemma 3.9 we conclude that $\alpha(g) \leq 1$. But this implies that $\nu(f; B_r) \geq 1$ for $r > 0$ which may be seen as follows.

Let $h : \overline{B}_r \to X$ be a map satisfying $\alpha(h) < 1$ and $h(x) \equiv 0$ for $x \in S_r$. Putting $x = g(y)$ we obtain $h(g(y)) = 0$ for $y \in S_r$, since g is a weak isometry. This shows that the map $h \circ g$ maps the closed ball $\overline{B}_r$ into itself and satisfies $\alpha(h \circ g) \leq \alpha(h)\alpha(g) < 1$. From Darbo's fixed point theorem [5] it follows that there is some $\hat{y} \in B_r$ with $\hat{y} = h(g(\hat{y}))$. Putting now $\hat{x} := g(\hat{y})$ we see that $f(\hat{x}) = f(g(\hat{y})) = \hat{y} = h(g(\hat{y})) = h(\hat{x})$, which shows that $\hat{x}$ satisfies the coincidence equation (1.5). We conclude that $\nu(f; B_r) \geq 1$, hence $\nu_0(f) \geq 1 > 0$, and so f is indeed cfv-regular. □

4. Some interconnections and examples. In this section we illustrate our abstract results with some more examples and indicate possible applications. First of all, we summarize Propositions 2.5, 3.1, and 3.4 with the following Table 1.

CFV-regular	$\Rightarrow$	locally V-regular	$\Rightarrow$	V-regular
$\Downarrow$		$\Downarrow$		$\Downarrow$
cfv-regular	$\Rightarrow$	locally v-regular	$\Rightarrow$	v-regular

Table 1: A comparison of regularity properties

Observe that every locally V-regular [resp. v-regular] map f satisfying $d_0(f) > 0$ is CFV-regular [resp. cfv-regular]; thus, for maps with $d_0(f) > 0$ some horizontal implications in Table 1 are actually equivalences. Similarly, every cfv-regular [resp. locally v-regular resp. v-regular] map f satisfying $\omega(f) > 0$ is CFV-regular [resp. locally V-regular resp. V-regular]; consequently, for maps with $\omega(f) > 0$ all vertical implications in Table 1 are actually equivalences. In particular, the "column regularities" are equivalent in finite dimensional spaces, and the "row regularities" are equivalent for homogeneous operators f satisfying $d_0(f) > 0$.

Our counterexamples considered so far show that none of the implications in Table 1 may in general be reversed. Example 2.2 shows that V-regularity does not imply local V-regularity, and v-regularity does not imply local v-regularity either. Example 2.3 shows that local V-regularity does not imply CFV-regularity, while Example 3.8 shows that local v-regularity does not imply cfv-regularity. Finally, Example 3.10 shows that cfv-regularity does not imply CFV-regularity, and local v-regularity does not imply local V-regularity either. It remains to show that v-regularity does not imply V-regularity; for the sake of completeness and the reader's ease we briefly recall (without proof) a corresponding counterexample, which is contained as Example 6.9 in the monograph [3].

Example 4.1. Let $X = C[0,1]$ be equipped with the usual max norm, and let $f : X \to X$ be defined by

$$f(x)(t) := \begin{cases} x(\frac{1}{2}t) & \text{if } \|x\| \le \frac{1}{2}, \\ x(\|x\|t) & \text{if } \frac{1}{2} < \|x\| < 1, \\ x(t) & \text{if } \|x\| \ge 1. \end{cases}$$

Then f maps the (noncompact) set $M := \{x \in X : \|x\| \le \frac{1}{2},\ x(t) \equiv 0 \text{ if } 0 \le t \le \frac{1}{2}\}$ into 0, and so $\omega(f) = 0$, i.e., f cannot be V-regular. On the other hand, it is shown by a sophisticated homotopy argument in [3] that f is k-epi for $0 \le k < \frac{1}{4}$, and so f is in fact v-regular. □

Observe that the operator f in Example 4.1 is constructed similarly as the operator f in Example 3.10, but now by means of the auxiliary function

$$\phi_x(t) := \begin{cases} \frac{1}{2}t & \text{if } \|x\| \le \frac{1}{2}, \\ \|x\| & \text{if } \frac{1}{2} < \|x\| < 1, \\ t & \text{if } \|x\| \ge 1. \end{cases}$$

In contrast to (3.8), for this function we have $\phi_{\tau x} \ne \phi_x$ in general, and so the operator f in Example 4.1 is not homogeneous.

From all these examples we conclude that all inclusions between the spectra contained in the following Table 2 may in general be strict.

$\sigma_v(f)$	$\subseteq$	$\sigma_V(f)$
$\cap\!\vert$		$\cap\!\vert$
$\sigma_{cfv}(f)$	$\subseteq$	$\sigma_{CFV}(f)$

Table 2: A comparison of spectra

It is interesting to check what the different kinds of regularity in Table 1 mean in case of a *linear* map. The following theorem shows that, in a certain sense, these regularity definitions are rather natural.

Theorem 4.2. *For $L : X \to X$ bounded and linear, the 6 regularity properties from Table 1 are all mutually equivalent. Moreover, they are equivalent to the fact that L is an isomorphism.*

Proof: As Table 1 shows, it suffices to prove that every v-regular linear map is an isomorphism, and every isomorphism is CFV-regular.

So let first $L : X \to X$ be v-regular which means that L is strictly epi on the closure $\overline{\Omega}$ of some set $\Omega \in \mathfrak{OBC}(X)$. Then L must be injective, since otherwise $Lx = 0$ for some $x \in \partial\Omega$, contradicting the fact that L has no zeros on $\partial\Omega$. But L has to be surjective as well, since otherwise we find $y_0 \in X \setminus \{0\}$ such that $Lx \neq \lambda y_0$ for all $\lambda \neq 0$. The map $h : \overline{\Omega} \to X$ defined by $h(x) := \text{dist}(x, \partial\Omega) y_0$ is continuous and compact and vanishes on $\partial\Omega$. Since L is epi on $\overline{\Omega}$, there exists a solution $\hat{x} \in \Omega$ of the equation $Lx = h(x) = \lambda y_0$ with $\lambda = \text{dist}(\hat{x}, \partial\Omega)$. But this implies that $\lambda = 0$, hence $\hat{x} \in \partial\Omega$, contradicting the fact that $\hat{x} \in \Omega$. So we have proved that L is a bijection, hence a linear isomorphism, by the closed graph theorem (or the bounded inverse theorem).

The fact that every linear isomorphism is CFV-regular has been proved in [4, Prop. 4.3]. □

From Theorem 4.2 it follows, in particular, that all spectra given in Table 2 reduce to the familiar spectrum in the case of a bounded linear operator. This is of course precisely what one should expect in nonlinear spectral theory.

In view of applications it is interesting to see which of the equivalences in Theorem 4.2 still hold for homogeneous operators, see (3.10). As observed before, in this case CFV-regularity, local V-regularity and V-regularity are mutually equivalent, and so are cfv-regularity, local v-regularity and v-regularity. Moreover, since in the proof of surjectivity of a linear v-regular map L in Theorem 4.2 we did not use the additivity of L, the same holds for homogeneous maps. So in the homogeneous case we have

$$\sigma_V(f) = \sigma_{CFV}(f), \quad \sigma_v(f) = \sigma_{cfv}(f), \tag{4.1}$$

while Example 3.10 shows that even in the homogeneous case the spectra $\sigma_{CFV}(f)$ and $\sigma_{cfv}(f)$ may be different. Of course, in case of a homogeneous operator f, the condition $N(f; r) = \{0\}$, see (1.11), is *independent* of $r > 0$. As was shown in [4, Prop. 2.9], in case of a homogeneous map f which satisfies $\omega(f) > 0$, this condition is actually equivalent to the apparently stronger condition $d(f) > 0$. On the other hand, the simple example $f(x) := \|x\|e$, with some fixed element $e \in S_1$, shows that a homogeneous map f may satisfy $N(f; r) \equiv \{0\}$ and $d(f) > 0$, but have the property that $\omega(f) = 0$. We shall return to this map in Example 4.5 below.

Here are some more examples, some of them in the scalar case $X = \mathbb{R}$ or $X = \mathbb{C}$, the last one in an arbitrary infinite dimensional Banach space.

Example 4.3. Let $f, g : \mathbb{R} \to \mathbb{R}$ be defined by

$$f(x) := \sqrt{|x|}, \qquad g(x) := (\text{sign}\, x)\sqrt{|x|}, \tag{4.2}$$

respectively. Then

$$\sigma_{CFV}(f) = \sigma_{cfv}(f) = \mathbb{R}, \quad \sigma_V(f) = \sigma_v(f) = \{0\},$$

but

$$\sigma_{CFV}(g) = \sigma_{cfv}(g) = \sigma_V(g) = \sigma_v(g) = \emptyset.$$

So even a small modification of the map may change its spectra drastically. □

Example 4.4. Let $f : \mathbb{C} \to \mathbb{C}$ be defined by $f(z) := |z| + i \operatorname{Im} z$. In [4] it is shown that $\sigma_{CFV}(f)$ is the "kidney shaped" region bounded by the closed curve $\Gamma := \{a + bi : (a, b) \in \mathbb{R}^2, (a-1)^2+b^2 = (a^2+b^2-a)^2\}$ in the complex plane (see Fig. 1 in [4]). Since f is a homogeneous map in a finite dimensional space, from our observations after Table 1 we conclude that all four spectra from Table 2 coincide in this case. □

Example 4.5. Let X be any infinite dimensional real Banach space, and let $f : X \to X$ and $g : X \to X$ be defined by

$$f(x) = \|x\|x, \qquad g(x) = \|x\|e, \tag{4.3}$$

where $e \in S_1$ is some fixed element. From (1.10) it follows that $\lambda = 0$ belongs to all spectra given in Table 2. In fact, in [3] it is shown that $\sigma_V(f) = \sigma_v(f) = \{0\}$. Since f admits a Fréchet derivative at zero, by [4, Cor. 4.23] and Theorem 3.7 we have $\sigma_{CFV}(f) = \sigma_{cfv}(f) = \sigma(f'(0)) = \{0\}$ as well.

Concerning the map g, we cannot apply Corollary 4.23 of [4], since g is not differentiable at zero. On the other hand, in contrast to f, the map g is compact which makes some calculations easier. For instance, it is clear that $\omega(\lambda I - g) = |\lambda|$, and so $\lambda = 0$ again belongs to the spectra $\sigma_V(g)$ and $\sigma_v(g)$. More precisely, one can show that

$$\sigma_V(g) = \sigma_v(g) = \sigma_{CFV}(g) = \sigma_{cfv}(g) = [-1, 1]. \tag{4.4}$$

By Table 2, to prove this it suffices to show that $[-1, 1] \subseteq \sigma_v(g)$ and $\sigma_{CFV}(g) \subseteq [-1, 1]$. For $r > 0$, consider the map $h : \overline{B}_r \to X$ defined by $h(x) := (r - \|x\|)e$. Obviously, this map is compact and vanishes on S_r. If $\hat{x} \in B_r$ is a solution of the coincidence equation $\lambda x - g(x) = h(x)$, then $\lambda\hat{x} = re$, and taking norms we obtain

$$r = \|re\| = \|\lambda\hat{x}\| < |\lambda|r,$$

which yields a contradiction for $|\lambda| \leq 1$. So $\lambda I - g$ cannot be epi and, in particular, $\lambda \in \sigma_v(g)$ for $\lambda \in [-1, 1]$. On the other hand, $\omega_0(\lambda I - g) = |\lambda| > 0$ for $\lambda \neq 0$, and $d_0(\lambda I - g) = |\lambda| - 1 > 0$ for $\lambda \neq [-1, 1]$. Now let $h : \overline{B}_r \to X$ be an arbitrary compact map satisfying $h(x) \equiv 0$ on S_r. Since the map λI is epi on $\overline{B}_r$, by Schauder's fixed point theorem, and for $\|x\| = r$ and $|\lambda| > 1$ we have the estimate

$$\|h(x) + \|x\|e\| = r < |\lambda|r = \|\lambda x\|,$$

from [4, Cor. 3.4] we conclude that $\lambda I - g$ is epi on $\overline{B}_r$ for $|\lambda| > 1$, and so $\lambda \notin \sigma_{CFV}(g)$.

Of course, if X is a complex Banach space, the same reasoning shows that (4.4) holds with the intervall $[-1, 1]$ replaced by the closed unit disc $\{z \in \mathbb{C} : |z| \leq 1\}$. □

One could also ask if our results on homogeneous maps still hold true for the larger class of *θ-homogeneous maps* $f : X \to X$, i.e., maps which satisfy instead of (3.10) the more general condition

$$f(\tau x) = \tau^\theta f(x) \qquad (\tau > 0) \tag{4.5}$$

for some $\theta > 0$. The following example shows that this is not true.

Example 4.6. Given an infinite dimensional Banach space X and $\theta > 0$, consider the map $f : X \to X$ defined by

$$f(x) = \|x\|^{\theta-1}x. \tag{4.6}$$

Obviously, this map is homogeneous of degree θ, i.e., satisfies (4.5). Like the map (1.8) (which is obtained from (4.6) for the special choice $\theta = 2$), also the map (4.6) is a *homeomorphism* on X with continuous inverse

$$f^{-1}(y) = \begin{cases} \|y\|^{-(\theta-1)/\theta} y & \text{if } y \neq 0, \\ 0 & \text{if } y = 0, \end{cases} \tag{4.7}$$

which is homogeneous of degree $1/\theta$. Now, since f is a bijection between the sphere S_r and the sphere S_{r^θ}, in case $\theta > 1$ the equalities (1.10) still hold. In case $0 < \theta < 1$ we have

$$\omega(f) = |f| = d(f) = 0, \quad \alpha(f) = \alpha_0(f) = \omega_0(f) = |f|_0 = d_0(f) = \infty, \tag{4.8}$$

while in case $\theta = 1$ the map f is simply the identity, and so all characteristics are equal to 1. We claim that

$$\sigma_V(f) = \sigma_v(f) = \sigma_{CFV}(f) = \sigma_{cfv}(f) = \begin{cases} \{0\} & \text{if } \theta > 1, \\ \emptyset & \text{if } \theta < 1 \end{cases} \tag{4.9}$$

for the map (4.6). Indeed, since f is not v-regular in case $\theta \neq 1$, we have $\{0\} \subseteq \sigma_v(f)$. On the other hand, for $\lambda \neq 0$ the restriction of the map $\lambda I - f$ to the ball $B_{|\lambda|/2}$ is open, injective, epi, and satisfies $d_0(\lambda I - f) = \omega_0(\lambda I - f) > 0$. So $\lambda I - f$ is CFV-regular for $\lambda \neq 0$ which shows that $\sigma_{CFV}(f) \subseteq \{0\}$. From Table 2 we conclude that the first chain of equalities in (4.9) is true. Observe that for $\theta > 1$ we could have deduced the equality $\sigma_{CFV}(f) = \sigma_{cfv}(f) = \{0\}$ also from the fact that f admits the Fréchet derivative $f'(0) = 0$ and use Theorem 3.7. The second chain of equalities in (4.9) follows from (4.8). □

5. Concluding remarks. To conclude, let us make some comments on eigenvalues. As was pointed out in [3], whenever one tries to define a new spectrum for some class of nonlinear operators, it is a useful device to introduce a corresponding notion of eigenvalues, i.e., a point spectrum or an approximate point spectrum. For example, in [4] the authors introduce and study the subset $\sigma_{CFV}^o(f)$ of $\sigma_{CFV}(f)$ consisting of all scalars $\lambda \in \mathbb{K}$ satisfying $d_0(\lambda I - f) = 0$ or $\omega_0(\lambda I - f) = 0$, and call this set the *approximate point spectrum of* f (at zero). This name is motivated and justified by the fact that, in case of a bounded linear operator L, $\sigma_{CFV}^o(L)$ clearly coincides with the familiar approximate point spectrum $\sigma_{ap}(L)$ of all $\lambda \in \mathbb{K}$ for which there exists a sequence $(x_n)_n$ in $S_1(X)$ such that $\|\lambda x_n - L x_n\| \to 0$ as $n \to \infty$.

An appropriate notion of eigenvalues associated with our spectrum (3.4) seems to be the following. Given a map $f : X \to X$ with $f(0) = 0$, let us call the subset $\sigma_{cfv}^o(f)$ of $\sigma_{cfv}(f)$ consisting of all scalars $\lambda \in \mathbb{K}$ satisfying $d_0(\lambda I - f) = 0$, the *point spectrum of* f (at zero). So by definition we have $\lambda \in \sigma_{cfv}^o(f)$ if and only if

$$\liminf_{\|x\| \to 0} \frac{\|\lambda x - f(x)\|}{\|x\|} = 0.$$

Now, in rather the same way as in [3, Th. 8.10], one may show then that every $\lambda \in \sigma_{cfv}(f) \setminus \{0\}$ actually belongs to $\sigma_{cfv}^o(f)$, provided that f is *compact, odd* (i.e., $f(-x) = -f(x)$), and θ-homogeneous (i.e., satisfies (4.5) for some $\theta > 0$). This generalizes the well-known result that every nonzero spectral value of a compact linear operator is an eigenvalue.

We briefly sketch how this may be applied to the eigenvalue problem with Dirichlet boundary condition of the form

$$\begin{cases} -\Delta_p u(x) = \mu|u(x)|^{p-2}u(x) & \text{in } G, \\ u(x) \equiv 0 & \text{on } \partial G. \end{cases} \tag{5.1}$$

Here Δ_p denotes, for $1 < p < \infty$, the so-called *p-Laplace operator* defined by

$$\Delta_p u = \operatorname{div}(|\nabla u|^{p-2}\nabla u), \tag{5.2}$$

$G \subset \mathbb{R}^N$ is a sufficiently regular bounded domain, and $\mu \in \mathbb{R} \setminus \{0\}$ is a scalar. The eigenvalue problem (5.1) consists in finding those scalars μ for which there exists a nontrivial solution u and arises in many fields of applied mathematics and mechanics, see e.g. [14]. Of course, in case $p = 2$ this problem just reduces to the classical *linear* eigenvalue problem for the Laplace operator $-\Delta$ which has been studied over and over in the last 150 years.

It is well-known that the operator (5.2) acts from the Sobolev space $X = W_0^{1,p}(G)$ to its dual, $X^* = W^{-1,p'}(G)$, where $p' = p/(p-1)$. More precisely, if we denote by J the differential operator defined by $-\Delta_p$ in the weak form, i.e.,

$$\langle Ju, v\rangle = \int_\Omega |\nabla u(x)|^{p-2}\nabla u(x)\nabla v(x)\,dx \qquad (u, v \in W_0^{1,p}(G)), \tag{5.3}$$

and by F the Nemytskij operator generated by the nonlinearity on the right hand side of (2), also in weak form, i.e.,

$$\langle Fu, v\rangle = \int_\Omega |u(x)|^{p-2}u(x)v(x)\,dx \qquad (u, v \in W_0^{1,p}(G)), \tag{5.4}$$

we obtain two operators acting from X to its dual X^*. Now, since the operator (5.3) is continuous, coercive, and strictly monotone between X and X^*, from Minty's celebrated existence theorem for monotone operators ([16], see also [24]) it follows that J is a homeomorphism. Consequently, the eigenvalue problem (5.1) may be rewritten, for $\mu \neq 0$ and $\lambda = 1/\mu$, equivalently as operator equation

$$f(u) := J^{-1}(F(u)) = \lambda u. \tag{5.5}$$

Since both operators (5.3) and (5.4) are odd and θ-homogeneous for $\theta = p-1$, i.e., $J(\tau u) = \tau^{p-1}J(u)$ and $F(\tau u) = \tau^{p-1}F(u)$, the map f defined in (5.5) is homogeneous of degree 1, i.e., fulfills (3.10). Moreover, from Krasnosel'skij's theorem on Nemytskij operators between Lebesgue spaces [13] and standard imbedding theorems for Sobolev spaces it follows that the operator (5.4) is also compact.

So by what we have observed above, every nonzero spectral value of f is actually an eigenvalue or, vice versa, any $\mu \neq 0$ for which we have uniqueness in (5.1) gives rise to a scalar which does not belong to any of the spectra in Table 2. In this way, one may obtain some kind of nonlinear Fredholm alternative for the problem (5.1) in the spirit of [6-8], but with stronger regularity. Details will be given in a subsequent paper.

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