Some bifurcation results for a semilinear elliptic equation

Francesca Gladiali

University of Sassari, Italy, fgladiali@uniss.it

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joint work with M.Grossi, F.Pacella and P.N.Srikanth.

Our problem

We consider the problem

$$\begin{cases} -\Delta u = u^p & \text{in } \Omega \\ u > 0 & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases} \tag{1}$$

where either $\Omega = A := \{x \in \mathbb{R}^N : a < |x| < b\}$, $b > a > 0$, is an annulus, $N \geq 2$, $p \in (1, +\infty)$, or $\Omega = \mathbb{R}^N \setminus B_1(0)$, is the exterior of a ball, $N \geq 3$ and $p > \frac{N+2}{N-2}$.

The case of the annulus

We consider first the case of the annulus $\Omega = A$.
Problem (1) has a radial solution $u = u(A, p)$ [Kazdan-Warner (1975)], and this radial solution is unique [Ni-Nussbaum (1985)].

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The first step in studying the bifurcation is to analyze the possible degeneracy of the radial solution u depending on the annulus or on the exponent, i.e. see if the linearized operator $L_u := -\Delta - pu^{p-1}I$ admits zero as an eigenvalue.

Radial Nondegeneracy

The Linearized Problem is

$$\begin{cases} -\Delta v - p u^{p-1} v = 0 & \text{in } A, \\ v = 0 & \text{on } \partial A \end{cases} \tag{2}$$

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Lemma

The linearized problem does not admit any nontrivial radial solution.

The radial Morse index of u is 1.

The Linearized Problem

It is easy to see that solving $L_u(v) = 0$, i.e.

$$\begin{cases} -\Delta v - pu^{p-1}v = 0 & \text{in } A, \\ v = 0 & \text{on } \partial A, \end{cases}$$

is equivalent to show that the linear operator

$$\widetilde{L}_u := |x|^2 \left(-\Delta - pu^{p-1}I\right), \ x \in A \tag{3}$$

has zero as an eigenvalue with the same boundary conditions.

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has zero as an eigenvalue with the same boundary conditions. Consider the 1-dimensional operator

$$\widehat{L}_u(v) := r^2 \left(-v'' - \frac{N-1}{r} v' - pu^{p-1} v\right) \qquad r \in (a, b) \tag{4}$$

with the same boundary conditions.

The Linearized Problem

The spectra of these operators are related by

$$\sigma(\widetilde{L}_u) = \sigma(\widehat{L}_u) + \sigma\left(-\Delta_{S^{N-1}}\right)$$

where $-\Delta_{S^{N-1}}$ is the Laplace-Beltrami operator on the sphere S^{N-1}.

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Let us denote by $\alpha_j = \alpha_j(A, p)$ the eigenvalues of $\widehat{L}_u$ and by $\lambda_k = k(k+N-2)$ the eigenvalues of $-\Delta_{S^{N-1}}$, the question is whether there exists j and k such that

$$0 = \alpha_j(A, p) + \lambda_k.$$

The Linearized Problem

Theorem

The linearized equation $L_u(v) = 0$ (2) has a nontrivial solution $\psi(x)$ if and only if

$$\alpha_1(A, p) + \lambda_k = 0 \tag{5}$$

for some $k \geq 1$. Moreover these solutions have the form $\psi(x) = w_1(|x|)\phi_k(\frac{x}{|x|})$.

Here α_1 and w_1 are the first eigenvalue and the first eigenfunction of the radial operator $\widehat{L}_u$ and ϕ_k is an eigenfunction of the Laplace-Beltrami operator relative to the eigenvalue λ_k.

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Recent results by [T.Bartsch-M.Clapp-M.Grossi-F.Pacella(2010), F.G.-M.Grossi-F.Pacella-P.N.Srikanth(2010)]

Varying the exponent p

Now we fix the annulus $A = \{x \in \mathbb{R}^N \,:\, a < |x| < b\}$, and let the exponent p vary. So we write $u = u_p$ and $\alpha_1 = \alpha_1(p)$.

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Now we fix the annulus $A = \{x \in \mathbb{R}^N : a < |x| < b\}$, and let the exponent p vary. So we write $u = u_p$ and $\alpha_1 = \alpha_1(p)$.
The solution u_p admits a limiting problem as $p \mapsto +\infty$.

Theorem (M.Grossi(2006))

Let u_p be the unique radial solution of (1). Then as $p \to +\infty$

$$u_p(|x|) \to \frac{4(N-2)}{a^{2-N} - b^{2-N}} G(r, r_0) \quad \text{in } C^0(\bar{A})$$

and also in $H^1_{0,r}(A)$, where $r_0 \in (a, b)$ and $G(r, s)$ is the Green's function of the operator $-(r^{N-1}u')'$, $\quad r \in (a, b)$ with Dirichlet boundary conditions. Moreover

$$\|u_p\|_\infty = 1 + \frac{\log p}{p} + \frac{\gamma}{p} + o(\frac{1}{p}), \quad \gamma > 0, \text{ as } p \to +\infty.$$

Varying the exponent p

Moreover

Theorem (M.Grossi(2006))

Letting

$$\tilde{u}_p(r) = \frac{p}{\|u_p\|_\infty}\left(u_p(\epsilon_p r + r_p) - \|u_p\|_\infty\right),$$

where $u_p(r_p) = \|u_p\|_\infty$ *and* $p\epsilon_p^2\|u_p\|^{p-1} = 1$*, we have that*

$$\tilde{u}_p \to U \quad \text{in } C^1_{loc}(\mathbb{R}), \tag{6}$$

where U *is the unique solution of*

$$\begin{cases} -U'' = e^U & \text{in } \mathbb{R} \\ U(0) = 0 & U'(0) = 0 \end{cases}$$

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- $\alpha_1(p) \to 0$ as $p \to 1$.

The analyticity of $\alpha_1(p)$ implies that for any $k \geq 1$ the equation $\alpha_1(p) + \lambda_k = 0$ has at most finitely many solutions and from the behavior at 1 and at $+\infty$ we get that for any $k \geq 1$ there exists p_k such that

$$\alpha_1(p_k) + \lambda_k = 0$$

$p_k \to +\infty$ as $k \to +\infty$ all roots of the equation behave like

$$p_k = \sqrt{\frac{-k(k+N-2)}{\beta}} + o(1)$$

as $k \to +\infty$, $\beta = -\frac{1}{2}\gamma r_0^2$

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Theorem (F.G.-M.Grossi-F.Pacella-P.N.Srikanth (2010))

For every $k \geq 1$ there exists at least one exponent p_k such that nonradial bifurcation occurs at (u_{p_k}, p_k), $p_k \to +\infty$.

If k is even we have $[\frac{N}{2}]$ nonradial solutions emanating from (u_{p_k}, p_k).

Sketch of the proof

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To prove the bifurcation result we need the dimension of the corresponding eigenspace to be odd.
To this end we consider the subspace of $C^{1,\alpha}(\bar{A})$ given by functions which are $O(N-1)$-invariant, i.e. such that

$$v(x_1,\dots,x_N) = v(g(x_1,\dots,x_{N-1}),x_N) \quad \text{for any } g \in O(N-1).$$

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The eigenspace of the linearized operator is then 1-dimensional (Smoller-Wasserman(1990)).
This implies that when crossing p_k the Morse index of the radial solution increases exactly by one.
This implies a change in the topological degree of a certain associated map and induces bifurcation by standard results.

Multiple solutions

We can obtain multiple bifurcating solutions considering some suitable subgroups $\mathcal{G}$ of $O(N)$ such that for k even the eigenspace relative to the eigenvalue λ_k, restricted to the function invariant by the action of $\mathcal{G}$, has dimension 1.

For example we can consider groups $\mathcal{G}_h = O(h) \times O(N-h)$. The number of this subgroups, if k is even, is $[\frac{N}{2}]$.

The global bifurcation result

In this case we can say something more. We let (u_{p_k}, p_k) be a bifurcation point and we let $\mathcal{C}(p_k)$ be the closed connected component that bifurcates from (u_{p_k}, p_k). Then we have

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Theorem (F.G.(2010))

- *either* $\mathcal{C}(p_k)$ *is unbounded in* $(1, +\infty) \times C^{1,\alpha}(\bar{A})$*;*
- *or* $\mathcal{C}(p_k)$ *intersects the curve of the radial positive solutions of (1) in another Morse index changing point.*

The proof relies on a careful use of the homotopy invariance of the degree of a certain map related to our problem, and other properties.

The case of the exterior of a ball

Here we consider the problem

$$\begin{cases} -\Delta u = u^p & \text{in } \mathbb{R}^N \setminus B_1(0) \\ u > 0 & \text{in } \mathbb{R}^N \setminus B_1(0) \\ u = 0 & \text{on } \partial B_1(0) \end{cases} \tag{7}$$

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with $N \geq 3$ and $p > \frac{N+2}{N-2}$. For $p > \frac{N+2}{N-2}$ there exists only one radial solution u_p with fast decay at infinity, i.e. such that

$$\limsup_{|x| \to +\infty} u_p(x)|x|^{N-2} < +\infty$$

there are also many slow decay radial solutions (Davila-Del Pino-Musso(2007)).

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The degeneracy of the fast decay radial solution u_p has been studied in Del Pino-Wei (2007). As in the case of the annulus the eigenvalue problem for the linearized operator at the radial solution u_p can be splitted into the radial and the angular part. In this way we can characterize the exponents at which the corresponding fast decay solutions of (7) are degenerate as solutions of the equation

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But then we cannot apply the standard bifurcation theory (using the Leray-Schauder degree) because unboundedness of the domain induces a lack of compactness.
We proceed in another way and get the

Theorem (F.G.-F.Pacella(2011))

There exists a sequence of exponents $\{p_k\}$, $p_k > \frac{2N}{N-2}$, $p_k \to +\infty$ such that nonradial bifurcation occurs at (u_{p_k}, p_k).

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The idea of the proof is to study the *"limit"* of the bifurcation branches in the annuli $A_R = \{x \in \mathbb{R}^N \, : \, 1 < |x| < R\}$, as $R \to +\infty$.

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We divide the proof in some steps:

- Step I

We study the asymptotic behavior of the radial solution u_p^R (in the annulus A_R corresponding to the nonlinearity with exponent p) as $R \to +\infty$:

if $p_n \to \bar{p}$, $\bar{p} > \frac{2N}{N-2}$ and $R_n \to +\infty$ then

$$u_{p_n}^{R_n} \to u_{\bar{p}} \quad \text{ as } n \to +\infty$$

in the space $D_0^{1,2}(\Omega) \cap L^\infty(\Omega)$, where $u_{\bar{p}}$ is the radial fast decay solution of (7).

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in the space $D_0^{1,2}(\Omega) \cap L^\infty(\Omega)$, where $u_{\bar{p}}$ is the radial fast decay solution of (7).

The proof of this step requires several estimates on the norms of u_p^R some of which hold for $p > \frac{2N}{N-2}$ (which explains the technical assumption $p > \frac{2N}{N-2} > \frac{N+2}{N-2}$).

The case of the exterior of a ball

- Step II

Show some *"convergence property"* of the spectrum of the linearized operator at the radial solution u_p^R in A_R to the spectrum of the linearized operator at the fast decay solution of (7). The aim is to show that radial degenerate solutions of the problem in A_R converge to a fast decay radial degenerate solution u_p of (7) and a change in the Morse index of u_p induces a change in the Morse index of the approximating solutions u_p^R, for R large.

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The proof of this part relies on the fact that the first eigenvalue $\alpha_1(p, R)$ related to the 1-dimensional operator $\widehat{L}_{u_p^R}$ (defined for the problem in the annulus) converges to the first eigenvalue $\alpha_1(p)$ of the 1-dimensional operator

$$\widehat{L}_{u_p}(\psi) = r^2(-\psi'' - \frac{N-1}{r}\psi' - pu_p^{p-1}\psi) \text{ in } (1, +\infty)$$

$\psi(1) = \psi(+\infty) = 0$ related to the problem in $\Omega = \mathbb{R}^N \setminus B_1(0)$.

The case of the exterior of a ball

- Step III

Show that the bifurcation branches for the problem in A_R, emanating from the radial solutions u_p^R *"converge"* in a suitable sense, as $R \to +\infty$, to some limit sets.

The case of the exterior of a ball

- Step III

Show that the bifurcation branches for the problem in A_R, emanating from the radial solutions u_p^R *"converge"* in a suitable sense, as $R \to +\infty$, to some limit sets.
This point uses a topological lemma (already used by Ambrosetti-Gamez (1997)) which is based on showing a precompactness property of the set given by the union of all branches.

The case of the exterior of a ball

- Step IV

The final step is to prove that these limit sets are really branches of nonradial solutions bifurcating from fast decay degenerate radial solutions of (7).

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We show that :
–the limit sets are nonempty and contain a point (u_p, p) with u_p fast decay degenerate radial solution of (7),
–they do not reduce to the point (u_p, p),
–they do not coincide with the set of the radial solutions of (7).

REMARK: Bifurcation is global and all solutions on the branches are fast decay solutions (by construction).

Open problems

- Are there other branches of solutions bifurcating from the radial one? For example, solutions with other symmetry properties?
- What about the Morse index of these bifurcating solutions;
- Does secondary bifurcation occur?
- Is it true that the equation $\alpha_1 + \lambda_k = 0$ has only one solution for any $k \geq 1$?

Some References on bifurcation

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