

# A DEGREE THEORY FOR A CLASS OF PERTURBED FREDHOLM MAPS

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We define a notion of degree for a class of perturbations of nonlinear Fredholm maps of index zero between infinite dimensional real Banach spaces. Our notion extends the degree introduced by Nussbaum for locally  $\alpha$ -contractive perturbations of the identity, as well as the recent degree for locally compact perturbations of Fredholm maps of index zero defined by the first and third authors in [3].

## 1. INTRODUCTION

In this paper we define a concept of degree for a special class of perturbations of (nonlinear) Fredholm maps of index zero between (infinite dimensional real) Banach spaces, called  $\alpha$ -Fredholm maps. The definition is based on the following two numbers (see e.g. [10]) associated with a map  $f: \Omega \rightarrow F$  from an open subset of a Banach space  $E$  into a Banach space  $F$ :

$$\alpha(f) = \sup \left\{ \frac{\alpha(f(A))}{\alpha(A)} : A \subseteq \Omega \text{ bounded, } \alpha(A) > 0 \right\},$$
$$\omega(f) = \inf \left\{ \frac{\alpha(f(A))}{\alpha(A)} : A \subseteq \Omega \text{ bounded, } \alpha(A) > 0 \right\},$$

where  $\alpha$  is the Kuratowski measure of noncompactness (in [10]  $\omega(f)$  is denoted by  $\beta(f)$ , however, since  $\omega$  is the last letter of the Greek alphabet, we prefer the notation  $\omega(f)$  as in [8]).

Roughly speaking, the  $\alpha$ -Fredholm maps are of the type  $f = g - k$ , where  $g$  is Fredholm of index zero and  $k$  satisfies, locally, the inequality

$$\alpha(k) < \omega(g).$$

These maps include locally compact perturbations of Fredholm maps (called *quasi-Fredholm maps*, for short) since, when  $g$  is Fredholm and  $k$  is locally compact, one has  $\alpha(k) = 0$  and  $\omega(g) > 0$ , locally. Moreover, they also contain the  $\alpha$ -contractive perturbations of the identity (called  *$\alpha$ -contractive vector fields*), where, following Darbo [5], a map  $k$  is  $\alpha$ -contractive if  $\alpha(k) < 1$ .

The degree obtained in this paper is a generalization of the degree for quasi-Fredholm maps defined for the first time in [14] by means of the Elworthy–Tromba theory. The latter degree has been recently redefined in [3] avoiding the use of the Elworthy–Tromba construction and using as a main tool a natural concept of orientation for nonlinear Fredholm maps introduced in [1] and [2]. Our construction is based on this new definition.

The paper ends by showing that for  $\alpha$ -contractive vector fields our degree coincides with the degree defined by Nussbaum in [12] and [13].

## 2. ORIENTABILITY FOR FREDHOLM MAPS

In this section we give a summary of the notion of orientability for nonlinear Fredholm maps of index zero between Banach spaces introduced in [1] and [2].

The starting point is a preliminary definition of a concept of orientation for linear Fredholm operators of index zero between real vector spaces (at this level no topological structure is needed).

Recall that, given two real vector spaces  $E$  and  $F$ , a linear operator  $L: E \rightarrow F$  is said to be (*algebraic*) *Fredholm* if the spaces  $\text{Ker } L$  and  $\text{coKer } L = F/\text{Im } L$  are finite dimensional. The *index* of  $L$  is the integer

$$\text{ind } L = \dim \text{Ker } L - \dim \text{coKer } L.$$

Given a Fredholm operator of index zero  $L$ , a linear operator  $A: E \rightarrow F$  is called a *corrector* of  $L$  if

- i)  $\text{Im } A$  has finite dimension,
- ii)  $L + A$  is an isomorphism.

We denote by  $\mathcal{C}(L)$  the nonempty set of correctors of  $L$  and we define in  $\mathcal{C}(L)$  the following equivalence relation. Given  $A, B \in \mathcal{C}(L)$ , consider the automorphism

$$T = (L + B)^{-1}(L + A) = I - (L + B)^{-1}(B - A)$$

of  $E$ . Clearly, the image of  $K = (L + B)^{-1}(B - A)$  is finite dimensional. Hence, given any finite dimensional subspace  $E_0$  of  $E$  containing the image of  $K$ , the restriction of  $T$  to  $E_0$  is an automorphism of  $E_0$ . Therefore, its determinant is well defined and nonzero. It is easy to check that this value does not depend on  $E_0$  (see [1]). Thus, the *determinant* of  $T$ ,  $\det T$  in symbols, is well defined as the determinant of the restriction of  $T$  to any finite dimensional subspace of  $E$  containing the image of  $K$ .

We say that  $A$  is *equivalent* to  $B$  or, more precisely,  $A$  is *L-equivalent* to  $B$ , if

$$\det((L + B)^{-1}(L + A)) > 0.$$

In [1] it is shown that this is actually an equivalence relation on  $\mathcal{C}(L)$  with two equivalence classes. This equivalence relation provides a concept of orientation of a linear Fredholm operator of index zero.

**Definition 2.1.** Let  $L$  be a linear Fredholm operator of index zero between two real vector spaces. An *orientation* of  $L$  is the choice of one of the two equivalence classes of  $\mathcal{C}(L)$ , and  $L$  is *oriented* when an orientation is chosen.

Given an oriented operator  $L$ , the elements of its orientation are called the *positive correctors* of  $L$ .

**Definition 2.2.** An oriented isomorphism  $L$  is said to be *naturally oriented* if the trivial operator is a positive corrector, and this orientation is called the *natural orientation* of  $L$ .

We now consider the notion of orientation in the framework of Banach spaces. From now on, and throughout the paper,  $E$  and  $F$  denote two real Banach spaces,  $L(E, F)$  is the Banach space of bounded linear operators from  $E$  into  $F$ , and  $\Phi_0(E, F)$  is the open subset of  $L(E, F)$  of the Fredholm operators of index zero. Given  $L \in \Phi_0(E, F)$ , the symbol  $\mathcal{C}(L)$  now denotes, with an abuse of notation, the set of bounded correctors of  $L$ , which is still nonempty.

Of course, the definition of orientation of  $L \in \Phi_0(E, F)$  can be given as the choice of one of the two equivalence classes of bounded correctors of  $L$ , according to the equivalence relation previously defined.

In the context of Banach spaces, an orientation of a linear Fredholm operator of index zero induces, by a sort of stability, an orientation to any sufficiently close operator. Precisely, consider  $L \in \Phi_0(E, F)$  and a corrector  $A$  of  $L$ . Since the set of the isomorphisms from  $E$  into  $F$  is open in  $L(E, F)$ ,  $A$  is a corrector of every  $T$  in a suitable neighborhood  $W$  of  $L$ . If, in addition,  $L$  is oriented and  $A$  is a positive corrector of  $L$ , then any  $T$  in  $W$  can be oriented by taking  $A$  as a positive corrector. This fact leads us to the following notion of orientation for a continuous map with values in  $\Phi_0(E, F)$ .

**Definition 2.3.** Let  $X$  be a topological space and  $h: X \rightarrow \Phi_0(E, F)$  be continuous. An *orientation* of  $h$  is a continuous choice of an orientation  $\alpha(x)$  of  $h(x)$  for each  $x \in X$ , where ‘continuous’ means that for any  $x \in X$  there exists  $A \in \alpha(x)$  which is a positive corrector of  $h(x')$  for any  $x'$  in a neighborhood of  $x$ . A map is *orientable* when it admits an orientation and *oriented* when an orientation is chosen.

**Remark 2.4.** It is possible to prove (see [2, Proposition 3.4]) that two equivalent correctors  $A$  and  $B$  of a given  $L \in \Phi_0(E, F)$  remain  $T$ -equivalent for any  $T$  in a neighborhood of  $L$ . This implies that the notion of ‘continuous choice of an orientation’ in Definition 2.3 is equivalent to the following one:

- for any  $x \in X$  and any  $A \in \alpha(x)$ , there exists a neighborhood  $W$  of  $x$  such that  $A \in \alpha(x')$  for all  $x' \in W$ .

As a straightforward consequence of Definition 2.3, if  $h: X \rightarrow \Phi_0(E, F)$  is orientable and  $g: Y \rightarrow X$  is any continuous map, then the composition  $hg$  is orientable as well. In particular, if  $h$  is oriented, then  $hg$  inherits in a natural way an orientation from the orientation of  $h$ . Thus, if

$$H: X \times [0, 1] \rightarrow \Phi_0(E, F)$$

is an oriented homotopy and  $t \in [0, 1]$  is given, the partial map  $H_t = Hi_t$ , where  $i_t(x) = (x, t)$ , inherits an orientation from  $H$ .

The following proposition shows an important property of the notions of orientation and orientability for maps into  $\Phi_0(E, F)$ . Such a property may be regarded as a sort of continuous transport of the orientation along a homotopy (see [2, Theorem 3.14]).

**Proposition 2.5.** *Let  $X$  be a topological space and consider a homotopy*

$$H: X \times [0, 1] \rightarrow \Phi_0(E, F).$$

*Assume that for some  $t \in [0, 1]$  the partial map  $H_t = H(\cdot, t)$  is oriented. Then there exists and is unique an orientation of  $H$  such that the orientation of  $H_t$  is inherited from that of  $H$ .*

Definition 2.3 and Remark 2.4 allow us to define a notion of orientability for Fredholm maps of index zero between Banach spaces. Recall that, given an open subset  $\Omega$  of  $E$ , a map  $g: \Omega \rightarrow F$  is *Fredholm* if it is  $C^1$  and its Fréchet derivative,  $g'(x)$ , is a Fredholm operator for all  $x \in \Omega$ . The *index* of  $g$  at  $x$  is the index of  $g'(x)$  and  $g$  is said to be of *index  $n$*  if it is of index  $n$  at any point of its domain.

**Definition 2.6.** An *orientation* of a Fredholm map of index zero  $g: \Omega \rightarrow F$  is an orientation of the derivative  $g': \Omega \rightarrow \Phi_0(E, F)$ , and  $g$  is *orientable*, or *oriented*, if so is  $g'$  according to Definition 2.3.

The notion of orientability of Fredholm maps of index zero is mainly discussed in [1] and [2], where the reader can find examples of orientable and nonorientable maps and a comparison with an earlier notion given by Fitzpatrick, Pejsachowicz and Rabier in [9]. Here we recall a property (Theorem 2.8 below) that is the analogue for Fredholm maps of the continuous transport of an orientation along a homotopy stated in Proposition 2.5. We need first the following definition.

**Definition 2.7.** Let  $\Omega$  be an open subset of  $E$  and  $G: \Omega \times [0, 1] \rightarrow F$  a  $C^1$  homotopy. Assume that any partial map  $G_t$  is Fredholm of index zero. An *orientation* of  $G$  is an orientation of the partial derivative

$$\partial_1 G: \Omega \times [0, 1] \rightarrow \Phi_0(E, F), \quad (x, t) \mapsto (G_t)'(x),$$

and  $G$  is *orientable*, or *oriented*, if so is  $\partial_1 G$  according to Definition 2.3.

From the above definition it follows immediately that if  $G$  is oriented, any partial map  $G_t$  inherits an orientation from  $G$ .

Theorem 2.8 below is a straightforward consequence of Proposition 2.5.

**Theorem 2.8.** *Let  $G: \Omega \times [0, 1] \rightarrow F$  be a  $C^1$  homotopy and assume that any  $G_t$  is a Fredholm map of index zero. If a given  $G_t$  is orientable, then  $G$  is orientable. If, in addition,  $G_t$  is oriented, then there exists and is unique an orientation of  $G$  such that the orientation of  $G_t$  is inherited from that of  $G$ .*

We conclude this section by showing how the orientation of a Fredholm map  $g$  is related to the orientations of domain and codomain of suitable restrictions of  $g$ . This argument will be crucial in the definition of the degree for quasi-Fredholm maps.

Let  $g: \Omega \rightarrow F$  be an oriented map and  $Z$  a finite dimensional subspace of  $F$  transverse to  $g$ . By classical transversality results,  $M = g^{-1}(Z)$  is a differentiable manifold of the same dimension as  $Z$ . In addition,  $M$  is orientable (see [1, Remark 2.5 and Lemma 3.1]). Here we show how the orientation of  $g$  and a chosen orientation of  $Z$  induce an orientation on any tangent space  $T_x M$ .

Let  $Z$  be oriented. Choose any  $x \in M$  and let  $A$  be any positive corrector of  $g'(x)$  with image contained in  $Z$  (the existence of such a corrector is ensured by the transversality of  $Z$  to  $g$ ). Then, orient the tangent space  $T_x M$  in such a way that the isomorphism

$$(g'(x) + A)|_{T_x M}: T_x M \rightarrow Z$$

is orientation preserving. As proved in [3], the orientation of  $T_x M$  does not depend on the choice of the positive corrector  $A$ , but just on the orientation of  $Z$  and  $g'(x)$ . With this orientation, we call  $M$  the *oriented Fredholm  $g$ -preimage* of  $Z$ .

### 3. ORIENTABILITY AND DEGREE FOR QUASI-FREDHOLM MAPS

In this section we summarize the main ideas in the construction of a topological degree for quasi-Fredholm maps. See [3] for details. We start by recalling the construction of an orientation for this class of maps.

As before,  $E$  and  $F$  are real Banach spaces, and  $\Omega$  is an open subset of  $E$ . A map  $k: \Omega \rightarrow F$  is called locally compact if for any  $x_0 \in \Omega$  the restriction of  $k$  to a convenient neighborhood of  $x_0$  is a compact map (that is, a map whose image is contained in a compact subset of  $F$ ).

**Definition 3.1.** A map  $f: \Omega \rightarrow F$  is said to be *quasi-Fredholm* provided that  $f = g - k$ , where  $g$  is Fredholm of index zero and  $k$  is locally compact. The map  $g$  is called a *smoothing map* of  $f$ .

The following definition provides an extension to quasi-Fredholm maps of the concept of orientability.

**Definition 3.2.** A quasi-Fredholm map  $f: \Omega \rightarrow F$  is *orientable* if it has an orientable smoothing map.

If  $f$  is an orientable quasi-Fredholm map, any smoothing map of  $f$  is orientable. Indeed, given two smoothing maps  $g^0$  and  $g^1$  of  $f$ , consider the homotopy

$$G(x, t) = (1 - t)g^0(x) + tg^1(x), \quad (x, t) \in \Omega \times [0, 1]. \quad (3.1)$$

Notice that any  $G_t$  is Fredholm of index zero, since it differs from  $g^0$  by a  $C^1$  locally compact map. By Theorem 2.8, if  $g^0$  is orientable, then  $g^1$  is orientable as well.

Let  $f: \Omega \rightarrow F$  be an orientable quasi-Fredholm map. To define a notion of orientation of  $f$ , consider the set  $\mathcal{S}(f)$  of the oriented smoothing maps of  $f$ . We introduce in  $\mathcal{S}(f)$  the following equivalence relation. Given  $g^0, g^1$  in  $\mathcal{S}(f)$ , consider, as in formula (3.1), the straight-line homotopy  $G$  joining  $g^0$  and  $g^1$ . We say that  $g^0$  is *equivalent* to  $g^1$  if their orientations are inherited from the same orientation of  $G$ , whose existence is ensured by Theorem 2.8. It is immediate to verify that this is an equivalence relation.

**Definition 3.3.** Let  $f: \Omega \rightarrow F$  be an orientable quasi-Fredholm map. An *orientation* of  $f$  is the choice of an equivalence class in  $\mathcal{S}(f)$ .

In the sequel, if  $f$  is an oriented quasi-Fredholm map, the elements of the chosen class of  $\mathcal{S}(f)$  will be called *positively oriented smoothing maps* of  $f$ .

As for the case of Fredholm maps of index zero, the orientation of quasi-Fredholm maps verifies a homotopy invariance property, stated in Theorem 3.6 below. We need first some definitions.

**Definition 3.4.** A map  $H: \Omega \times [0, 1] \rightarrow F$  of the type

$$H(x, t) = G(x, t) - K(x, t)$$

is called a *homotopy of quasi-Fredholm maps* provided that  $G$  is  $C^1$ , any  $G_t$  is Fredholm of index zero, and  $K$  is locally compact. In this case  $G$  is said to be a *smoothing homotopy* of  $H$ .

We need a concept of orientability for homotopies of quasi-Fredholm maps. The definition is analogous to that given for quasi-Fredholm maps. Let  $H: \Omega \times [0, 1] \rightarrow F$  be a homotopy of quasi-Fredholm maps. Let  $\mathcal{S}(H)$  be the set of oriented smoothing homotopies of  $H$ . Assume that  $\mathcal{S}(H)$  is nonempty and define on this set an equivalence relation as follows. Given  $G^0$  and  $G^1$  in  $\mathcal{S}(H)$ , consider the map

$$\mathcal{G}: \Omega \times [0, 1] \times [0, 1] \rightarrow F$$

defined as

$$\mathcal{G}(x, t, s) = (1 - s)G^0(x, t) + sG^1(x, t).$$

We say that  $G^0$  is *equivalent* to  $G^1$  if their orientations are inherited from an orientation of the map

$$(x, t, s) \mapsto \partial_1 \mathcal{G}(x, t, s).$$

The reader can easily verify that this is actually an equivalence relation on  $\mathcal{S}(H)$ .

**Definition 3.5.** A homotopy of quasi-Fredholm maps  $H: \Omega \times [0, 1] \rightarrow F$  is said to be *orientable* if  $\mathcal{S}(H)$  is nonempty. An *orientation* of  $H$  is the choice of an equivalence class of  $\mathcal{S}(H)$ .

The following homotopy invariance property of the orientation of quasi-Fredholm maps is the analogue of Theorem 2.8 and a straightforward consequence of Proposition 2.5.

**Theorem 3.6.** *Let  $H: \Omega \times [0, 1] \rightarrow F$  be a homotopy of quasi-Fredholm maps. If a partial map  $H_t$  is oriented, then there exists and is unique an orientation of  $H$  such that the orientation of  $H_t$  is inherited from that of  $H$ .*

Let us now summarize the construction of the degree.

**Definition 3.7.** Let  $f: \Omega \rightarrow F$  be an oriented quasi-Fredholm map and  $U$  an open subset of  $\Omega$ . The triple  $(f, U, 0)$  is said to be *qF-admissible* provided that  $f^{-1}(0) \cap U$  is compact.

The degree is defined as a map from the set of all qF-admissible triples into  $\mathbb{Z}$ . The construction is divided in two steps. In the first one we consider triples  $(f, U, 0)$  such that  $f$  has a smoothing map  $g$  with  $(f - g)(U)$  contained in a finite dimensional subspace of  $F$ . In the second step this assumption is removed, the degree being defined for general qF-admissible triples.

*Step 1.* Let  $(f, U, 0)$  be a qF-admissible triple and let  $g$  be a positively oriented smoothing map of  $f$  such that  $(f - g)(U)$  is contained in a finite dimensional subspace of  $F$ . As  $f^{-1}(0) \cap U$  is compact, there exist a finite dimensional subspace  $Z$  of  $F$  and an open subset  $W$  of  $U$  containing  $f^{-1}(0) \cap U$  and such that  $g$  is transverse to  $Z$  in  $W$ . We may assume that  $Z$  contains  $(f - g)(U)$ . Choose any orientation of  $Z$  and, as in Section 2, let the manifold  $M = g^{-1}(Z) \cap W$  be the oriented Fredholm  $g|_W$ -preimage of  $Z$ . One can easily verify that  $(f|_M)^{-1}(0) = f^{-1}(0) \cap U$ . Thus  $(f|_M)^{-1}(0)$  is compact, and the Brouwer degree of the triple  $(f|_M, M, 0)$  is well defined.

**Definition 3.8.** Let  $(f, U, 0)$  be a qF-admissible triple and let  $g$  be a positively oriented smoothing map of  $f$  such that  $(f - g)(U)$  is contained in a finite dimensional subspace of  $F$ . Let  $Z$  be a finite dimensional subspace of  $F$  and  $W \subseteq U$  an open neighborhood of  $f^{-1}(0) \cap U$  such that

- (1)  $Z$  contains  $(f - g)(U)$ ,
- (2)  $g$  is transverse to  $Z$  in  $W$ .

Assume  $Z$  oriented and let  $M$  be the oriented Fredholm  $g|_W$ -preimage of  $Z$ . Then, the degree of  $(f, U, 0)$  is defined as

$$\deg_{qF}(f, U, 0) = \deg(f|_M, M, 0), \quad (3.2)$$

where the right hand side of the above formula is the Brouwer degree of the triple  $(f|_M, M, 0)$ .

In [3] it is proved that the above definition is well posed, in the sense that the right hand side of (3.2) is independent of the choice of the smoothing map  $g$ , the open set  $W$  and the oriented subspace  $Z$ .

*Step 2.* Let us now extend the definition of degree to general  $qF$ -admissible triples.

**Definition 3.9** (General definition of degree). Let  $(f, U, 0)$  be a  $qF$ -admissible triple. Consider:

- (1) a positively oriented smoothing map  $g$  of  $f$ ;
- (2) an open neighborhood  $V$  of  $f^{-1}(0) \cap U$  such that  $\bar{V} \subseteq U$ ,  $g$  is proper on  $\bar{V}$  and  $(f - g)|_{\bar{V}}$  is compact;
- (3) a continuous map  $\xi: \bar{V} \rightarrow F$  having bounded finite dimensional image and such that

$$\|g(x) - f(x) - \xi(x)\| < \rho, \quad \forall x \in \partial V,$$

where  $\rho$  is the distance in  $F$  between 0 and  $f(\partial V)$ .

Then, the degree of  $(f, U, 0)$  is given by

$$\deg_{qF}(f, U, 0) = \deg_{qF}(g - \xi, V, 0). \quad (3.3)$$

Observe that the right hand side of (3.3) is well defined since the triple  $(g - \xi, V, 0)$  is  $qF$ -admissible. Indeed,  $g - \xi$  is proper on  $\bar{V}$  and thus  $(g - \xi)^{-1}(0)$  is a compact subset of  $\bar{V}$  which is actually contained in  $V$  by assumption (3). Moreover, as shown in [3], Definition 3.9 is well posed since  $\deg_{qF}(g - \xi, V, 0)$  does not depend on  $g$ ,  $\xi$  and  $V$ .

Theorem 3.10 below collects the most important properties of the degree for quasi-Fredholm maps (see [3] for the proof).

**Theorem 3.10.** *The following properties of the degree hold:*

1. (Normalization) *If the identity  $I$  of  $E$  is naturally oriented, then*

$$\deg_{qF}(I, E, 0) = 1.$$

2. (Additivity) *Given a  $qF$ -admissible triple  $(f, U, 0)$  and two disjoint open subsets  $U_1, U_2$  of  $U$  such that  $f^{-1}(0) \cap U \subseteq U_1 \cup U_2$ , one has*

$$\deg_{qF}(f, U, 0) = \deg_{qF}(f, U_1, 0) + \deg_{qF}(f, U_2, 0).$$

3. (Excision) *If  $(f, U, 0)$  is  $qF$ -admissible and  $U_1$  is an open subset of  $U$  containing  $f^{-1}(0) \cap U$ , then*

$$\deg_{qF}(f, U, 0) = \deg_{qF}(f, U_1, 0).$$

4. (Existence) *If  $(f, U, 0)$  is  $qF$ -admissible and*

$$\deg_{qF}(f, U, 0) \neq 0,$$

*then the equation  $f(x) = 0$  has a solution in  $U$ .*

5. (Homotopy invariance) *Let  $H: U \times [0, 1] \rightarrow F$  be an oriented homotopy of quasi-Fredholm maps. If  $H^{-1}(0)$  is compact, then  $\deg_{qF}(H_t, U, 0)$  does not depend on  $t \in [0, 1]$ .*

4. MEASURES OF NONCOMPACTNESS

In this section we recall the definition and properties of the Kuratowski measure of noncompactness [11], together with some related concepts. For general reference, see e.g. Deimling [6].

From now on the spaces  $E$  and  $F$  are assumed to be infinite dimensional. As before  $\Omega$  is an open subset of  $E$ .

The *Kuratowski measure of noncompactness*  $\alpha(A)$  of a bounded subset  $A$  of  $E$  is defined as the infimum of the real numbers  $d > 0$  such that  $A$  admits a finite covering by sets of diameter less than  $d$ . If  $A$  is unbounded, we set  $\alpha(A) = +\infty$ . We summarize the following properties of the measure of noncompactness. Given  $A \subseteq E$ , by  $\overline{\text{co}} A$  we denote the closed convex hull of  $A$ .

**Proposition 4.1.** *Let  $A, B \subseteq E$ . Then*

- (1)  $\alpha(A) = 0$  if and only if  $\overline{A}$  is compact;
- (2)  $\alpha(\lambda A) = |\lambda|\alpha(A)$  for any  $\lambda \in \mathbb{R}$ ;
- (3)  $\alpha(A + B) \leq \alpha(A) + \alpha(B)$ ;
- (4) if  $A \subseteq B$ , then  $\alpha(A) \leq \alpha(B)$ ;
- (5)  $\alpha(A \cup B) = \max\{\alpha(A), \alpha(B)\}$ ;
- (6)  $\alpha(\overline{\text{co}} A) = \alpha(A)$ .

Properties (1)–(5) are straightforward consequences of the definition, while the last one is due to Darbo [5].

Given a continuous map  $f: \Omega \rightarrow F$ , let  $\alpha(f)$  and  $\omega(f)$  be as in the Introduction. It is important to observe that  $\alpha(f) = 0$  if and only if  $f$  is completely continuous (that is, the restriction of  $f$  to any bounded subset of  $\Omega$  is a compact map) and  $\omega(f) > 0$  only if  $f$  is proper on bounded closed sets. For a complete list of properties of  $\alpha(f)$  and  $\omega(f)$  we refer to [10]. We need the following one concerning linear operators.

**Proposition 4.2.** *Let  $L: E \rightarrow F$  be a bounded linear operator. Then  $\omega(L) > 0$  if and only if  $\text{Im } L$  is closed and  $\dim \text{Ker } L < +\infty$ .*

As a consequence of Proposition 4.2 one gets that a bounded linear operator  $L: E \rightarrow F$  is Fredholm if and only if  $\omega(L) > 0$  and  $\omega(L^*) > 0$ , where  $L^*$  is the adjoint of  $L$ .

Let  $f$  be as above and fix  $p \in \Omega$ . We recall the definitions of  $\alpha_p(f)$  and  $\omega_p(f)$  given in [4]. Let  $B(p, r)$  denote the open ball in  $E$  centered at  $p$  with radius  $r$ . Suppose that  $B(p, r) \subseteq \Omega$  and consider

$$\alpha(f|_{B(p,r)}) = \sup \left\{ \frac{\alpha(f(A))}{\alpha(A)} : A \subseteq B(p, r), \alpha(A) > 0 \right\}.$$

This is nondecreasing as a function of  $r$ . Hence, we can define

$$\alpha_p(f) = \lim_{r \rightarrow 0} \alpha(f|_{B(p,r)}).$$

Clearly  $\alpha_p(f) \leq \alpha(f)$  for any  $p \in \Omega$ . In an analogous way, we define

$$\omega_p(f) = \lim_{r \rightarrow 0} \omega(f|_{B(p,r)}),$$

and we have  $\omega_p(f) \geq \omega(f)$  for any  $p$ . It is easy to show that the main properties of  $\alpha$  and  $\omega$  hold, with minor changes, as well for  $\alpha_p$  and  $\omega_p$  (see [4]).

**Proposition 4.3.** *Let  $f: \Omega \rightarrow F$  be continuous and  $p \in \Omega$ . Then*

- (1) if  $f$  is locally compact,  $\alpha_p(f) = 0$ ;
- (2) if  $\omega_p(f) > 0$ ,  $f$  is locally proper at  $p$ .

Clearly, for a bounded linear operator  $L: E \rightarrow F$ , the numbers  $\alpha_p(L)$  and  $\omega_p(L)$  do not depend on the point  $p$  and coincide, respectively, with  $\alpha(L)$  and  $\omega(L)$ . Furthermore, for the  $C^1$  case we get the following result.

**Proposition 4.4** ([4]). *Let  $f: \Omega \rightarrow F$  be of class  $C^1$ . Then, for any  $p \in \Omega$  we have  $\alpha_p(f) = \alpha(f'(p))$  and  $\omega_p(f) = \omega(f'(p))$ .*

Observe that if  $f: \Omega \rightarrow F$  is a Fredholm map, as a straightforward consequence of Propositions 4.2 and 4.4, we obtain  $\omega_p(f) > 0$  for any  $p \in \Omega$ .

As an application of Proposition 4.4 one could deduce the following result.

**Proposition 4.5** ([4]). *Let  $g: \Omega \rightarrow F$  and  $\varphi: \Omega \rightarrow \mathbb{R}$  be of class  $C^1$ , with  $\varphi(x) \geq 0$ . Consider the product map  $f: \Omega \rightarrow F$  defined by  $f(x) = \varphi(x)g(x)$ . Then, for any  $p \in \Omega$  we have  $\alpha_p(f) = \varphi(p)\alpha_p(g)$  and  $\omega_p(f) = \varphi(p)\omega_p(g)$ .*

By means of Proposition 4.5 one can easily find examples of maps  $f$  such that  $\alpha(f) = \infty$  and  $\alpha_p(f) < \infty$  for any  $p$ , and examples of maps  $f$  with  $\omega(f) = 0$  and  $\omega_p(f) > 0$  for any  $p$  (see [4]). Moreover, in [4] there is an example of a map  $f$  such that  $\alpha(f) > 0$  and  $\alpha_p(f) = 0$  for any  $p$ .

In the sequel we will deal with maps  $G$  defined on the product space  $E \times \mathbb{R}$ . In order to define  $\alpha_{(p,t)}(G)$ , we consider the norm

$$\|(p, t)\| = \max\{\|p\|, |t|\}.$$

The natural projection of  $E \times \mathbb{R}$  onto the first factor will be denoted by  $\pi_1$ .

**Remark 4.6.** With the above norm,  $\pi_1$  is nonexpansive. Therefore  $\alpha(\pi_1(X)) \leq \alpha(X)$  for any subset  $X$  of  $E \times \mathbb{R}$ . More precisely, since  $\mathbb{R}$  is finite dimensional, if  $X \subseteq E \times \mathbb{R}$  is bounded, we have  $\alpha(\pi_1(X)) = \alpha(X)$ .

## 5. DEFINITION OF DEGREE

This section is devoted to the construction of a concept of degree for a class of triples that we shall call  $\alpha$ -admissible. We start with two definitions.

**Definition 5.1.** Let  $g: \Omega \rightarrow F$  be an oriented map,  $k: \Omega \rightarrow F$  a continuous map and  $U$  an open subset of  $\Omega$ . The triple  $(g, U, k)$  is said to be  $\alpha$ -admissible if

- i)  $\alpha_p(k) < \omega_p(g)$  for any  $p \in U$ ;
- ii) the solution set  $S = \{x \in U : g(x) = k(x)\}$  is compact.

**Definition 5.2.** Let  $(g, U, k)$  be an  $\alpha$ -admissible triple and  $\mathcal{V} = \{V_1, \dots, V_N\}$  a finite covering of open balls of its solution set  $S$ . We say that  $\mathcal{V}$  is an  $\alpha$ -covering of  $S$  (relative to  $(g, U, k)$ ) if for any  $i \in \{1, \dots, N\}$  the following properties hold:

- i) the ball  $\tilde{V}_i$  of double radius and same center as  $V_i$  is contained in  $U$ ;
- ii)  $\alpha(k|_{\tilde{V}_i}) < \omega(g|_{\tilde{V}_i})$ .

Let  $(g, U, k)$  be an  $\alpha$ -admissible triple and  $\mathcal{V} = \{V_1, \dots, V_N\}$  an  $\alpha$ -covering of the solution set  $S$ . We define the following sequence  $\{C_n\}$  of convex closed subsets of  $E$ :

$$C_1 = \overline{\text{co}} \left( \bigcup_{i=1}^N \{x \in V_i : g(x) \in k(\tilde{V}_i)\} \right),$$

and, inductively,

$$C_n = \overline{\text{co}} \left( \bigcup_{i=1}^N \{x \in V_i : g(x) \in k(\tilde{V}_i \cap C_{n-1})\} \right), \quad n \geq 2.$$

Observe that, by induction,  $C_{n+1} \subseteq C_n$  and  $S \subseteq C_n$  for any  $n \geq 1$ . Then the set

$$C_\infty = \bigcap_{n \geq 1} C_n$$

turns out to be closed, convex, and containing  $S$ . Consequently, if  $S$  is nonempty, so is  $C_\infty$ . To emphasize the fact that the set  $C_\infty$  is uniquely determined by the covering  $\mathcal{V}$ , sometimes it will be denoted by  $C_\infty^\mathcal{V}$ . Let us prove two other crucial properties of  $C_\infty$ :

- (1)  $\{x \in V_i : g(x) \in k(\tilde{V}_i \cap C_\infty)\} \subseteq C_\infty$ , for any  $i = 1, \dots, N$ ;
- (2)  $C_\infty$  is compact.

To verify the first one, fix  $i \in \{1, \dots, N\}$  and let  $x \in V_i$  be such that  $g(x) \in k(\tilde{V}_i \cap C_\infty)$ . In particular, it follows  $g(x) \in k(\tilde{V}_i)$  and, consequently,  $x \in C_1$ . Moreover, for any  $n \geq 1$  we have  $g(x) \in k(\tilde{V}_i \cap C_n)$  and this implies  $x \in C_{n+1}$ . Hence,  $x \in C_\infty$ , and the first property holds.

To check the compactness of  $C_\infty$ , we prove that  $\alpha(C_n) \rightarrow 0$  as  $n \rightarrow \infty$ . Let  $n \geq 2$  be fixed. By the properties of the measure of noncompactness (see Section 4) we have

$$\begin{aligned} \alpha(C_n) &= \alpha\left(\bigcup_{i=1}^N \{x \in V_i : g(x) \in k(\tilde{V}_i \cap C_{n-1})\}\right) \\ &= \max_{1 \leq i \leq N} \alpha\left(\{x \in V_i : g(x) \in k(\tilde{V}_i \cap C_{n-1})\}\right). \end{aligned}$$

Fix  $i \in \{1, \dots, N\}$ , and denote

$$A_{n,i} = \{x \in V_i : g(x) \in k(\tilde{V}_i \cap C_{n-1})\}.$$

Since  $A_{n,i} \subseteq \tilde{V}_i$ , by definition we have  $\alpha(A_{n,i})\omega(g|_{\tilde{V}_i}) \leq \alpha(g(A_{n,i}))$ . Moreover,  $g(A_{n,i}) \subseteq k(\tilde{V}_i \cap C_{n-1})$ . Therefore, as  $\omega(g|_{\tilde{V}_i}) > 0$ , we have

$$\alpha(A_{n,i}) \leq \frac{1}{\omega(g|_{\tilde{V}_i})} \alpha(g(A_{n,i})) \leq \frac{1}{\omega(g|_{\tilde{V}_i})} \alpha(k(\tilde{V}_i \cap C_{n-1})).$$

On the other hand, by definition,  $\alpha(k(\tilde{V}_i \cap C_{n-1})) \leq \alpha(k|_{\tilde{V}_i})\alpha(\tilde{V}_i \cap C_{n-1})$ , thus

$$\alpha(A_{n,i}) \leq \frac{\alpha(k|_{\tilde{V}_i})}{\omega(g|_{\tilde{V}_i})} \alpha(\tilde{V}_i \cap C_{n-1}) = \mu_i \alpha(\tilde{V}_i \cap C_{n-1}) \leq \mu_i \alpha(C_{n-1}),$$

where by assumption  $\mu_i = \alpha(k|_{\tilde{V}_i})/\omega(g|_{\tilde{V}_i}) < 1$ . Finally,

$$\alpha(C_n) = \max_{1 \leq i \leq N} \alpha(A_{n,i}) \leq \max_{1 \leq i \leq N} \mu_i \alpha(C_{n-1}) = \mu \alpha(C_{n-1}),$$

where  $\mu = \max_i \mu_i < 1$ . Hence,  $\alpha(C_n) \rightarrow 0$ , and this implies that the set  $C_\infty$  is compact, as claimed.

**Definition 5.3.** Let  $(g, U, k)$  be an  $\alpha$ -admissible triple,  $\mathcal{V} = \{V_1, \dots, V_N\}$  an  $\alpha$ -covering of the solution set  $S$  and  $C$  a compact convex set. We say that  $(\mathcal{V}, C)$  is an  $\alpha$ -pair (relative to  $(g, U, k)$ ) if the following properties hold:

- (1)  $U \cap C \neq \emptyset$ ;
- (2)  $C_\infty^\mathcal{V} \subseteq C$ ;
- (3)  $\{x \in V_i : g(x) \in k(\tilde{V}_i \cap C)\} \subseteq C$  for any  $i = 1, \dots, N$ .

**Remark 5.4.** Given any  $\alpha$ -admissible triple  $(g, U, k)$ , it is always possible to find an  $\alpha$ -pair  $(\mathcal{V}, C)$ . Indeed, fix an  $\alpha$ -covering  $\mathcal{V}$  of the solution set  $S$ . If the corresponding compact set  $C_\infty^\mathcal{V}$  is nonempty,

then, clearly, the pair  $(\mathcal{V}, C_\infty^\mathcal{V})$  verifies properties (1)–(3). If  $C_\infty^\mathcal{V} = \emptyset$  (this can happen only if  $S = \emptyset$ ), we may assume without loss of generality that

$$U \setminus \bigcup_{i=1}^N \tilde{V}_i \neq \emptyset.$$

One can check that, given any  $p \in U \setminus \bigcup_{i=1}^N \tilde{V}_i$ , the pair  $(\mathcal{V}, \{p\})$  satisfies properties (1)–(3).

Let now  $(\mathcal{V}, C)$  be an  $\alpha$ -pair. Consider a retraction  $r: E \rightarrow C$ , whose existence is ensured by Dugundji's Extension Theorem [7]. Denote  $V = \bigcup_{i=1}^N V_i$ , and let  $W$  be a (possibly empty) open subset of  $V$  containing  $S$  such that, for any  $i$ ,  $x \in W \cap V_i$  implies  $r(x) \in \tilde{V}_i$ . For example, if  $\rho$  denotes the minimum of the radii of the balls  $V_i$ , one may take as  $W$  the set

$$\{x \in V : \|x - r(x)\| < \rho\}.$$

Observe that property (3) above implies that the two equations  $g(x) = k(x)$  and  $g(x) = k(r(x))$  have the same solution set in  $W$  (notice that the composition  $kr$  is defined in  $r^{-1}(U)$ ). The map  $kr$  is locally compact (even if not necessarily compact), hence the triple  $(g - kr, W, 0)$  is admissible for the degree for quasi-Fredholm maps. We define the degree of  $(g, U, k)$  as follows:

$$\deg(g, U, k) = \deg_{qF}(g - kr, W, 0),$$

where the right hand side is the degree defined in Section 3.

The following definition summarizes the above construction.

**Definition 5.5.** Let  $(g, U, k)$  be an  $\alpha$ -admissible triple and  $(\mathcal{V}, C)$  an  $\alpha$ -pair. Consider a retraction  $r: E \rightarrow C$ . Let  $\mathcal{V} = \{V_1, \dots, V_N\}$ , denote  $V = \bigcup_{i=1}^N V_i$ , and let  $W$  be an open subset of  $V$  containing  $S$  such that, for any  $i$ ,  $x \in W \cap V_i$  implies  $r(x) \in \tilde{V}_i$ . We put

$$\deg(g, U, k) = \deg_{qF}(g - kr, W, 0).$$

In order to show that this definition is well posed, we have to prove that it is independent of the choice of the  $\alpha$ -pair  $(\mathcal{V}, C)$ , of the retraction  $r$  and of the open set  $W$ . This is the purpose of the following proposition.

**Proposition 5.6.** Let  $(\mathcal{V}, C)$  and  $(\mathcal{V}', C')$  be two  $\alpha$ -pairs relative to an  $\alpha$ -admissible triple  $(g, U, k)$ , where

$$\mathcal{V} = \{V_1, \dots, V_N\} \quad \text{and} \quad \mathcal{V}' = \{V'_1, \dots, V'_M\}.$$

Consider two retractions  $r: E \rightarrow C$  and  $r': E \rightarrow C'$ . Denote  $V = \bigcup_{i=1}^N V_i$ , and let  $W$  be an open subset of  $V$  containing  $S$  such that, for any  $i$ ,  $x \in W \cap V_i$  implies  $r(x) \in \tilde{V}_i$ . Analogously, denote  $V' = \bigcup_{j=1}^M V'_j$ , and let  $W'$  be an open subset of  $V'$  containing  $S$  such that, for any  $j$ ,  $x' \in W' \cap V'_j$  implies  $r'(x') \in \tilde{V}'_j$ . Then

$$\deg_{qF}(g - kr, W, 0) = \deg_{qF}(g - kr', W', 0).$$

*Proof.* Consider a third covering  $\mathcal{V}'' = \{V''_1, \dots, V''_T\}$  of the solution set  $S$  of open balls such that for any  $l \in \{1, \dots, T\}$  there exist  $i$  and  $j$  such that  $V''_l \subseteq V_i \cap V'_j$ . In particular,  $\mathcal{V}''$  is still an  $\alpha$ -covering of  $S$ . Consider the compact convex set  $C_\infty^{\mathcal{V}''}$ . We distinguish two different cases.

i)  $C_\infty^{\mathcal{V}''} = \emptyset$ . In this case  $S = \emptyset$  and, consequently, by the existence property of the degree for quasi-Fredholm maps we have

$$\deg_{qF}(g - kr, W, 0) = 0 \quad \text{and} \quad \deg_{qF}(g - kr', W', 0) = 0.$$

ii)  $C_\infty^{\mathcal{V}''} \neq \emptyset$ . In this case,  $(\mathcal{V}'', C_\infty^{\mathcal{V}''})$  is an  $\alpha$ -pair. To simplify the notations, denote  $C''_\infty = C_\infty^{\mathcal{V}''}$ . Consider a retraction  $r'': E \rightarrow C''_\infty$ . Denote  $V'' = \bigcup_{l=1}^T V''_l$ , and let  $W''$  be an open subset of  $V''$

containing  $S$  such that, for any  $l$ ,  $x \in W'' \cap V_l''$  implies  $r''(x) \in \tilde{V}_l''$ . Clearly, to prove the assertion it is sufficient to show that

$$\deg_{qF}(g - kr, W, 0) = \deg_{qF}(g - kr'', W'', 0).$$

Now, denote  $C_\infty = C_\infty^\vee$  and let  $\{C_n\}$  and  $\{C_n''\}$  be the sequences of sets defining  $C_\infty$  and  $C_\infty''$ , respectively. Since  $C_n'' \subseteq C_n$  for any  $n \geq 1$ , it follows  $C_\infty'' \subseteq C_\infty$ . In particular,  $C_\infty'' \subseteq C$ . Moreover, without loss of generality, we can assume that the open set  $W''$  is contained in  $W$ . Thus, by the excision property of the degree for quasi-Fredholm maps we have

$$\deg_{qF}(g - kr, W, 0) = \deg_{qF}(g - kr'', W'', 0). \quad (5.1)$$

Consider the following homotopy:

$$H: W'' \times [0, 1] \rightarrow F,$$

$$H(x, t) = g(x) - k(tr(x) + (1-t)r''(x)).$$

Let  $x \in W''$ , let  $V_l''$  contain  $x$  for some  $l$ , and let  $V_i$  contain  $V_l''$  for some  $i$ . Since  $x \in W'' \subseteq W$ , we have  $r(x) \in \tilde{V}_i$  and  $r''(x) \in \tilde{V}_l''$ . Hence, as  $\tilde{V}_l'' \subseteq \tilde{V}_i$ , it follows  $r''(x) \in \tilde{V}_i$  and, consequently,  $H$  is well defined.

Let now  $(x, t) \in W'' \times [0, 1]$  be a pair such that  $H(x, t) = 0$ . If  $x \in V_l''$  for some  $l$ , and  $V_l'' \subseteq V_i$  for some  $i$ , then both  $r(x)$  and  $r''(x)$  belong to  $\tilde{V}_i \cap C$ , since  $r''(x) \in C_\infty''$  and  $C_\infty'' \subseteq C$ . Thus,  $tr(x) + (1-t)r''(x) \in \tilde{V}_i \cap C$  and, in particular,  $g(x) \in k(\tilde{V}_i \cap C)$ . This implies  $x \in C$  and, consequently,  $r(x) = x$ .

We want to show that, actually,  $x \in C_\infty''$ . Since  $r(x) = x$ , we have

$$tx + (1-t)r''(x) \in \tilde{V}_l'' \cap C$$

and, in particular,  $g(x) \in k(\tilde{V}_l'')$ . Consequently,  $x \in C_1''$ . As  $C_\infty'' \subseteq C_1''$ , we have  $r''(x) \in C_1''$ , and  $tx + (1-t)r''(x) \in \tilde{V}_l'' \cap C_1''$  since this is convex. Thus,  $g(x) \in k(\tilde{V}_l'' \cap C_1'')$ , and this implies  $x \in C_2''$ . Inductively, we get  $x \in C_n''$  for any  $n \geq 1$ . Hence,  $x \in C_\infty''$  and, consequently,  $r''(x) = x$ .

Finally,  $g(x) = k(x)$ , that is,  $x \in S$ . Therefore, the solution set

$$\{(x, t) \in W'' \times [0, 1] : H(x, t) = 0\}$$

coincides with  $S \times [0, 1]$ . Hence, we can apply the homotopy invariance of the degree for quasi-Fredholm maps to get

$$\deg_{qF}(g - kr, W'', 0) = \deg_{qF}(g - kr'', W'', 0),$$

and the assertion follows taking into account formula (5.1).  $\square$

## 6. PROPERTIES OF THE DEGREE

**Theorem 6.1.** *The following properties of the degree hold:*

1. (Normalization) *Let the identity  $I$  of  $E$  be naturally oriented. Then*

$$\deg(I, E, 0) = 1.$$

2. (Additivity) *Given an  $\alpha$ -admissible triple  $(g, U, k)$  and two disjoint open subsets  $U^1, U^2$  of  $U$ , assume that  $S = \{x \in U : g(x) = k(x)\}$  is contained in  $U^1 \cup U^2$ . Then*

$$\deg(g, U, k) = \deg(g, U^1, k) + \deg(g, U^2, k).$$

3. (Homotopy invariance) *Let  $H: U \times [0, 1] \rightarrow F$  be a homotopy of the form  $H(x, t) = G(x, t) - K(x, t)$ , where  $G$  is of class  $C^1$ , any  $G_t = G(\cdot, t)$  is Fredholm of index zero,  $K$  is continuous, and  $\alpha_{(p,t)}(K) < \omega_{(p,t)}(G)$  for any pair  $(p, t) \in U \times [0, 1]$ . Assume that  $G$  is oriented and that  $H^{-1}(0)$  is compact. Then  $\deg(G_t, U, K_t)$  is well defined and does not depend on  $t \in [0, 1]$ .*

*Proof.* 1. (*Normalization*) It follows easily from the normalization property of the degree for quasi-Fredholm maps.

2. (*Additivity*) Let  $S^1 = S \cap U^1$  and  $S^2 = S \cap U^2$ , so that  $S = S^1 \cup S^2$ . The fact that the triples  $(g, U^1, k)$  and  $(g, U^2, k)$  are  $\alpha$ -admissible is clear from the definition.

Let  $\mathcal{V}^1 = \{V_1^1, \dots, V_N^1\}$  and  $\mathcal{V}^2 = \{V_1^2, \dots, V_M^2\}$  be two  $\alpha$ -coverings of  $S^1$  (relative to  $(g, U^1, k)$ ) and of  $S^2$  (relative to  $(g, U^2, k)$ ), respectively. For simplicity, denote  $C_\infty^1 = C_\infty^{\mathcal{V}^1}$  and  $C_\infty^2 = C_\infty^{\mathcal{V}^2}$ . Then, consider the family

$$\mathcal{V} = \{V_1^1, \dots, V_N^1, V_1^2, \dots, V_M^2\}.$$

Note that  $\mathcal{V}$  is an  $\alpha$ -covering of  $S$ . Consider the compact convex set  $C_\infty = C_\infty^{\mathcal{V}}$ . By definition,  $C_\infty$  contains both  $C_\infty^1$  and  $C_\infty^2$ ; moreover, it has the following properties:

$$\{x \in V_i^1 : g(x) \in k(\tilde{V}_i^1 \cap C_\infty)\} \subseteq C_\infty, \quad i = 1, \dots, N;$$

and

$$\{x \in V_j^2 : g(x) \in k(\tilde{V}_j^2 \cap C_\infty)\} \subseteq C_\infty, \quad j = 1, \dots, M.$$

We distinguish two different cases.

i) If  $C_\infty = \emptyset$ , then  $S = \emptyset$ , hence  $S^1 = \emptyset$  and  $S^2 = \emptyset$ . Consequently, applying Definition 5.5, by the existence property of the degree for quasi-Fredholm maps it follows

$$\deg(g, U, k) = 0; \quad \deg(g, U^1, k) = 0; \quad \deg(g, U^2, k) = 0.$$

ii) If  $C_\infty \neq \emptyset$ , consider a retraction  $r: E \rightarrow C_\infty$ . Denote  $V^1 = \bigcup_{i=1}^N V_i^1$ ,  $V^2 = \bigcup_{j=1}^M V_j^2$  and  $V = V^1 \cup V^2$ . Let  $W$  be an open subset of  $V$  containing  $S$  such that, for any  $i$ ,  $x \in W \cap V_i^1$  implies  $r(x) \in \tilde{V}_i^1$  and, for any  $j$ ,  $x' \in W \cap V_j^2$  implies  $r(x') \in \tilde{V}_j^2$ . By definition we have

$$\deg(g, U, k) = \deg_{qF}(g - kr, W, 0).$$

Since  $W$  is an open neighborhood of  $S$  in  $V$ , and  $V$  is the disjoint union of  $V^1$  and  $V^2$ , we can assume  $W = W^1 \cup W^2$ , where  $W^1 \subseteq V^1$  and  $W^2 \subseteq V^2$ . The open sets  $W^1$  and  $W^2$  are disjoint. In addition,  $W^1$  contains  $S^1$ , and  $W^2$  contains  $S^2$ . Therefore, by the additivity property of the degree for quasi-Fredholm maps, we have

$$\deg_{qF}(g - kr, W, 0) = \deg_{qF}(g - kr, W^1, 0) + \deg_{qF}(g - kr, W^2, 0).$$

Now, observe that  $(\mathcal{V}^\lambda, C_\infty)$  is an  $\alpha$ -pair relative to  $(g, U^\lambda, k)$ , for  $\lambda = 1, 2$ . Consequently,

$$\deg(g, U^\lambda, k) = \deg_{qF}(g - kr, W^\lambda, 0), \quad \lambda = 1, 2, \tag{6.1}$$

and the assertion follows.

3. (*Homotopy invariance*) For  $t \in [0, 1]$ , let  $\Sigma^t$  denote the compact set  $\{x \in U : G_t(x) = K_t(x)\}$ . Given any  $t$ , the fact that the triple  $(G_t, U, K_t)$  is  $\alpha$ -admissible follows easily from the compactness of  $\Sigma^t$  and observing that  $\alpha_p(K_t) \leq \alpha_{(p,t)}(K)$  and  $\omega_p(G_t) \geq \omega_{(p,t)}(G)$  for all  $p \in U$ . Consequently, it is sufficient to show that the integer-valued function

$$t \mapsto \deg(G_t, U, K_t)$$

is locally constant. To this purpose, fix  $\tau \in [0, 1]$  and, given  $\delta > 0$ , let  $I_\delta$  denote the interval  $[\tau - \delta, \tau + \delta] \cap [0, 1]$ . It is possible to find  $\delta > 0$  and a finite family of open balls  $\mathcal{V} = \{V_1, \dots, V_N\}$  with the following properties:

- i)  $V = \bigcup_{i=1}^N V_i$  contains  $\Sigma^t$  for any  $t \in I_\delta$ ;
- ii) the ball  $\tilde{V}_i$  of double radius and same center as  $V_i$  is contained in  $U$ ;
- iii)  $\alpha(K|_{\tilde{V}_i \times I_\delta}) < \omega(G|_{\tilde{V}_i \times I_\delta})$ , for any  $i = 1, \dots, N$ .

In particular it follows that, for any  $t \in I_\delta$ ,  $\mathcal{V}$  is an  $\alpha$ -covering of  $\Sigma^t$ . As in the construction of the sequence  $\{C_n\}$  in Section 5, for any fixed  $t \in I_\delta$  we define the following sequence of sets:

$$C_1^t = \overline{\text{co}} \left( \bigcup_{i=1}^N \{x \in V_i : G_t(x) \in K_t(\tilde{V}_i)\} \right),$$

and, inductively,

$$C_n^t = \overline{\text{co}} \left( \bigcup_{i=1}^N \{x \in V_i : G_t(x) \in K_t(\tilde{V}_i \cap C_{n-1}^t)\} \right), \quad n \geq 2.$$

Then we set  $C_\infty^t = \bigcap_{n \geq 1} C_n^t$ . We observe that  $C_\infty^t$  is compact and convex, moreover it has the following property:

$$\{x \in V_i : G_t(x) \in K_t(\tilde{V}_i \cap C_\infty^t)\} \subseteq C_\infty^t, \quad i = 1, \dots, N.$$

Now, we define the following sequence  $\{\hat{C}_n\}$  of convex closed subsets of  $E$  independent of  $t$ :

$$\hat{C}_1 = \overline{\text{co}} \left( \pi_1 \left( \bigcup_{i=1}^N \{(x, t) \in V_i \times I_\delta : G(x, t) \in K(\tilde{V}_i \times I_\delta)\} \right) \right),$$

and, inductively,

$$\hat{C}_n = \overline{\text{co}} \left( \pi_1 \left( \bigcup_{i=1}^N \{(x, t) \in V_i \times I_\delta : G(x, t) \in K((\tilde{V}_i \cap \hat{C}_{n-1}) \times I_\delta)\} \right) \right), \quad n \geq 2.$$

Observe that, by induction,  $\hat{C}_{n+1} \subseteq \hat{C}_n$  for any  $n \geq 1$ . Then the set

$$\hat{C}_\infty = \bigcap_{n \geq 1} \hat{C}_n$$

is closed and convex. We claim that the following properties of  $\hat{C}_\infty$  hold:

- (1)  $\hat{C}_\infty$  is compact;
- (2)  $\hat{C}_\infty$  contains  $C_\infty^t$  for any  $t \in I_\delta$ ;
- (3)  $\{x \in V_i : G_t(x) \in K_t(\tilde{V}_i \cap \hat{C}_\infty)\} \subseteq \hat{C}_\infty$  for any  $i = 1, \dots, N$  and  $t \in I_\delta$ .

Let us prove that  $\hat{C}_\infty$  is compact. For simplicity, for any  $n \geq 2$  and  $i \in \{1, \dots, N\}$  we denote

$$\hat{A}_{n,i} = \{(x, t) \in V_i \times I_\delta : G(x, t) \in K((\tilde{V}_i \cap \hat{C}_{n-1}) \times I_\delta)\},$$

and we set  $\hat{A}_n = \bigcup_{i=1}^N \hat{A}_{n,i}$ . Let  $n \geq 2$  be fixed. Since  $\hat{A}_n \subseteq \hat{C}_n \times I_\delta$ , by Remark 4.6 we have  $\alpha(\hat{A}_n) \leq \alpha(\hat{C}_n \times I_\delta) = \alpha(\hat{C}_n)$ . On the other hand,

$$\alpha(\hat{C}_n) = \alpha(\overline{\text{co}}(\pi_1(\hat{A}_n))) = \alpha(\pi_1(\hat{A}_n)) \leq \alpha(\hat{A}_n),$$

the last inequality due to the fact that  $\pi_1$  is nonexpansive. Consequently, we have

$$\alpha(\hat{C}_n) = \alpha(\hat{A}_n) = \alpha \left( \bigcup_{i=1}^N \hat{A}_{n,i} \right) = \max_{1 \leq i \leq N} \alpha(\hat{A}_{n,i}).$$

Now, fix  $i \in \{1, \dots, N\}$ . Since  $\hat{A}_{n,i} \subseteq \tilde{V}_i \times I_\delta$ , by definition we have

$$\alpha(\hat{A}_{n,i}) \omega(G|_{\tilde{V}_i \times I_\delta}) \leq \alpha(G(\hat{A}_{n,i})).$$

Moreover,  $G(\hat{A}_{n,i}) \subseteq K((\tilde{V}_i \cap \hat{C}_{n-1}) \times I_\delta)$ . Therefore,

$$\alpha(\hat{A}_{n,i}) \leq \frac{1}{\omega(G|_{\tilde{V}_i \times I_\delta})} \alpha(G(\hat{A}_{n,i})) \leq \frac{1}{\omega(G|_{\tilde{V}_i \times I_\delta})} \alpha(K((\tilde{V}_i \cap \hat{C}_{n-1}) \times I_\delta)).$$

On the other hand, by definition we have

$$\alpha(K((\tilde{V}_i \cap \hat{C}_{n-1}) \times I_\delta)) \leq \alpha(K|_{\tilde{V}_i \times I_\delta}) \alpha((\tilde{V}_i \cap \hat{C}_{n-1}) \times I_\delta),$$

and, by Remark 4.6,  $\alpha((\tilde{V}_i \cap \hat{C}_{n-1}) \times I_\delta) = \alpha(\tilde{V}_i \cap \hat{C}_{n-1})$ . Hence

$$\alpha(\hat{A}_{n,i}) \leq \frac{\alpha(K|_{\tilde{V}_i \times I_\delta})}{\omega(G|_{\tilde{V}_i \times I_\delta})} \alpha(\tilde{V}_i \cap \hat{C}_{n-1}) = \nu_i \alpha(\tilde{V}_i \cap \hat{C}_{n-1}) \leq \nu_i \alpha(\hat{C}_{n-1}),$$

where by assumption  $\nu_i = \alpha(K|_{\tilde{V}_i \times I_\delta})/\omega(G|_{\tilde{V}_i \times I_\delta}) < 1$ . Finally,

$$\alpha(\hat{C}_n) = \max_{1 \leq i \leq N} \alpha(\hat{A}_{n,i}) \leq \max_{1 \leq i \leq N} \nu_i \alpha(\hat{C}_{n-1}) \leq \nu \alpha(\hat{C}_{n-1}),$$

where  $\nu = \max_i \nu_i < 1$ . Thus,  $\alpha(\hat{C}_n) \rightarrow 0$  as  $n \rightarrow \infty$ , and this implies that the set  $\hat{C}_\infty$  is compact, as claimed.

For any fixed  $t \in I_\delta$ , the inclusion  $C_\infty^t \subseteq \hat{C}_\infty$  follows immediately from the fact that  $C_n^t \subseteq \hat{C}_n$  for any  $n \geq 1$ .

To verify the third property, fix  $i \in \{1, \dots, N\}$  and  $t \in I_\delta$ , and let  $x \in V_i$  be such that  $G_t(x) \in K_t(\tilde{V}_i \cap \hat{C}_\infty)$ . In particular, we have  $G_t(x) \in K_t(\tilde{V}_i)$ , and this implies  $x \in \hat{C}_1$ . Moreover, for any  $n \geq 2$  we have  $G_t(x) \in K_t(\tilde{V}_i \cap \hat{C}_{n-1})$ . It follows  $(x, t) \in \hat{A}_{n,i}$ , and, consequently,  $x \in \pi_1(\hat{A}_{n,i})$ . Therefore,  $x \in \hat{C}_n$  for any  $n \geq 2$ . Hence,  $x \in \hat{C}_\infty$ , and property (3) holds.

Since  $\tau \in [0, 1]$  is arbitrary, the assertion follows if we show that  $\deg(G_t, U, K_t)$  is independent of  $t \in I_\delta$ . We distinguish two different cases.

i)  $\hat{C}_\infty = \emptyset$ . In this case  $C_\infty^t = \emptyset$  for any  $t \in I_\delta$ , hence  $\Sigma^t = \emptyset$  for any  $t$ . Consequently, applying Definition 5.5, by the existence property of the degree for quasi-Fredholm maps we have  $\deg(G_t, U, K_t) = 0$  for any  $t \in I_\delta$ .

ii)  $\hat{C}_\infty \neq \emptyset$ . In this case, as properties (1)–(3) of  $\hat{C}_\infty$  hold, for any fixed  $t \in I_\delta$  the pair  $(\mathcal{V}, \hat{C}_\infty)$  is an  $\alpha$ -pair relative to the triple  $(G_t, U, K_t)$ . Consider a retraction  $r: E \rightarrow \hat{C}_\infty$ . Let  $W$  be an open subset of  $V$  containing  $V \cap \hat{C}_\infty$  such that, for any  $i$ ,  $x \in W \cap V_i$  implies  $r(x) \in \tilde{V}_i$ . In particular, for any fixed  $t \in I_\delta$  the open set  $W$  contains  $\Sigma^t$ . Thus, by definition we have

$$\deg(G_t, U, K_t) = \deg_{qF}(G_t - K_t r, W, 0), \quad t \in I_\delta.$$

Consider the following homotopy:

$$\hat{H}: W \times I_\delta \rightarrow F$$

$$\hat{H}(x, t) = G(x, t) - K(r(x), t).$$

This is a homotopy of quasi-Fredholm maps, since it is continuous and the map  $(x, t) \mapsto K(r(x), t)$  is locally compact. Moreover,  $\hat{H}^{-1}(0)$  is compact, as it is closed in the compact set  $H^{-1}(0)$ . Then, the homotopy invariance property of the degree for quasi-Fredholm maps implies that  $\deg_{qF}(G_t - K_t r, W, 0)$  does not depend on  $t$ . Hence,  $\deg(G_t, U, K_t)$  is independent of  $t \in I_\delta$ , and we are done.  $\square$

## 7. COMPARISON WITH OTHER DEGREE THEORIES

The purpose of this section is to show that our concept of degree extends the degree for quasi-Fredholm maps summarized in Section 3, and that it agrees with the Nussbaum degree [13] for the class of locally  $\alpha$ -contractive vector fields.

**7.1. Degree for quasi-Fredholm maps.** Let  $f: \Omega \rightarrow F$  be an oriented quasi-Fredholm map and  $U$  an open subset of  $\Omega$ . We recall that the triple  $(f, U, 0)$  is  $qF$ -admissible provided that  $f^{-1}(0) \cap U$  is compact.

Let  $(f, U, 0)$  be a  $qF$ -admissible triple and let  $f = g - k$ , where  $g$  is a positively oriented smoothing map of  $f$  and  $k$  is locally compact. As pointed out in Section 4, we have  $\omega_p(g) > 0$  and  $\alpha_p(k) = 0$  for any  $p \in U$ . Hence, the triple  $(g, U, k)$  is  $\alpha$ -admissible. We claim that

$$\deg(g, U, k) = \deg_{qF}(f, U, 0).$$

Indeed, let  $\mathcal{V} = \{V_1, \dots, V_N\}$  be an  $\alpha$ -covering of  $S = \{x \in U : g(x) = k(x)\}$  relative to the triple  $(g, U, k)$ , and consider the compact convex set  $C_\infty = C_\infty^\mathcal{V}$ . We distinguish two different cases.

i) If  $C_\infty = \emptyset$ , then  $S = \emptyset$ . Consequently, by the existence property of the degree for quasi-Fredholm maps and by Definition 5.5, we have

$$\deg_{qF}(f, U, 0) = 0 \quad \text{and} \quad \deg(g, U, k) = 0.$$

ii) If  $C_\infty \neq \emptyset$ , consider a retraction  $r: E \rightarrow C_\infty$ . Denote  $V = \bigcup_{i=1}^N V_i$ , and let  $W$  be a (possibly empty) open subset of  $V$  containing  $S$  such that, for any  $i$ ,  $x \in W \cap V_i$  implies  $r(x) \in \tilde{V}_i$ . By definition we have

$$\deg(g, U, k) = \deg_{qF}(g - kr, W, 0).$$

On the other hand, as  $S \subseteq W$ , by the excision property of the degree for quasi-Fredholm maps we have

$$\deg_{qF}(f, U, 0) = \deg_{qF}(f, W, 0).$$

Consider the following homotopy:

$$H: W \times [0, 1] \rightarrow F,$$

$$H(x, t) = g(x) - k(tr(x) + (1-t)x).$$

Let  $x \in W$ , and let  $V_i$  contain  $x$  for some  $i$ . Since  $r(x) \in \tilde{V}_i$  and  $x \in \tilde{V}_i$ , it follows  $tr(x) + (1-t)x \in \tilde{V}_i$  for any  $t \in [0, 1]$ , and this shows that  $H$  is well defined.

As in the proof of Proposition 5.6 one gets

$$H^{-1}(0) \cap (W \times [0, 1]) = S \times [0, 1].$$

Hence, we can apply the homotopy invariance of the degree for quasi-Fredholm maps, obtaining

$$\deg_{qF}(g - kr, W, 0) = \deg_{qF}(g - k, W, 0),$$

and the claim follows.

**7.2. Degree for locally  $\alpha$ -contractive vector fields.** Let  $f: \Omega \rightarrow F$  be a continuous map from an open subset of  $E$  into  $F$ . We recall the following definitions. The map  $f$  is said to be  $\alpha$ -Lipschitz if  $\alpha(f(A)) \leq \mu\alpha(A)$  for some  $\mu \geq 0$  and any  $A \subseteq \Omega$ . If the  $\alpha$ -Lipschitz constant  $\mu$  is less than 1, then  $f$  is called  $\alpha$ -contractive. The map  $f$  is said to be  $\alpha$ -condensing if  $\alpha(f(A)) < \alpha(A)$  for any  $A \subseteq \Omega$  such that  $0 < \alpha(A) < +\infty$ . If for any  $p \in \Omega$  there exists a neighborhood  $V_p$  of  $p$  such that  $f|_{V_p}$  is  $\alpha$ -contractive (resp.  $\alpha$ -condensing), the map  $f$  is said to be *locally  $\alpha$ -contractive* (resp. *locally  $\alpha$ -condensing*).

In [12] and [13], Nussbaum developed a degree theory for triples of the form  $(I - k, U, 0)$ , where  $k$  is locally  $\alpha$ -condensing. In particular, let  $U$  be an open subset of  $\Omega$  and  $k: \Omega \rightarrow E$  a locally  $\alpha$ -condensing map. Assume that the set  $S = \{x \in U : (I - k)(x) = 0\}$  is compact. Then, the triple  $(I - k, U, 0)$  is admissible for the Nussbaum degree ( $N$ -admissible, for short). We will denote by  $\deg_N(I - k, U, 0)$  the Nussbaum degree of an  $N$ -admissible triple.

We want to show that, in a sense to be specified, our degree and the Nussbaum degree coincide on the class of  $N$ -admissible triples of the form  $(I - k, U, 0)$ , where  $k$  is locally  $\alpha$ -contractive.

Let  $(I - k, U, 0)$  be a  $N$ -admissible triple and assume that the map  $k$  is locally  $\alpha$ -contractive. Clearly, provided that  $I$  is oriented, the triple  $(I, U, k)$  is  $\alpha$ -admissible. We claim that, if we assign the natural orientation to  $I$ , it follows

$$\deg(I, U, k) = \deg_N(I - k, U, 0).$$

Indeed, let  $\mathcal{V} = \{V_1, \dots, V_N\}$  be an  $\alpha$ -covering of  $S$  relative to the triple  $(I, U, k)$ , and consider the (possibly empty) compact convex set  $C_\infty = C_\infty^\mathcal{V}$ .

Denote  $\tilde{V} = \bigcup_{i=1}^N \tilde{V}_i$ . As  $S$  is contained in  $\tilde{V}$ , by the excision property of the Nussbaum degree we have

$$\deg_N(I - k, U, 0) = \deg_N(I - k, \tilde{V}, 0).$$

Consider the following sequence  $\{\tilde{C}_n\}$  of convex closed subsets of  $E$ :

$$\tilde{C}_1 = \overline{\text{co}} \left( k(\tilde{V}) \right),$$

and, inductively,

$$\tilde{C}_n = \overline{\text{co}} \left( k(\tilde{V} \cap \tilde{C}_{n-1}) \right), \quad n \geq 2.$$

Then the set

$$\tilde{C}_\infty = \bigcap_{n \geq 1} \tilde{C}_n$$

turns out to be closed, convex, and containing  $S$ . Moreover, the fact that  $k$  is locally  $\alpha$ -contractive implies that  $\tilde{C}_\infty$  is compact. We observe that the following properties of  $\tilde{C}_\infty$  hold:

- (1)  $\tilde{C}_\infty$  contains  $C_\infty$ ;
- (2)  $\{x \in V_i : x \in k(\tilde{V}_i \cap \tilde{C}_\infty)\} \subseteq \tilde{C}_\infty$  for any  $i = 1, \dots, N$ .

The inclusion  $C_\infty \subseteq \tilde{C}_\infty$  follows immediately from the fact that  $C_n \subseteq \tilde{C}_n$  for any  $n \geq 1$ , where  $\{C_n\}$  is the sequence of sets which defines  $C_\infty$ , as in Section 5. On the other hand, property (2) follows from the trivial inclusion

$$\{x \in V_i : x \in k(\tilde{V}_i \cap \tilde{C}_n)\} \subseteq k(\tilde{V} \cap \tilde{C}_n),$$

which holds for any  $n \geq 1$  and  $i \in \{1, \dots, N\}$ .

To prove the assertion, we distinguish two different cases.

i)  $\tilde{C}_\infty = \emptyset$ . In this case,  $C_\infty = \emptyset$  by (1), and  $S = \emptyset$ . Consequently, by the existence property of the Nussbaum degree and by Definition 5.5, we have

$$\deg_N(I - k, U, 0) = 0 \quad \text{and} \quad \deg(I, U, k) = 0.$$

ii)  $\tilde{C}_\infty \neq \emptyset$ . In this case, as properties (1) and (2) of  $\tilde{C}_\infty$  hold,  $(\mathcal{V}, \tilde{C}_\infty)$  is an  $\alpha$ -pair relative to the triple  $(I, U, k)$ . Consider a retraction  $r: E \rightarrow \tilde{C}_\infty$ . Denote  $V = \bigcup_{i=1}^N V_i$ , and let  $W$  be a (possibly empty) open subset of  $V$  containing  $S$  such that, for any  $i$ ,  $x \in W \cap V_i$  implies  $r(x) \in \tilde{V}_i$ . By definition we have

$$\deg(I, U, k) = \deg_{qF}(I - kr, W, 0).$$

On the other hand (see [12] and [13]), we have

$$\deg_N(I - k, \tilde{V}, 0) = \deg_{LS}(I - kr, r^{-1}(\tilde{V}) \cap \tilde{V}, 0).$$

Finally, let  $W' = W \cap r^{-1}(\tilde{V}) \cap \tilde{V}$ . As  $S$  is contained in  $W'$ , by the excision property of the Leray–Schauder degree we have

$$\deg_{LS}(I - kr, r^{-1}(\tilde{V}) \cap \tilde{V}, 0) = \deg_{LS}(I - kr, W', 0),$$

and by the excision property of the degree for quasi-Fredholm maps we have

$$\deg_{qF}(I - kr, W, 0) = \deg_{qF}(I - kr, W', 0).$$

The claim now follows from the fact that the degree for quasi-Fredholm maps is an extension of the Leray–Schauder degree (see [3]).

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