

Sobolev embeddings and concentration-compactness alternative for fractional Sobolev spaces

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(with [A. Pisante](#)) Sobolev embeddings and concentration-compactness alternative
for fractional Sobolev spaces, *submitted paper*, 2010.

1. Introduction
2. Concentration-compactness alternative
3. Final remarks

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Problems involving the fractional powers of the Laplacian

Let $s > 0$.

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- Natural spaces

$H_0^s(\mathbb{R}^N)$, the completion of $C_0^\infty(\mathbb{R}^N)$ w.r.t. the norm

$$\|u\|_{H_0^s}^2 = \|(-\Delta)^{\frac{s}{2}} u\|_{L^2}^2 = \int_{\mathbb{R}^N} |\xi|^{2s} |\hat{u}(\xi)|^2 d\xi.$$

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The fractional powers of the Laplacian are experiencing impressive applications in different subjects:

thin obstacle problems ([Silvestre 2007](#), [Milakis-Silvestre 2008](#))

financial market problems ([Cont-Tankov 2004](#))

phase transitions ([Alberti et al. 1998](#), [Cabr -SolaMorales 2005](#), [Sire-Valdinoci 2009](#), [Farina et al. 2011](#))

water waves ([Stoker 1957](#), [Whitham 1974](#), [Craig-Nicholls 2004](#), [De La Llave-Valdinoci 2009](#))

dislocations in crystals ([Toland 1997](#), [Gonzalez-Monneau 2011](#))

soft thin films ([Kurzke 2006](#))

semipermeable membranes and flame propagation ([Caffarelli-Mellet-Sire 2011](#))

quasi-geostrophic flows ([Majda-Takab 1996](#), [Cordoba 1998](#), [Caffarelli-Vasseur 2010](#))

minimal surfaces ([Caffarelli-Roquejoffre-Savin 2010](#), [Caffarelli-Valdinoci 2011](#))

anomalous diffusion ([Metzler-Klafter 2000](#))

ultra-relativistic limits of quantum mechanics ([Fefferman-De La Llave 1986](#))

multiple scattering ([Duistermaat-Guillemin 1975](#), [Colton-Cress 1998](#), [Grote-Kirsch 2004](#))

etc...

Fractional Sobolev embeddings

If $0 < s < N/2$ and $2^* = 2N/(N - 2s)$, the Sobolev critical exponent, the following Sobolev inequality is valid for some positive constant $S^* = S^*(N, s)$

$$(\star) \quad \|u\|_{L^{2^*}(\mathbb{R}^N)}^{2^*} \leq S^* \|(-\Delta)^{\frac{s}{2}} u\|_{L^2(\mathbb{R}^N)}^{2^*} \quad \forall u \in H_0^s(\mathbb{R}^N).$$

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Cotsiolis-Tavoularis (2004): (\star) is attained iff $u(x) = \frac{c}{(\lambda^2 + |x - x_0|^2)^{\frac{N-2s}{2}}} \quad \forall x \in \mathbb{R}^N$,
where $c \in \mathbb{R} \setminus \{0\}$, $\lambda > 0$ and $x_0 \in \mathbb{R}^N$ are fixed constants.

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$\Omega \subset \mathbb{R}^N$ bounded open set

$$S_\Omega^* := \sup \left\{ F_\Omega(u) : u \in H_0^s(\Omega), \int_{\mathbb{R}^N} |(-\Delta)^{\frac{s}{2}} u|^2 dx \leq 1 \right\} \quad \text{with} \quad F_\Omega(u) := \int_{\Omega} |u|^{2^*} dx.$$

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Theorem 1 [G.P., A. Pisante, 2010]

Let (u_n) be a sequence in $H_0^s(\Omega)$ weakly converging to u such that

$$|(-\Delta)^{\frac{s}{2}} u_n|^2 dx \xrightarrow{*} \mu \quad \text{and} \quad |u_n|^{2^*} dx \xrightarrow{*} \nu \quad \text{in } \mathcal{M}(\mathbb{R}^N).$$

Then, either $u_n \rightarrow u$ in $L_{\text{loc}}^{2^*}(\mathbb{R}^N)$ or there exists a finite set of distinct points x_1, \dots, x_k in $\overline{\Omega}$ and positive numbers ν_1, \dots, ν_k such that we have

$$\nu = |u|^{2^*} dx + \sum_{j=1}^k \nu_j \delta_{x_j}, \quad (S^*)^{1 - \frac{2^*}{2}} \leq \nu_j.$$

If in addition Ω is bounded, there exist a positive measure $\tilde{\mu} \in \mathcal{M}(\mathbb{R}^N)$ with $\text{spt } \tilde{\mu} \subset \overline{\Omega}$ and positive numbers μ_1, \dots, μ_k such that

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Concentration-compactness alternative for fractional Sobolev spaces

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$s = 1, 2m$ Standard C-C-A P.L. Lions (1985)

Corollary 1 (concentration of the optimizing sequences)

$\Omega \subset \mathbb{R}^N$ bounded open set.

Corollary 1

Let $(u_n) \in H_0^s(\Omega)$ be a maximizing sequence for the critical Sobolev inequality

$$\|u\|_{L^{2^*}(\Omega)}^{2^*} \leq S^* \|(-\Delta)^{\frac{s}{2}} u\|_{L^2(\Omega)}^{2^*}.$$

Then (u_n) concentrates at one point $x_0 \in \overline{\Omega}$.

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We have $\int_{\Omega} |u_n|^{2^*} dx \rightarrow S^*$

and so $|u_n|^{2^*} dx \xrightarrow{*} \nu \in \mathcal{M}(\mathbb{R}^N)$ with $\nu(\Omega) = S^*$.

$$\mu = |(-\Delta)^{\frac{s}{2}} u|^2 + \tilde{\mu} + \sum_{i=0}^{\infty} \mu_i \delta_{x_i}$$

We have $S^* = \nu(\Omega)$

Corollary 1 (concentration of the optimizing sequences) - proof

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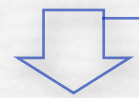
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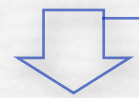
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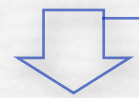
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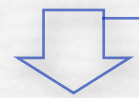
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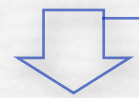
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Sobolev inequality is not attained on bounded domains $\Rightarrow u$ is zero.

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The function $t \mapsto t^{\frac{2^*}{2}}$ is strictly convex \Rightarrow Only one of the μ_i 's can be nonzero.

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Corollary 1 (concentration of the optimizing sequences) - proof

$$\underline{\mu = \delta_{x_i}}$$

We have $S^* = \nu(\Omega)$

Theorem 1 (C-C-A)

$$= \int_{\Omega} |u|^{2^*} dx + \sum_{i \in I} \nu_i$$

Sobolev inequality + C-C-A

$$\leq S^* \left(\int_{\Omega} |(-\Delta)^{\frac{s}{2}} u|^2 dx \right)^{\frac{2^*}{2}} + S^* \sum_{i \in I} \mu_i^{\frac{2^*}{2}}$$

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Let $\Omega \subset \mathbb{R}^N$ a bounded open set and let $\varphi \in C_0^\infty(\mathbb{R}^N)$. Then

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whenever $u_n \rightharpoonup 0$ in $H_0^s(\Omega)$ as $n \rightarrow \infty$,

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By conjugation with Fourier transform, we have

$$Lu = \mathcal{F}^{-1} \circ M_{|\xi|^s} \circ \mathcal{F}(u), \quad L_\varepsilon u = \mathcal{F}^{-1} \circ M_{(|\xi|^2 + \varepsilon)^{s/2}} \circ \mathcal{F}(u).$$

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Estimating the norm in $\mathcal{L}(H^s, L^2)$

$$\|L_\varepsilon - L\| \leq \sup_{\xi} \frac{|(\varepsilon + |\xi|^2)^{s/2} - |\xi|^s|}{(1 + |\xi|^2)^{\frac{s}{2}}} \xrightarrow{\varepsilon \rightarrow 0} 0.$$

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Hence $L_\varepsilon \in OP\mathcal{BS}_{1,1}^s$ and, since $0 < s < \frac{N}{2}$, according to [Taylor \(2002\)](#), we have the following commutator estimate

$$\|[L_\varepsilon, \varphi]u\|_{L^2(\mathbb{R}^N)} \leq C \|\varphi\|_{H^\sigma(\mathbb{R}^N)} \|u\|_{H^{s-1}(\mathbb{R}^N)}$$

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Since $\varphi \in C_0^\infty(\mathbb{R}^N)$ and the embedding $H_0^s(\Omega) \hookrightarrow H^{s-1}(\mathbb{R}^N)$ is compact for all $s \in (0, \frac{N}{2})$, we conclude that $[L_\varepsilon, \varphi] : H_0^s(\Omega) \rightarrow L^2(\mathbb{R}^N)$ is compact. \square

Subcritical approximation

$\Omega \subset \mathbb{R}^N$ bounded open set.

For any $0 < \varepsilon < 2^* - 2$ consider the following variational problems

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What happens when $\varepsilon \rightarrow 0$ (both to the energy functional
and to the corresponding maximizers u_{ε}) ?

Nonlinear elliptic equations involving critical Sobolev exponent

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$$F(u, \mu) = \int_{\Omega} |u|^{2^*} dx + S^* \sum_{i=0}^{\infty} \mu_i^{\frac{2^*}{2}}.$$

Theorem 3 [G.P., A. Pisante, 2010]

As $\varepsilon \rightarrow 0$,

(i) $S_\varepsilon^* \rightarrow S^*$.

(ii) Let $u_\varepsilon \in H_0^s(\Omega)$ be a maximizer for S_ε^* . Then (up to subsequences) $u_\varepsilon \rightharpoonup 0$ in $H_0^s(\Omega)$ and it concentrates at some point $x_0 \in \overline{\Omega}$ both in L^{2^*} and in H^s , i.e.

$$|u_\varepsilon|^{2^*} dx \xrightarrow{*} S^* \delta_{x_0} \quad \text{and} \quad |(-\Delta)^{\frac{s}{2}} u_\varepsilon|^2 dx \xrightarrow{*} \delta_{x_0} \quad \text{in } \mathcal{M}(\mathbb{R}^N).$$

(iii) $\exists x_\varepsilon \rightarrow x_0$ and $\lambda_\varepsilon \searrow 0$ s. t. the function \tilde{u}_ε defined by $\tilde{u}_\varepsilon(x) = \lambda_\varepsilon^{N-2s/2} u_\varepsilon(x_\varepsilon + \lambda_\varepsilon x)$ converges to u , maximizer for S^* , in H_0^s (and in L^{2^*}).

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