Variations on the $p$–Laplacian

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Topics

\( p \)-harmonic functions

\[-\Delta_p u = 1 \]

overdetermined problems

open problems
Laplace operator

\[ \Delta u = u_{x_1 x_1} + \ldots + u_{x_n x_n} = u_{\nu \nu} + u_\nu \text{ div}(\nu) \]

where \( \nu(x) = -\frac{\nabla u(x)}{|\nabla u(x)|} \) is direction of steepest descent. In fact,

\[ \text{div}(\nu) = -\frac{\Delta u}{|\nabla u|} + \frac{u_{x_i} u_{x_j} u_{x_i x_j}}{|\nabla u|^2} = -\frac{\Delta u}{|\nabla u|} + \frac{u_{\nu \nu}}{|\nabla u|} \]

so that \( \Delta u = u_{\nu \nu} - |\nabla u| \text{ div}(\nu) = u_{\nu \nu} + u_\nu \text{ div}(\nu) \) or

\[ \Delta u = u_{\nu \nu} + u_\nu (n - 1)H \]

with \( H \) denoting mean curvature of a level set of \( u \).

For radial \( u \) recall \( \Delta u = u_{rr} + \frac{n-1}{r} u_r \).
For $p \in (1, \infty)$ one can write the \textbf{$p$-Laplace operator} as

$$\Delta_p u = \text{div} \left( |\nabla u|^{p-2} \nabla u \right) = |\nabla u|^{p-2} [\Delta u + (p - 2) u_{\nu\nu}]$$

$$= |\nabla u|^{p-2} [(p - 1) u_{\nu\nu} + (n - 1) Hu_\nu]$$

and the \textbf{normalized or game-theoretic $p$-Laplace operator} as

$$\Delta^N_p u = \frac{1}{p} |\nabla u|^{2-p} \text{div} \left( |\nabla u|^{p-2} \nabla u \right)$$

$$= \frac{p-1}{p} u_{\nu\nu} + \frac{1}{p} (n - 1) Hu_\nu = \frac{p-1}{p} \Delta^N_\infty u + \frac{1}{p} \Delta^N_1 u .$$

Observe $\Delta^N_\infty u = u_{\nu\nu}$, $\Delta^N_2 u = \frac{1}{2} \Delta u$ and $\Delta^N_1 u = |\nabla u| \text{div}(\frac{\nabla u}{|\nabla u|})$. 
$p$-harmonic functions

Given $\Omega \subset \mathbb{R}^n$ bounded, $\partial \Omega$ of class $C^{2,\alpha}$ and $g(x) \in W^{1,p}(\Omega)$

$$-\Delta_p u = 0 \quad \text{in } \Omega,$$
$$u(x) = g(x) \quad \text{on } \partial \Omega.$$  \hfill (1) \hfill (2)

$u$ can be characterized as the unique (weak) solution of the strictly convex variational problem

$$\text{Minimize } I_p(v) = \|\nabla v\|_{L^p(\Omega)} \quad \text{on } g(x) + W^{1,p}_0(\Omega),$$  \hfill (3)

so that

$$\int_{\Omega} |\nabla u|^{p-2}\nabla u \nabla \phi \, dx = 0 \text{ for every } \phi \in W^{1,p}_0(\Omega).$$  \hfill (4)

It is well known, that weak solutions are locally of class $C^{1,\alpha}$. They are even of class $C^\infty$ wherever their gradient does not vanish.

Bernd Kawohl
Variations on the $p$–Laplacian
One can show (Juutinen, Lindqvist, Manfredi 2001) that weak solutions are also viscosity solutions of the associated Euler equation

\[ F_p(Du, D^2 u) = -|Du|^{p-4} (|Du|^2 \text{trace} D^2 u + \langle D^2 u Du, Du \rangle) = 0 \]

Incidentally, only for \( p \in (1, 2) \) does this imply that they are also viscosity solutions of the normalized equation

\[ F_p^N(Du, D^2 u) = -\frac{1}{p} \text{trace} D^2 u - \frac{p-2}{p} \frac{\langle D^2 u Du, Du \rangle}{|Du|^2} = 0 \]
What happens as $p \to \infty$? For $g \in W^{1,\infty}(\Omega)$ the family $u_p$ is uniformly bounded in $W^{1,p}$ because $I_p(u_p) \leq I_p(g) \leq ||\nabla g||_{\infty} |\Omega|$.

Wolog $|\Omega| := 1$. For $q > n$ fixed and $p > q$ one finds

$$||\nabla u_p||_{q} \leq ||\nabla u_p||_{p} |\Omega|^{(p-q)/pq} \leq ||\nabla g||_{\infty} |\Omega|^{1+1/q+1}$$
as $p \to \infty$,

so $u_p \to u_\infty$ in some $C^\alpha$.

By the stability theorem for viscosity solutions $u_\infty$ should be viscosity solution to a limit equation $F_\infty(Du, D^2u) = 0$.

What is this equation? Let us check the condition for subsolutions.
Let $\varphi$ be a $C^2$ testfunction s.th. $\varphi - u_\infty$ has a min at $x_\infty$ and $\nabla \varphi(x_\infty) \neq 0$. Then wolog $\varphi - u_p$ has a min at $x_p$ near $x_\infty$ and $x_p \to x_\infty$ as $p \to \infty$. Since $u_p$ is viscosity subsolution

$$-|D\varphi|^{p-4} [ |D\varphi|^2 \Delta \varphi + (p-2) \langle D^2\varphi D\varphi, D\varphi \rangle ](x_p) \leq 0,$$

or

$$-\frac{p-2}{p} \langle D^2\varphi D\varphi, D\varphi \rangle(x_p) \leq \frac{1}{p} |D\varphi|^2 \Delta \varphi(x_p).$$

$p \to \infty$ gives $\langle D^2\varphi D\varphi, D\varphi \rangle(x_\infty) := -\Delta_\infty \varphi \leq 0$.

... Thus $u_\infty$ is (unique) viscosity solution of $-\Delta_\infty u = 0$ in $\Omega$,

$$u = g \text{ on } \partial \Omega.$$
It is worth noting that the variational problem

$$\text{Minimize } I_\infty(v) = \|\nabla v\|_{L^\infty(\Omega)} \text{ on } g(x) + W^{1,\infty}_0(\Omega),$$

(5)
can have many solutions,
It is worth noting that the variational problem

$$\text{Minimize } I_{\infty}(v) = ||\nabla v||_{L^\infty(\Omega)} \text{ on } g(x) + W^{1,\infty}_0(\Omega),$$

(5)

can have many solutions, e.g. the minimum of two cones (not $C^1$) or $u_\infty \in C^{1,\alpha}$ (Savin).

Kawohl, Shagholian 2005
What happens to $p$-harmonic functions as $p \to 1$? I. g. no uniform convergence, but Juutinen (2005) found sufficient conditions:

If $g \in C(\overline{\Omega})$ and $\Omega$ convex, then $u_p \to u_1$ uniformly as $p \to 1$. Moreover, $u_1$ is unique minimizer of

$$E_1(v) = \sup \left\{ \int_{\Omega} u \, \text{div}\sigma \, dx; \, \sigma \in C_0^\infty(\Omega, \mathbb{R}^n), \, |\sigma(x)| \leq 1 \text{ in } \Omega \right\}$$

on $\{v \in BV(\Omega) \cap C(\overline{\Omega}), \, v = g \text{ on } \partial \Omega\}$.

Here the limiting variational problem has a unique solution, while the limiting Euler equation can have many viscosity solutions.
Heuristic reason for **uniqueness** of minimizer of the TV-functional:

If there are two minimizers $u$ and $v$ (for simplicity in $W^{1,1}(\Omega)$) of $E_1$, then any convex combination $w = tu + (1 - t)v$ would also be minimizer, hence level lines of $u$ are also level lines of $v$, $\nabla u \parallel \nabla v$, $v = f(u)$. Dirichlet cond. implies $f(g) = g$, so that $f = ld$ on $\text{range}_{\partial \Omega}(g)$. But since $\min_{\partial \Omega} g \leq u, v \leq \max_{\partial \Omega} g$ in $\Omega$ we find $f(u) = u$ in $\Omega$. 
Nonuniqueness of viscosity solutions to the Dirichlet problem

\[-\Delta_1 u = 0 \text{ in } \Omega, \ u = g \text{ on } \partial \Omega.\]

Sternberg, Ziemer (1994) gave counterexample: \( \Omega = B(0, 1) \in \mathbb{R}^2, \ g(x_1, x_2) = \cos(2\varphi) \) has a whole family \( u_\lambda \) of viscosity solutions, \( \lambda \in [-1, 1] \), but only \( u_0 \) minimizes \( E_1 \). In fact,

\[
 u_\lambda(x_1, x_2) = \begin{cases} 
 2x_1^2 - 1 & \text{left and right of rectangle} \\
 \lambda & \text{in the rectangle generated by } \cos(2\varphi) = \lambda \\
 1 - 2x_2^2 & \text{on top and bottom}
\end{cases}
\]

is viscosity sol. of both \(-\Delta_1 u = 0\) and \(-\Delta_1^N u = \kappa|\nabla u| = 0 \text{ in } \Omega.\)
Aufgabe 1: 
Sei $B$ die Einheitskreisscheibe. Für $\lambda \in [-1, 1]$ wird eine Funktion $u_\lambda : B \rightarrow \mathbb{R}$ mit folgenden Eigenschaften definiert:

(a) Auf dem Rand $\partial B$ gilt $u_\lambda(1, \varphi) = \cos 2\varphi =: \psi(\varphi)$ für $\varphi \in [0, 2\pi)$, wobei hier $u_\lambda(r, \varphi)$ die Darstellung der Funktion $u_\lambda$ in Polarkoordinaten ist.

(b) Die Niveaulinien von $u_\lambda$ sehen entsprechend der linken Abbildung aus.

Diese Funktion löst formell das Randwertproblem $\Delta u = 0$ in $B$, $u = \psi$ auf $\partial B$.

Geben Sie $u_\lambda$ explizit an und berechnen Sie $f(\lambda) := \int_B |\nabla u_\lambda| \, dx$.
definition of viscosity solutions for discontinuous $F$

$u$ is a **viscosity solution** of $F(Du, D^2u) = 0$, iff sub- and supersol.

$u$ is **subsol.** if for every $x \in \Omega$ and $\varphi \in C^2$ s.th. $\varphi - u$ has min at $x$ the ineq. $F_*(D\varphi, D^2\varphi) \leq 0$ holds. Here $F_* = \text{lsc hull of } F$.

$u$ is **supersol.** if for every $x \in \Omega$ and $\varphi \in C^2$ s.th. $\varphi - u$ has max at $x$ the ineq. $F^*(D\varphi, D^2\varphi) \geq 0$ holds. Here $F^* = \text{usc hull of } F$.

$$F_p^N(q, X) = \begin{cases} -\frac{1}{p} \left( \delta_{ij} + (p - 1) \frac{q_i q_j}{|q|^2} \right) X_{ij} & \text{if } q \neq 0 \\ ? & \text{if } q = 0 \end{cases}$$
Symm. matrix $X$ has eigenvalues $\lambda_1(X) \leq \lambda_2(X) \leq \ldots \leq \lambda_n(X)$

$$F^N_{p^*}(0, X) = \begin{cases} 
  -\frac{1}{p} \sum_{i=1}^{n-1} \lambda_i - \frac{p-1}{p} \lambda_n & \text{if } p \in [2, \infty] \\
  -\frac{1}{p} \sum_{i=2}^{n} \lambda_i - \frac{p-1}{p} \lambda_1 & \text{if } p \in [1, 2]
\end{cases}$$

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Symm. matrix $X$ has eigenvalues $\lambda_1(X) \leq \lambda_2(X) \leq \ldots \leq \lambda_n(X)$

$$F^N_{p\ast}(0, X) = \begin{cases} -\frac{1}{p} \sum_{i=1}^{n-1} \lambda_i - \frac{p-1}{p} \lambda_n & \text{if } p \in [2, \infty] \\ -\frac{1}{p} \sum_{i=2}^{n} \lambda_i - \frac{p-1}{p} \lambda_1 & \text{if } p \in [1, 2] \end{cases}$$

$$F^{N\ast}_{p\ast}(0, X) = \begin{cases} -\frac{1}{p} \sum_{i=2}^{n} \lambda_i - \frac{p-1}{p} \lambda_1 & \text{if } p \in [2, \infty] \\ -\frac{1}{p} \sum_{i=1}^{n-1} \lambda_i - \frac{p-1}{p} \lambda_n & \text{if } p \in [1, 2] \end{cases}$$

In particular, for $n = 2$, $F^N_{1\ast}(0, X) = -\lambda_2$ and $F^{N\ast}_{1\ast}(0, X) = -\lambda_1$. 
Symm. matrix $X$ has eigenvalues $\lambda_1(X) \leq \lambda_2(X) \leq \ldots \leq \lambda_n(X)$

$$F_N^p(0, X) = \begin{cases} 
- \frac{1}{p} \sum_{i=1}^{n-1} \lambda_i - \frac{p-1}{p} \lambda_n & \text{if } p \in [2, \infty] \\
- \frac{1}{p} \sum_{i=2}^{n} \lambda_i - \frac{p-1}{p} \lambda_1 & \text{if } p \in [1, 2]
\end{cases}$$

$$F_N^{p*}(0, X) = \begin{cases} 
- \frac{1}{p} \sum_{i=2}^{n} \lambda_i - \frac{p-1}{p} \lambda_1 & \text{if } p \in [2, \infty] \\
- \frac{1}{p} \sum_{i=1}^{n-1} \lambda_i - \frac{p-1}{p} \lambda_n & \text{if } p \in [1, 2]
\end{cases}$$

In particular, for $n = 2$, $F_N^{1*}(0, X) = -\lambda_2$ and $F_N^{1*}(0, X) = -\lambda_1$, so that we require $-\lambda_2(D^2 \varphi) \leq 0$ for subsols. if $\nabla \varphi(x) = 0$ and $-\lambda_1(D^2 \varphi) \geq 0$ for supersols. if $\nabla \varphi(x) = 0$
\[-\Delta_p u = 1\]

The Dirichlet problem \(-\Delta_p u_p = 1\) in \(\Omega\), \(u_p = 0\) on \(\partial\Omega\) can be treated in a similar way. Again surprises as \(p \to \infty\) or 1.

\[
\lim_{p \to \infty} u_p(x) = d(x, \partial\Omega) \quad \text{(Kawohl 1990)}
\]

and the limiting deq and bvp is \(|Du| = 1\) in \(\Omega\), \(u = 0\) on \(\partial\Omega\) (Bhattacharya, DiBenedetto, Manfredi 1991)

It has a unique viscosity solution, but many distributional solutions.
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It has a unique viscosity solution, but many distributional solutions.

\[
\lim_{p \to 1} u_p(x) = \begin{cases} 
0 & \text{if } \Omega \text{ is small} \\
\text{discontinuous} & \text{if } \Omega \text{ is inbetween} \\
+\infty & \text{if } \Omega \text{ is large}
\end{cases}
\]
Why this strange behaviour as $p \to 1$?

The limiting equation $-\Delta_1 u = 1$ reads $(n - 1)H = 1$ or $H = \frac{1}{n-1}$ in intrinsic coordinates.
Why this strange behaviour as $p \to 1$?

The limiting equation $-\Delta_1 u = 1$ reads $(n - 1)H = 1$ or $H = \frac{1}{n-1}$ in intrinsic coordinates.

Level surfaces satisfying this curvature condition in $\Omega$ are boundaries of so-called Cheeger sets.

A set $C_\Omega$ is a Cheeger set of $\Omega$ if it infimizes $|\partial D|/|D|$ among all smooth subsets of $\Omega$, ... this would be an extra talk.
\(-\Delta_p u = 1\)

What about \(-\Delta^N_p u_p = 1\) in \(\Omega\), \(u_p = 0\) on \(\partial\Omega\)?

For \(p \in (1, \infty]\) there exists a unique viscosity solution (Lu Wang 2008), details in the published version of this talk.

For \(p = \infty\) the equation reads \(-u_{\nu\nu} = 1\) in \(\Omega\),

and for \(p = 1\) it is \(|\nabla u|(n-1)H = 1\) in \(\Omega\).

They are degenerate elliptic in the sense of viscosity solutions.
Serrin and Weinberger

proved in 1971 that the following overdet. bvp cannot have a solution in a smooth simply connected domain unless $\Omega$ is a ball.

$$-\Delta u = 1 \quad \text{in } \Omega,$$

$$u = 0 \quad \text{and} \quad -\frac{\partial u}{\partial \nu} = a = \text{const.} \quad \text{on } \partial \Omega.$$  

Physical interpretation: Laminar flow in a noncircular pipe cannot have constant shear stress on the wall of the pipe.
Serrin’s proof uses the moving plane method and applies to positive classical solutions of autonomous strongly elliptic equations

\[-\sum_{i,j=1}^{n} a_{ij}(u, |\nabla u|)u_{x_i x_j} = f(u, |\nabla u|),\]

while Weinberger’s proof is given only for \(-\Delta u = 1\) and uses both variational methods and (other) maximum principles.

Does the proof at least extend to \(-\Delta_p u = 1\)?
Serrin’s proof uses the moving plane method and applies to positive classical solutions of autonomous strongly elliptic equations

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Does the proof at least extend to \(-\Delta_p u = 1\)?

Yes (Farina Kawohl 2008)
1) \( P(x) := \frac{2(p-1)}{p} |\nabla u(x)|^p + \frac{2}{n} u(x) \) attains max. over \( \bar{\Omega} \) on \( \partial \Omega \), and thus \( P(x) \leq \frac{2(p-1)}{p} a^p =: c \) in \( \Omega \).
1) $P(x) := \frac{2(p-1)}{p}|\nabla u(x)|^p + \frac{2}{n} u(x)$ attains max. over $\Omega$ on $\partial \Omega$, and thus $P(x) \leq \frac{2(p-1)}{p} a^p =: c$ in $\Omega$.

2) Show that $\int_{\Omega} P(x)dx = c|\Omega|$, then $P(x) \equiv c$ on $\overline{\Omega}$.
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2) Show that \( \int_{\Omega} P(x) \, dx = c|\Omega| \), then \( P(x) \equiv c \) on \( \overline{\Omega} \).

3) Show that \( P \equiv c \) in \( \Omega \) implies radial symmetry.
1) \( P(x) := \frac{2(p-1)}{p} |\nabla u(x)|^p + \frac{2}{n} u(x) \) attains max. over \( \Omega \) on \( \partial \Omega \), and thus \( P(x) \leq \frac{2(p-1)}{p} a^p =: c \) in \( \Omega \).

2) Show that \( \int_{\Omega} P(x) \, dx = c |\Omega| \), then \( P(x) \equiv c \) on \( \overline{\Omega} \).

3) Show that \( P \equiv c \) in \( \Omega \) implies radial symmetry.

Caution with 1) and 2):

1) \( u \notin C^3 \), so \( -\Delta P + \ldots \leq 0 \) is problematic. Regularize

2) \( u \notin C^2 \), so classical Pohožaev identities need adjustments to \( C^1 \)-functions by Degiovanni, Musesti, Squassina (2003).
Proof that $P \equiv c$ in $\Omega$:

Testing $-\Delta_p u = 1$ with $u$ gives

$$\int_\Omega |\nabla u|^p \, dx = \int_\Omega u \, dx \quad (1)$$
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test with $(x, \nabla u)$:

$$-\int_{\Omega} \Delta_p u(x, \nabla u) = \int_{\Omega} (x, \nabla u) = -n \int_{\Omega} u \quad (2)$$
Proof that $P \equiv c$ in $\Omega$:

Testing $-\Delta_p u = 1$ with $u$ gives
\[ \int_\Omega |\nabla u|^p \, dx = \int_\Omega u \, dx \quad (1) \]

Test with $(x, \nabla u)$:
\[ -\int_\Omega \Delta_p u(x, \nabla u) = \int_\Omega (x, \nabla u) = -n \int_\Omega u \quad (2) \]

Lhs of (2) = \[ \int_\Omega |\nabla u|^{p-2} \nabla u \nabla (x, \nabla u) - \int_{\partial \Omega} a^{p-2} u_\nu (x, \nabla u) \]
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Testing $-\Delta_p u = 1$ with $u$ gives
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Lhs of (2) =
$$\int_{\Omega} |\nabla u|^{p-2} \nabla u \nabla (x, \nabla u) - \int_{\partial \Omega} a^{p-2} u \nu(x, \nabla u)$$

$$= \int_{\Omega} |\nabla u|^{p-2} \left[ |\nabla u|^2 + (x, \nabla (|\nabla u|^2/2)) \right] - \int_{\partial \Omega} a^p(x, \nu)$$
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Lhs of (2) =

$$\int_{\Omega} |\nabla u|^{p-2} \nabla u \nabla (x, \nabla u) - \int_{\partial \Omega} a^{p-2} u \nu (x, \nabla u)$$

$$= \int_{\Omega} |\nabla u|^{p-2} \left[ |\nabla u|^2 + (x, \nabla (\frac{|\nabla u|^2}{2})) \right] - \int_{\partial \Omega} a^p (x, \nu)$$

$$= \int_{\Omega} |\nabla u|^p + (x, \nabla (\frac{|\nabla u|^p}{p})) \, dx - a^p \, n \, |\Omega|$$

$$= \int_{\Omega} |\nabla u|^p - n \frac{|\nabla u|^p}{p} \, dx + \int_{\partial \Omega} \frac{a^p}{p} (x, \nu) \, ds - a^p \, n \, |\Omega|$$
Proof that $P \equiv c$ in $\Omega$:

Testing $-\Delta_p u = 1$ with $u$ gives

$$\int_\Omega |\nabla u|^p dx = \int_\Omega udx \quad (1)$$

Test with $(x, \nabla u)$:

$$-\int_\Omega \Delta_p u(x, \nabla u) = \int_\Omega (x, \nabla u) = -n \int_\Omega u \quad (2)$$

Lhs of (2) = $\int_\Omega |\nabla u|^{p-2} \nabla u \nabla (x, \nabla u) - \int_{\partial \Omega} a^{p-2} u \nu (x, \nabla u)$

$$= \int_\Omega |\nabla u|^{p-2} \left[ |\nabla u|^2 + (x, \nabla (\frac{|\nabla u|^2}{2})) \right] - \int_{\partial \Omega} a^p (x, \nu)$$

$$= \int_\Omega |\nabla u|^p + (x, \nabla \left( \frac{|\nabla u|^p}{p} \right)) dx - a^p n |\Omega|$$

$$= \int_\Omega |\nabla u|^p - n \frac{|\nabla u|^p}{p} dx + \int_{\partial \Omega} \frac{a^p}{p} (x, \nu) ds - a^p n |\Omega|$$

$$= \int_\Omega n \left[ \frac{1}{n} |\nabla u|^p - \frac{|\nabla u|^p}{p} \right] dx - \frac{p-1}{p} a^p n |\Omega|$$. 

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Proof that \( P \equiv c \) in \( \Omega \):

Testing \( -\Delta_p u = 1 \) with \( u \) gives
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Test with \( (x, \nabla u) \):
\[-\int_\Omega \Delta_p u(x, \nabla u) = \int_\Omega (x, \nabla u) = -n \int_\Omega u \quad (2)\]

Lhs of (2) = \[
\int_\Omega |\nabla u|^{p-2} \nabla u \nabla (x, \nabla u) - \int_{\partial\Omega} a^{p-2} u_\nu (x, \nabla u)
\]
\[
= \int_\Omega |\nabla u|^{p-2} \left[ |\nabla u|^2 + (x, \nabla \left( \frac{|\nabla u|^2}{2} \right)) \right] - \int_{\partial\Omega} a^p (x, \nu)
\]
\[
= \int_\Omega |\nabla u|^p + (x, \nabla \left( \frac{|\nabla u|^p}{p} \right)) \, dx - a^p \, n \, |\Omega|
\]
\[
= \int_\Omega |\nabla u|^p - n \frac{|\nabla u|^p}{p} \, dx + \int_{\partial\Omega} \frac{a^p}{p} (x, \nu) \, ds - a^p \, n \, |\Omega|
\]
\[
= \int_\Omega n \left[ \frac{1}{n} |\nabla u|^p - \frac{|\nabla u|^p}{p} \right] \, dx - \frac{p-1}{p} a^p \, n \, |\Omega|
\]

Now \( \frac{2}{n} \) (2) = \[
\int_\Omega \frac{2}{n} |\nabla u|^p - \frac{2}{p} |\nabla u|^p \, dx - c |\Omega| = -2 \int_\Omega u
\]
Proof that $P \equiv c$ in $\Omega$:

Testing $-\Delta_p u = 1$ with $u$ gives

$$\int_\Omega |\nabla u|^p \, dx = \int_\Omega u \, dx \quad (1)$$

test with $(x, \nabla u)$:

$$-\int_\Omega \Delta_p u(x, \nabla u) = \int_\Omega (x, \nabla u) = -n \int_\Omega u \quad (2)$$

Lhs of (2) =

$$\int_\Omega |\nabla u|^{p-2} \nabla u \nabla (x, \nabla u) - \int_{\partial \Omega} a^{p-2} u \nu (x, \nabla u)$$

$$= \int_\Omega |\nabla u|^{p-2} \left[ |\nabla u|^2 + (x, \nabla (\frac{|\nabla u|^2}{2})) \right] - \int_{\partial \Omega} a^p (x, \nu)$$

$$= \int_\Omega |\nabla u|^p + (x, \nabla (\frac{|\nabla u|^p}{p})) \, dx - a^p n |\Omega|$$

$$= \int_\Omega |\nabla u|^p - n \frac{|\nabla u|^p}{p} \, dx + \int_{\partial \Omega} \frac{a^p}{p} (x, \nu) \, ds - a^p n |\Omega|$$

$$= \int_\Omega n \left[ \frac{1}{n} |\nabla u|^p - \frac{|\nabla u|^p}{p} \right] \, dx - \frac{p-1}{p} a^p n |\Omega|$$

now $\frac{2}{n} (2) = \int_\Omega \frac{2}{n} |\nabla u|^p - \frac{2}{p} |\nabla u|^p \, dx - c|\Omega| = -2 \int_\Omega u$

so by (1):

$$\int_\Omega \frac{2}{n} u + \frac{2(p-1)}{p} |\nabla u|^p \, dx = c|\Omega| \quad (= \int_\Omega P(x) \, dx)$$
\[ P \equiv c \text{ in } \Omega \text{ implies symmetry:} \]

a) If \( \partial \Omega \in C^{2,\alpha} \), then \( P_\nu = 0 \) on \( \partial \Omega \) implies \( H = \frac{1}{n} a^{1-p} \), because

\[
P_\nu = 2(p-1)|u_\nu|^p u_\nu u_{\nu\nu} + \frac{2}{n} u_\nu = \left[ (p-1)|u_\nu|^{p-2} u_{\nu\nu} + \frac{1}{n} \right] 2u_\nu = 0
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and \( \Delta_{p}u = -1 = (p-1)|u_{\nu}|^{p-2}u_{\nu\nu} + (n-1)H|u_{\nu}|^{p-2}u_{\nu} \) imply \( H = \frac{1}{n} a^{1-p} \) on \( \partial \Omega \). Done.
$P \equiv c$ in $\Omega$ implies symmetry:

a) If $\partial \Omega \in C^{2,\alpha}$, then $P_\nu = 0$ on $\partial \Omega$ implies $H = \frac{1}{n} a^{1-p}$, because

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b) If $\partial \Omega$ is not smooth, consider $\Gamma = \{x | u(x) = \varepsilon\}$. 
\[ P \equiv c \text{ in } \Omega \text{ implies symmetry:} \]

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b) If \( \partial \Omega \) is not smooth, consider \( \Gamma = \{ x \mid u(x) = \varepsilon \} \).

\( u \in C^{1,\beta}(\Omega) \) & \( u_\nu = -a \) on \( \partial \Omega \) imply \( \nabla u \neq 0 \) and \( u \in C^{2,\beta} \) near \( \Gamma \).

Bernd Kawohl

Variations on the \( p \)-Laplacian
\[ P \equiv c \text{ in } \Omega \text{ implies symmetry:} \]

a) If \( \partial \Omega \in C^{2,\alpha} \), then \( P_\nu = 0 \) on \( \partial \Omega \) implies \( H = \frac{1}{n} a^{1-p} \), because

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b) If \( \partial \Omega \) is not smooth, consider \( \Gamma = \{ x \mid u(x) = \varepsilon \} \).

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Thus \( \Gamma \in C^{2,\alpha} \) and \( P_\nu = 0 \) on \( \Gamma \), i.e. \( \left[ (p-1)|u_\nu|^{p-2}u_{\nu \nu} + \frac{1}{n} \right] = 0 \)
$P \equiv c$ in $\Omega$ implies symmetry:

a) If $\partial \Omega \in C^{2,\alpha}$, then $P_\nu = 0$ on $\partial \Omega$ implies $H = \frac{1}{n}a^{1-p}$, because

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b) If $\partial \Omega$ is not smooth, consider $\Gamma = \{x \mid u(x) = \varepsilon\}$. $u \in C^{1,\beta}(\Omega)$ & $u_\nu = -a$ on $\partial \Omega$ imply $\nabla u \neq 0$ and $u \in C^{2,\beta}$ near $\Gamma$.

Thus $\Gamma \in C^{2,\alpha}$ and $P_\nu = 0$ on $\Gamma$, i.e. $\left[(p-1)|u_\nu|^{p-2}u_{\nu\nu} + \frac{1}{n}\right] = 0$

Now we get $-1 - (n-1)H|u_\nu|^{p-1} + \frac{1}{n} = 0$ or $H = h(|u_\nu|)$ on $\Gamma$. 
\( P \equiv c \text{ in } \Omega \) implies symmetry:

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\( H = \frac{1}{n} a^{1-p} \) on \( \partial \Omega \). Done.

b) If \( \partial \Omega \) is not smooth, consider \( \Gamma = \{ x \mid u(x) = \varepsilon \} \).

\( u \in C^{1,\beta} (\Omega) \& u_\nu = -a \) on \( \partial \Omega \) imply \( \nabla u \neq 0 \) and \( u \in C^{2,\beta} \) near \( \Gamma \).

Thus \( \Gamma \in C^{2,\alpha} \) and \( P_\nu = 0 \) on \( \Gamma \), i.e.

\[
\left( (p-1)|u_\nu|^{p-2}u_{\nu\nu} + \frac{1}{n} \right) = 0
\]

Now we get \( -1 - (n-1)H|u_\nu|^{p-1} + \frac{1}{n} = 0 \) or \( H = h(|u_\nu|) \) on \( \Gamma \).

But since \( P \equiv c \), \( |\nabla u| = g(u) \) and \( H = h(g(\varepsilon)) = \text{const.} \) on \( \Gamma \).  \( \Box \)
There is also an anisotropic version of the Serrin/Weinberger result

**Theorem** (A. Cianchi & P. Salani, Dec 2008) *Suppose that* $H$ *is a norm with a strictly convex unit ball and that* $u$ *is a minimizer of*

$$
\int_{\Omega} \left( \frac{1}{2} H(\nabla v)^2 - v \right) \, dx \text{ in } W^{1,2}_0(\Omega), \quad \text{and } H(\nabla u) = a \text{ on } \partial \Omega.
$$

*Then* $\Omega$ *is a ball in the dual norm* $H_0$ *to* $H$ *of suitable radius* $r$ *and*

$$
u(x) = \frac{r^2 - H_0(x)^2}{2n}.
$$

The proof of Cianchi and Salani uses entirely different methods. Independently, in May 2009 Guofang Wang and Chao Xia gave another proof that follows our method.
What about $-\Delta_p u_p = 1$ in $\Omega$, $u_p = 0$ AND $|\nabla u| = a$ on $\partial \Omega$?

For $p = 1$ we look at

$$|\nabla u|(n - 1)H = 1 \text{ in } \Omega, \quad |\nabla u| = a \text{ and } u = 0 \text{ on } \partial \Omega.$$ 

So a $C^2$ solution on a smooth domain satisfies $H \equiv 1/(a(n - 1))$ on $\partial \Omega$. Apply Alexandrov to see that $\Omega = \text{ball of radius } (n - 1)a$.

For $p = \infty$ the overdetermined bvp.

$$-u_{\nu\nu} = 1 \text{ in } \Omega, \quad |\nabla u| = a \text{ and } u = 0 \text{ on } \partial \Omega$$

can have $C^1$ viscosity solutions on special (non-ball) domains, e.g. stadium domains.

Buttazzzo Kawohl 2011
For fixed $p \in (1, \infty)$ consider the second eigenfunction

$$\Delta_p u_2 + \lambda_2 |u_2|^{p-2} u_2 = 0 \text{ in } \Omega, \quad u = 0 \text{ on } \partial\Omega.$$ 

It changes sign, it has two nodal domains, it can be characterized as a mountain pass going from $u_1$ to $-u_1$. (Cuesta, de Figuereido, Gossez 1999)
For $p \in (1, \infty)$ and $\Omega \subset \mathbb{R}^2$ the eigenfunction $u_2$ has a nodal line.

Conjectures:

a) For $\Omega$ a disk, the nodal line is a diameter.

b) For $\Omega$ a square the nodal line is diagonal if $p \in (2, \infty)$ and horizontal or vertical if $p \in (1, 2)$.

Conjectures a) and b) hold for $p = 1$ (Enea Parini 2009), $p = 2$, and $p = \infty$ (Juutinen & Lindqvist 2005). Moreover, they are supported for general $p$ by numerical evidence of Jiří Horáček (2009).
\( \Omega \) a disk, \( p = 1.1 \), courtesy of J. Horák
\( p \)-harmonic functions \( -\Delta_p u = 1 \) overetermined problems open problems

\( \Omega \) a square, \( p = 5 \), courtesy of J. Horák
\[ -\Delta_p u = 1 \]

\( \Omega \) a square, \( p = 1.1 \), courtesy of J. Horák