Aubry-Mather Theory for Geometric PDEs

Abstract: Aubry-Mather theory studies minimal orbits of convex Hamiltonian systems. It can be considered a very weak version of KAM theory that has the advantage of being valid also for arbitrarily large perturbations of an integrable system. The results of Aubry-Mather theory are particularly strong if the configuration space is two-dimensional, see e.g. [1] and [2]. Aubry-Mather theory is closely related to geometric measure theory, cf. [3], and to the mass transportation problem, cf. [4] and [5].

The lectures will start with an introduction to the basics of Aubry-Mather theory which will be explained for the case of geodesics of arbitrary Riemannian metrics on the 2-torus and on compact orientable surfaces of genus greater than one, cf. [6].

In 1986 Jürgen Moser laid the ground for a PDE version of Aubry-Mather theory. In [7] he treated the problem of minimizing a $\mathbb{Z}^{n+1}$-periodic variational integral for graphs of functions $u : \mathbb{R}^n \to \mathbb{R}$. This led to a continuing lively research activity with important contributions by P. Rabinowitz and E. Stredulinsky, R. de la Llave and E. Valdinoci, U. Bessi and many others.

Here, I will present a variant of this theory valid for minimal hypersurfaces in compact, orientable Riemannian manifolds, and its relations to geometric measure theory, cf. the announcement [8].

In 1994 Jürgen Moser proved a KAM-type result for foliations of an almost complex $(2n)$-torus by holomorphic lines, and asked if, in this situation, there are also global (i.e. non-pertubative) results in the spirit of Aubry-Mather theory. This turns out to be a hard problem, and I will explain the little that is known on this question. Note that holomorphic curves in an almost complex manifold are not characterized by a variational principle, but rather as solutions of a first order elliptic system of Cauchy-Riemann type.

Finally, I will address some recent rigidity results which will bring us back to geodesics on surfaces. In general, a rigidity result characterizes a standard variational problem within a class of variational problems by properties of its solutions. In our case, the prototypical result is E. Hopf's famous theorem that a Riemannian 2-torus is flat if all of its geodesics are minimal, cf. [10]. This was generalized to tori of arbitrary dimensions by D. Burago and S. Ivanov [11]. Analogous results in the world of PDEs seem to be hard to find, see the work by M. Bialy and R.S. McKay [12]. Here, I will present a recent rigidity theorem for complete Riemannian cylinders $S^1 \times \mathbb{R}$ all of whose geodesics are minimal, cf. [13].

References

[1] Bangert, V.: Mather sets for twist maps and geodesics on tori. Dynamics reported, Vol. 1, 1-56, Dynam. Report. Ser. Dynam. Systems Appl., 1, Wiley, Chichester, 1988.

[2] Mather, John N.; Forni, Giovanni: Action minimizing orbits in Hamiltonian systems. Transition to chaos in classical and quantum mechanics (Montecatini Terme, 1991), 92-186, Lecture Notes in Math., 1589, Springer, Berlin, 1994.

[3] Bangert, V.: Minimal measures and minimizing closed normal one-currents. Geom. Funct. Anal. 9 (1999), no. 3, 413-427.

[4] De Pascale, Luigi; Gelli, Maria Stella; Granieri, Luca: Minimal measures, one-dimensional currents and the Monge-Kantorovich problem. Calc. Var. Partial Differential Equations 27 (2006), no. 1, 1-23.

[5] Bernard, Patrick; Buffoni, Boris: Optimal mass transportation and Mather theory. J. Eur. Math. Soc. (JEMS) 9 (2007), no. 1, 85-121.

[6] Massart, D.: Stable norms of surfaces: local structure of the unit ball at rational directions. Geom. Funct. Anal. 7 (1997), no. 6, 996-1010.

[7] Moser, Jürgen: Minimal solutions of variational problems on a torus. Ann. Inst. H. Poincaré Anal. Non Linéaire 3 (1986), no. 3, 229-272.

[8] Auer, Franz; Bangert, Victor: Minimising currents and the stable norm in codimension one. C. R. Acad. Sci. Paris Sér. I Math. 333 (2001), no. 12, 1095-1100,

[9] Moser, Jürgen: On the persistence of pseudo-holomorphic curves on an almost complex torus (with an appendix by Jürgen Pöschel). Invent. Math. 119 (1995), no. 3, 401-442.

[10] Hopf, Eberhard: Closed surfaces without conjugate points. Proc. Nat. Acad. Sci. U. S. A. 34, (1948). 47-51.

[11] Burago, D.; Ivanov, S.: Riemannian tori without conjugate points are flat. Geom. Funct. Anal. 4 (1994), no. 3, 259-269.

[12] Bialy, M. L.; MacKay, R. S.: Variational properties of a nonlinear elliptic equation and rigidity. Duke Math. J. 102 (2000), no. 3, 391-401.

[13] Bangert, V., Emmerich, P.: Riemannian cylinders without conjugate points and with subquadratic area growth are flat. Preprint, arXiv: 1201.2883