Nonlinear aspects of Calderón-Zygmund theory

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Overture: The standard CZ theory

- **Consider the model case**

$$\triangle u = f \qquad \text{in } \mathbb{R}^n$$

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$$f \in L^q \quad \text{implies} \quad D^2 u \in L^q \qquad 1 < q < \infty$$

with natural failure in the borderline cases $q = 1, \infty$

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- **As a consequence (Sobolev embedding)**

$$Du \in L^{\frac{nq}{n-q}} \qquad q < n$$

The singular integral approach

- **Representation via Green's function**

$$u(x) \approx \int G(x,y) f(y)\, dy$$

with

$$G(x,y) = \begin{cases} |x-y|^{2-n} & \text{if } n > 2 \\ -\log|x-y| & \text{if } n = 2 \end{cases}$$

- **Differentiation yields**

$$D^2 u(x) = \int K(x,y) f(y)\, dy$$

and $K(x,y)$ is a singular integral kernel, and the conclusion follows

Singular kernels with cancellations

- **Initial boundedness assumption**

$$\|\hat{K}\|_{L^\infty} \leq B\,,$$

where $\hat{K}$ denotes the Fourier transform of $K(\cdot)$

- **Hörmander cancelation condition**

$$\int_{|x|\geq 2|y|} |K(x-y) - K(x)|\, dx \leq B \qquad \text{for every} \quad y \in \mathbb{R}^n$$

Another linear case

- **Higher order right hand side**

$$\triangle u = \text{div } Du = \text{div } F$$

Then

$$F \in L^q \Longrightarrow Du \in L^q \qquad q > 1$$

just "simplify" the divergence operator!!

Part 1: Gradient integrability theory

The fundamentals

Theorem (Iwaniec, Studia Math. 83)

$$\operatorname{div}\,(|Du|^{p-2}Du) = \operatorname{div}\,(|F|^{p-2}F) \qquad \textit{in } \mathbb{R}^n$$

Then it holds that

$$F \in L^q \Longrightarrow Du \in L^q \qquad p \leq q < \infty$$

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The local estimate

$$\left(\fint_{B_R} |Du|^q\, dz\right)^{\frac{1}{q}} \leq c\left(\fint_{B_{2R}} |Du|^p\, dz\right)^{\frac{1}{p}} + c\left(\fint_{B_{2R}} |F|^q\, dz\right)^{\frac{1}{q}}$$

General elliptic problems

- **In the same way the non-linear result of Iwaniec extends to all elliptic equations in divergence form of the type**

$$\text{div } a(Du) = \text{div } (|F|^{p-2}F)$$

  where $a(\cdot)$ is p-monotone in the sense of the previous slides
- **and to all systems with special structure**

$$\text{div } (g(|Du|)Du) = \text{div } (|F|^{p-2}F)$$

General systems - the elliptic case

- **The previous result cannot hold for general systems**

$$\text{div } a(Du) = \text{div } (|F|^{p-2}F)$$

  with $a(\cdot)$ being a general p-monotone in the sense of the previous slide. The failure of the result, which happens already in the case $p = 2$, can be seen as follows

- **Consider the homogeneous case**

$$\text{div } a(Du) = 0$$

  The validity of the result would imply $Du \in L^q$ for every $q < \infty$, and, ultimately, that

$$u \in L^\infty$$

- **But Sverák & Yan (Proc. Natl. Acad. Sci. USA 02) recently proved the existence of unbounded solutions, even when $a(\cdot)$ is non-degenerate and smooth**

The up-to-a-certain-extent CZ theory

- For general elliptic systems it holds

Theorem (Kristensen & Min., ARMA 06)

$$\operatorname{div}\, a(Du) = \operatorname{div}\, (|F|^{p-2}F) \qquad \textit{in } \Omega$$

for $a(Du)$ being a p-monotone vector field and

$$p \leq q < p + \frac{2p}{n-2} + \delta \qquad n > 2$$

Then it holds that

$$F \in L^q_{\mathrm{loc}} \Longrightarrow Du \in L^q_{\mathrm{loc}}$$

- **Applications to singular sets estimates follow**

The up-to-a-certain-extent CZ theory

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$$F \in L^q_{\text{loc}} \Longrightarrow Du \in L^q_{\text{loc}}$$

The Dirichlet problem

Theorem (Kristensen & Min., ARMA 06)

Let

$$\begin{cases} -\text{div } a(Du) = 0 & \text{in } \Omega \\ u = v & \text{on } \partial\Omega \end{cases}$$

with Ω being suitably regular (say $C^{1,\alpha}$). Moreover, let

$$p \leq q < p + \frac{2p}{n-2} + \delta \qquad n > 2\,.$$

Then it holds that

$$\int_\Omega |Du|^q \, dx \leq c \int_\Omega (|Dv|^q + 1) \, dx$$

There is some evidence that the assumed bound on q is sharp

The parabolic case

Theorem (Acerbi & Min., Duke Math. J. 07)

$$u_t - \text{div}\,(|Du|^{p-2}Du) = \text{div}\,(|F|^{p-2}F) \qquad \textit{in } \Omega \times (0, T)$$

for

$$p > \frac{2n}{n+2}$$

Then it holds that

$$F \in L^q_{\text{loc}} \Longrightarrow Du \in L^q_{\text{loc}} \qquad \textit{for} \qquad p \le q < \infty$$

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The lower bound

$$p > \frac{2n}{n+2}$$

is optimal

The parabolic case

- **The elliptic approach via maximal operators only works in the case $p = 2$**
- **The result also works for systems, that is when $u(x,t) \in \mathbb{R}^N$, $N \geq 1$**
- **First Harmonic Analysis free approach to non-linear Calderón-Zygmund estimates**

The parabolic case

- **The result is new already in the case of equations i.e. $N = 1$, the difficulty being in the lack of homogenous scaling of parabolic problems with $p \neq 2$, and not being caused by the degeneracy of the problem, but rather by the polynomial growth.**
- **The result extends to all parabolic equations of the type**

$$u_t - \text{div } a(Du) = \text{div } (|F|^{p-2}F)$$

with $a(\cdot)$ being a monotone operator with p-growth. More precisely we assume

$$\begin{cases} \nu(s^2 + |z_1|^2 + |z_2|^2)^{\frac{p-2}{2}}|z_2 - z_1|^2 \leq \langle a(z_2) - a(z_1), z_2 - z_1 \rangle \\ |a(z)| \leq L(s^2 + |z|^2)^{\frac{p-1}{2}} , \end{cases}$$

The parabolic case

- **The result also holds for systems with a special structure (sometimes called Uhlenbeck structure). This means**

$$u_t - \text{div } a(Du) = \text{div } (|F|^{p-2}F)$$

with $a(\cdot)$ being p-monotone in the sense of the previous slide, and satisfying the structure assumption

$$a(Du) = g(|Du|)Du$$

- **The p-Laplacean system is an instance of such a structure**

Part 2: Pointwise estimates via nonlinear potentials

- **Consider the model case**

$$-\triangle u = \mu \qquad \text{in } \mathbb{R}^n$$

- **Then, if we define**

$$I_\beta(\mu)(x) := \int_{\mathbb{R}^n} \frac{d\mu(y)}{|x-y|^{n-\beta}}\,, \qquad\qquad \beta \in (0,n]$$

we have

$$|u(x)| \leq cI_2(|\mu|)(x)\,, \qquad \text{and} \qquad |Du(x)| \leq cI_1(|\mu|)(x)$$

Local versions

- **In bounded domains one uses**

$$\mathbf{I}^{\mu}_{\beta}(x,R) := \int_0^R \frac{\mu(B(x,\varrho))}{\varrho^{n-\beta}}\,\frac{d\varrho}{\varrho} \qquad\qquad \beta \in (0,n]$$

since

$$\begin{aligned}\mathbf{I}^{\mu}_{\beta}(x,R) &\lesssim \int_{B_R(x)} \frac{d\mu(y)}{|x-y|^{n-\beta}}\\ &= I_\beta(\mu \llcorner B(x,R))(x)\\ &\leq I_\beta(\mu)(x)\end{aligned}$$

for non-negative measures

What happens in the nonlinear case?

- **For instance for nonlinear equations with linear growth**

$$-\text{div } a(Du) = \mu$$

that is equations well posed in $W^{1,2}$ (p-growth and $p = 2$)

- **And degenerate ones like**

$$-\text{div } (|Du|^{p-2}Du) = \mu$$

The setting

- **We consider equations**

$$-\text{div}\ a(Du) = \mu$$

- **under the assumptions**

$$\begin{cases} |a(z)| + |a_z(z)|(|z|^2+s^2)^{\frac{1}{2}} \leq L(|z|^2+s^2)^{\frac{p-1}{2}} \\ \nu^{-1}(|z|^2+s^2)^{\frac{p-2}{2}}|\lambda|^2 \leq \langle a_z(x,z)\lambda,\lambda\rangle \end{cases}$$

with

$$p \geq 2$$

this last bound is assumed in order to keep the exposition brief

Non-linear potentials

- **The nonlinear Wolff potential is defined by**

$$\mathbf{W}^{\mu}_{\beta,p}(x,R) := \int_0^R \left(\frac{|\mu|(B(x,\varrho))}{\varrho^{n-\beta p}}\right)^{\frac{1}{p-1}} \frac{d\varrho}{\varrho} \qquad \beta \in (0, n/p]$$

which for $p = 2$ reduces to the usual Riesz potential

$$\mathbf{I}^{\mu}_{\beta}(x,R) := \int_0^R \frac{\mu(B(x,\varrho))}{\varrho^{n-\beta}} \frac{d\varrho}{\varrho} \qquad \beta \in (0, n]$$

- **The nonlinear Wolff potential** plays in nonlinear potential theory the same role the Riesz potential plays in the linear one

A fundamental estimate

For solutions to div $(|Du|^{p-2}Du) = \mu$ with $p \leq n$ we have

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Theorem (Kilpeläinen-Malý, Acta Math. 94)

$$|u(x)| \leq c\mathbf{W}^{\mu}_{1,p}(x,R) + c\left(\fint_{B(x,R)} |u|^{p-1}\,dy\right)^{\frac{1}{p-1}}$$

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For $p = 2$ we have $\mathbf{W}^{\mu}_{1,p} = \mathbf{I}^{\mu}_{2}$

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For $p = 2$ we have $\mathbf{W}^{\mu}_{1,p} = \mathbf{I}^{\mu}_{2}$

Another approach to this result has been given by Trudinger & Wang (Amer. J. Math. 02)

Integral estimates follow via Wolff inequalities

- **We have**

$$\mu \in L^q \Longrightarrow \mathbf{W}^{\mu}_{\beta,p} \in L^{\frac{nq(p-1)}{n-qp\beta}} \qquad q \in (1,n)$$

with related explicit estimates, also in Marcinkiewicz spaces

- **Such a property allows to reduce the study of integrability of solutions to that of nonlinear potentials**
- **The key is the following inequality:**

$$\int_0^\infty \left(\frac{|\mu|(B(x,\varrho))}{\varrho^{n-\beta p}}\right)^{\frac{1}{p-1}} \frac{d\varrho}{\varrho} \le c I_\beta \left\{[I_\beta(|\mu|)]^{\frac{1}{p-1}}\right\}(x)$$

the last quantity is called Havin-Maz'ja potential

The potential gradient estimate for $p = 2$

Theorem (Min., JEMS 11)

$$|D_\xi u(x)| \leq c\mathbf{I}_1^{|\mu|}(x,R) + c \fint_{B(x,R)} |D_\xi u| \, dy$$

holds for almost every point x *and* $\xi \in \{1, \ldots, n\}$

The general potential gradient estimate

Theorem (Duzaar & Min., Amer. J. Math. 201?)

$$|Du(x)| \leq c\mathbf{W}^{\mu}_{1/p,p}(x,R) + c \fint_{B(x,R)} (|Du| + s)\, dy$$

holds for almost every point x

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Theorem (Duzaar & Min., Amer. J. Math. 201?)

$$|Du(x)| \leq c\mathbf{W}^{\mu}_{1/p,p}(x,R) + c\fint_{B(x,R)} (|Du| + s)\, dy$$

holds for almost every point x

This means

$$|Du(x)| \leq c\int_0^R \left(\frac{|\mu|(B(x,\varrho))}{\varrho^{n-1}}\right)^{\frac{1}{p-1}} \frac{d\varrho}{\varrho} + c\fint_{B(x,R)} (|Du| + s)\, dx$$

Integral estimates follow as a corollary

- **For the model case equation** $\text{div}\,(|Du|^{p-2}Du) = \mu$ **the previous estimate implies for instance all those included in the papers**
- Iwaniec, Studia Math. 83
- Di Benedetto & Manfredi, Amer. J. Math. 93
- Boccardo & Gallöuet, J. Funct. Anal. 87, Comm PDE 92
- Talenti, Ann. SNS Pisa 76
- Kilpelainen & Li, Diff. Int. Equ. 00
- Dolzmann & Hungerbühler & Müller, Crelle J. 00
- Alvino & Ferone & Trombetti, Ann. Mat. Pura Appl. 00
- Boccardo, Ann. Mat. Pura Appl. 08
- **Moreover, the borderline cases which appeared as open problems in some of the above papers papers now follow as a corollary**

The two estimates

- **The two potential estimates are**

$$|u(x)| \leq c\mathbf{W}^{\mu}_{1,p}(x,R) + c \fint_{B(x,R)} (|u| + Rs)\, dy$$

and

$$|Du(x)| \leq c\mathbf{W}^{\mu}_{1/p,p}(x,R) + c \fint_{B(x,R)} (|Du| + s)\, dy$$

- **They basically provide size estimates on** u **and** Du
- **The aim is now to provide estimates on the oscillations of solutions and/or alternatively, on intermediate derivatives**

- **The following definition is due to DeVore & Sharpley (Mem. AMS, 1982)**
- Let $\alpha \in (0,1]$, $q \geq 1$, and let $\Omega \subset \mathbb{R}^n$ be a bounded open subset. A measurable function v, finite a.e. in Ω, belongs to the Calderón space $C_q^\alpha(\Omega)$ if and only if there exists a nonnegative function $m \in L^q(\Omega)$ such that

$$|v(x) - v(y)| \leq [m(x) + m(y)]|x - y|^\alpha$$

holds for almost every couple $(x, y) \in \Omega \times \Omega$.

Calderón spaces of DeVore & Sharpley

- In other words

$$m(x) \approx \partial^{\alpha} v(x)$$

- Indeed DeVore & Sharpley take

$$M^{\alpha}_{\#} v(x) = \sup_{B(x,\varrho)} \varrho^{-\alpha} \fint_{B(x,\varrho)} |v(y) - (v)_{B(x,\varrho)}| \, dy$$

- For $\alpha \in (0,1)$ and $q > 1$ we have

$$W^{\alpha,q} \subset C^{\alpha,q} \subset W^{\alpha-\varepsilon,q}$$

therefore such spaces, although not being of interpolation type, are just another way to say "fractional differentiabiltiy"

Classical regularity review

- For equations as

$$\operatorname{div}\left(\gamma(x)|Du|^{p-2}Du\right) = 0$$

we have

- If $\gamma(x)$ is measurable then $u \in C^{0,\alpha_m}$ for some $\alpha_m > 0$
- If $\gamma(x)$ is VMO then $u \in C^{0,\alpha}$ for every $\alpha < 1$
- If $\gamma(x)$ is Dini then $u \in C^{0,1}$
- If $\gamma(x)$ is Hölder the $Du \in C^{0,\alpha_M}$ for some $\alpha_M > 0$

General quasilinear equations

- The previous and forthcoming results hold for general quasilinear equations

$$\operatorname{div} a(x, Du) = 0 \qquad -\operatorname{div} a(x, Du) = \mu$$

where

$$x \to a(x, \cdot) \qquad \text{is just measurable/VMO/Dini}$$

The homogeneous case

- The classical estimate

$$|u(x)-u(y)| \leq \fint_{B_R} (|u| + Rs)\, d\xi \cdot \left(\frac{|x-y|}{R}\right)^{\alpha_m}$$

when coefficients are measurable

The homogeneous case

- The classical estimate

$$|u(x)-u(y)| \leq \fint_{B_R} (|u| + Rs)\, d\xi \cdot \left(\frac{|x-y|}{R}\right)^{\alpha}$$

when coefficients are VMO-regular

The homogeneous case

- The classical estimate

$$|u(x) - u(y)| \leq \fint_{B_R} (|u| + Rs)\, d\xi \cdot \left(\frac{|x-y|}{R}\right)^{\alpha}$$

holds for $\alpha \in (0, \alpha_m], (0,1), (0,1]$ depending on the regularity of coefficients

- Moreover

$$|Du(x) - Du(y)| \leq \fint_{B_R} (|Du| + s)\, d\xi \cdot \left(\frac{|x-y|}{R}\right)^{\alpha}$$

whenever $\alpha \leq \alpha_M$

The first universal potential estimate

- In the case of measurable coefficients we have

Theorem (Kuusi & Min.)

The estimate

$$
\begin{aligned}
&|u(x) - u(y)| \\
&\le c \left[\mathbf{W}^{\mu}_{1-\frac{\alpha(p-1)}{p},p}(x,R) + \mathbf{W}^{\mu}_{1-\frac{\alpha(p-1)}{p},p}(y,R) \right] |x-y|^{\alpha} \\
&\qquad + c \fint_{B_R} (|u| + Rs)\, d\xi \cdot \left(\frac{|x-y|}{R} \right)^{\alpha}
\end{aligned}
$$

holds uniformly in every compact subset of $\alpha \in [0, \alpha_m)$*, whenever* $x, y \in B_{R/4}$

- The case $\alpha = 0$ gives back the known potential estimate of Kilpeläinen & Malý as endpoint case

The second universal potential estimate

- In the case of VMO coefficients we have

Theorem (Kuusi & Min.)

The estimate

$$
\begin{aligned}
&|u(x) - u(y)| \\
&\leq c \left[\mathbf{W}^{\mu}_{1-\frac{\alpha(p-1)}{p},p}(x,R) + \mathbf{W}^{\mu}_{1-\frac{\alpha(p-1)}{p},p}(y,R) \right] |x-y|^{\alpha} \\
&\qquad + c \fint_{B_R} (|u| + Rs)\, d\xi \cdot \left(\frac{|x-y|}{R} \right)^{\alpha}
\end{aligned}
$$

holds uniformly in every compact subset of $\alpha \in [0,1)$, *whenever* $x, y \in B_{R/4}$

The third universal potential estimate

- Finally, in the case of Dini coefficients we have

Theorem (Kuusi & Min.)

The estimate

$$
\begin{aligned}
&|u(x) - u(y)| \\
&\leq c \left[\mathbf{W}^{\mu}_{1-\frac{\alpha(p-1)}{p},p}(x,R) + \mathbf{W}^{\mu}_{1-\frac{\alpha(p-1)}{p},p}(y,R) \right] |x-y|^{\alpha} \\
&\qquad + c \fint_{B_R} (|u| + Rs)\, d\xi \cdot \left(\frac{|x-y|}{R} \right)^{\alpha}
\end{aligned}
$$

holds uniformly in $\alpha \in [0,1]$, *whenever* $x, y \in B_{R/4}$

- The cases $\alpha = 0$ and $\alpha = 1$ give back the two known potential estimates as endpoint cases

The fourth universal potential estimate

- In the case coefficients are Hölder continuous we have

Theorem (Kuusi & Min.)

The estimate

$$
\begin{aligned}
&|Du(x) - Du(y)| \\
&\leq c \left[\mathbf{W}^{\mu}_{1-\frac{(1+\alpha)(p-1)}{p},p}(x,R) + \mathbf{W}^{\mu}_{1-\frac{(1+\alpha)(p-1)}{p},p}(y,R) \right] |x-y|^{\alpha} \\
&\qquad + c \fint_{B_R} (|Du| + s)\, d\xi \cdot \left(\frac{|x-y|}{R} \right)^{\alpha}
\end{aligned}
$$

uniformly α belong to any compact subset of $[0, \alpha_M)$, whenever $x, y \in B_{R/4}$

- The case $\alpha = 0$ gives back the gradient potential estimate

The universal potential estimate

- With the previous terminology we have

$$\partial^{\alpha} u \leq c\mathbf{W}^{\mu}_{1-\frac{\alpha(p-1)}{p},p}$$

  and

$$\partial^{\alpha} Du \leq c\mathbf{W}^{\mu}_{1-\frac{(1+\alpha)(p-1)}{p},p}$$

- As a corollary we have optimal regularity criteria in oscillation spaces, such Hölder spaces, Fractional spaces, and so on, both for u and for the gradient Du

Part 3: A fully fractional approach

The setting for $p = 2$

- **We consider equations**

$$-\text{div}\ a(Du) = \mu$$

- **under the assumptions**

$$\begin{cases} |a(z)| + |a_z(z)||z| \leq L|z| \\ \nu^{-1}|\lambda|^2 \leq \langle a_z(z)\lambda, \lambda\rangle \end{cases}$$

The setting for $p = 2$

Theorem (Min., JEMS 11)

$$|D_\xi u(x)| \leq c\mathbf{I}_1^{|\mu|}(x,R) + c \fint_{B(x,R)} |D_\xi u| \, dx$$

for every $\xi \in \{1, \ldots, n\}$*, where*

$$\mathbf{I}_\beta^\mu(x,R) := \int_0^R \frac{\mu(B(x,\varrho))}{\varrho^{n-1}} \, \frac{d\varrho}{\varrho}$$

Classical Gradient estimates

- **Consider energy solutions to** $\operatorname{div} a(Du) = 0$ for $p = 2$
- **First prove** $Du \in W^{1,2}$
- **Then use that** $v = D_\xi u$ solves

$$\operatorname{div}(A(x)Dv) = 0 \qquad\qquad A(x) := a_z(Du(x))$$

- **The boundedness of** $D_\xi u$ follows by Standard DeGiorgi's theory
- **This is a consequence of Caccioppoli's inequalities of the type**

$$\int_{B_{R/2}} |D(D_\xi u - k)_+|^2 \, dy \leq \frac{c}{R^2} \int_{B_R} |(D_\xi u - k)_+|^2 \, dy$$

where

$$(D_\xi u - k)_+ := \max\{D_\xi u - k, 0\}$$

Recall the definition

- **We have**

$$v \in W^{\sigma,1}(\Omega')$$

iff $v \in L^1(\Omega')$ and

$$[v]_{\sigma,1;\Omega'} = \int_{\Omega'} \int_{\Omega'} \frac{|v(x) - v(y)|}{|x-y|^{n+\sigma}} \, dx \, dy < \infty$$

There is a differentiability problem

For solutions to

$$\operatorname{div} a(Du) = \mu \quad \text{in general} \quad Du \notin W^{1,1}$$

but nevertheless it holds

Theorem (Min., Ann. SNS Pisa 07)

$$Du \in W^{1-\varepsilon,1}_{\mathrm{loc}}(\Omega, \mathbb{R}^n) \qquad \textit{for every } \varepsilon \in (0,1)$$

This means that

$$[Du]_{1-\varepsilon,1;\Omega'} = \int_{\Omega'} \int_{\Omega'} \frac{|Du(x) - Du(y)|}{|x-y|^{n+1-\varepsilon}}\, dx\, dy < \infty$$

holds for every $\varepsilon \in (0,1)$, and every subdomain $\Omega' \Subset \Omega$

Step 1: A non-local Caccioppoli inequality

Theorem (Min., JEMS 11)

Let

$$w = D_\xi u \quad \textit{with} \quad -\operatorname{div} a(Du) = \mu$$

where $\xi \in \{1, \ldots, n\}$ *then*

$$[(|w| - k)_+]_{\sigma,1;B_{R/2}} \leq \frac{c}{R^\sigma} \int_{B_R} (|w| - k)_+ \, dy + \frac{cR|\mu|(B_R)}{R^\sigma}$$

holds for every $\sigma < 1/2$

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where $\xi \in \{1, \dots, n\}$ *then*

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holds for every $\sigma < 1/2$

Compare with the usual one for $\operatorname{div} a(Du) = 0$, that is

$$[(w - k)_+]^2_{1,2;B_{R/2}} \equiv \int_{B_{R/2}} |D(w - k)_+|^2 \, dy \leq \frac{c}{R^2} \int_{B_R} (w - k)_+^2 \, dy$$

Step 1: A non-local Caccioppoli inequality

- **This approach reveal the robustness of energy inequalities, which hold below the natural growth exponent 2, and for fractional order of differentiability, although the equation has integer order**
- **Classical VS fractional**

	classical	fractional
spaces	$L^2 - L^2$	$L^1 - L^1$
differentiability	$0 \longrightarrow 1$	$0 \longrightarrow \sigma$

Step 2: Fractional De Giorgi's iteration

Theorem (Min., JEMS 11)

Let w be an L^1-function w satisfying the fractional Caccioppoli's inequality

$$[(|w|-k)_+]_{\sigma,1;B_{R/2}} \leq \frac{L}{R^\sigma} \int_{B_R} (|w|-k)_+ \, dy + \frac{LR|\mu|(B_R)}{R^\sigma}$$

for some $\sigma > 0$ and every $k \geq 0$. Then it holds that

$$|w(x)| \leq c\mathbf{I}_1^{|\mu|}(x,R) + c \fint_{B(x,R)} |w| \, dy$$

for every Lebesgue point x of w

Step 2: Fractional De Giorgi's iteration

Theorem (Min., JEMS 11)

Let w be an L^1-function w satisfying the fractional Caccioppoli's inequality

$$[(|w|-k)_+]_{\sigma,1;B_{R/2}} \leq \frac{L}{R^\sigma}\int_{B_R}(|w|-k)_+\,dy + \frac{LR|\mu|(B_R)}{R^\sigma}$$

for some $\sigma > 0$ and every $k \geq 0$. Then it holds that

$$|w(x)| \leq c\mathbf{I}_1^{|\mu|}(x,R) + c\,⨍_{B(x,R)}|w|\,dy$$

for every Lebesgue point x of w

Proof of the gradient potential estimate: apply the previous result to $w \equiv D_\xi u$