# Variations on the p-Laplacian 

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## Topics

p-harmonic functions

$$
-\Delta_{p} u=1
$$

overdetermined problems
open problems

## Laplace operator

$$
\Delta u=u_{x_{1} x_{1}}+\ldots+u_{x_{n} x_{n}}=u_{\nu \nu}+u_{\nu} \operatorname{div}(\nu)
$$

where $\nu(x)=-\frac{\nabla u(x)}{|\nabla u(x)|}$ is direction of steepest descent. In fact,

$$
\operatorname{div}(\nu)=-\frac{\Delta u}{|\nabla u|}+\frac{u_{x_{i}} u_{x_{j}} u_{x_{i} x_{j}}}{|\nabla u|^{3}}=-\frac{\Delta u}{|\nabla u|}+\frac{u_{\nu \nu}}{|\nabla u|}
$$

so that $\Delta u=u_{\nu \nu}-|\nabla u| \operatorname{div}(\nu)=u_{\nu \nu}+u_{\nu} \operatorname{div}(\nu)$ or

$$
\Delta u=u_{\nu \nu}+u_{\nu}(n-1) H
$$

with $H$ denoting mean curvature of a level set of $u$.
For radial $u$ recall $\quad \Delta u=u_{r r}+\frac{n-1}{r} u_{r}$.

For $p \in(1, \infty)$ one can write the $p$-Laplace operator as

$$
\begin{aligned}
\Delta_{p} u & =\operatorname{div}\left(|\nabla u|^{p-2} \nabla u\right)=|\nabla u|^{p-2}\left[\Delta u+(p-2) u_{\nu \nu}\right] \\
& =|\nabla u|^{p-2}\left[(p-1) u_{\nu \nu}+(n-1) H u_{\nu}\right]
\end{aligned}
$$

and the normalized or game-theoretic $p$-Laplace operator as

$$
\begin{aligned}
\Delta_{p}^{N} u & =\frac{1}{p}|\nabla u|^{2-p} \operatorname{div}\left(|\nabla u|^{p-2} \nabla u\right) \\
& =\frac{p-1}{p} u_{\nu \nu}+\frac{1}{p}(n-1) H u_{\nu}=\frac{p-1}{p} \Delta_{\infty}^{N} u+\frac{1}{p} \Delta_{1}^{N} u .
\end{aligned}
$$

Observe $\Delta_{\infty}^{N} u=u_{\nu \nu}, \Delta_{2}^{N} u=\frac{1}{2} \Delta u$ and $\Delta_{1}^{N} u=|\nabla u| \operatorname{div}\left(\frac{\nabla u}{|\nabla u|}\right)$.

## p-harmonic functions

Given $\Omega \subset \mathbb{R}^{n}$ bounded, $\partial \Omega$ of class $C^{2, \alpha}$ and $g(x) \in W^{1, p}(\Omega)$

$$
\begin{align*}
-\Delta_{p} u & =0 & \text { in } \Omega,  \tag{1}\\
u(x) & =g(x) & \text { on } \partial \Omega . \tag{2}
\end{align*}
$$

$u$ can be charactzerized as the unique (weak) solution of the strictly convex variational problem

$$
\begin{equation*}
\text { Minimize } \quad I_{p}(v)=\|\nabla v\|_{L^{p}(\Omega)} \quad \text { on } g(x)+W_{0}^{1, p}(\Omega) \text {, } \tag{3}
\end{equation*}
$$

so that

$$
\begin{equation*}
\int_{\Omega}|\nabla u|^{p-2} \nabla u \nabla \phi d x=0 \text { for every } \phi \in W_{0}^{1, p}(\Omega) . \tag{4}
\end{equation*}
$$

It is well known, that weak solutions are locally of class $C^{1, \alpha}$. They are even of class $C^{\infty}$ wherever their gradient does not vanish.

One can show (Juutinen, Lindqvist, Manfredi 2001) that weak solutions are also viscosity solutions of the associated Euler equation

$$
F_{p}\left(D u, D^{2} u\right)=-|D u|^{p-4}\left(|D u|^{2} \operatorname{trace} D^{2} u+\left\langle D^{2} u D u, D u\right\rangle\right)=0
$$

Incidentally, only for $p \in(1,2)$ does this imply that they are also viscosity solutions of the normalized equation

$$
F_{p}^{N}\left(D u, D^{2} u\right)=-\frac{1}{p} \operatorname{trace} D^{2} u-\frac{p-2}{p} \frac{\left\langle D^{2} u D u, D u\right\rangle}{|D u|^{2}}=0
$$

What happens as $p \rightarrow \infty$ ? For $g \in W^{1, \infty}(\Omega)$ the family $u_{p}$ is uniformly bounded in $W^{1, p}$ because $I_{p}\left(u_{p}\right) \leq I_{p}(g) \leq\|\nabla g\|_{\infty}|\Omega|$.
Wolog $|\Omega|:=1$. For $q>n$ fixed and $p>q$ one finds
$\left\|\nabla u_{p}\right\|_{q} \leq\left\|\nabla u_{p}\right\|_{p}|\Omega|^{(p-q) / p q} \leq\|\nabla g\|_{\infty}|\Omega|^{1+1 / q}+1 \quad$ as $p \rightarrow \infty$,
so $u_{p} \rightarrow u_{\infty}$ in some $C^{\alpha}$.
By the stability theorem for viscosity solutions $u_{\infty}$ should be viscosity solution to a limit equation $F_{\infty}\left(D u, D^{2} u\right)=0$.
What is this equation? Let us check the condition for subsolutions.

Let $\varphi$ be a $C^{2}$ testfunction s.th. $\varphi-u_{\infty}$ has a min at $x_{\infty}$ and $\nabla \varphi\left(x_{\infty}\right) \neq 0$. Then wolog $\varphi-u_{p}$ has a min at $x_{p}$ near $x_{\infty}$ and $x_{p} \rightarrow x_{\infty}$ as $p \rightarrow \infty$. Since $u_{p}$ is viscosity subsolution

$$
-|D \varphi|^{p-4}\left[|D \varphi|^{2} \Delta \varphi+(p-2)\left\langle D^{2} \varphi D \varphi, D \varphi\right\rangle\right]\left(x_{p}\right) \leq 0
$$

or

$$
-\frac{p-2}{p}\left\langle D^{2} \varphi D \varphi, D \varphi\right\rangle\left(x_{p}\right) \leq \frac{1}{p}|D \varphi|^{2} \Delta \varphi\left(x_{p}\right) .
$$

$p \rightarrow \infty$ gives
$\left\langle D^{2} \varphi D \varphi, D \varphi\right\rangle\left(x_{\infty}\right):=-\Delta_{\infty} \varphi \leq 0$.
$\ldots$ Thus $u_{\infty}$ is (unique) viscosity solution of $-\Delta_{\infty} u=0$ in $\Omega$,

$$
u=g \text { on } \partial \Omega
$$

It is worth noting that the variational problem

$$
\begin{equation*}
\text { Minimize } \quad I_{\infty}(v)=\|\nabla v\|_{L^{\infty}(\Omega)} \quad \text { on } g(x)+W_{0}^{1, \infty}(\Omega) \tag{5}
\end{equation*}
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can have many solutions,

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can have many solutions,
e.g. the minimum of two cones (not $C^{1}$ ) or $u_{\infty} \in C^{1, \alpha}$ (Savin).


What happens to $p$-harmonic functions as $p \rightarrow 1$ ? I. g. no uniform convergence, but Juutinen (2005) found sufficient conditions:
If $g \in C(\bar{\Omega})$ and $\Omega$ convex, then $u_{p} \rightarrow u_{1}$ uniformly as $p \rightarrow 1$.
Moreover, $u_{1}$ is unique minimizer of

$$
\begin{aligned}
& E_{1}(v)=\sup \left\{\int_{\Omega} u \operatorname{div} \sigma d x ; \sigma \in C_{0}^{\infty}\left(\Omega, \mathbb{R}^{n}\right),|\sigma(x)| \leq 1 \text { in } \Omega\right\} \\
& \text { on }\{v \in B V(\Omega) \cap C(\bar{\Omega}), v=g \text { on } \partial \Omega\}
\end{aligned}
$$

Here the limiting variational problem has a unique solution, while the limiting Euler equation can have many viscosity solutions.

Heuristic reason for uniqueness of minimizer of the TV-functional:

If there are two minimizers $u$ and $v$ (for simplicity in $W^{1,1}(\Omega)$ ) of $E_{1}$, then any convex combination $w=t u+(1-t) v$ would also be minimizer,
hence level lines of $u$ are also level lines of $v, \nabla u \| \nabla v, v=f(u)$.
Dirichlet cond. implies $f(g)=g$, so that $f=I d$ on range ${ }_{\partial \Omega}(g)$.
But since $\min _{\partial \Omega} g \leq u, v \leq \max _{\partial \Omega} g$ in $\Omega$ we find $f(u)=u$ in $\Omega$.

Nonuniqueness of viscosity solutions to the Dirichlet problem

$$
-\Delta_{1} u=0 \text { in } \Omega, u=g \text { on } \partial \Omega
$$

Sternberg, Ziemer (1994) gave counterexample: $\Omega=B(0,1) \in \mathbb{R}^{2}$, $g\left(x_{1}, x_{2}\right)=\cos (2 \varphi)$ has a whole family $u_{\lambda}$ of viscosity solutions, $\lambda \in[-1,1]$, but only $u_{0}$ minimizes $E_{1}$. In fact,
$u_{\lambda}\left(x_{1}, x_{2}\right)=\left\{\begin{array}{cl}2 x_{1}^{2}-1 & \text { left and right of rectangle } \\ \lambda & \text { in the rectangle generated by } \cos (2 \varphi)=\lambda \\ 1-2 x_{2}^{2} & \text { on top and bottom }\end{array}\right.$
is viscosity sol. of both $-\Delta_{1} u=0$ and $-\Delta_{1}^{N} u=\kappa|\nabla u|=0$ in $\Omega$.

(level) plot of $u_{\lambda}$

## definition of viscosity solutions for discontinuous $F$

$u$ is a viscosity solution of $F\left(D u, D^{2} u\right)=0$, iff sub- and supersol.
$u$ is subsol. if for every $x \in \Omega$ and $\varphi \in C^{2}$ s.th. $\varphi-u$ has min at $x$ the ineq. $F_{*}\left(D \varphi, D^{2} \varphi\right) \leq 0$ holds. Here $F_{*}=\mathrm{Isc}$ hull of $F$. $u$ is supersol. if for every $x \in \Omega$ and $\varphi \in C^{2}$ s.th. $\varphi-u$ has max at $x$ the ineq. $F^{*}\left(D \varphi, D^{2} \varphi\right) \geq 0$ holds. Here $F^{*}=$ usc hull of $F$.

$$
F_{p}^{N}(q, X)=\left\{\begin{array}{cl}
-\frac{1}{p}\left(\delta_{i j}+(p-1) \frac{q_{i} q_{j}}{|q|^{2}}\right) X_{i j} & \text { if } q \neq 0 \\
? & \text { if } q=0
\end{array}\right.
$$

Symm. matrix $X$ has eigenvalues $\lambda_{1}(X) \leq \lambda_{2}(X) \leq \ldots \leq \lambda_{n}(X)$

$$
\begin{aligned}
& F_{p *}^{N}(0, X)= \begin{cases}-\frac{1}{p} \sum_{i=1}^{n-1} \lambda_{i}-\frac{p-1}{p} \lambda_{n} & \text { if } p \in[2, \infty] \\
-\frac{1}{p} \sum_{i=2}^{n} \lambda_{i}-\frac{p-1}{p} \lambda_{1} & \text { if } p \in[1,2]\end{cases} \\
& F_{p}^{N^{*}}(0, X)= \begin{cases}-\frac{1}{p} \sum_{i=2}^{n} \lambda_{i}-\frac{p-1}{p} \lambda_{1} & \text { if } p \in[2, \infty] \\
-\frac{1}{p} \sum_{i=1}^{n-1} \lambda_{i}-\frac{p-1}{p} \lambda_{n} & \text { if } p \in[1,2]\end{cases}
\end{aligned}
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In particular, for $n=2, F_{1}^{N}(0, X)=-\lambda_{2}$ and $F_{1}^{N^{*}}(0, X)=-\lambda_{1}$,

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\end{aligned}
$$

In particular, for $n=2, F_{1}^{N}(0, X)=-\lambda_{2}$ and $F_{1}^{N^{*}}(0, X)=-\lambda_{1}$, so that we require $-\lambda_{2}\left(D^{2} \varphi\right) \leq 0$ for subsols. if $\nabla \varphi(x)=0$ and $-\lambda_{1}\left(D^{2} \varphi\right) \geq 0$ for supersols. if $\nabla \varphi(x)=0$

## $-\Delta_{p} u=1$

The Dirichlet problem $-\Delta_{p} u_{p}=1$ in $\Omega, u_{p}=0$ on $\partial \Omega$ can be treated in a similar way. Again surprises as $p \rightarrow \infty$ or 1 .

$$
\lim _{p \rightarrow \infty} u_{p}(x)=d(x, \partial \Omega) \quad(\text { Kawohl 1990) }
$$

and the limiting deq and bvp is $|D u|=1$ in $\Omega, u=0$ on $\partial \Omega$
(Bhattacharya, DiBenedetto, Manfredi 1991)
It has a unique viscosity solution, but many distributional solutions.

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$$
\lim _{p \rightarrow 1} u_{p}(x)=\left\{\begin{array}{cl}
0 & \text { if } \Omega \text { is small } \\
\text { discontinuous } & \text { if } \Omega \text { is inbetween } \\
+\infty & \text { if } \Omega \text { is large }
\end{array}\right.
$$

Why this strange behaviour as $p \rightarrow 1$ ?
The limiting equation $-\Delta_{1} u=1$ reads $(n-1) H=1$ or $H=\frac{1}{n-1}$ in intrinsic coordinates.

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Level surfaces satisfying this curvature condition in $\Omega$ are boundaries of so-called Cheeger sets.

A set $C_{\Omega}$ is a Cheeger set of $\Omega$ if it infimizes $|\partial D| /|D|$ among all smooth subsets of $\Omega, \ldots$ this would be an extra talk.
$-\Delta_{p}^{N} u=1$

What about $-\Delta_{p}^{N} u_{p}=1$ in $\Omega, u_{p}=0$ on $\partial \Omega$ ?
For $p \in(1, \infty]$ there exists a unique viscosity solution (Lu Wang 2008), details in the published version of this talk.

For $p=\infty$ the equation reads and for $p=1$ it is

$$
\begin{aligned}
-u_{\nu \nu} & =1 \text { in } \Omega, \\
|\nabla u|(n-1) H & =1 \text { in } \Omega .
\end{aligned}
$$

They are degenerate elliptic in the sense of viscosity solutions.

## Serrin and Weinberger

proved in 1971 that the following overdet. bvp cannot have a solution in a smooth simply connected domain unless $\Omega$ is a ball.

$$
\begin{gathered}
-\Delta u=1 \quad \text { in } \Omega \\
u=0 \quad \text { and } \quad-\frac{\partial u}{\partial \nu}=a=\text { const. on } \partial \Omega .
\end{gathered}
$$

Physical interpretation: Laminar flow in a noncircular pipe cannot have constant shear stress on the wall of the pipe.

Serrin's proof uses the moving plane method and applies to positive classical solutions of autonomous strongly elliptic equations

$$
-\sum_{i, j=1}^{n} a_{i j}(u,|\nabla u|) u_{x_{i} x_{j}}=f(u,|\nabla u|)
$$

while Weinberger's proof is given only for $-\Delta u=1$ and uses both variational methods and (other) maximum principles.

Does the proof at least extend to $-\Delta_{p} u=1$ ?

Serrin's proof uses the moving plane method and applies to positive classical solutions of autonomous strongly elliptic equations

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while Weinberger's proof is given only for $-\Delta u=1$ and uses both variational methods and (other) maximum principles.

Does the proof at least extend to $-\Delta_{p} u=1$ ?
Yes (Farina Kawohl 2008)

1) $P(x):=\frac{2(p-1)}{p}|\nabla u(x)|^{p}+\frac{2}{n} u(x)$ attains max. over $\bar{\Omega}$ on $\partial \Omega$, and thus $P(x) \leq \frac{2(p-1)}{p} a^{p}=: c$ in $\Omega$.
2) $P(x):=\frac{2(p-1)}{p}|\nabla u(x)|^{p}+\frac{2}{n} u(x)$ attains max. over $\bar{\Omega}$ on $\partial \Omega$, and thus $P(x) \leq \frac{2(p-1)}{p} a^{p}=: c$ in $\Omega$.
3) Show that $\int_{\Omega} P(x) d x=c|\Omega|$, then $P(x) \equiv c$ on $\bar{\Omega}$.
4) $P(x):=\frac{2(p-1)}{p}|\nabla u(x)|^{p}+\frac{2}{n} u(x)$ attains max. over $\bar{\Omega}$ on $\partial \Omega$, and thus $P(x) \leq \frac{2(p-1)}{p} a^{p}=: c$ in $\Omega$.
5) Show that $\int_{\Omega} P(x) d x=c|\Omega|$, then $P(x) \equiv c$ on $\bar{\Omega}$.
6) Show that $P \equiv c$ in $\Omega$ implies radial symmetry.
7) $P(x):=\frac{2(p-1)}{p}|\nabla u(x)|^{p}+\frac{2}{n} u(x)$ attains max. over $\bar{\Omega}$ on $\partial \Omega$, and thus $P(x) \leq \frac{2(p-1)}{p} a^{p}=: c$ in $\Omega$.
8) Show that $\int_{\Omega} P(x) d x=c|\Omega|$, then $P(x) \equiv c$ on $\bar{\Omega}$.
9) Show that $P \equiv c$ in $\Omega$ implies radial symmetry.

Caution with 1) and 2):

1) $u \notin C^{3}$, so $-\Delta P+\ldots \leq 0$ is problematic. Regularize
2) $u \notin C^{2}$, so classical Pohožaev identities need adjustments to $C^{1}$-functions by Degiovanni, Musesti, Squassina (2003).

## Proof that $P \equiv c$ in $\Omega$ :

Testing $-\Delta_{p} u=1$ with $u$ gives

$$
\begin{equation*}
\int_{\Omega}|\nabla u|^{p} d x=\int_{\Omega} u d x \tag{1}
\end{equation*}
$$

Proof that $P \equiv c$ in $\Omega$ :
Testing $-\Delta_{p} u=1$ with $u$ gives $\quad \int_{\Omega}|\nabla u|^{p} d x=\int_{\Omega} u d x$ test with $(x, \nabla u):-\int_{\Omega} \Delta_{p} u(x, \nabla u)=\int_{\Omega}(x, \nabla u)=-n \int_{\Omega} u$

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lhs of $(2)=\int_{\Omega}|\nabla u|^{p-2} \nabla u \nabla(x, \nabla u)-\int_{\partial \Omega} a^{p-2} u_{\nu}(x, \nabla u)$

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$$
\text { Ihs of } \begin{align*}
(2) & =\int_{\Omega}|\nabla u|^{p-2} \nabla u \nabla(x, \nabla u)-\int_{\partial \Omega} a^{p-2} u_{\nu}(x, \nabla u)  \tag{2}\\
& =\int_{\Omega}|\nabla u|^{p-2}\left[|\nabla u|^{2}+\left(x, \nabla\left(\frac{\left.\Omega u\right|^{2}}{2}\right)\right)\right]-\int_{\partial \Omega} a^{p}(x, \nu)
\end{align*}
$$

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& =\int_{\Omega}|\nabla u|^{p-2}\left[|\nabla u|^{2}+\left(x, \nabla\left(\frac{|\nabla u|^{2}}{2}\right)\right)\right]-\int_{\partial \Omega} a^{p}(x, \nu) \\
& =\int_{\Omega}|\nabla u|^{p}+\left(x, \nabla\left(\frac{|\nabla u|^{p}}{p}\right)\right) d x-a^{p} n|\Omega|
\end{align*}
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& =\int_{\Omega}|\nabla u|^{p}-n \frac{|\nabla u|^{p}}{p} d x+\int_{\partial \Omega} \frac{a^{p}}{p}(x, \nu) d s-a^{p} n|\Omega|
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$=\int_{\Omega}|\nabla u|^{p-2}\left[|\nabla u|^{2}+\left(x, \nabla\left(\frac{|\nabla u|^{2}}{2}\right)\right)\right]-\int_{\partial \Omega} a^{p}(x, \nu)$
$=\int_{\Omega}|\nabla u|^{p}+\left(x, \nabla\left(\frac{|\nabla u|^{p}}{p}\right)\right) d x-a^{p} n|\Omega|$
$=\int_{\Omega}|\nabla u|^{p}-n \frac{|\nabla u|^{p}}{p} d x+\int_{\partial \Omega} \frac{a^{p}}{p}(x, \nu) d s-a^{p} n|\Omega|$
$=\int_{\Omega} n\left[\frac{1}{n}|\nabla u|^{p}-\frac{|\nabla u|^{p}}{p}\right] d x-\frac{p-1}{p} a^{p} n|\Omega|$

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$=\int_{\Omega} n\left[\frac{1}{n}|\nabla u|^{p}-\frac{|\nabla u|^{p}}{p}\right] d x-\frac{p-1}{p} a^{p} n|\Omega|$
now $\frac{2}{n}(2)=\int_{\Omega} \frac{2}{n}|\nabla u|^{p}-\frac{2}{p}|\nabla u|^{p} d x-c|\Omega|=-2 \int_{\Omega} u$

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lhs of $(2)=\int_{\Omega}|\nabla u|^{p-2} \nabla u \nabla(x, \nabla u)-\int_{\partial \Omega} a^{p-2} u_{\nu}(x, \nabla u)$

$$
=\int_{\Omega}|\nabla u|^{p-2}\left[|\nabla u|^{2}+\left(x, \nabla\left(\frac{|\nabla u|^{2}}{2}\right)\right)\right]-\int_{\partial \Omega} a^{p}(x, \nu)
$$

$$
=\int_{\Omega}|\nabla u|^{p}+\left(x, \nabla\left(\frac{|\nabla u|^{p}}{p}\right)\right) d x-a^{p} n|\Omega|
$$

$$
=\int_{\Omega}|\nabla u|^{p}-n \frac{|\nabla u|^{p}}{p} d x+\int_{\partial \Omega} \frac{a^{p}}{p}(x, \nu) d s-a^{p} n|\Omega|
$$

$$
=\int_{\Omega} n\left[\frac{1}{n}|\nabla u|^{p}-\frac{|\nabla u|^{p}}{p}\right] d x-\frac{p-1}{p} a^{p} n|\Omega|
$$

now $\frac{2}{n}(2)=\int_{\Omega} \frac{2}{n}|\nabla u|^{p}-\frac{2}{p}|\nabla u|^{p} d x-c|\Omega|=-2 \int_{\Omega} u$
so by (1): $\quad \int_{\Omega} \frac{2}{n} u+\frac{2(p-1)}{p}|\nabla u|^{p} d x=c|\Omega| \quad\left(=\int_{\Omega} P(x) d x\right)$
$P \equiv c$ in $\Omega$ implies symmetry:
a) If $\partial \Omega \in C^{2, \alpha}$, then $P_{\nu}=0$ on $\partial \Omega$ implies $H=\frac{1}{n} a^{1-p}$, because

$$
P_{\nu}=2(p-1)\left|u_{\nu}\right|^{p-2} u_{\nu} u_{\nu \nu}+\frac{2}{n} u_{\nu}=\left[(p-1)\left|u_{\nu}\right|^{p-2} u_{\nu \nu}+\frac{1}{n}\right] 2 u_{\nu}=0
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and $\Delta_{p} u=-1=(p-1)\left|u_{\nu}\right|^{p-2} u_{\nu \nu}+(n-1) H\left|u_{\nu}\right|^{p-2} u_{\nu}$ imply $H=\frac{1}{n} a^{1-p}$ on $\partial \Omega$. Done.
$P \equiv c$ in $\Omega$ implies symmetry:
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b) If $\partial \Omega$ is not smooth, consider $\Gamma=\{x \mid u(x)=\varepsilon\}$.
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b) If $\partial \Omega$ is not smooth, consider $\Gamma=\{x \mid u(x)=\varepsilon\}$. $u \in C^{1, \beta}(\Omega) \& u_{\nu}=-a$ on $\partial \Omega$ imply $\nabla u \neq 0$ and $u \in C^{2, \beta}$ near $\Gamma$.
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Now we get $-1-(n-1) H\left|u_{\nu}\right|^{p-1}+\frac{1}{n}=0$ or $H=h\left(\left|u_{\nu}\right|\right)$ on $\Gamma$.
But since $P \equiv c,|\nabla u|=g(u)$ and $H=h(g(\varepsilon))=$ const. on $\Gamma$.

There is also an anisotropic version of the Serrin/Weinberger result Theorem (A. Cianchi \& P. Salani, Dec 2008) Suppose that $H$ is a norm with a strictly convex unit ball and that $u$ is a minimizer of

$$
\int_{\Omega}\left(\frac{1}{2} H(\nabla v)^{2}-v\right) d x \text { in } W_{0}^{1,2}(\Omega), \quad \text { and } H(\nabla u)=a \text { on } \partial \Omega .
$$

Then $\Omega$ is a ball in the dual norm $H_{0}$ to $H$ of suitable radius $r$ and

$$
u(x)=\frac{r^{2}-H_{0}(x)^{2}}{2 n}
$$

The proof of Cianchi and Salani uses entirely different methods. Independently, in May 2009 Guofang Wang and Chao Xia gave another proof that follows our method.

What about $-\Delta_{p}^{N} u_{p}=1$ in $\Omega, u_{p}=0$ AND $|\nabla u|=a$ on $\partial \Omega$ ?
For $p=1$ we look at

$$
|\nabla u|(n-1) H=1 \text { in } \Omega, \quad|\nabla u|=a \text { and } u=0 \text { on } \partial \Omega .
$$

So a $C^{2}$ solution on a smooth domain satisfies $H \equiv 1 /(a(n-1))$ on $\partial \Omega$. Apply Alexandrov to see that $\Omega=$ ball of radius $(n-1)$ a.

For $p=\infty$ the overdetermined bvp.

$$
-u_{\nu \nu}=1 \text { in } \Omega, \quad|\nabla u|=a \text { and } u=0 \text { on } \partial \Omega
$$

can have $C^{1}$ viscosity solutions on special (non-ball) domains, e.g. stadium domains.

## open problems

For fixed $p \in(1, \infty)$ consider the second eigenfunction

$$
\Delta_{p} u_{2}+\lambda_{2}\left|u_{2}\right|^{p-2} u_{2}=0 \text { in } \Omega, \quad u=0 \text { on } \partial \Omega
$$

It changes sign, it has two nodal domains, it can be characterized as a mountain pass going from $u_{1}$ to $-u_{1}$. (Cuesta, de Figuereido, Gossez 1999)

For $p \in(1, \infty)$ and $\Omega \subset \mathbb{R}^{2}$ the eigenfunction $u_{2}$ has a nodal line.
Conjectures:
a) For $\Omega$ a disk, the nodal line is a diameter.
b) For $\Omega$ a square the nodal line is diagonal if $p \in(2, \infty)$ and horizontal or vertical if $p \in(1,2)$.

Conjectures a) and b) hold for $p=1$ (Enea Parini 2009), $p=2$, and $p=\infty$ (Juutinen \& Lindqvist 2005). Moreover, they are supported for general $p$ by numerical evidence of Jiří Horák (2009).

$\Omega$ a disk, $p=1.1$, courtesy of J. Horák

$\Omega$ a square, $p=5$, courtesy of J. Horák

$\Omega$ a square, $p=1.1$, courtesy of J. Horák

Bernd Kawohl
Variations on the $p$-Laplacian

