Some remarks on the variational methods

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Some remarks on the classical Ambrosetti-Rabinowitz theorem are presented. In particular, it is observed that the geometry of the mountain pass, if the function is bounded from below, is equivalent to the existence of at least two local minima, while, when the function is unbounded from below, it is equivalent to the existence of at least one local minimum.

So, the Ambrosetti-Rabinowitz theorem actually ensures three or two distinct critical points, according to the function is bounded from below or not.

HISTORICAL NOTES

Let X be a real Banach space, $I : X \to \mathbb{R}$ a continuously Gâteaux differentiable function which verifies (PS).

1973

THE AMBROSETTI-RABINOWITZ THEOREM

Assume that

(G) *there are* $u_0, u_1 \in X$ *and* $r \in \mathbb{R}$, *with* $0 < r < \|u_1 - u_0\|$, *such that*

$$\inf_{\|u-u_0\|=r} I(u) > \max\{I(u_0), I(u_1)\}.$$

Then, I *admits a critical value* c *characterized by*

$$c = \inf_{\gamma\in\Gamma} \max_{t\in[0,1]} I(\gamma(t))$$

where

$$\Gamma = \{\gamma \in C([0,1], X) : \gamma(0) = u_0; \gamma(1) = u_1\}.$$

1985

THE PUCCI-SERRIN THEOREM

(G') *there are* $u_0, u_1 \in X$ *and* $r, R \in \mathbb{R}$, *with* $0 < r < R < \|u_1 - u_0\|$, *such that*

$$\inf_{r<\|u-u_0\|<R} I(u) \geq \max\{I(u_0), I(u_1)\}.$$

Corollary. *If* I *admits two local minima, then* I *admits a third critical point.*

1989

THE GHOUSSOUB-PREISS THEOREM

$$(MG)\quad \textit{there are } u_0, u_1 \in X \textit{ and } r \in \mathbb{R}, \textit{ with } 0 < r < \|u_1 - u_0\|, \textit{ such that}$$

$$\inf_{\|u-u_0\|=r} I(u) \geq \max\{I(u_0), I(u_1)\}.$$

A REMARK

Theorem. *Let X be a real Banach space, $I : X \to \mathbb{R}$ a continuously Gâteaux differentiable function which verifies (PS) and it is bounded from below. Then, the following assertions are equivalent:*

(MG) *there are $u_0, u_1 \in X$ and $r \in \mathbb{R}$, with $0 < r < \|u_1 - u_0\|$, such that*

$$\inf_{\|u-u_0\|=r} I(u) \geq \max\{I(u_0), I(u_1)\};$$

(L) *I admits at least two distinct local minima.*

So, the Ambrosetti-Rabinowitz theorem, when the function is bounded from below actually ensures three distinct critical points.

In fact, in this case the mountain pass geometry implies the existence of two local minima and the Pucci-Serrin theorem ensures the third critical point.

In a similar way it is possible to see that, when the function is unbounded from below, the mountain pass geometry is equivalent to the existence of at least one local minimum.

In order to apply the Ambrosetti-Rabinowitz theorem, it is important to establish the existence of a local minimum which is not a strict global minimum.

THE TONELLI-WEIERSTRASS THEOREM

The existence of a global minimum can be obtained owing to the classical theorem of direct methods in the variational calculus where the key assumptions are the sequential weak lower semicontinuity and the coercivity.

Here, the version for differentiable functions is recalled.

Let X be a real Banach space, $I : X \to \mathbb{R}$ a continuously Gâteaux differentiable function which verifies (PS) and it is bounded from below. Then, it admits a global minimum.

A LOCAL MINIMUM THEOREM

Our aim is to present a local minimum theorem for functions of the type:

Φ $-$ Ψ

$$\Phi - \Psi$$

An existence theorem of a local minimum for continuously Gâteaux differentiable functions, possibly unbounded from below, is presented.

The approach is based on Ekeland's Variational Principle applied to a non-smooth variational framework by using also a novel type of Palais-Smale condition which is more general than the classical one.

PALAIS-SMALE CONDITION

Let X be a real Banach space, we say that a Gâteaux differentiable function

$$I : X \to \boldsymbol{R}$$

verifies the Palais-Smale condition (in short (PS)-condition) if any sequence $\{u_n\}$ *such that*

(α) $\{I(u_n)\}$ is bounded,

(β) $\lim_{n\to+\infty} \|I'(u_n)\|_{X^*} = 0,$

has a convergent subsequence.

Let X be a real Banach space and let $\Phi : X \to \mathbf{R}$, $\Psi : X \to R$ two Gâteaux differentiable functions. Put $I = \Phi - \Psi$.

Fix r_1, $r_2 \in [-\infty;+\infty]$, with $r_1 < r_2$, we say that the function I verifies the Palais-Smale condition cut off lower at r_1 and upper at r_2 (in short ${}^{[r_1]}(PS)^{[r_2]}$-condition) if any sequence $\{u_n\}$ such that

(α) $\{I(u_n)\}$ is bounded,

(β) $\lim_{n\to+\infty} \|I'(u_n)\|_{X^*} = 0,$

(γ) $r_1 < \Phi(u_n) < r_2 \ \forall n \in \mathbf{N},$

has a convergent subsequence.

Clearly, if $r_1 = -\infty$ *and* $r_2 = +\infty$ *it coincides with the classical (PS)-condition.*
Moreover, if $r_1 = -\infty$ *and* $r_2 \in \mathbf{R}$ *we denote it by* $(PS)^{[r_2]}$, *while if* $r_1 \in \mathbf{R}$ *and* $r_2 = +\infty$ *we denote it by* ${}^{[r_1]}(PS)$.

In particular,

If $I = \Phi - \Psi$ satisfies (PS)-condition, then it satisfies $^{[r_1]}(PS)^{[r_2]}$-condition for all $r_1, r_2 \in [-\infty, +\infty]$ with $r_1 < r_2$.

Proposition. *Let X be a reflexive real Banach space; $\Phi : X \to \mathbb{R}$ be a sequentially weakly lower semicontinuous, coercive and continuously Gâteaux differentiable function whose Gâteaux derivative admits a continuous inverse on X^*, $\Psi : X \to \mathbb{R}$ be a continuously Gâteaux differentiable function whose Gâteaux derivative is compact.*

Then, for all $r_1, r_2 \in [-\infty, +\infty[$, with $r_1 < r_2$, the functional $\Phi - \Psi$ satisfies the ${}^{[r_1]}(PS)^{[r_2]}$-condition.

CLARKE

CHANG

To prove the local minimum theorem we use the theory for locally Lipschitz functionals investigated by K.C. Chang, which is based on the Nonsmooth Analysis by F.H. Clarke, and generalizes the study on the variational inequalities as given by A. Szulkin .

SZULKIN

PANAGIOTOPOULOS

This theory is applied to study variational and variational-hemivariational inequalities . In particular, for instance, differential inclusions and equations with discontinuous nonlinearities are investigated.

MOTREANU

PAPAGEORGIOU

RADULESCU

Here, by using the nonsmooth theory we obtain results for smooth functions.

THE EKELAND VARIATIONAL PRINCIPLE

Arguing in a classical way of the smooth analysis (as, for instance, Ghossoub), but using the definitions and properties of the non-smooth analysis (as, for instance, Motreanu-Radulescu, the following consequence of the Ekeland variational Principle can be obtained.

A CONSEQUENCE OF THE EKELAND VARIATIONAL PRINCIPLE IN THE NONSMOOTH ANALYSIS FRAMEWORK

Lemma. *Let X be a real Banach space and $I : X \to \mathbf{R}$ a locally Lipschitz function bounded from below. Then, for all minimizing sequence of I, $\{u_n\}_{n\in\mathbf{N}} \subseteq X$, there exists a minimizing sequence of I, $\{v_n\}_{n\in\mathbf{N}} \subseteq X$, such that*

$$I(v_n) \le I(u_n) \quad \forall n \in \mathbf{N},$$

$$I^\circ(v_n; h) \ge -\varepsilon_n \|h\| \quad \forall h \in X, \ \forall n \in \mathbf{N}, \qquad \text{where } \varepsilon_{\mathrm{n}} \to 0^+.$$

A LOCAL MINIMUM THEOREM

Let X be a real Banach space and $\Phi, \Psi : X \to \mathbb{R}$ two continuously Gâteaux differentiable functions. Put

$$I = \Phi - \Psi$$

and assume that there are $x_0 \in X$ and $r_1, r_2 \in \mathbb{R}$, with $r_1 < \Phi(x_0) < r_2$, such that

$$\sup_{u \in \Phi^{-1}(]r_1, r_2[)} \Psi(u) \leq r_2 - \Phi(x_0) + \Psi(x_0), \quad (1)$$

$$\sup_{u \in \Phi^{-1}(]-\infty, r_1])} \Psi(u) \leq r_1 - \Phi(x_0) + \Psi(x_0). \quad (2)$$

Moreover, assume that I satisfies ${}^{[r_1]}(PS)^{[r_2]}$-condition.

Then, there is $u_0 \in \Phi^{-1}(]r_1, r_2[)$ such that $I(u_0) \leq I(u)$ for all $u \in \Phi^{-1}(]r_1, r_2[)$ and $I'(u_0) = 0$.

Proof. *Put*

$$M = r_2 - \Phi(x_0) + \Psi(x_0),$$

$$\Psi_M(u) = \begin{cases} \Psi(u) & \text{if } \Psi(u) < M \\ M & \text{if } \Psi(u) \geq M, \end{cases}$$

$$\Phi^{r_1}(u) = \begin{cases} \Phi(u) & \text{if } \Phi(u) > r_1 \\ r_1 & \text{if } \Phi(u) \leq r_1, \end{cases}$$

$$J = \Phi^{r_1} - \Psi_M.$$

Clearly, J is locally Lipschitz and bounded from below. Hence, Lemma and a suitable computation ensure the conclusion.

TWO SPECIAL CASES

First

Let X be a real Banach space and $\Phi, \Psi : X \to \mathbb{R}$ two continuously Gâteaux differentiable functions with Φ bounded from below. Put

$$I = \Phi - \Psi$$

and assume that there are $x_0 \in X$ and $r \in \mathbb{R}$ with $r > \Phi(x_0)$ such that

$$\sup_{u \in \Phi^{-1}(]-\infty, r[)} \Psi(u) \leq r - \Phi(x_0) + \Psi(x_0). \quad (1)$$

Moreover, assume that I satisfies $(PS)^{[r]}$-condition.

Then, there is $u_0 \in \Phi^{-1}(]-\infty, r[)$ such that $I(u_0) \leq I(u)$ for all $u \in \Phi^{-1}(]-\infty, r[)$ and $I'(u_0) = 0$.

Second

A SPECIAL CASE OF THE LOCAL MINIMUM THEOREM

Let X be a real Banach space and $\Phi, \Psi : X \to \mathbb{R}$ two continuously Gâteaux differentiable functions. Put

$$I = \Phi - \Psi$$

and assume that I is bounded from below and there are $x_1 \in X$ and $r \in \mathbb{R}$, with $r < \Phi(x_1)$, such that

$$\sup_{u \in \Phi^{-1}(]-\infty, r])} \Psi(u) \leq r - \Phi(x_1) + \Psi(x_1). \quad \textbf{(2)}$$

Moreover, assume that I satisfies ${}^{[r]}(PS)$-condition.

Then, there is $u_1 \in \Phi^{-1}(]r, +\infty[)$ such that $I(u_1) \leq I(u)$ for all $u \in \Phi^{-1}(]r, +\infty[)$ and $I'(u_1) = 0$.

A THREE CRITICAL POINTS THEOREM

From the preceding two variants of the local minimum theorem, a three critical points theorem is obtained. Here a special case is pointed out.

A THREE CRITICAL POINTS THEOREM

Let X be a real Banach space and $\Phi, \Psi : X \to \mathbb{R}$ two continuously Gâteaux differentiable functionals with Φ bounded from below. Assume that $\Phi(0) = \Psi(0) = 0$ and there are $r > 0$ and $\overline{x} \in X$, with $r < \Phi(\overline{x})$, such that

$$\frac{\sup\limits_{u \in \Phi^{-1}(]-\infty, r])} \Psi(u)}{r} < \frac{\Psi(\overline{x})}{\Phi(\overline{x})}. \qquad (3)$$

Further assume that, for each

$$\lambda \in \Lambda := \left] \frac{\Phi(\overline{x})}{\Psi(\overline{x})}, \frac{r}{\sup\limits_{u \in \Phi^{-1}(]-\infty, r])} \Psi(u)} \right[,$$

the functional $I_\lambda = \Phi - \lambda\Psi$ is bounded from below and satisfies (PS)-condition.

Then, for each $\lambda \in \Lambda$ the functional I_λ admits at least three critical points.

NONLINEAR DIFFERENTIAL PROBLEMS

A TWO-POINT BOUNDARY VALUE PROBLEM

Consider the following two point boundary value problem

$$(D_\lambda) \qquad \begin{cases} -u'' = \lambda f(u) \ \text{ in } \ ]0,1[\\ u(0) = u(1) = 0, \end{cases}$$

where $f : \mathbf{R} \to \mathbf{R}$ is a continuous function and is λ a positive real parameter.

Moreover, put

$$F(\xi) = \int_0^{\xi} f(t)dt$$

for all $\xi \in \mathbf{R}$ *and assume, for clarity, that* f *is nonnegative.*

Theorem. *Assume that*
there are two positive constants c and d, with $c < d$, such that

(1) $$\frac{F(c)}{c^2} < \frac{1}{4}\frac{F(d)}{d^2}$$

and there are two positive constants a and s, with $s < 2$, such that

(2) $$F(\xi) \leq a(1 + |\xi|^s) \qquad \forall \xi \in \mathbb{R}.$$

Then, for each $\lambda \in \left]8\frac{d^2}{F(d)}, 2\frac{c^2}{F(c)}\right[$, *problem* (D_λ) *admits at least three (nonnegative) classical solutions.*

TWO-POINT BOUNDARY VALUE PROBLEMS

$$(D_\lambda) \qquad \begin{cases} -(pu')' + qu = \lambda f(x,u) \ \text{ in } \ ]\mathrm{a},\mathrm{b}[\\ u(a) = u(b) = 0 \end{cases}$$

there exist two positive constants c,d, with c < d, such that

$$\frac{\int_a^b \max_{|\xi|\le c} F(x,\xi)dx}{c^2} < K \frac{\int_{a+\frac{1}{4}(b-a)}^{b-\frac{1}{4}(b-a)} F(x,d)dx}{d^2}$$

$$K := \frac{6p_0}{12\|p\|_\infty + (b-a)^2\|q\|_\infty}$$

Fix p>1.

$$(D_\lambda) \quad \begin{cases} -(|u'|^{p-2}u')' = \lambda f(x,u) \ \text{ in } \ ]\mathrm{a},\mathrm{b}[\\ u(a) = u(b) = 0, \end{cases}$$

there exist two positive constants c,d, with c < d, such that

$$\frac{\displaystyle\int_a^b \max_{|\xi|\le c} F(x,\xi)dx}{c^p} < M \frac{\displaystyle\int_{a+\frac{1}{4}(b-a)}^{b-\frac{1}{4}(b-a)} F(x,d)dx}{d^p}$$

there exist two positive constants a,s, with s < p, such that

$$F(x,\xi) \le a(1+|\xi|^s) \qquad \forall (x,\xi) \in [a,b] \times \mathbb{R}.$$

NEUMANN BOUNDARY VALUE PROBLEMS

$$(N_\lambda) \qquad \begin{cases} -(pu')' + qu = \lambda f(x,u) \ \text{ in } \ ]\mathrm{a},\mathrm{b}[\\ u'(a) = u'(b) = 0, \end{cases}$$

MIXED BOUNDARY VALUE PROBLEMS

$$(M_\lambda) \qquad \begin{cases} -(pu')' + qu = \lambda f(x,u) \ \text{ in } \ ]\mathrm{a},\mathrm{b}[\\ u(a) = u'(b) = 0, \end{cases}$$

STURM-LIOUVILLE BOUNDARY VALUE PROBLEMS

$$(SL_\lambda) \quad \begin{cases} -(\rho\Phi_p(u'))' + s\Phi_p(u) = \lambda f(x,u) & x \in [a,b] \\ \alpha u'(a) - \beta u(a) = A, \quad \gamma u'(b) + \sigma u(b) = B, \end{cases}$$

where $p > 1$, $\Phi_p(u) = |u|^{p-2}u$, $\rho, s \in L^\infty([a,b])$, with $\operatorname{ess\,inf}_{[a,b]} \rho > 0$ and $\operatorname{ess\,inf}_{[a,b]} s > 0$, $A, B \in \mathbb{R}, \alpha, \beta, \gamma, \sigma > 0$, $f : [a,b] \times \mathbb{R} \to \mathbb{R}$ is an L^1-Carathéodory function and λ is a positive real parameter.

HAMILTONIAN SYSTEMS

$$\begin{cases} -u'' + A(t)u = \lambda b(t)\nabla G(u) & \text{a.e. in } [0,T], \\ u(T) - u(0) = u'(T) - u'(0) = 0. \end{cases}$$

FOURTH-ORDER ELASTIC BEAM EQUATIONS

$$\begin{cases} u^{iv} + Au'' + Bu = \lambda f(t,u) \quad \text{in } [0,1], \\ u(0) = u(1) = 0, \\ u''(0) = u''(1) = 0, \end{cases}$$

BOUNDARY VALUE PROBLEMS ON THE HALF_LINE

$$\begin{cases} -y'' + m^2 y = \lambda f(t,y) \\ y(0) = 0 \\ \lim_{t\to\infty} y(t) = 0, \end{cases}$$

NONLINEAR DIFFERENCE PROBLEMS

$$\begin{cases} -\Delta(\phi_p(\Delta u(k-1))) = \lambda f(k, u(k)), & k \in [1, T], \\ u(0) = u(T+1) = 0, \end{cases}$$

where T is a fixed positive integer, $[1, T]$ is the discrete interval $\{1, \ldots, T\}$, λ is a positive real parameter, $\Delta u(k) := u(k+1) - u(k)$ is the forward difference operator, $\phi_p(s) = |s|^{p-2}s$, $1 < p < +\infty$ and $f : [1, T] \times \mathbb{R} \to \mathbb{R}$ is a continuous function.

ELLIPTIC DIRICHLET PROBLEMS INVOLVING THE P-LAPLACIAN WITH P>N

$$\begin{cases} \Delta_p u + \lambda f(x, u) = 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

$\Omega \subset \mathbf{R}^N$ $(N \geqslant 1)$ is a nonempty bounded open set with a boundary $\partial\Omega$ of class C^1, $p > N$, $\lambda > 0$ and $f : \Omega \times \mathbf{R} \to \mathbf{R}$ is a Carathéodory function. $\Delta_p u = \mathrm{div}(|\nabla u|^{p-2} \nabla u)$

ELLIPTIC NEUMANN PROBLEMS INVOLVING THE P-LAPLACIAN WITH P>N

$$\begin{cases} -\Delta_p u + a(x)\,|u|^{p-2}\,u = \lambda f(x,u) & \text{in } \Omega \\ \dfrac{\partial u}{\partial \upsilon} = 0 & \text{on } \partial\Omega \end{cases}$$

ELLIPTIC SYSTEMS

$$\begin{cases} -\Delta_{p_1} u_1 + q_1(x)|u_1|^{p_1-2}u_1 = \lambda F_{u_1}(x, u_1, \ldots, u_m) & \text{in } \Omega, \\ -\Delta_{p_2} u_2 + q_2(x)|u_2|^{p_2-2}u_2 = \lambda F_{u_2}(x, u_1, \ldots, u_m) & \text{in } \Omega, \\ \vdots \\ -\Delta_{p_m} u_m + q_1(x)|u_m|^{p_m-2}u_m = \lambda F_{u_m}(x, u_1, \ldots, u_m) & \text{in } \Omega, \\ u_i|_{\partial\Omega} = 0 \quad (1 \le i \le m), \end{cases}$$

ELLIPTIC PROBLEMS INVOLVING THE p-LAPLACIAN WITH p≤N

$$\begin{cases} -\Delta u = \lambda f(x,u) & \text{in } \Omega, \\ u|_{\partial\Omega} = 0, \end{cases}$$

(h_1) *there exist two non-negative constants* a_1, a_2 *and* $q \in]1, 2N/(N-2)[$ *such that*

$$|f(x,t)| \leqslant a_1 + a_2|t|^{q-1},$$

for every $(x,t) \in \Omega \times \mathbb{R}$.

there exist two positive constants γ *and* δ, *with* $\delta > \gamma\kappa$ *such that*

$$\frac{\inf_{x\in\Omega} F(x,\delta)}{\delta^2} > a_1\frac{K_1}{\gamma} + a_2 K_2 \gamma^{q-2}, \qquad (1)$$

where a_1, a_2 *are given in* (h_1) *and* κ, K_1, K_2 *are given by*

$$\kappa := \frac{D\sqrt{2}}{2\pi^{N/4}}\left(\frac{\Gamma(1+N/2)}{D^N-(D/2)^N}\right)^{1/2} \quad K_1 := \frac{2\sqrt{2}c_1(2^N-1)}{D^2} \quad K_2 := \frac{2^{\frac{q+2}{2}}c_q^q(2^N-1)}{qD^2}.$$

ELLIPTIC PROBLEMS INVOLVING THE p(x)-LAPLACIAN

NONLINEAR EIGENVALUE PROBLEMS IN ORLICS-SOBOLEV SPACES

NONLINEAR ELLIPTIC PROBLEMS ON THE SIERPI NSKI GASKET

GENERALIZED YAMABE EQUATIONS ON RIEMANNIAN MANIFOLDS

FURTHER APPLICATIONS OF THE LOCAL MINIMUM THEOREM

A VARIANT OF THREE CRITICAL POINT THEOREM FOR FUNCTIONS UNBOUNDED FROM BELOW

If we apply two times the first special case of the local minimum theorem and owing to a novel version of the mountain pass theorem where the (PS) cut off upper at r is assumed we can give a variant of the three critical theorem. In the applications it became

Theorem. *Assume that there are three positive constants c_1, d and c_2, with $c_1 < d < \frac{\sqrt{2}}{2} c_2$, such that*

$$\frac{F(c_1)}{c_1^2} < \frac{1}{6} \frac{F(d)}{d^2} \qquad \textbf{(1)}$$

and

$$\frac{F(c_2)}{c_2^2} < \frac{1}{12} \frac{F(d)}{d^2}. \qquad \textbf{(2)}$$

Then, for each $\lambda \in \left] 12 \frac{d^2}{F(d)}, \min\left\{ 2 \frac{c_1^2}{F(c_1)}, \frac{c_2^2}{F(c_2)} \right\} \right[$, problem (D_λ) admits at least three (nonnegative) classical solutions u_i, $i = 1, 2, 3$, such that

$$\max_{x \in [0,1]} |u_i(x)| < c_2, \qquad i = 1, 2, 3.$$

INFINITELY MANY CRITICAL POINTS THEOREM

If we apply iteratively the first special case of the local minimum theorem in a suitable way, we obtain an infinitely many critical points theorem. As an example of application, here, we present the following result.

Theorem. *Assume that*

(1) $$\liminf_{\xi\to+\infty}\frac{F(\xi)}{\xi^2}<\frac{1}{4}\limsup_{\xi\to+\infty}\frac{F(\xi)}{\xi^2}.$$

Then, for each $\lambda\in\left]\dfrac{8}{\limsup\limits_{\xi\to+\infty}\dfrac{F(\xi)}{\xi^2}},\dfrac{2}{\liminf\limits_{\xi\to+\infty}\dfrac{F(\xi)}{\xi^2}}\right[$,

the problem (D_λ) *admits a sequence of pairwise distinct positive classical solutions.*

PERTURBED PROBLEMS

Previous results can be applied to perturbed problems, obtaining for instance, results of the following type.

Theorem. *Assume that there exist three positive constants* c_1, c_2, d , *with* $c_1 < d < \frac{c_2}{2}$ *such that*

$$f(\xi) \geq 0 \ \textit{for each} \ \xi \in [0, c_2];$$

$$\frac{F(c_1)}{c_1^2} < \frac{F(d)}{6d^2};$$

$$\frac{F(c_2)}{c_2^2} < \frac{F(d)}{12d^2};$$

Then, for every $\lambda \in \Lambda :=]\frac{12d^2}{F(d)}, \min\{\frac{2c_1^2}{F(c_1)}, \frac{c_2^2}{F(c_2)}\}[$ *and* *for every positive continuous function* $g : [0,1] \times \mathbb{R} \to \mathbb{R}$, *there exists* $\delta^*_{\lambda,g} > 0$ *given by*

$$\delta^*_{\lambda,g} := \min\left\{\frac{2c_1^2 - \lambda F(c_1)}{\int_0^1 G(x,c_1)dx}, \frac{c_2^2 - \lambda F(c_2)}{\int_0^1 G(x,c_2)dx}\right\}.$$

such that, for each $\mu \in]0, \delta^*_{\lambda,g}[$, *the problem*

$$\begin{cases} -u'' = \lambda f(u) + \mu g(x,u) \quad \text{in }]0,1[\\ u(0) = u(1) = 0, \end{cases}$$

has at least three classical solutions u_i, $i = 1, 2, 3$ *such that*

$$0 < u_i(x) < c_2 \quad \forall x \in [0,1], \qquad i = 1, 2, 3.$$

VARIATIONAL-HEMIVARIATIONAL INEQUALITIES

In a natural way the previous results have been also obtained in the framework of the non-smooth Analysis. As example, here, the following problem is considered.

Let Ω be a non-empty, bounded, open subset of the Euclidian space $\mathbb{R}^N$, $N \geq 1$, with C^1-boundary $\partial\Omega$, let $p \in]N, +\infty[$, and let $q \in L^\infty(\Omega)$ satisfy $\operatorname{ess\,inf}_{x\in\Omega} q(x) > 0$.

Problem: Find $u \in K$ such that, for all $v \in K$,

$$\int_\Omega |\nabla u(x)|^{p-2}\nabla u(x) \cdot \nabla(v(x) - u(x))dx + \int_\Omega q(x)|u(x)|^{p-2}u(x)(v(x) - u(x))dx$$
$$+ \int_\Omega \lambda\alpha(x)F^\circ(u(x); v(x) - u(x))dx + \int_{\partial\Omega} \mu\beta(x)G^\circ(\gamma u(x); \gamma v(x) - \gamma u(x))d\sigma \geq 0,$$

where K is a closed convex subset of $W^{1,p}(\Omega)$ containing the constant functions, and $\alpha \in L^1(\Omega)$, $\beta \in L^1(\partial\Omega)$, with $\alpha(x) \geq 0$ for a.a. $x \in \Omega$, $\alpha \not\equiv 0$, $\beta(x) \geq 0$ for a.a. $x \in \partial\Omega$, and λ, μ are real parameters, with $\lambda > 0$ and $\mu \geq 0$. Here, F° and G° stand for Clarke's generalized directional derivatives of locally Lipschitz functions $F, G : \mathbb{R} \to \mathbb{R}$ given by $F(\xi) = \int_0^\xi f(t)dt$, $G(\xi) = \int_0^\xi g(t)dt$, $\xi \in \mathbb{R}$, with $f, g : \mathbb{R} \to \mathbb{R}$ locally essentially bounded functions, and $\gamma : W^{1,p}(\Omega) \to L^p(\partial\Omega)$ denotes the trace operator.

A prototype of the previous problem for $K = W^{1,p}(\Omega)$ is the following boundary value problem with nonsmooth potential and nonhomogeneous, nonsmooth Neumann boundary condition

$$\begin{cases} \Delta_p u - q(x)|u|^{p-2}u \in \lambda\alpha(x)\partial F(u) & \text{in } \Omega, \\ |\nabla u|^{p-2}\dfrac{\partial u}{\partial \nu} \in -\mu\beta(x)\partial G(\gamma u) & \text{on } \partial\Omega. \end{cases}$$

Theorem. *Let* $f : \mathbb{R} \to \mathbb{R}$ *be a non-negative, locally essentially bounded function and set* $F(\xi) = \int_0^\xi f(t)\,dt$ *for all* $\xi \in \mathbb{R}$. *Assume that*

(1)
$$\liminf_{\xi \to +\infty} \frac{F(\xi)}{\xi^2} = 0 \quad \textit{and} \quad \limsup_{\xi \to +\infty} \frac{F(\xi)}{\xi^2} = +\infty.$$

Then, for each non-negative, continuous function $g : \mathbb{R} \to \mathbb{R}$ *such that*

$$g_\infty := \lim_{t \to +\infty} \frac{g(t)}{t} < +\infty,$$

there is a sequence of pairwise distinct functions $\{u_n\} \subset W^{2,2}(]0,1[)$ *such that for all* $n \in \mathbb{N}$ *one has*

$$\begin{cases} -u_n''(x) + u_n(x) \in [f^-(u_n(x)), f^+(u_n(x))] & \text{for a.a. } x \in]0,1[\\ u_n'(0) = \mu g(u_n(0)) \\ u_n'(1) = -\mu g(u_n(1)), \end{cases}$$

where $f^-(t) = \lim_{\delta \to 0^+} \operatorname{ess\,inf}_{|t-z|<\delta} f(z)$ *and* $f^+(t) = \lim_{\delta \to 0^+} \operatorname{ess\,sup}_{|t-z|<\delta} f(z)$ *for all* $t \in \mathbb{R}$.

A DIRECT APPLICATION OF THE LOCAL MINIMUM THEOREM

The local minimum theorem can be directly applied to obtain the existence of at least one solution.

$$\begin{cases} -u'' = \lambda\alpha(x)f(u) & x \in]0,1[\\ u(0) = u(1) = 0. \end{cases}$$

Theorem. *Assume that f is nonnegative and there exist two positive constants c, d, with $\sqrt{2}d < c$, such that*

$$\frac{F(c)}{c^2} < \left(\frac{\int_{\frac{1}{4}}^{\frac{3}{4}} \alpha(x)dx}{2\|\alpha\|_1} \right) \frac{F(d)}{d^2}. \qquad (1)$$

Then, for each $\lambda \in \left] \frac{4}{\int_{\frac{1}{4}}^{\frac{3}{4}} \alpha(x)dx} \frac{d^2}{F(d)}, \frac{2}{\|\alpha\|_1} \frac{c^2}{F(c)} \right[$, *the problem* (D_λ) *admits at least one positive weak solution* $\bar{u}$ *such that*

$$|\bar{u}(x)| < c \text{ for all } x \in [0,1].$$

Theorem *Assume that*

$$\lim_{t \to 0^+} \frac{f(t)}{t} = +\infty, \qquad \textbf{(1')}$$

fix $\delta > 0$ *such that* $f(t) > 0$ *for all* $t \in]0, \delta[$, *and put*

$$\lambda^* = \frac{2}{\|\alpha\|_1} \sup_{c \in]0,\delta[} \frac{c^2}{F(c)}.$$

Then, for each $\lambda \in \,]0, \lambda^*[$, *the problem* (D_λ) *admits at least one positive weak solution* $\bar{u}$ *such that*

$|\bar{u}(x)| < \delta$ *for all* $x \in [0, 1]$.

Consider the following problem

$$\begin{cases} -\Delta_p u = \lambda f(u) & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

Theorem. *Assume that*

$$\limsup_{t\to 0^+} \frac{F(t)}{t^p} = +\infty. \qquad \textbf{(1)}$$

Then, for each $\lambda \in]0, \lambda_0[$*, where* $\lambda_0 = k_0 \sup_{c\in]0,+\infty[} \frac{c^p}{F(c)}$ *and*

$$k_0 = \frac{N\pi^{\frac{p}{2}}}{p\left[m(\Omega)\Gamma(1+\frac{N}{2})\right]^{\frac{p}{N}}} \left(\frac{p-N}{p-1}\right)^{p-1},$$

the problem (D_λ) *admits at least one positive weak solution. Further, assume that* $f(0) \neq 0$ *and*

$$0 < \mu F(t) \leq t f(t) \qquad \textbf{(AR)}$$

for all $|t| \geq r$*, for some* $r > 0$ *and for some* $\mu > p$*. Then, for each* $\lambda \in]0, \lambda_0[$*, the problem* (D_λ) *admits at least two positive weak solutions.*

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These results have been obtained in the setting of
PRIN 2007-ORDINARY DIFFERENTIAL EQUATIONS AND APPLICATIONS
(National Scientific Project Manager: **Fabio Zanolin**)
from the Unit of MESSINA.

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Thank you very much for your kind attention