On the non-occurrence of the Lavrentiev phenomenon

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In 1927 a remarkable paper by N. Lavrentiev presented an example of a variational functional over the interval (a, b), with boundary conditions $u(a) = \alpha$, $u(b) = \beta$, whose infimum over the set of absolutely continuos functions was strictly lower than the infimum of the same functional over the set of Lipschitzean functions satisfying the same boundary conditions. Since then, this phenomenon is called the Lavrentiev phenomenon. In 1993,Alberti and Serra Cassano did show that the phenomenon does not occurr for autonomous integrands over a one-dimensional integration set. It is a problem of lower-semicontinuity

The following example of Manià is well known.

Consider the problem of minimizing the functional

$$\int_0^1 [t - x(t)^3]^2 [x'(t)]^6 dt, \quad x(0) = 0, x(1) = 1.$$

Then the infimum taken over the space of absolutely continuous functions (assumed in $x(t) = \sqrt[3]{t}$) is strictly lower than the infimum taken over the space of Lipschitzian functions. Comment: it has been modified so as to make the functional coercive.

Boundary condition: To change or not to change?

If we allow the boundary conditions to be $x(0) = \varepsilon, x(1) = 1)$, the problem disappears.

We will consider the Lavrentiev phenomenon in its strict sense, with fixed boundary conditions.

1 The multi-dimensional case

When the integration set is a subset Ω of $\mathbb{R}^N$, the boundary condition is described by the inclusion $u - u^0 \in W_0^{1,1}(\Omega)$ and, in order for the problem of the occurrence of the Lavrentiev phenomenon to make sense, u^0 is a Lipschitzean function on $\overline{\Omega}$; in 1999 Esposito, Leonetti and Mingione did show that the phenomenon does not occurr for functionals of the form

$$\int_\Omega f(\nabla v(x))\,dx$$

provided that Ω is the unit ball, f is a convex $C^2(\mathbb{R}^N)$ function and the growth of f is of the $(p-q)$ type, i.e., $m|z|^p \leq f(z) \leq L(1+|z|)^q$, with $2 \leq q < p < 2+q$; in addition, some further growth conditions on the first and second derivatives of f are assumed.

2 The difficulties of the problem

When I explain Manià's example I usually present first a simpler computation: I approximate the solution $x(t) = (t)^{\frac{1}{3}}$ by interpolating linearly $(0,0)$ and $(\alpha, \alpha^{\frac{1}{3}})$ and, (in general, at the audience's surprise) show that the difference of the value of the functional on the solution and the value on the approximating functions, goes to $+\infty$.

A computation *impossible to perform*

; a model case: the minimal area problem on an annulus.

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Being able to perform this computation is only the first step: it is by no means sufficient to establish the non-occurrence of the Lavrentiev phenomenon!

3 Tools

Being the integrand convex, a first idea would be to smooth the solution u by convolution, i.e., having fixed $\delta > 0$, consider

$$\tilde{u}(y) = \int_{B(0,\delta)} u(y-x)\rho(x)\,dx$$

where ρ is a mollifier having support in $B(0,\delta)$; when y is close to the boundary, the convolution has to be well defined; this means the the function u has to be extended to a neighborhood of Ω.

A first approach: assume that Ω is star-shaped w.r.t. the origin, and define

$$u_\varepsilon(x) = u\left(\frac{x}{1+\varepsilon}\right);$$

this operation extends u to a neighborhood of Ω, and one can show that the value of the functional actually decreases; however, the boundary conditions are changed. (this tool is used by Buttazzo and Belloni)

A second approach, used by Esposito, Leonetti and Mingione :

do the opposite.

Extend the boundary conditions to the inside to a δ neighborhhod of ∂B and set

$$u^\delta(x) = u(\frac{1}{1-\delta}x)$$

for $|x| \leq 1-\delta$, to be u^0 otherwise.

A problem with this approach

At a point x, $\nabla u^\delta(x) = \frac{1}{1-\delta}\nabla u(x)$, i.e., the gradient increases, hence

$$L(\nabla u^\delta(x)) \geq L(\nabla u(x)).$$

Outside of polynomial growth, there is, in general, no way to limit how much L becomes larger!

The condition needed to control this increase of l is often expressed requiring that

$$L(2\xi) \leq \sigma L(\xi).$$

4 Our contribution

The following is our result, an approximation result that, in particular, guarantees the non-occurrence of the Lavrentiev phenomenon.

Theorem 1 *Let* $\Omega \subset \mathbb{R}^N$ *be an open bounded set, with* $\partial\Omega \in C^2$*; let* $u^0 \in C^2(\overline{\Omega})$*; let* $L : [0, \infty) \to [0, \infty)$ *be convex and such that* $L(0) = 0$*. Let* $u \in u^0 + W^{1,1}(\Omega)$ *be bounded on* Ω *and such that*

$$\int_\Omega L(|\nabla u(x)|)\, dx < \infty.$$

Then, given $\varepsilon > 0$*, there exists* $u_\varepsilon \in u^0 + W^{1,1}(\Omega)$*, with* u_ε *Lipschitzean on* $\overline{\Omega}$*, such that*

$$\int_\Omega L(|\nabla u_\varepsilon(x)|)\, dx \le \int_\Omega L(|\nabla u(x)|)\, dx + \varepsilon.$$

The previous Theorem contains neither regularity nor growth assumptions on the Lagrangean L, besides its being convex. When u is a solution, the boundedness of u follows from the boundedness of u^0.

5 A first step: the impossible computation.

We wish to define special Lipschitzean functions, having the same boundary conditions as u; For $x \in \Omega$, set

$$w_+^h(x) = \min\{u^0(z) + h|z - x| : z \in \partial\Omega\} \tag{(5.1)}$$

Notice: if we had $u^0 = 0$, then $w_+^h(x) = h \cdot \mathrm{dist}(x, \partial\Omega)$

The function w_+^h is Lipschitzean, but this property by itself would not be sufficient for our computation:

The real property of this function is:

given x, let $y = y(x)$ the (unique) point where the minimum is attained; then, the gradient of w_+^h is constant of norm h along the line segment (y, x) and is directed as $x - y$.

Define also

$$m_+^h(x) = \min\{w_+^h(x), u(x)\}.$$

Finally, set

$$M^h(x) = \begin{cases} m_+^h(x) & \text{when } u(x) > m_+^h(x) \\ u(x) & \text{when } m_-^h(x) \le u(x) \le m_+^h(x) \\ m_-^h(x) & \text{when } u(x) < m_-^h(x) \end{cases}$$

6 A local computation, in dimension two, through the coarea theorem

Let $P \in \partial\Omega$; we choose as coordinate system (depending on P) the one that has the origin in P and the ξ_2 axis in the direction of the normal to the inside of Ω, so that the ξ_1 axis is along the tangent. On this system, $\partial\Omega$ is described locally by $\xi_2 = \phi(\xi_1)$ with ϕ a smooth function such that $\phi(0) = \phi'(0) = 0$; given $\Phi \leq 1$, we shall call $I_\Phi(P)$ the maximal open interval such that, for $\xi_1 \in I_\Phi$ we have $|\phi'(\xi_1)| < \Phi$; 0 is in the interior of I_Φ.

Set $\Omega^+ = \{x : u(x) > m_+^{\tilde{h}}(x)\}$, $\Omega^- = \{x : u(x) < m_-^{\tilde{h}}(x)\}$ and $\Omega^0 = \{x : m_-^{\tilde{h}}(x) \le u(x) \le m_+^{\tilde{h}}(x)\}$. We have, almost everywhere in Ω,

$$|\nabla M^{\tilde{h}}| = \begin{cases} \tilde{h} & \text{for } x \in \Omega^- \cup \Omega^+ \\ |\nabla u| & \text{for } x \in \Omega^0 \end{cases}$$

so that

$$\int_\Omega L(|\nabla M^{\tilde{h}}(x)|)\, dx = \int_{\Omega^-} L(\tilde{h})\, dx + \int_{\Omega^+} L(\tilde{h})\, dx + \int_{\Omega^0} L(|\nabla u|)\, dx$$

We wish to show that

$$\int_\Omega L(|\nabla M^{\tilde{h}}(x)|)\, dx \le \int_\Omega L(|\nabla u(x)|)\, dx + \frac{\varepsilon}{2}; \tag{(6.2)}$$

it is enough to show that

$$\int_{\Omega^+} L(|\tilde{h}|)\, dx = \int_{\Omega^+} L(|\nabla M^{\tilde{h}}(x)|)\, dx \le \int_{\Omega^+} L(|\nabla u(x)|)\, dx + \frac{\varepsilon}{4} \tag{(6.3)}$$

and

$$\int_{\Omega^-} L(|\tilde{h}|)\, dx = \int_{\Omega^-} L(|\nabla M^{\tilde{h}}(x)|)\, dx \le \int_{\Omega^-} L(|\nabla u(x)|)\, dx + \frac{\varepsilon}{4}. \tag{(6.4)}$$

We shall proceed locally.

Definition 2 *For given* h*,* Φ*,* δ*, and for* $P \in \partial\Omega$*, set,*

$$V^{+}_{h,\Phi,\delta}(P) = \left\{ x \in \Omega : \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \right.$$

$$\left. = \begin{pmatrix} \xi_1 \\ \phi(\xi_1) \end{pmatrix} + \ell \frac{\nabla \tilde{w}^h_+(\xi_1)}{h}; \xi_1 \in I_\Phi(P); \ell \in (0, \ell^*); d(y + \ell^* \frac{\nabla \tilde{w}^h_+(\xi)}{h}) = \delta \right\}.$$

We shall call V this set.

7 The map g

Consider the map g, from $\Omega_\delta \subset \mathbb{R}^2$ to $\mathbb{R}$, given by

$$g(x) = g(x_1, x_2) = y(x)_1 = \xi$$

We wish to prove

$$\int_{\Omega^+\cap V} L(|\nabla m^{\tilde{h}}(x)|)\,dx \le (1+\varepsilon)\int_{\Omega^+\cap V} L(|\nabla u(x)|)\,dx.$$

Apply the coarea theorem to the set $\Omega^+ \cap V$ to obtain

$$\int_{\Omega^+\cap V} L(|\nabla u(x)|)\,dx = \int_I [\int_{\{g(x)=\xi_1\}\cap(\Omega^+\cap V)} \frac{L(|\nabla u(x)|)}{J(g(x))} dH^1] d\xi_1;$$

Consider the line segment

$$L_{\xi_1} = \{y_{\xi_1} + \ell\frac{\nabla \tilde{w}^{\tilde{h}}(\xi_1)}{\tilde{h}} : \ell \in (0,\ell^*); d(y + \ell^* \frac{\nabla \tilde{w}^{\tilde{h}}(\xi)}{\tilde{h}}) = d^*\} :$$

we have that
$\{g(x) = \xi_1\} \cap (\Omega^+ \cap V) = L_{\xi_1} \cap \Omega^+$.

Consider the two maps

$$\tilde{u}_{\xi_1}(\ell) = u(y_{\xi_1} + \ell\frac{\nabla\tilde{w}^{\tilde{h}}(\xi_1)}{\tilde{h}}),$$

$$\tilde{w}^{\tilde{h}}(\ell) = w^{\tilde{h}}(y_{\xi_1} + \ell\frac{\nabla\tilde{w}^{\tilde{h}}(\xi_1)}{\tilde{h}});$$

there are at most countably many open intervals (a_j, b_j) such that $S_{\xi_1} = \cup(a_j, b_j)$ and $\tilde{u}_{\xi_1}(a_j) - w^{\tilde{h}}(a_j) = \tilde{u}_{\xi_1}(b_j) - w^{\tilde{h}}(b_j) = 0$ while, for $\ell \in (a_j, b_j)$, $\tilde{u}_{\xi_1}(\ell) > w^{\tilde{h}}(\ell)$. Fix one such (a_j, b_j). The problem of minimizing

$$\int_{a_j}^{b_j} L(|v'(\ell)|)\, d\ell; \quad v(a_j) = \tilde{u}_{\xi_1}(a_j); v(b_j) = \tilde{u}_{\xi_1}(b_j)$$

admits the solution $w^{\tilde{h}}$, so that, in particular,

$$\int_{a_j}^{b_j} L(\tilde{h})\, d\ell \le \int_{a_j}^{b_j} L(|\tilde{u}'_{\xi_1}(\ell)|) d\ell = \int_{a_j}^{b_j} L(|\langle \nabla u(y_{\xi_1} + \ell\frac{\nabla\tilde{w}^{\tilde{h}}(\xi)}{\tilde{h}}), \frac{\nabla\tilde{w}^{h}_{+}(\xi)}{h}\rangle|)\, d\ell.$$

Recall that $|\frac{\nabla \tilde{w}^{\tilde{h}}(\xi)}{\tilde{h}}| = 1$; since L is non-decreasing, we obtain that

$$L(|\langle \nabla u(y_{\xi_1} + \ell \frac{\nabla \tilde{w}_+^h(\xi)}{h}), \frac{\nabla \tilde{w}_+^h(\xi)}{h} \rangle|) \leq L(|\nabla u(y_{\xi_1} + \ell \frac{\nabla \tilde{w}_+^h(\xi)}{h})|),$$

hence that

$$\int_{a_j}^{b_j} L(\tilde{h})\, d\ell \leq \int_{a_j}^{b_j} L(|\nabla u(y_{\xi_1} + \ell \frac{\nabla \tilde{w}^{\tilde{h}}(\xi)}{\tilde{h}}|)\, d\ell$$

Since the restriction to $L_{\xi_1} \cap \Omega^+$ of the gradient of $m^{\tilde{h}}$ is $\tilde{h}\frac{\nabla \tilde{w}^{\tilde{h}}(\xi_1)}{\tilde{h}}$ when ℓ belongs to the intervals (a_j, b_j), the previous inequality implies

$$\int_{\{g(x)=\xi_1\} \cap (V_{Z_j} \cap \Omega^+)} L(|\nabla m^{\tilde{h}}|)\, dH^1 \leq \int_{\{g(x)=\xi_1\} \cap (V_{Z_j} \cap \Omega^+)} L(|\nabla u|)\, dH^1.$$

Assume that we have proved that, by choosing h large and Φ small, we have

$$\frac{1-\eta}{1+\eta} \leq \sqrt{g_{x_1}^2 + g_{x_2}^2} \leq \frac{\sqrt{1+\eta^2}}{(1-\eta)}.$$

Then:

$$\begin{aligned}
\int_{V\cap\Omega^+} L(|\nabla u(x)|)\, dx &= \int_I [\int_{\{g(x)=\xi_1\}\cap(V\cap\Omega^+)} \frac{L(|\nabla u(x)|)}{J(g(x))} dH^1] d\xi_1 \\
&\geq \frac{(1-\eta)}{\sqrt{1+\eta^2}} \int_{Z_j} [\int_{\{g(x)=\xi_1\}\cap(V_{Z_j}\cap\Omega^+)} L(|\nabla u(x)|) dH^1] d\xi_1 \\
&\geq \frac{(1-\eta)}{\sqrt{1+\eta^2}} \int_{Z_j} [\int_{\{g(x)=\xi_1\}\cap(V_{Z_j}\cap\Omega^+)} L(|\nabla m^{\tilde{h}}|)\, dH^1]\, d\xi_1 \\
&\geq \frac{(1-\eta)}{\sqrt{1+\eta^2}} \int_I [\int_{\{g(x)=\xi_1\}\cap(V\cap\Omega^+)} \frac{1-\eta}{1+\eta} \frac{L(|\nabla m^{\tilde{h}}(x)|)}{J(g(x))} dH^1]\, d\xi_1 \\
&= \frac{(1-\eta)^2}{(1+\eta)\sqrt{1+\eta^2}} \int_{V\cap\Omega^+} L(|\nabla m^{\tilde{h}}(x)|)\, dx.
\end{aligned}$$

We have obtained

$$\int_{V\cap\Omega^+} L(|\nabla m^{\tilde{h}}(x)|)\, dx \leq (1+\varepsilon^1) \int_{V\cap\Omega^+} L(|\nabla u(x)|)\, dx.$$

Performing this computation on a covering of $\partial\Omega$, we obtain that the functional, computed on the function M instead of u, increases only by ε.

This is only the first step, since the function M is far from being Lipschitzean. It has the property, that u didn't have, that calling y^* the point at the boundary nearest to x,

$$|M(y^*) - M(x)| \leq D|y^* - x|$$

8 Smoothing

The smoothing is obtained by a moving average of radius d, i.e. using

$$\rho(x) = \frac{1}{d^N \omega_N} \chi_{B(0,d)}(x).$$

Step two

To preserve the boundary condition, near $\partial\Omega$ we use a radius $d = d(x) = \mathrm{dist}(x, \partial\Omega)$. This device works well for the boundary condition; however, in general is not suitable, since, when computing the lipschitz constant, δ goes at the denominator so that the Lipschitz constant would go to ∞ as x tends to the boundary.

We show that, in our case, this does not happen, because of the property of M.

Hence we show that, smoothing M by the moving average with radius $d(x) = \mathrm{dist}(x, \partial\Omega)$, we obtain a Lipschitzean function, of some Lipschitz constant D. Since we have D, we can compute $L(D)$ and choose δ^* so small, that the

$$\int_{\Omega_{\delta^*}} L(D) dx \leq \varepsilon$$

9 Last step

Since we have δ^*, when $\mathrm{dist}(x, \partial\Omega) \geq \delta^*$ we take the usual moving average with (fixed) radius δ^*; the convexity of L implies that we are not increasing the value of the functional.