Some remarks on the variational methods

Gabriele Bonanno

University of Messina
Some remarks on the classical Ambrosetti-Rabinowitz theorem are presented. In particular, it is observed that the geometry of the mountain pass, if the function is bounded from below, is equivalent to the existence of at least two local minima, while, when the function is unbounded from below, it is equivalent to the existence of at least one local minimum.
So, the Ambrosetti-Rabinowitz theorem actually ensures three or two distinct critical points, according to the function is bounded from below or not.
Let $X$ be a real Banach space, $I : X \to \mathbb{R}$ a continuously Gâteaux differentiable function which verifies (PS).
Assume that

\[(G) \text{ there are } u_0, u_1 \in X \text{ and } r \in \mathbb{R}, \text{ with } 0 < r < \|u_1 - u_0\|, \text{ such that} \]

\[
\inf_{\|u - u_0\| = r} I(u) > \max\{I(u_0), I(u_1)\}.
\]

Then, \( I \) admits a critical value \( c \) characterized by

\[
c = \inf_{\gamma \in \Gamma} \max_{t \in [0, 1]} I(\gamma(t))
\]

where

\[
\Gamma = \{\gamma \in C([0, 1], X) : \gamma(0) = u_0; \gamma(1) = u_1\}.
\]
(G') there are $u_0, u_1 \in X$ and $r, R \in \mathbb{R}$, with $0 < r < R < \|u_1 - u_0\|$, such that

$$\inf_{r < \|u-u_0\| < R} I(u) \geq \max\{I(u_0), I(u_1)\}.$$ 

Corollary. If $I$ admits two local minima, then $I$ admits a third critical point.
(MG) there are $u_0, u_1 \in X$ and $r \in \mathbb{R}$, with $0 < r < \|u_1 - u_0\|$, such that

$$\inf_{\|u - u_0\| = r} I(u) \geq \max\{I(u_0), I(u_1)\}.$$
Theorem. Let $X$ be a real Banach space, $I : X \to \mathbb{R}$ a continuously Gâteaux differentiable function which verifies (PS) and it is bounded from below. Then, the following assertions are equivalent:

(MG) there are $u_0, u_1 \in X$ and $r \in \mathbb{R}$, with $0 < r < \|u_1 - u_0\|$, such that

$$\inf_{\|u - u_0\| = r} I(u) \geq \max\{I(u_0), I(u_1)\};$$

(L) $I$ admits at least two distinct local minima.
So, the Ambrosetti-Rabinowitz theorem, when the function is bounded from below actually ensures three distinct critical points.

In fact, in this case the mountain pass geometry implies the existence of two local minima and the Pucci-Serrin theorem ensures the third critical point.
In a similar way it is possible to see that, when the function is unbounded from below, the mountain pass geometry is equivalent to the existence of at least one local minimum.

In order to apply the Ambrosetti-Rabinowitz theorem, it is important to establish the existence of a local minimum which is not a strict global minimum.
The existence of a global minimum can be obtained owing to the classical theorem of direct methods in the variational calculus where the key assumptions are the sequential weak lower semicontinuity and the coercivity.
Here, the version for differentiable functions is recalled.

Let $X$ be a real Banach space, $I : X \to \mathbb{R}$ a continuously Gâteaux differentiable function which verifies $(PS)$ and it is bounded from below. Then, it admits a global minimum.
Our aim is to present a local minimum theorem for functions of the type:

\[ \Phi - \Psi \]
An existence theorem of a local minimum for continuously Gâteaux differentiable functions, possibly unbounded from below, is presented.

The approach is based on Ekeland’s Variational Principle applied to a non-smooth variational framework by using also a novel type of Palais-Smale condition which is more general than the classical one.
Let $X$ be a real Banach space, we say that a Gâteaux differentiable function $I : X \rightarrow \mathbb{R}$ verifies the Palais-Smale condition (in short $(PS)$-condition) if any sequence $\{u_n\}$ such that
(α) \( \{ I(u_n) \} \) is bounded,

(β) \( \lim_{n \to +\infty} \| I'(u_n) \|_{X^*} = 0, \)

has a convergent subsequence.

Let \( X \) be a real Banach space and let \( \Phi : X \to \mathbb{R}, \Psi : X \to \mathbb{R} \) two Gâteaux differentiable functions. Put \( I = \Phi - \Psi \).
Fix \( r_1, r_2 \in [−\infty; +\infty] \), with \( r_1 < r_2 \), we say that the function \( I \) verifies the Palais-Smale condition cut off lower at \( r_1 \) and upper at \( r_2 \) (in short \([r_1](PS)[r_2]\) condition) if any sequence \( \{u_n\} \) such that

\[
(\alpha) \quad \{I(u_n)\} \text{ is bounded},
\]
\[
(\beta) \quad \lim_{n \to +\infty} \|I'(u_n)\|_{X^*} = 0,
\]
\[
(\gamma) \quad r_1 < \Phi(u_n) < r_2 \quad \forall n \in \mathbb{N},
\]
has a convergent subsequence.

Clearly, if \( r_1 = -\infty \) and \( r_2 = +\infty \) it coincides with the classical \((PS)\)-condition.

Moreover, if \( r_1 = -\infty \) and \( r_2 \in \mathbb{R} \) we denote it by \((PS)^{[r_2]}\), while if \( r_1 \in \mathbb{R} \) and \( r_2 = +\infty \) we denote it by \( [r_1](PS) \).
In particular,

If $I = \Phi - \Psi$ satisfies $(PS)$-condition, then it satisfies $[r_1](PS)^{[r_2]}$-condition for all $r_1, r_2 \in [-\infty, +\infty]$ with $r_1 < r_2$. 
Proposition. Let $X$ be a reflexive real Banach space; $\Phi : X \to \mathbb{R}$ be a sequentially weakly lower semicontinuous, coercive and continuously Gâteaux differentiable function whose Gâteaux derivative admits a continuous inverse on $X^*$, $\Psi : X \to \mathbb{R}$ be a continuously Gâteaux differentiable function whose Gâteaux derivative is compact.

Then, for all $r_1, r_2 \in [-\infty, +\infty]$, with $r_1 < r_2$, the functional $\Phi - \Psi$ satisfies the $[r_1](PS)[r_2]$-condition.
To prove the local minimum theorem we use the theory for locally Lipschitz functionals investigated by K.C. Chang, which is based on the Nonsmooth Analysis by F.H. Clarke, and generalizes the study on the variational inequalities as given by A. Szulkin.
This theory is applied to study variational and variational-hemivariational inequalities. In particular, for instance, differential inclusions and equations with discontinuous nonlinearities are investigated.
Here, by using the nonsmooth theory we obtain results for smooth functions.

THE EKELAND VARIATIONAL PRINCIPLE

Arguing in a classical way of the smooth analysis (as, for instance, Ghossoub), but using the definitions and properties of the non-smooth analysis (as, for instance, Motreanu-Radulescu, the following consequence of the Ekeland variational Principle can be obtained.
**A CONSEQUENCE OF THE EKELAND VARIATIONAL PRINCIPLE IN THE NONSMOOTH ANALYSIS FRAMEWORK**

**Lemma.** Let $X$ be a real Banach space and $I : X \to \mathbb{R}$ a locally Lipschitz function bounded from below. Then, for all minimizing sequence of $I$, $\{u_n\}_{n \in \mathbb{N}} \subseteq X$, there exists a minimizing sequence of $I$, $\{v_n\}_{n \in \mathbb{N}} \subseteq X$, such that

\[
I(v_n) \leq I(u_n) \quad \forall n \in \mathbb{N},
\]

\[
I^*(v_n; h) \geq -\varepsilon_n \|h\| \quad \forall h \in X, \forall n \in \mathbb{N}, \quad \text{where } \varepsilon_n \to 0^+.
\]
A LOCAL MINIMUM THEOREM

Let $X$ be a real Banach space and $\Phi, \Psi : X \to \mathbb{R}$ two continuously Gâteaux differentiable functions. Put

$$ I = \Phi - \Psi $$

and assume that there are $x_0 \in X$ and $r_1, r_2 \in \mathbb{R}$, with $r_1 < \Phi(x_0) < r_2$, such that

$$ \sup_{u \in \Phi^{-1}([r_1, r_2])} \Psi(u) \leq r_2 - \Phi(x_0) + \Psi(x_0), \quad (1) $$

$$ \sup_{u \in \Phi^{-1}([\infty, r_1])} \Psi(u) \leq r_1 - \Phi(x_0) + \Psi(x_0). \quad (2) $$

Moreover, assume that $I$ satisfies $[r_1](PS)^{[r_2]}$-condition.

Then, there is $u_0 \in \Phi^{-1}([r_1, r_2])$ such that $I(u_0) \leq I(u)$ for all $u \in \Phi^{-1}([r_1, r_2])$ and $I'(u_0) = 0$. 
**Proof.** Put

\[ M = r_2 - \Phi(x_0) + \Psi(x_0), \]

\[ \Psi_M(u) = \begin{cases} 
\Psi(u) & \text{if } \Psi(u) < M \\
M & \text{if } \Psi(u) \geq M,
\end{cases} \]

\[ \Phi^{r_1}(u) = \begin{cases} 
\Phi(u) & \text{if } \Phi(u) > r_1 \\
r_1 & \text{if } \Phi(u) \leq r_1,
\end{cases} \]

\[ J = \Phi^{r_1} - \Psi_M. \]

Clearly, \( J \) is locally Lipschitz and bounded from below. Hence, Lemma and a suitable computation ensure the conclusion.
Let $X$ be a real Banach space and $\Phi, \Psi : X \to \mathbb{R}$ two continuously Gâteaux differentiable functions with $\Phi$ bounded from below. Put

$$I = \Phi - \Psi$$

and assume that there are $x_0 \in X$ and $r \in \mathbb{R}$ with $r > \Phi(x_0)$ such that

$$\sup_{u \in \Phi^{-1}(]-\infty,r[)} \Psi(u) \leq r - \Phi(x_0) + \Psi(x_0). \quad (1)$$

Moreover, assume that $I$ satisfies $(PS)^{[r]}$-condition. Then, there is $u_0 \in \Phi^{-1}(]-\infty,r[)$ such that $I(u_0) \leq I(u)$ for all $u \in \Phi^{-1}(]-\infty,r[)$ and $I'(u_0) = 0$. 

First

TWO SPECIAL CASES
Let $X$ be a real Banach space and $\Phi, \Psi : X \to \mathbb{R}$ two continuously Gâteaux differentiable functions. Put

$$ I = \Phi - \Psi $$

and assume that $I$ is bounded from below and there are $x_1 \in X$ and $r \in \mathbb{R}$, with $r < \Phi(x_1)$, such that

$$ \sup_{u \in \Phi^{-1}([-\infty,r])} \Psi(u) \leq r - \Phi(x_1) + \Psi(x_1). \quad (2) $$

Moreover, assume that $I$ satisfies $[r](PS)$-condition.

Then, there is $u_1 \in \Phi^{-1}([r, +\infty[)$ such that $I(u_1) \leq I(u)$ for all $u \in \Phi^{-1}([r, +\infty[)$ and $I'(u_1) = 0$. 

Gabriele Bonanno, University of Messina, Some remarks on the variational methods 28/60
A THREE CRITICAL POINTS THEOREM

From the preceding two variants of the local minimum theorem, a three critical points theorem is obtained. Here a special case is pointed out.
Let $X$ be a real Banach space and $\Phi, \Psi : X \to \mathbb{R}$ two continuously Gâteaux differentiable functionals with $\Phi$ bounded from below. Assume that $\Phi(0) = \Psi(0) = 0$ and there are $r > 0$ and $\overline{x} \in X$, with $r < \Phi(\overline{x})$, such that

\[
\sup_{u \in \Phi^{-1}([-\infty,r])} \frac{\Psi(u)}{r} < \frac{\Psi(\overline{x})}{\Phi(\overline{x})}. \tag{3}
\]

Further assume that, for each

\[
\lambda \in \Lambda := \left[ \frac{\Phi(\overline{x})}{\Psi(\overline{x})}, \frac{r}{\sup_{u \in \Phi^{-1}([-\infty,r])} \Psi(u)} \right],
\]

the functional $I_\lambda = \Phi - \lambda \Psi$ is bounded from below and satisfies $(PS)$-condition.

Then, for each $\lambda \in \Lambda$ the functional $I_\lambda$ admits at least three critical points.
Consider the following two point boundary value problem

\[(D_{\lambda}) \quad \begin{cases} 
-u'' = \lambda f(u) \quad \text{in } [0, 1] \\
u(0) = u(1) = 0,
\end{cases}\]

where \( f : \mathbb{R} \to \mathbb{R} \) is a continuous function and \( \lambda \) is a positive real parameter.
Moreover, put

\[ F(\xi) = \int_0^\xi f(t)dt \]

for all \( \xi \in \mathbb{R} \) and assume, for clarity, that \( f \) is nonnegative.
Theorem. Assume that there are two positive constants $c$ and $d$, with $c < d$, such that

$$\frac{F(c)}{c^2} < \frac{1}{4} \frac{F(d)}{d^2} \tag{1}$$

and there are two positive constants $a$ and $s$, with $s < 2$, such that

$$F(\xi) \leq a(1 + |\xi|^s) \quad \forall \xi \in \mathbb{R}. \tag{2}$$

Then, for each $\lambda \in \left[\frac{d^2}{2F(d)}, \frac{c^2}{2F(c)}\right]$, problem $(D_\lambda)$ admits at least three (nonnegative) classical solutions.
TWO-POINT BOUNDARY VALUE PROBLEMS

\[
(D_{\lambda}) \quad \begin{cases}
-(pu')' + qu = \lambda f(x, u) \text{ in } [a, b] \\
u(a) = u(b) = 0
\end{cases}
\]

there exist two positive constants \(c, d\), with \(c < d\), such that

\[
\int_{a}^{b} \max_{|\xi| \leq c} F(x, \xi) \, dx \quad < \quad K \int_{a + \frac{1}{4} (b-a)}^{b} F(x, d) \, dx
\]

\[
K \,:= \, \frac{6p_0}{12 \|p\|_\infty + (b-a)^2 \|q\|_\infty}
\]
Fix $p > 1$.

\[
(D_\lambda) \quad \begin{cases}
- (|u'|^{p-2} u')' = \lambda f(x, u) \text{ in } ]a, b[ \\
u(a) = u(b) = 0,
\end{cases}
\]

there exist two positive constants $c, d$, with $c < d$, such that

\[
\int_a^b \max_{|\xi| \leq c} F(x, \xi) \, dx \cdot \frac{c^p}{d^p} < M \int_{a+\frac{1}{4}(b-a)}^{b-\frac{1}{4}(b-a)} F(x, d) \, dx
\]

there exist two positive constants $a, s$, with $s < p$, such that

\[
F(x, \xi) \leq a(1 + |\xi|^s) \quad \forall (x, \xi) \in [a, b] \times \mathbb{R}.
\]
(N_\lambda) \quad \begin{cases} 
-(pu')' + qu = \lambda f(x, u) & \text{in } ]a, b[ \\
u'(a) = u'(b) = 0,
\end{cases}

(M_\lambda) \quad \begin{cases} 
-(pu')' + qu = \lambda f(x, u) & \text{in } ]a, b[ \\
u(a) = u'(b) = 0,
\end{cases}
STURM-LIOUVILLE BOUNDARY VALUE PROBLEMS

\[
(SL_\lambda) \quad \begin{cases}
-(\rho \Phi_p(u'))' + s\Phi_p(u) = \lambda f(x, u) & x \in [a, b] \\
\alpha u'(a) - \beta u(a) = A, & \gamma u'(b) + \sigma u(b) = B,
\end{cases}
\]

where \( p > 1, \Phi_p(u) = |u|^{p-2}u, \rho, s \in L^\infty([a, b]), \) with \( \text{ess inf}_{[a,b]} \rho > 0 \) and \( \text{ess inf}_{[a,b]} s > 0, A, B \in \mathbb{R}, \alpha, \beta, \gamma, \sigma > 0, \) \( f : [a, b] \times \mathbb{R} \to \mathbb{R} \) is an \( L^1 \)-Carathéodory function and \( \lambda \) is a positive real parameter.

HAMILTONIAN SYSTEMS

\[
\begin{cases}
-u'' + A(t)u = \lambda b(t) \nabla G(u) & \text{a.e. in } [0, T], \\
u(T) - u(0) = u'(T) - u'(0) = 0.
\end{cases}
\]
FOURTH-ORDER ELASTIC BEAM EQUATIONS

\[
\begin{align*}
\nu^{iv} + Au'' + Bu &= \lambda f(t, u) \quad \text{in } [0, 1], \\
u(0) &= u(1) = 0, \\
u''(0) &= u''(1) = 0,
\end{align*}
\]

BOUNDARY VALUE PROBLEMS ON THE HALF_LINE

\[
\begin{align*}
-y'' + m^2 y &= \lambda f(t, y) \\
y(0) &= 0 \\
\lim_{t \to \infty} y(t) &= 0,
\end{align*}
\]
NONLINEAR DIFFERENCE PROBLEMS

\[
\begin{aligned}
-\Delta(\phi_p(\Delta u(k - 1))) &= \lambda f(k, u(k)), & k \in [1, T], \\
u(0) &= u(T + 1) = 0,
\end{aligned}
\]

where $T$ is a fixed positive integer, $[1, T]$ is the discrete interval $\{1, \ldots, T\}$, $\lambda$ is a positive real parameter, $\Delta u(k) := u(k + 1) - u(k)$ is the forward difference operator, $\phi_p(s) = |s|^{p-2}s$, $1 < p < +\infty$ and $f : [1, T] \times \mathbb{R} \to \mathbb{R}$ is a continuous function.

ELLiptic Dirichlet problems

involving the p-laplacian with $p > N$

\[
\begin{aligned}
\Delta_p u + \lambda f(x, u) &= 0 & \text{in } \Omega, \\
u &= 0 & \text{on } \partial\Omega.
\end{aligned}
\]

$\Omega \subset \mathbb{R}^N$ ($N \geq 1$) is a nonempty bounded open set with a boundary $\partial\Omega$ of class $C^1$, $p > N$, $\lambda > 0$ and $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function. $\Delta_p u = \text{div}(|\nabla u|^{p-2}\nabla u)$.
ELLIPOTIC NEUMANN PROBLEMS
IN INVOLVING THE P-LAPLACIAN WITH P>N

\[
\begin{aligned}
-\Delta_p u + a(x)|u|^{p-2}u &= \lambda f(x,u) & \text{in } \Omega \\
\frac{\partial u}{\partial \nu} &= 0 & \text{on } \partial \Omega
\end{aligned}
\]

ELLIPOTIC SYSTEMS

\[
\begin{aligned}
-\Delta_{p_1} u_1 + q_1(x)|u_1|^{p_1-2}u_1 &= \lambda F_{u_1}(x,u_1,\ldots,u_m) & \text{in } \Omega, \\
-\Delta_{p_2} u_2 + q_2(x)|u_2|^{p_2-2}u_2 &= \lambda F_{u_2}(x,u_1,\ldots,u_m) & \text{in } \Omega, \\
& \vdots & \\
-\Delta_{p_m} u_m + q_1(x)|u_m|^{p_m-2}u_m &= \lambda F_{u_m}(x,u_1,\ldots,u_m) & \text{in } \Omega, \\
u_i|_{\partial \Omega} &= 0 & (1 \leq i \leq m),
\end{aligned}
\]
\[
\begin{aligned}
-\Delta u &= \lambda f(x, u) \quad \text{in } \Omega, \\
u|_{\partial \Omega} &= 0,
\end{aligned}
\]

(h_1) there exist two non-negative constants \(a_1, a_2\) and \(q \in ]1, 2N/(N - 2)[\) such that
\[
|f(x, t)| \leq a_1 + a_2|t|^{q-1},
\]
for every \((x, t) \in \Omega \times \mathbb{R}.

there exist two positive constants \(\gamma, \delta, \kappa\) such that
\[
\inf_{x \in \Omega} \frac{F(x, \delta)}{\delta^2} > a_1 \frac{K_1}{\gamma} + a_2 K_2 \gamma^{q-2}, \tag{1}
\]
where \(a_1, a_2\) are given in (h_1) and \(\kappa, K_1, K_2\) are given by
\[
\kappa := \frac{D \sqrt{2}}{2 \pi^{N/4}} \left( \frac{\Gamma(1+N/2)}{D^N - (D/2)^N} \right)^{1/2}, \quad K_1 := \frac{2 \sqrt{2} c_1 (2^N - 1)}{D^2}, \quad K_2 := \frac{2^{q+2} c_q^q (2^N - 1)}{q D^2}.
\]
Some remarks on the variational methods
FURTHER APPLICATIONS OF THE LOCAL MINIMUM THEOREM

A VARIANT OF THREE CRITICAL POINT THEOREM FOR FUNCTIONS UNBOUNDED FROM BELOW

If we apply two times the first special case of the local minimum theorem and owing to a novel version of the mountain pass theorem where the (PS) cut off upper at $r$ is assumed we can give a variant of the three critical theorem. In the applications it became
Theorem. Assume that there are three positive constants $c_1$, $d$ and $c_2$, with $c_1 < d < \frac{\sqrt{2}}{2} c_2$, such that

$$\frac{F(c_1)}{c_1^2} < \frac{1}{6} \frac{F(d)}{d^2} \quad (1)$$

and

$$\frac{F(c_2)}{c_2^2} < \frac{1}{12} \frac{F(d)}{d^2}. \quad (2)$$

Then, for each $\lambda \in \left[ 12 \frac{d^2}{F(d)} , \min \left\{ 2 \frac{c_1^2}{F(c_1)}, \frac{c_2^2}{F(c_2)} \right\} \right]$, problem $(D_\lambda)$ admits at least three (nonnegative) classical solutions $u_i$, $i = 1, 2, 3$, such that

$$\max_{x \in [0,1]} |u_i(x)| < c_2, \quad i = 1, 2, 3.$$
If we apply iteratively the first special case of the local minimum theorem in a suitable way, we obtain an infinitely many critical points theorem. As an example of application, here, we present the following result.
Theorem. Assume that

\[ \liminf_{\xi \to +\infty} \frac{F(\xi)}{\xi^2} < \frac{1}{4} \limsup_{\xi \to +\infty} \frac{F(\xi)}{\xi^2}. \]  

(1)

Then, for each \( \lambda \in \left[ \frac{8}{\limsup_{\xi \to +\infty} \frac{F(\xi)}{\xi^2}}, \frac{2}{\liminf_{\xi \to +\infty} \frac{F(\xi)}{\xi^2}} \right], \)

the problem \( (D_\lambda) \) admits a sequence of pairwise distinct positive classical solutions.
Previous results can be applied to perturbed problems, obtaining for instance, results of the following type.

**Theorem.** Assume that there exist three positive constants $c_1, c_2, d$, with $c_1 < d < \frac{c_2}{2}$, such that

\[
f(\xi) \geq 0 \text{ for each } \xi \in [0, c_2];
\]

\[
\frac{F(c_1)}{c_2^2} < \frac{F(d)}{6d^2};
\]

\[
\frac{F(c_2)}{c_2^2} < \frac{F(d)}{12d^2};
\]

Then, for every $\lambda \in \Lambda := \left] \frac{12d^2}{F(d)}, \min \left\{ \frac{2c_1^2}{F(c_1)}, \frac{c_2^2}{F(c_2)} \right\} \right[ \text{ and } \ldots
for every positive continuous function \( g : [0, 1] \times \mathbb{R} \to \mathbb{R} \),
there exists \( \delta^{*}_{\lambda, g} > 0 \) given by

\[
\delta^{*}_{\lambda, g} := \min \left\{ \frac{2c_1^2 - \lambda F(c_1)}{\int_0^1 G(x, c_1) \, dx}, \frac{c_2^2 - \lambda F(c_2)}{\int_0^1 G(x, c_2) \, dx} \right\}.
\]

such that, for each \( \mu \in ]0, \delta^{*}_{\lambda, g}[ \), the problem

\[
\begin{cases}
-u'' = \lambda f(u) + \mu g(x, u) & \text{in } ]0, 1[ \\
u(0) = u(1) = 0,
\end{cases}
\]

has at least three classical solutions \( u_i, i = 1, 2, 3 \) such that

\[ 0 < u_i(x) < c_2 \quad \forall x \in [0, 1], \quad i = 1, 2, 3. \]
In a natural way the previous results have been also obtained in the framework of the non-smooth Analysis. As example, here, the following problem is considered.

Let $\Omega$ be a non-empty, bounded, open subset of the Euclidian space $\mathbb{R}^N$, $N \geq 1$, with $C^1$-boundary $\partial \Omega$, let $p \in ]N, +\infty[$, and let $q \in L^\infty(\Omega)$ satisfy $\text{ess inf}_{x \in \Omega} q(x) > 0$.

**Problem:** Find $u \in K$ such that, for all $v \in K$,

\[
\int_{\Omega} |\nabla u(x)|^{p-2} \nabla u(x) \cdot \nabla (v(x) - u(x)) \, dx + \int_{\Omega} q(x) |u(x)|^{p-2} u(x)(v(x) - u(x)) \, dx
\]

\[+ \int_{\Omega} \lambda \alpha(x) F^\circ(u(x); v(x) - u(x)) \, dx + \int_{\partial \Omega} \mu \beta(x) G^\circ(\gamma u(x); v(x) - \gamma u(x)) \, d\sigma \geq 0,
\]

where $K$ is a closed convex subset of $W^{1,p}(\Omega)$ containing the constant functions, and $\alpha \in L^1(\Omega)$, $\beta \in L^1(\partial \Omega)$, with $\alpha(x) \geq 0$ for a.a. $x \in \Omega$, $\alpha \not\equiv 0$, $\beta(x) \geq 0$ for a.a. $x \in \partial \Omega$, and $\lambda, \mu$ are real parameters, with $\lambda > 0$ and $\mu \geq 0$. Here, $F^\circ$ and $G^\circ$ stand for Clarke’s generalized directional derivatives of locally Lipschitz
functions $F, G : \mathbb{R} \rightarrow \mathbb{R}$ given by $F(\xi) = \int_0^\xi f(t)\,dt$, $G(\xi) = \int_0^\xi g(t)\,dt$, $\xi \in \mathbb{R}$, with $f, g : \mathbb{R} \rightarrow \mathbb{R}$ locally essentially bounded functions, and $\gamma : W^{1,p}(\Omega) \rightarrow L^p(\partial\Omega)$ denotes the trace operator.

A prototype of the previous problem for $K = W^{1,p}(\Omega)$ is the following boundary value problem with nonsmooth potential and nonhomogeneous, nonsmooth Neumann boundary condition

$$
\begin{cases}
\Delta_p u - q(x)|u|^{p-2}u \in \lambda\alpha(x)\partial F(u) & \text{in } \Omega, \\
|\nabla u|^{p-2}\frac{\partial u}{\partial \nu} \in -\mu\beta(x)\partial G(\gamma u) & \text{on } \partial\Omega.
\end{cases}
$$
Theorem. Let $f : \mathbb{R} \to \mathbb{R}$ be a non-negative, locally essentially bounded function and set $F(\xi) = \int_0^\xi f(t) \, dt$ for all $\xi \in \mathbb{R}$. Assume that

$$
\liminf_{\xi \to +\infty} \frac{F(\xi)}{\xi^2} = 0 \quad \text{and} \quad \limsup_{\xi \to +\infty} \frac{F(\xi)}{\xi^2} = +\infty.
$$

Then, for each non-negative, continuous function $g : \mathbb{R} \to \mathbb{R}$ such that

$$
g_\infty := \lim_{t \to +\infty} \frac{g(t)}{t} < +\infty,
$$

there is a sequence of pairwise distinct functions $\{u_n\} \subset W^{2,2}([0,1])$ such that for all $n \in \mathbb{N}$ one has

$$
\begin{cases}
-u_n''(x) + u_n(x) \in [f^-(u_n(x)), f^+(u_n(x))] & \text{for a.a. } x \in [0,1[ \\
u_n'(0) = \mu g(u_n(0)) \\
u_n'(1) = -\mu g(u_n(1)),
\end{cases}
$$

where $f^-(t) = \lim_{\delta \to 0^+} \text{ess inf}_{|t-z|<\delta} f(z)$ and $f^+(t) = \lim_{\delta \to 0^+} \text{ess sup}_{|t-z|<\delta} f(z)$ for all $t \in \mathbb{R}$.
The local minimum theorem can be directly applied to obtain the existence of at least one solution.

\[
\begin{align*}
-u'' &= \lambda \alpha(x)f(u) \quad x \in ]0, 1[ \\
u(0) &= u(1) = 0.
\end{align*}
\]
Theorem. Assume that $f$ is nonnegative and there exist two positive constants $c$, $d$, with $\sqrt{2}d < c$, such that

$$\frac{F(c)}{c^2} < \left( \frac{\int_{1/4}^{3/4} \alpha(x) \, dx}{2\|\alpha\|_1} \right) \frac{F(d)}{d^2}.$$  \hfill (1)

Then, for each $\lambda \in \left[ \frac{4}{\int_{1/4}^{3/4} \alpha(x) \, dx} \frac{d^2}{F(d)}, \frac{2}{\|\alpha\|_1} \frac{c^2}{F(c)} \right]$, the problem $(D_\lambda)$ admits at least one positive weak solution $\bar{u}$ such that $|\bar{u}(x)| < c$ for all $x \in [0, 1]$. 
Theorem Assume that
\[\lim_{t \to 0^+} \frac{f(t)}{t} = +\infty,\] (1')

fix \(\delta > 0\) such that \(f(t) > 0\) for all \(t \in]0, \delta[,\) and put
\[\lambda^* = \frac{2}{\|\alpha\|_1} \sup_{c \in ]0, \delta[} \frac{c^2}{F(c)}.
\]

Then, for each \(\lambda \in ]0, \lambda^*[\), the problem \((D_\lambda)\) admits at least one positive weak solution \(\bar{u}\) such that
\[|\bar{u}(x)| < \delta\] for all \(x \in [0, 1]\).

Consider the following problem

\[
\begin{align*}
-\Delta_p u &= \lambda f(u) & \text{in } \Omega \\
u &= 0 & \text{on } \partial\Omega,
\end{align*}
\]
Theorem. Assume that

$$\limsup_{t \to 0^+} \frac{F(t)}{t^p} = +\infty. \quad (1)$$

Then, for each $\lambda \in ]0, \lambda_0[$, where $\lambda_0 = k_0 \sup_{c \in ]0, +\infty[} \frac{c^p}{F(c)}$ and

$$k_0 = \frac{N \pi^{p/2}}{p \left[ m(\Omega) \Gamma(1 + \frac{N}{2}) \right]^{p/2} } \left( \frac{p - N}{p - 1} \right)^{p-1},$$

the problem $(D_\lambda)$ admits at least one positive weak solution. Further, assume that $f(0) \neq 0$ and

$$0 < \mu F(t) \leq tf(t) \quad (AR)$$

for all $|t| \geq r$, for some $r > 0$ and for some $\mu > p$. Then, for each $\lambda \in ]0, \lambda_0[$, the problem $(D_\lambda)$ admits at least two positive weak solutions.
These results have been obtained in the setting of **PRIN 2007-ORDINARY DIFFERENTIAL EQUATIONS AND APPLICATIONS** (National Scientific Project Manager: **Fabio Zanolin**)

from the **Unit of MESSINA**.

**CO-AUTHORS and REFERENCES**

- Nuccio MARANO (CT)
- Pasquale CANDITO (RC)
- Roberto LIVREA (RC)
- Pina BARLETTA (RC)
- Giovanni MOLICA BISCI (RC)

**Some remarks on the variational methods**
Some remarks on the variational methods

Others co-authors:

Antonella CHINNI’ (ME)
Beatrice DI BELLA (ME)
Giusy D’AGUI’ (ME)
Pasquale PIZZIMENTI (ME)
Angela SCIAMMETTA (ME)

Diego AVERNA (PA)
Nicola GIOVANNELLO (PA)
Elisa TORNATORE (PA)
Stefania BUCELLATO (PA)
Giusy RICCOBONO (PA)


Thank you very much for your kind attention