Sobolev embeddings and concentration-compactness alternative for fractional Sobolev spaces

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(with A. Pisante) **Sobolev embeddings and concentration-compactness alternative for fractional Sobolev spaces,** *submitted paper*, 2010.

1. Introduction
2. Concentration-compactness alternative
3. Final remarks

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- Natural spaces

$H_0^s(\mathbb{R}^N)$, the completion of $C_0^\infty(\mathbb{R}^N)$ w.r.t. the norm

$$\|u\|_{H_0^s}^2 = \|(-\Delta)^{\frac{s}{2}} u\|_{L^2}^2 = \int_{\mathbb{R}^N} |\xi|^{2s} |\hat{u}(\xi)|^2 d\xi \,.$$

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The fractional powers of the Laplacian are experiencing impressive applications in different subjects:

thin obstacle problems (Silvestre 2007, Milakis-Silvestre 2008)

financial market problems (Cont-Tankov 2004)

phase transitions (Alberti et al. 1998, Cabré-SolaMorales 2005, Sire-Valdinoci 2009, Farina et al. 2011)

water waves (Stoker 1957, Whitham 1974, Craig-Nicholls 2004, De La lLave-Valdinoci 2009)

dislocations in crystals (Toland 1997, Gonzalez-Monneau 2011)

soft thin films (Kurzke 2006)

semipermeable membranes and flame propagation (Caffarelli-Mellet-Sire 2011)

quasi-geostrophic flows (Majda-Takab 1996, Cordoba 1998, Caffarelli-Vasseur 2010)

minimal surfaces (Caffarelli-Roquejoffre-Savin 2010, Caffarelli-Valdinoci 2011)

anomalous diffusion (Metzler-Klafter 2000)

ultra-relativistic limits of quantum mechanics (Fefferman-De La lLave 1986)

multiple scattering (Duistermaat-Guillemin 1975, Colton-Cress 1998, Grote-Kirsch 2004)

etc...

Fractional Sobolev embeddings

If $0 < s < N/2$ and $2^* = 2N/(N-2s)$, the Sobolev critical exponent,

the following Sobolev inequality is valid for some positive constant $S^* = S^*(N,s)$

$$(\star) \qquad \|u\|^{2^*}_{L^{2^*}(\mathbb{R}^N)} \leq S^* \|(-\Delta)^{\frac{s}{2}} u\|^{2^*}_{L^2(\mathbb{R}^N)} \quad \forall u \in H^s_0(\mathbb{R}^N).$$

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Cotsiolis-Tavoularis (2004): $(\star)$ is attained *iff* $u(x) = \dfrac{c}{(\lambda^2 + |x-x_0|^2)^{\frac{N-2s}{2}}} \quad \forall x \in \mathbb{R}^N,$

where $c \in \mathbb{R} \setminus \{0\}$, $\lambda > 0$ *and* $x_0 \in \mathbb{R}^N$ *are fixed constants.*

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$\Omega \subset \mathbb{R}^N$ bounded open set

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Concentration-compactness alternative for fractional Sobolev spaces

$\Omega \subseteq \mathbb{R}^N$. If $0 < s < N/2$ and $2^* = 2N/(N-2s)$,

Theorem 1 [G. P., A. Pisante, 2010]

Let (u_n) be a sequence in $H^s_0(\Omega)$ weakly converging to u such that

$$|(-\Delta)^{\frac{s}{2}} u_n|^2 dx \overset{*}{\rightharpoonup} \mu \quad \text{and} \quad |u_n|^{2^*} dx \overset{*}{\rightharpoonup} \nu \ \text{ in } \mathcal{M}(\mathbb{R}^N).$$

Then, either $u_n \to u$ in $L^{2^*}_{\mathrm{loc}}(\mathbb{R}^N)$ or there exists a finite set of distinct points $x_1, \dots, x_k$ in $\overline{\Omega}$ and positive numbers $\nu_1, \dots, \nu_k$ such that we have

$$\nu = |u|^{2^*} dx + \sum_{j=1}^{k} \nu_j \delta_{x_j}, \qquad (S^*)^{1-\frac{2^*}{2}} \leq \nu_j.$$

If in addition Ω is bounded, there exist a positive measure $\tilde{\mu} \in \mathcal{M}(\mathbb{R}^N)$ with $spt\, \tilde{\mu} \subset \overline{\Omega}$ and positive numbers $\mu_1, \dots, \mu_k$ such that

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$s = 1$, 2m Standard C-C-A P. L. Lions (1985)

Corollary 1 (concentration of the optimizing sequences)

$\Omega \subset \mathbb{R}^N$ bounded open set.

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Let $(u_n) \in H^s_0(\Omega)$ be a maximizing sequence for the critical Sobolev inequality

$$\|u\|^{2^*}_{L^{2^*}(\Omega)} \leq S^* \|(-\Delta)^{\frac{s}{2}} u\|^{2^*}_{L^2(\Omega)} .$$

Then (u_n) concentrates at one point $x_0 \in \overline{\Omega}$.

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Proof.

We want to prove that $|(-\Delta)^{\frac{s}{2}} u_n|^2 dx \overset{*}{\rightharpoonup} \delta_{x_0}$ in $\mathcal{M}(\mathbb{R}^N)$.

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We have $\displaystyle\int_\Omega |u_n|^{2^*} dx \to S^*$

and so $|u_n|^{2^*} dx \overset{*}{\rightharpoonup} \nu \in \mathcal{M}(\mathbb{R}^N)$ with $\nu(\Omega) = S^*$.

Corollary 1 (concentration of the optimizing sequences) - proof

$$\mu = |(-\Delta)^{\frac{s}{2}} u|^2 + \tilde{\mu} + \sum_{i=0}^{\infty} \mu_i \delta_{x_i}$$

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&\Downarrow \quad \text{convexity of the function } t \mapsto t^{\frac{2^*}{2}} \\
&\leq S^* \left(\int_\Omega |(-\Delta)^{\frac{s}{2}} u|^2 dx + \sum_{i \in I} \mu_i \right)^{\frac{2^*}{2}} \\
&\Downarrow \quad \int_\Omega |(-\Delta)^{\frac{s}{2}} u|^2 dx + \sum_{i \in I} \mu_i \leq \mu(\mathbb{R}^N) \leq 1 \\
&\leq S^*
\end{aligned}
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Sobolev inequality is not attained on bounded domains $\Longrightarrow$ u is zero.

The function $t \mapsto t^{\frac{2^*}{2}}$ is strictly convex $\Longrightarrow$ Only one of the μ_i's can be nonzero.

Hence, concentration occurs at one point $x_0 \in \overline{\Omega}$. $\square$

Proof of Theorem 1: a suitable tool

Lemma [G.P., A. Pisante, 2010]

Let $\Omega \subset \mathbb{R}^N$ a bounded open set and let $\varphi \in C_0^\infty(\mathbb{R}^N)$. Then

$$\varphi((-\Delta)^{s/2} u_n) - (-\Delta)^{s/2}(\varphi u_n) \to 0 \quad \text{in} \quad L^2(\mathbb{R}^N)$$

whenever $u_n \rightharpoonup 0$ in $H_0^s(\Omega)$ as $n \to \infty$,

i.e., the commutator $[\varphi, (-\Delta)^{s/2}] : H_0^s(\Omega) \to L^2(\mathbb{R}^N)$ is a compact operator.

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$$Lu = \mathcal{F}^{-1} \circ M_{|\xi|^s} \circ \mathcal{F}(u)\,, \qquad L_\varepsilon u = \mathcal{F}^{-1} \circ M_{(|\xi|^2+\varepsilon)^{s/2}} \circ \mathcal{F}(u)\,.$$

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Estimating the norm in $\mathcal{L}(H^s, L^2)$

$$\|L_\varepsilon - L\| \leq \sup_\xi \frac{|(\varepsilon + |\xi|^2)^{s/2} - |\xi|^s|}{(1+|\xi|^2)^{\frac{s}{2}}} \xrightarrow{\varepsilon \to 0} 0\,.$$

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Hence $L_\varepsilon \in OP\mathcal{B}S^s_{1,1}$ and, since $0 < s < \frac{N}{2}$, according to Taylor (2002), we have the following commutator estimate

$$\|[L_\varepsilon, \varphi]u\|_{L^2(\mathbb{R}^N)} \leq C\|\varphi\|_{H^\sigma(\mathbb{R}^N)}\|u\|_{H^{s-1}(\mathbb{R}^N)}$$

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Since $\varphi \in C_0^\infty(\mathbb{R}^N)$ and the embedding $H_0^s(\Omega) \hookrightarrow H^{s-1}(\mathbb{R}^N)$ is compact for all $s \in (0, \frac{N}{2})$, we conclude that $[L_\varepsilon, \varphi] : H_0^s(\Omega) \to L^2(\mathbb{R}^N)$ is compact. $\square$

$\Omega \subset \mathbb{R}^N$ bounded open set.

For any $0 < \varepsilon < 2^* - 2$ consider the following variational problems

$$S^*_\varepsilon := \sup\left\{ F_\varepsilon(u) : \; u \in H^s_0(\Omega), \int_{\mathbb{R}^N} |(-\Delta)^{\frac{s}{2}} u|^2 dx \leq 1 \right\}$$

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What happens when $\varepsilon \to 0$ (both to the energy functional and to the corresponding maximizers u_ε) ?

Nonlinear elliptic equations involving critical Sobolev exponent

$\Omega \subset \mathbb{R}^N, \;\; 2^* = 2N/(N-2s), \;\; 0 < \varepsilon < 2^* - 2$

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Variational approach

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Theorem 2 [G. P., A. Pisante, 2010]

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$$\Big\downarrow \Gamma^+ (w\text{–}L^{2^*}(\Omega) \times \mathcal{M}(\mathbb{R}^N))$$

$$F(u,\mu) = \int_\Omega |u|^{2^*}dx + S^* \sum_{i=0}^{\infty} \mu_i^{\frac{2^*}{2}}.$$

Theorem 3 [G. P., A. Pisante, 2010]

As $\varepsilon \to 0$,

(i) $S^*_\varepsilon \to S^*$.

(ii) Let $u_\varepsilon \in H^s_0(\Omega)$ be a maximizer for S^*_ε. Then (up to subsequences) $u_\varepsilon \rightharpoonup 0$ in $H^s_0(\Omega)$ and it concentrates at some point $x_0 \in \overline{\Omega}$ both in L^{2^*} and in H^s, i.e.

$$|u_\varepsilon|^{2^*}dx \overset{*}{\rightharpoonup} S^*\delta_{x_0} \quad \text{and} \quad |(-\Delta)^{\frac{s}{2}}u_\varepsilon|^2 dx \overset{*}{\rightharpoonup} \delta_{x_0} \text{ in } \mathcal{M}(\mathbb{R}^N).$$

(iii) $\exists\, x_\varepsilon \to x_0$ and $\lambda_\varepsilon \searrow 0$ s. t. the function $\tilde{u}_\varepsilon$ defined by $\tilde{u}_\varepsilon(x) = \lambda_\varepsilon{}^{N-2s/2}u_\varepsilon(x_\varepsilon + \lambda_\varepsilon x)$ converges to u, maximizer for S^*, in H^s_0 (and in L^{2^*}).

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Nonlinear problems involving the fractional powers of the Laplacian, *work in progress.*

Sobolev embeddings and concentration-compactness alternative for fractional Sobolev spaces

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