Spectral Optimization Problems

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Spectral optimization

$$=$$

How to design your favourite drum

We deal with the optimization problem

$$\min\Big\{F(\Omega)\ :\ \Omega\in\mathcal{A}(D)\Big\}$$

where D is a given bounded domain of $\mathbf{R}^d$ and $\mathcal{A}(D)$ is a class of admissible choices made of subsets of D.

Typical examples are:

$F(\Omega)=\Phi\Big(\lambda(\Omega)\Big)$ where $\lambda(\Omega)$ is the spectrum of the Dirichlet Laplacian in Ω and $\Phi:\mathbf{R}^\mathbf{N}\to[0,+\infty]$ is a given function; for instance

$$F(\Omega)=\lambda_k(\Omega).$$

$F(\Omega) = \int_D j(x, u_\Omega)\, dx$ where j is a given integrand and u_Ω is the solution in $H_0^1(\Omega)$ of $-\Delta u = f$; for instance

$$j(x, u) = -a(x)u^p.$$

In both cases a very natural choice for admissible domains is

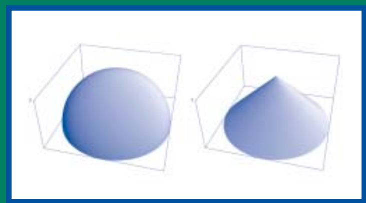
$$\mathcal{A}(D) = \Big\{\Omega \subset D,\ \Omega \text{ quasi-open},\ |\Omega| \le m\Big\}.$$

Some good sources for results and problems...

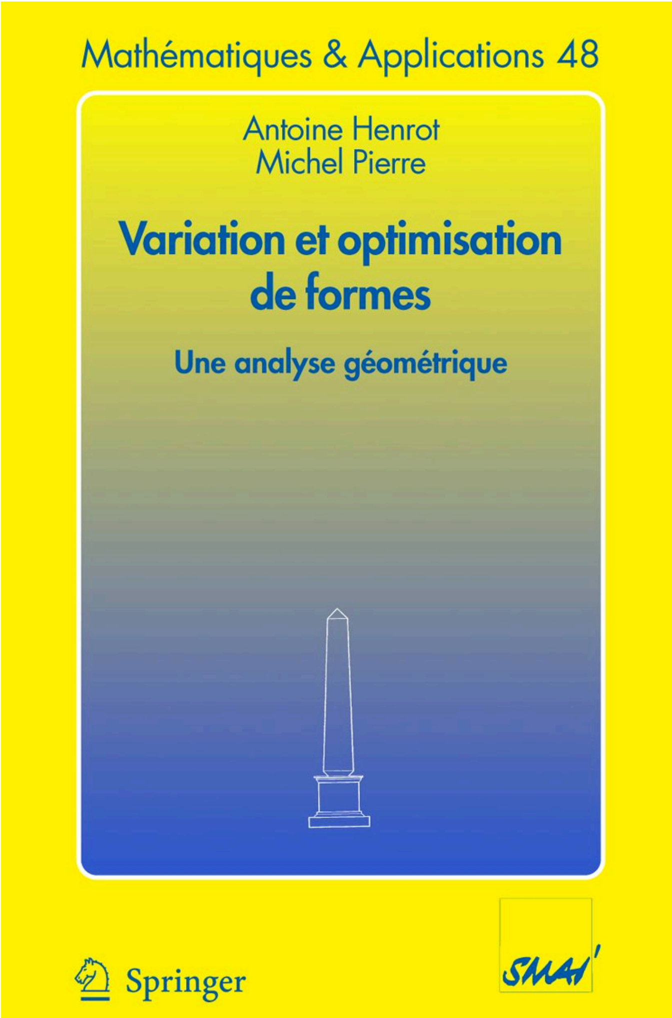
Progress in Nonlinear Differential Equations and Their Applications

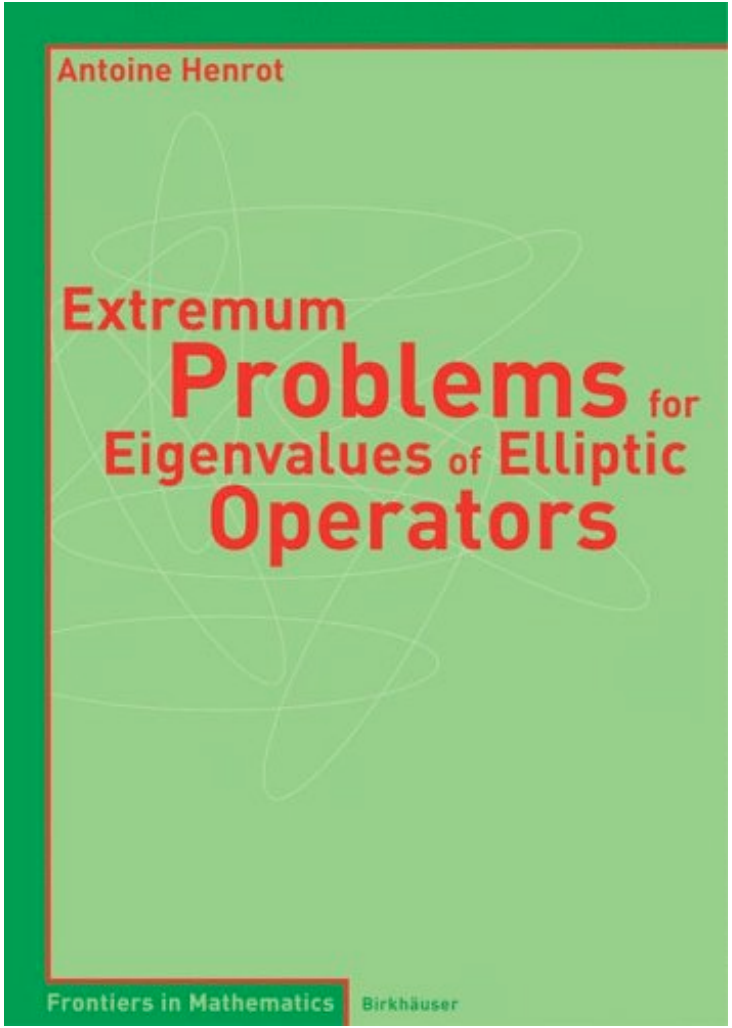
Dorin Bucur
Giuseppe Buttazzo

Variational Methods in Shape Optimization Problems



Birkhäuser


Mathématiques & Applications 48
Antoine Henrot
Michel Pierre
Variation et optimisation de formes
Une analyse géométrique
Springer
SMAI


Antoine Henrot
Extremum Problems for Eigenvalues of Elliptic Operators
Frontiers in Mathematics
Birkhäuser

One issue we intend to develop is the study of gradient flow evolutions for shape optimization problems, via minimizing movements.

The framework is a metric space (space of shapes) and a functional $\mathcal{F} : X \to [0, +\infty]$; via the Euler scheme of time step ε and initial condition $u_0 \in X$ we may construct a step function $u_\varepsilon(t) = w([t/\varepsilon])$ by setting $w(0) = u_0$ and

$$w(n+1) \in \operatorname{argmin}\left\{\mathcal{F}(v) + \frac{d^2\big(v, w(n)\big)}{2\varepsilon}\right\}.$$

The gradient flow $u(t)$ is then a limit of a sequence u_{ε_n} with $\varepsilon_n \to 0$.

In spectral optimization we take a suitable distance on the space of shapes and

$$\mathcal{F}(\Omega) = \Phi\Big(\lambda(\Omega)\Big).$$

The goal is to study existence and properties of spectral flows $\Omega(t)$. At this stage the only available results are obtained for debonding problems, where mushy regions may appear during the evolution.

D.Bucur, G.Buttazzo: J. Convex Anal. 2008
D.Bucur, G.Buttazzo, A.Lux: Arch. Rational Mech. Anal. 2008.

From now on we restrict to static shape optimization problems.

In general one cannot expect the existence of a solution; for instance, the problem

$$\min\left\{\int_D |u-c|^2\,dx\ :\ -\Delta u = 1 \text{ in } H^1_0(\Omega)\right\}$$

has no solution if c is small, and one has to deal with relaxed solutions that in this case are capacitary mesures μ, i.e. Borel countably additive set functions with values in $[0, +\infty]$ and such that

$$\mu(E) = 0 \text{ whenever } \operatorname{cap}(E) = 0.$$

On the other hand, adding some geometrical constraint to the admissible domains, gives extra compactness that provides the existence of a solution in a very large number of situation. For instance:

- The class $\mathcal{A}_{convex}$ of convex sets contained in D.
- The class $\mathcal{A}_{unif\,cone}$ of domains satisfying a uniform exterior cone property.
- The class $\mathcal{A}_{unif\,flat\,cone}$ of domains satisfying a uniform flat cone condition, i.e., as above, but with the weaker requirement that the cone may be flat, that is of dimension $d-1$.

• The class $\mathcal{A}_{cap\,density}$ of domains satisfying a uniform capacity density condition.

• The class $\mathcal{A}_{unif\,Wiener}$ of domains satisfying a uniform Wiener condition.

$$\mathcal{A}_{convex} \subset \mathcal{A}_{unif\,cone} \subset \mathcal{A}_{unif\,flat\,cone} \subset \mathcal{A}_{cap\,density} \subset \mathcal{A}_{unif\,Wiener}$$

• Another interesting class, which is only of topological type and is not contained in any of the previous ones, is (for $d = 2$) the class of domains for which the number of connected components of $D \setminus \Omega$ is uniformly bounded (Sverák 1993).

A powerful result which applies in the situations when no geometric constraints are imposed(Buttazzo-Dal Maso 1993):

Theorem Assume that
1. F is decreasing for the set inclusion;
2. F is l.s.c. for the γ-convergence.
Then the minimum problem

$$\min\Big\{F(\Omega)\ :\ |\Omega| \leq m,\ \Omega \subset D \text{ quasi open}\Big\}$$

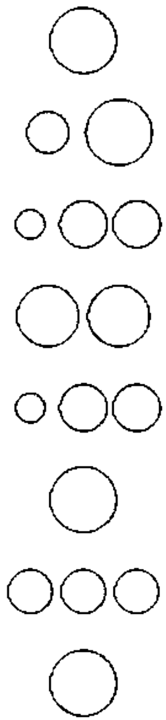
admits at least a solution Ω_{opt}.

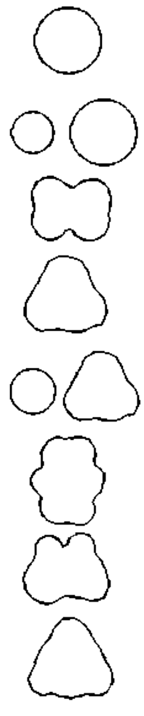
Example 1. Take $F(\Omega) = \Phi\big(\lambda(\Omega)\big)$ with Φ l.s.c. and increasing in each of its variables; then the assumptions are verified. For instance we may take $F(\Omega) = \lambda_k(\Omega)$.

For $k = 1$ (and large D) the best domain is a ball of volume m (Faber-Krahn 1923);

For $k = 2$ (and large D) the best domain is the union of two disjoint balls of volume $m/2$ (Pólya-Szegö 1955).

Here is a numerical computation of optimal shapes for $k = 3 - 10$ (Oudet)





k	best array of balls	best shape
3	46.125	46.125
4	64.293	64.293
5	82.462	**78.47**
6	92.250	**88.96**
7	110.42	**107.47**
8	127.88	**119.9**
9	138.37	**133.52**
10	154.62	**143.45**

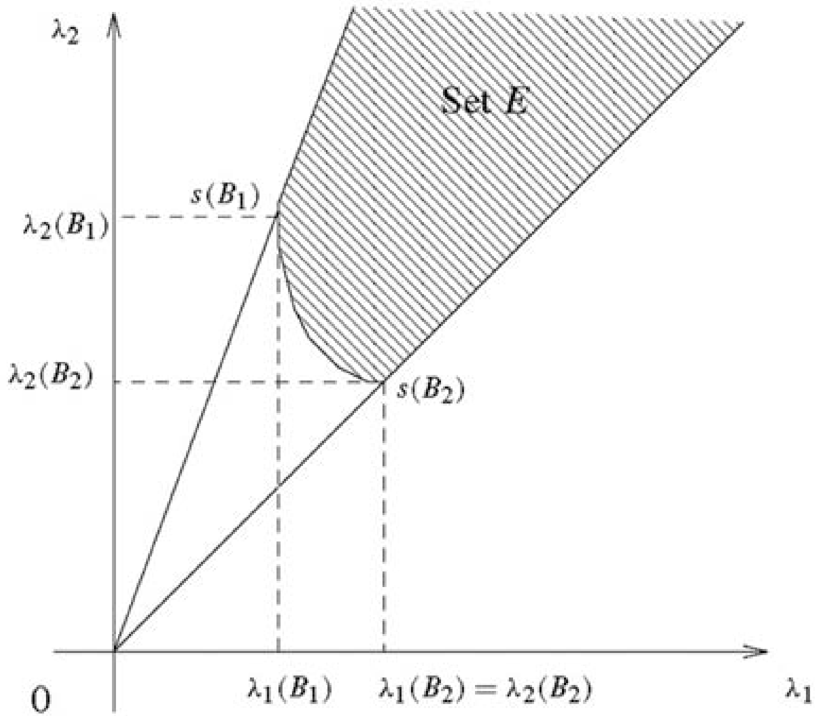
Very little is known on regularity of optimal domains; apart the case of λ_1 the proof that they are actually open sets is still missing.

For spectral optimization problems that do not fulfill the decreasing property of theorem above, the case of

$$\min\Big\{\Phi(\lambda_1,\lambda_2)\ :\ |\Omega|\le m,\ \Omega\subset D\Big\}$$

is interesting. Indeed, if D is large enough, it is possible to prove (Bucur-Buttazzo-Figueiredo 1999) that the existence of an optimal domain always occur, i.e. for every (continuous) function Φ.

Here is the plot of the set E of points of $\mathbf{R}^2$ that are of the form $\big(\lambda_1(\Omega), \lambda_2(\Omega)\big)$ for some $\Omega \subset D$ with $|\Omega| \le m$ (Keller-Wolf 1994).



Optimal partitions

We look for a partition $\Omega_1, \ldots, \Omega_N$ of Ω which minimizes a cost of the form

$$F(\Omega_1, \ldots, \Omega_N)$$

among all partitions of Ω (volume constraints $|\Omega_i| = m_i$ can be added). The cost F could be very general, for instance

$$F(\Omega_1, \ldots, \Omega_N) = \Phi\Big(\lambda_{k_1}(\Omega_1), \ldots, \lambda_{k_N}(\Omega_N)\Big).$$

General existence theorem

Bucur-B.-Henrot Adv. Math. Sci. Appl. '98

Relaxation
B., Timofte Adv. Math. Sci. Appl. '02

Regularity for $\lambda_1(\Omega_1) + \cdots + \lambda_1(\Omega_N)$
Caffarelli, Lin J. Sci. Comp. '06

Other variants (manifolds, nonlinear, ...)
Conti, Terracini, Verzini JFA '03, Calc.Var '05
Helffer, Hoffmann-Ostenhof Preprint

Conjecture: dim=2, optimal partitions for $\lambda_1(\Omega_1) + \cdots + \lambda_1(\Omega_N)$ approach as $N \to \infty$ a regular exagonal tiling.

Replacing the volume constraint $|\Omega| \leq m$ by a perimeter constraint $\text{Per}(\Omega) \leq L$ provides interesting variants to the spectral optimization problem (Bucur-B.-Henrot '09, Van den Berg-Iversen '09). Note that spectral functional are in general not l.s.c. for the L^1 convergence.

Theorem Assume that F is as above (decreasing and γ-l.s.c.). Then the minimum problem

$$\min\Big\{F(\Omega)\ :\ \text{Per}(\Omega) \leq L,\ \Omega \subset D\Big\}.$$

admits at least a solution Ω_{opt}.

For $F(\Omega) = \lambda_2(\Omega)$ and $D = \mathbf{R}^d$ we have:

• in the case $d = 2$ Ω_{opt} is convex, of class C^∞, its boundary does not contain any segment, does not contain any arc of circle, contains exactly two points where the curvature vanishes.

• In the case $d \geq 3$ Ω_{opt} is not convex, is connected, regularity of Ω_{opt} is not known (even not if it is an open set).

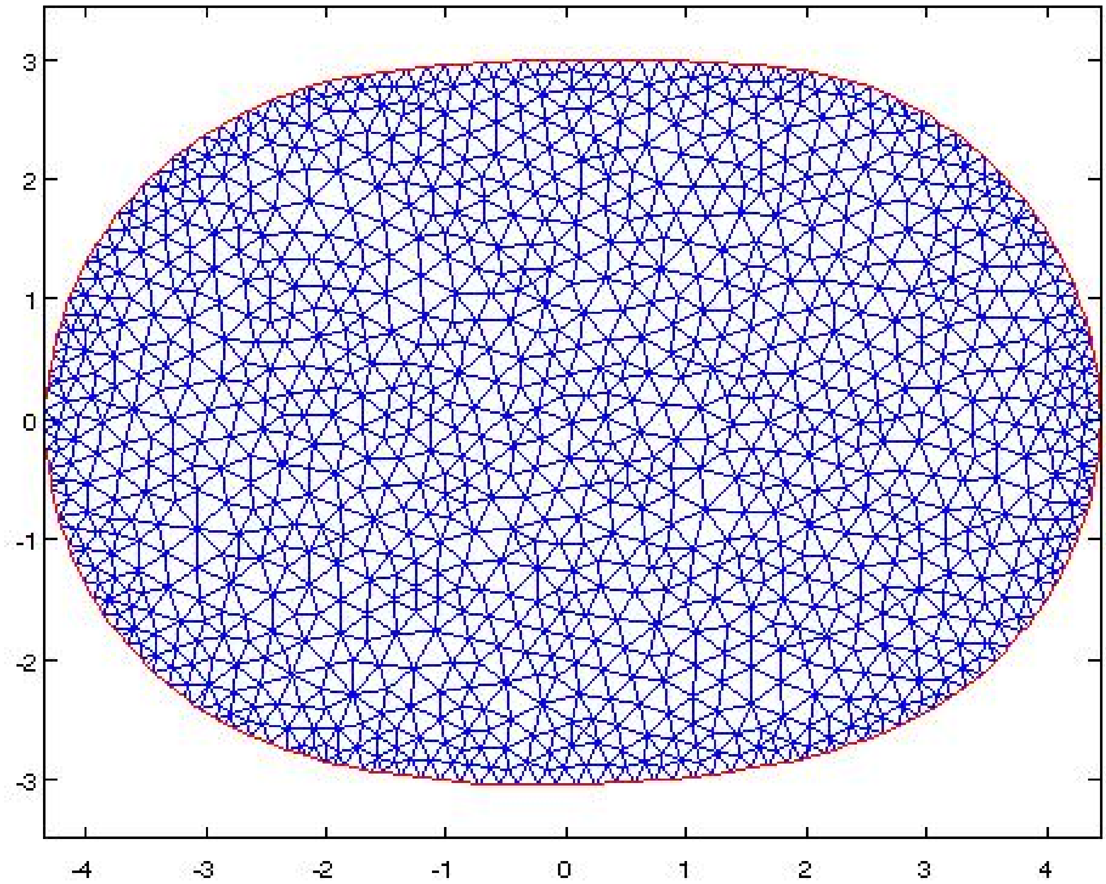
• A similar problem has been studied by Athanasopoulos, Caffarelli, Kenig, Salsa (CPAM 2001):

$$\min\Big\{E(\Omega) + \mathrm{Per}(\Omega) \ : \ \Omega \subset D\Big\}$$

where $E(\Omega)$ is the Dirichlet energy

$$E(\Omega) = \min\left\{ \int_D |Dv|^2\,dx \ : \ \begin{array}{l} v = 0 \text{ on } D \setminus \Omega \\ v = g \text{ on } \partial D \end{array} \right\}.$$

• numerical plot of Ω_{opt} for $d = 2$ (courtesy of E. Oudet).


3
2
1
0
-1
-2
-3
-4
-3
-2
-1
0
1
2
3
4

We consider now Cheeger-like rescaled shape optimization problems

$$\min\left\{M(\Omega)J(\Omega)\ :\ \Omega\subset D\right\}$$

where $M(\Omega)$ is a scaling factor and $J(\Omega)$ a shape functional. The coercivity condition

$$\lim_{M(\Omega)\to 0} M(\Omega)J(\Omega)=+\infty$$

is assumed, which means that the scaling is "above" the scaling invariance, as it happens in the Cheeger problem where

$$M(\Omega)=\mathrm{Per}(\Omega),\qquad J(\Omega)=|\Omega|^{-\alpha},\qquad \alpha>1-1/d.$$

Theorem (B.-Wagner) If M and J are non-negative, fulfill the coercivity condition above and

- J is γ-l.s.c. and decreasing for inclusion;
- M is $w\gamma$-l.s.c..

the minimum problem above has a solution.

Examples

- $|\Omega|^\alpha \lambda_k(\Omega), \quad \alpha < 2/d$
- $|\Omega|^\alpha \Big(C(\Omega)\Big)^{-1}, \quad \alpha < 1 + \frac{2}{d}$

where $C(\Omega)$ is the compliance functional $\int_\Omega f u_\Omega \, dx$

- $\Big(\text{Per}(\Omega)\Big)^\alpha \lambda_k(\Omega), \quad \alpha < 2/(d-1);$
- $\Big(\text{Per}(\Omega)\Big)^\alpha \Big(C(\Omega)\Big)^{-1}, \quad \alpha < 1 + \frac{2}{d-1}.$

Note that Per(Ω) is not a $w\gamma$-l.s.c. and the existence proof requires some extra devices.

Necessary conditions of optimality can be found; for instance in the case of $|\Omega|^\alpha\Big(C(\Omega)\Big)^{-1}$ if Ω is an optimal domain in D and u is the corresponding solution:

$|\nabla u|^2 = \alpha\frac{C(\Omega)}{|\Omega|}$ on $\partial\Omega \cap D$ (free boundary);

$|\nabla u|^2 \geq \alpha\frac{C(\Omega)}{|\Omega|}$ on $\partial\Omega\cap\partial D$ (common bound.).

For instance if D has a corner the optimal domain Ω cannot fill the entire D.

To finish, a challenging **open question**. Consider the problem

$$\min\left\{C(\Omega)\lambda_k^\alpha(\Omega)\ :\ \Omega\subset D\right\}$$

where α is above the scaling invariance $1+d/2$. Thanks to a result of Kohler-Jobin the minimum of $C(\Omega)\lambda_1^{1+d/2}(\Omega)$ is reached by a ball, and so when $\alpha>1+d/2$ the coercivity condition

$$\lim_{M(\Omega)\to 0} M(\Omega)J(\Omega)=+\infty$$

is fulfilled, taking

$$M(\Omega)=C(\Omega)\quad\text{and}\quad J(\Omega)=\lambda_k^\alpha(\Omega).$$

However, the other conditions are not fulfilled; in particular $C(\Omega)$ is not $w\gamma$-l.s.c. and, even if strongly expected, the existence of an optimal domain is still missing.

Once the existence of an optimal domain is established, the subsequent step would be the regularity of the free boundary and the necessary conditions of optimality.