BIFURCATION RESULTS FOR A CLASS OF PERTURBED FREDHOLM MAPS

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Abstract. We prove a global bifurcation result for an equation of the type $Lx + \lambda(h(x) + k(x)) = 0$, where $L : E \to F$ is a linear Fredholm operator of index zero between Banach spaces, and, given an open subset Ω of E, h, $k : \Omega \times [0, +\infty) \to F$ are C^1 and continuous, respectively. Under suitable conditions, we prove the existence of an unbounded connected set of nontrivial solutions of the above equation, i.e. solutions (x, λ) with $\lambda \neq 0$, whose closure contains a trivial solution $(\bar{x}, 0)$. The proof is based on a degree theory for noncompact perturbation of Fredholm maps of index zero developed by the authors with M. Furi.

1. Introduction

We study a bifurcation problem for the semilinear operator equation

$$Lx + \lambda(h(x) + k(x)) = 0 \tag{1.1}$$

in $\Omega \times [0, +\infty)$, where Ω is an open subset of a Banach space E, $L : E \to F$ is a linear Fredholm operator of index zero between real Banach spaces, and the maps $h : \Omega \to F$ and $k : \Omega \to F$ are of class C^1 and continuous, respectively. In addition we assume that, for any nonnegative real λ, the map $x \mapsto Lx+\lambda h(x)$ is a nonlinear Fredholm map of index zero.

The set of *trivial solutions* of (1.1) is obtained when $\lambda = 0$. It coincides with $(\Omega \cap \operatorname{Ker} L) \times \{0\}$, which we suppose not empty. One of the problems related to equation (1.1) is to establish under what conditions the set of nontrivial solutions is not empty and to determine topological properties of this set. One of them is the existence of a *bifurcation point*; that is, a point p in $\Omega \cap \operatorname{Ker} L$ such that $(p, 0)$ lies in the closure of the set of nontrivial solutions. The related bifurcation theory is sometimes called *cobifurcation* [13] or *atypical bifurcation* [17].

Independently, Furi and Pera [13] and Martelli [16] have studied an unperturbed equation of the form

$$Lx + \lambda h(x) = 0, \tag{1.2}$$

with L as in (1.1) and $h : \Omega \to F$ compact. These authors proved the existence of a connected bifurcating branch of nontrivial solutions of (1.2) that is either unbounded or its closure contains at least two bifurcation points. More recently an analogous result has been obtained by Benevieri, Furi, Martelli and Pera in [6] by removing the compactness assumption on h, but requiring that such a map is of class C^1. Their proof is based on a degree theory developed in [3] for the class of Fredholm maps of index zero.

A further extension has been obtained by Benevieri and Furi in [5]. They studied equation (1.1) assuming that the map h is C^1 and the perturbation k is locally compact. To tackle this type of problem they applied a topological degree theory for the class of compact perturbations of nonlinear Fredholm maps (*quasi-Fredholm maps* for short), which is introduced in [5] and generalizes that given in [3].

In this paper we extend the domain of investigation of equation (1.1) by assuming that the perturbation k is continuous and not necessarily compact. We require an additional condition on k, given in terms of measure of noncompactness. Roughly speaking, we suppose that the noncompactness of k is small with respect to a number depending on L and h. In Theorem 6.3 below we prove the existence of a

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connected bifurcating branch of nontrivial solutions of (1.1) as in [6]. The technique used here is based on a topological degree theory introduced in [1] (see also [2, 7, 8]) for a special class of noncompact perturbations of Fredholm maps, called *α-Fredholm maps*. Such a theory extends that defined in [5] (we will see below that any quasi-Fredholm map is also α-Fredholm).

The first part of the paper, namely Sections 2–5, is devoted to summarizing the definition of the degree for α-Fredholm maps. We think that this preliminary part, although quite long, represents an unavoidable help to the reader for a correct understanding of the paper.

Concerning the organization of the paper, in Section 2 we recall the notion (introduced in [3] and [4]) of orientability for nonlinear Fredholm maps. Following [5], in Section 3 we extend the notion of orientability to quasi-Fredholm maps. This concept is crucial in the definition of the degree for quasi-Fredholm maps. Section 4 is devoted to recall the definition of Kuratowski measure of noncompactness together with some related concepts. In Section 5 we sketch the construction of the degree for α-Fredholm maps (see [1]). Section 6 contains our main result, Theorem 6.3. In Section 7 we give an application to the study of T-periodic solutions of a boundary value problem depending on a parameter. For this problem we obtain a global bifurcation theorem generalizing analogous results in [6] and [5].

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2. Orientability for Fredholm maps

In this section we summarize the notion, introduced in [3] and [4], of orientability for nonlinear Fredholm maps of index zero between Banach spaces. Throughout the paper E and F will denote two real Banach spaces. The Banach space of bounded linear operators from E to F will be denoted by $L(E, F)$, and $\Phi_0(E, F)$ will be the (open) subset of Fredholm operators of index zero.

The starting point in our construction is a concept of orientation for an operator $L \in \Phi_0(E, F)$. A bounded linear operator $A : E \to F$ with finite dimensional image is called a *corrector* of L if $L + A$ is an isomorphism. On the (nonempty) set $\mathcal{C}(L)$ of correctors of L we define an equivalence relation as follows. Let $A, B \in \mathcal{C}(L)$ be given and consider the following automorphism of E:

$$T = (L + B)^{-1}(L + A) = I - (L + B)^{-1}(B - A).$$

The image $\operatorname{Im} K$ of $K = (L+B)^{-1}(B-A)$ has of course finite dimension. Hence, given any nontrivial finite dimensional subspace E_0 of E containing $\operatorname{Im} K$, the restriction of T to E_0 is an automorphism. Thus, its determinant is nonzero and independent of the choice of E_0. Then, one can define the *determinant* of T as the determinant of the restriction of T to any nontrivial finite dimensional subspace of E containing $\operatorname{Im} K$.

We say that A is *equivalent* to B or, more precisely, A is *L-equivalent* to B if

$$\det\big((L + B)^{-1}(L + A)\big) > 0.$$

As shown in [3], this is actually an equivalence relation on $\mathcal{C}(L)$ with two equivalence classes.

Definition 2.1. Let $L \in \Phi_0(E, F)$ be given. An *orientation* of L is the choice of one of the two equivalence classes of $\mathcal{C}(L)$, and L is *oriented* when an orientation is chosen.

Given an oriented operator L, the elements of its orientation are called *positive correctors* of L.

Definition 2.2. An oriented isomorphism L is said to be *naturally oriented* if the trivial operator is a positive corrector, and this orientation is called the *natural orientation* of L.

An orientation of L induces an orientation to any operator in a neighborhood of L. In fact, consider a corrector A of L. Since the set of the isomorphisms from E into F is open in $L(E, F)$, A turns out to be a corrector of every T close enough of L. Therefore, if L is oriented and A is a positive corrector of

L, any T close to L can be oriented taking A as a positive corrector of T. This fact allows us to define the following notion of orientation for a continuous map with values in $\Phi_0(E,F)$.

Definition 2.3. Let X be a topological space and $h : X \to \Phi_0(E,F)$ a continuous map. An *orientation* of h is a choice of an orientation $\mathcal{O}(x)$ of $h(x)$ for each $x \in X$, such that for any $x \in X$ there exists $A \in \mathcal{O}(x)$ which is a positive corrector of $h(x')$ for any x' in a neighborhood of x. A map is *orientable* when it admits an orientation and *oriented* when an orientation is chosen.

Remark 2.4. With an abuse of terminology we can say that, if a map h is oriented, the orientation $\mathcal{O}(x)$ of $h(x)$ depends continuously on x.

The following important property of the above notion of orientability is, roughly speaking, its stability along a homotopy (see [4, Theorem 3.14]).

Proposition 2.5. *Consider a homotopy $H : X \times [0,1] \to \Phi_0(E,F)$. Assume that, for some $\lambda \in [0,1]$, the partial map $H_\lambda = H(\cdot,\lambda)$ is oriented and call $\mathcal{O}$ its orientation. Then there exists a unique orientation of H, say β, such that $\beta(x,\lambda) = \mathcal{O}(x)$ for any $x \in X$.*

By Definition 2.3 we can give a notion of orientability for Fredholm maps of index zero between Banach spaces. Recall that, given an open subset Ω of E, a map $g : \Omega \to F$ is a *Fredholm map* if it is C^1 and its Fréchet derivative, $g'(x)$, is a Fredholm operator for all $x \in \Omega$. The map g is said to be of *index* n if $g'(x)$ is of index n for any $x \in \Omega$.

Definition 2.6. An *orientation* of a Fredholm map of index zero $g : \Omega \to F$ is an orientation of the continuous map $g' : x \mapsto g'(x)$, and g is *orientable*, or *oriented*, if so is g' according to Definition 2.3.

The notion of orientability of Fredholm maps of index zero is accurately discussed in [3] and [4], where the reader can find examples of orientable and nonorientable maps. Here we recall a property (Theorem 2.8 below) which is the analogue for Fredholm maps of the continuous transport of an orientation along a homotopy, as seen in Proposition 2.5. We need first the following definition.

Definition 2.7. Let $H : \Omega \times [0,1] \to F$ be a C^1 homotopy. Assume that any partial map H_λ is Fredholm of index zero. An *orientation* of H is an orientation of the derivative with respect to the first variable

$$\partial_1 H : \Omega \times [0,1] \to \Phi_0(E,F), \quad \partial_1 H(x,\lambda) = (H_\lambda)'(x),$$

and H is *orientable*, or *oriented*, if so is $\partial_1 H$ according to Definition 2.3.

Theorem 2.8. *Let $H : \Omega \times [0,1] \to F$ be C^1 and assume that any H_λ is a Fredholm map of index zero. Suppose that, for some $\lambda \in [0,1]$, the partial map H_λ is oriented and call $\mathcal{O}$ its orientation. Then there exists a unique orientation of H, say β, such that $\beta(x,\lambda) = \mathcal{O}(x)$ for any $x \in \Omega$.*

We conclude this section by showing that the orientation of a Fredholm map g is related to the orientations of domain and codomain of suitable restrictions of g. This argument is crucial in the definition of the degree for quasi-Fredholm maps, based on a finite-dimensional reduction process.

Let $g : \Omega \to F$ be an oriented map and Z a finite dimensional subspace of F, transverse to g. By classical transversality results, $M = g^{-1}(Z)$ is a differentiable manifold of the same dimension as Z. In addition, M is orientable (see [3, Remark 2.5 and Lemma 3.1]). We find worthwhile to show how, given any $x \in M$, the orientation of g and a chosen orientation of Z induce an orientation on the tangent space T_xM of M at x. Let Z be oriented. Consider $x \in M$ and a positive corrector A of $g'(x)$ with image contained in Z (the existence of such a corrector is ensured by the transversality of Z to g). Then, orient T_xM in such a way that the isomorphism

$$(g'(x) + A)|_{T_xM} : T_xM \to Z$$

is orientation preserving. As proved in [3], the orientation of T_xM does not depend on the choice of the positive corrector A, but only on the orientations of Z and $g'(x)$. With this orientation, we call M the *oriented g-preimage* of Z.

3. Orientability and degree for quasi-Fredholm maps

In this section we recall the concept of degree for quasi-Fredholm maps, defined in [5]. We find worthwhile to recall this topic as a preliminary step in the construction of the degree for α-Fredholm maps.

Definition 3.1. Let Ω be an open subset of E, $g : \Omega \to F$ a Fredholm map of index zero and $k : \Omega \to F$ a locally compact map. The map $f : \Omega \to F$, defined by $f = g - k$, is called a *quasi-Fredholm map* and g is a *smoothing map* of f.

The following definition provides an extension to quasi-Fredholm maps of the concept of orientability.

Definition 3.2. A quasi-Fredholm map $f : \Omega \to F$ is *orientable* if it has an orientable smoothing map.

If f is an orientable quasi-Fredholm map, any smoothing map of f is orientable. Indeed, given two smoothing maps g_0 and g_1 of f, consider the homotopy $H : \Omega \times [0, 1] \to F$, defined by

$$H(x, \lambda) = (1 - \lambda)g_0(x) + \lambda g_1(x). \tag{3.1}$$

Any H_λ is Fredholm of index zero, since it differs from g_0 by a C^1 locally compact map. By Theorem 2.8, if g_0 is orientable, then so is g_1.

Let $f : \Omega \to F$ be an orientable quasi-Fredholm map. To define an orientation on f we introduce an equivalence relation in the set $\mathcal{S}(f)$ of the oriented smoothing maps of f. Given g_0, g_1 in $\mathcal{S}(f)$, consider, as in formula (3.1), the straight-line homotopy H joining g_0 and g_1. We say that g_0 is *equivalent* to g_1 if their orientations induce the same orientation of H (recall Theorem 2.8). If the domain of f is connected, any smoothing map has two orientations and hence $\mathcal{S}(f)$ has exactly two equivalence classes.

Definition 3.3. Let $f : \Omega \to F$ be an orientable quasi-Fredholm map. An *orientation* of f is the choice of an equivalence class in $\mathcal{S}(f)$.

By the above argument, given an orientable quasi-Fredholm map f, an orientation of a smoothing map g determines uniquely an orientation of f. Therefore, in the sequel, if f is oriented, we will refer to a *positively oriented smoothing map* of f as an element in the chosen class of $\mathcal{S}(f)$.

As for Fredholm maps of index zero, the orientation of quasi-Fredholm maps verifies a homotopy invariance property, as said in Theorem 3.6 below. We need first two definitions.

Definition 3.4. Let $H : \Omega \times [0, 1] \to F$ be a map of the form

$$H(x, \lambda) = G(x, \lambda) - K(x, \lambda),$$

where G is C^1, any G_λ is Fredholm of index zero and K is locally compact. We call H a *homotopy of quasi-Fredholm maps* and G a *smoothing homotopy* of H.

A definition of orientability for homotopies of quasi-Fredholm maps can be given analogously to that for quasi-Fredholm maps. Let $H : \Omega \times [0, 1] \to F$ be a homotopy of quasi-Fredholm maps. Let $\mathcal{S}(H)$ be the set of oriented smoothing homotopies of H. Assume that $\mathcal{S}(H)$ is nonempty and define on this set an equivalence relation as follows. Given G_0 and G_1 in $\mathcal{S}(H)$, consider the map

$$\mathcal{H} : \Omega \times [0, 1] \times [0, 1] \to F, \quad \mathcal{H}(x, \lambda, s) = (1 - s)G_0(x, \lambda) + sG_1(x, \lambda).$$

We say that G_0 is *equivalent* to G_1 if their orientations induce the same orientation of the map

$$(x, \lambda, s) \mapsto \partial_1 \mathcal{H}(x, \lambda, s).$$

Definition 3.5. A homotopy of quasi-Fredholm maps $H : \Omega \times [0,1] \to F$ is said to be *orientable* if $\mathcal{S}(H)$ is nonempty. An *orientation* of H is the choice of an equivalence class of $\mathcal{S}(H)$.

Theorem 3.6 ([5])**.** *Let $H : \Omega\times[0,1] \to F$ be a homotopy of quasi-Fredholm maps. If, for some $\lambda \in [0,1]$, the partial map H_λ is oriented by an orientation $\mathcal{O}$, then there exists a unique orientation of H, say β, such that $\beta(x,\lambda) = \mathcal{O}(x)$ for any $x \in \Omega$.*

Let us now summarize the construction of the degree.

Definition 3.7. Let $f : \Omega \to F$ be an oriented quasi-Fredholm map and U an open subset of Ω. The triple $(f,U,0)$ is said to be *qF-admissible* provided that $f^{-1}(0) \cap U$ is compact.

The construction of the degree for qF-admissible triples is given in two steps.

In the first one we consider triples $(f,U,0)$ such that f has a smoothing map g with $(f-g)(U)$ contained in a finite dimensional subspace of F. In the second step we remove this assumption, defining the degree for all qF-admissible triples.

Step 1. Let $(f,U,0)$ be a qF-admissible triple and let g be a positively oriented smoothing map of f such that $(f-g)(U)$ is contained in a finite dimensional subspace of F. As $f^{-1}(0)\cap U$ is compact, there exist a finite dimensional subspace Z of F and an open neighborhood W of $f^{-1}(0)$ in U, such that g is transverse to Z in W. We may assume that Z contains $(f-g)(U)$. Let $M = g^{-1}(Z)\cap W$. As seen at the end of Section 2, let Z be oriented and orient M in such a way that it is the oriented $g|_W$-preimage of Z. One can easily verify that $(f|_M)^{-1}(0) = f^{-1}(0) \cap U$. Thus $(f|_M)^{-1}(0)$ is compact, and the Brouwer degree of the triple $(f|_M, M, 0)$ turns out to be well defined.

Definition 3.8. Let $(f,U,0)$ be a qF-admissible triple and let g be a positively oriented smoothing map of f such that $(f-g)(U)$ is contained in a finite dimensional subspace of F. Let Z be a finite dimensional subspace of F and W an open neighborhood of $f^{-1}(0)$ in U such that

(1) Z contains $(f-g)(U)$,
(2) g is transverse to Z in W.

Assume Z oriented and let M be the oriented $g|_W$-preimage of Z. Then, the degree of $(f,U,0)$ is defined as

$$\deg_{qF}(f,U,0) = \deg_B(f|_M, M, 0), \tag{3.2}$$

where the right hand side is the Brouwer degree of the triple $(f|_M, M, 0)$.

As proved in [5] the above definition is well posed in the sense that the right hand side of (3.2) is independent of the choice of the smoothing map g, the open set W and the subspace Z.

Step 2. Let us now extend the definition of degree to general qF-admissible triples.

Definition 3.9. Let $(f,U,0)$ be a qF-admissible triple. Consider:

(1) a positively oriented smoothing map g of f;
(2) an open neighborhood V of $f^{-1}(0)\cap U$ such that $\overline{V} \subseteq U$, g is proper on $\overline{V}$ and $(f-g)|_{\overline{V}}$ is compact;
(3) a continuous map $\xi : \overline{V} \to F$ having bounded finite dimensional image and such that

$$\|g(x) - f(x) - \xi(x)\| < \rho, \quad \forall x \in \partial V,$$

where ρ is the distance in F between 0 and $f(\partial V)$.

Then,

$$\deg_{qF}(f,U,0) = \deg_{qF}(g-\xi, V, 0). \tag{3.3}$$

Observe that the right hand side of (3.3) is well defined since the triple $(g-\xi, V, 0)$ is qF-admissible. Indeed, $g-\xi$ is proper on $\overline{V}$ and thus $(g-\xi)^{-1}(0)$ is a compact subset of $\overline{V}$ which is actually contained in V by assumption (3).

In [5] it is proved that Definition 3.9 is well posed since formula (3.3) does not depend on g, ξ and V.

The degree for quasi-Fredholm maps verifies classical properties in degree theory, as additivity and homotopy invariance. The reader can find details in [5].

4. Measures of noncompactness

In this section we recall the definition and properties of the Kuratowski measure of noncompactness together with some related concepts. For general reference see e.g. [10] or [15].

From now on the Banach spaces E and F are assumed to be infinite dimensional.

The *Kuratowski measure of noncompactness* $\alpha(A)$ of a bounded subset A of E is defined as the infimum of real numbers $d > 0$ such that A admits a finite covering by sets of diameter less than d. If A is unbounded, we set $\alpha(A) = +\infty$.

We summarize the following properties of the measure of noncompactness. Given a subset A of E, we denote by $\overline{\operatorname{co}}\, A$ the closed convex hull of A, and by $[0,1]A$ the set $\{\lambda x : \lambda \in [0,1], x \in A\}$.

Proposition 4.1. *Let A and B be subsets of E. Then*

1. $\alpha(A) = 0$ *if and only if* $\overline{A}$ *is compact;*
2. $\alpha(\lambda A) = |\lambda|\alpha(A)$ *for any* $\lambda \in \mathbb{R}$*;*
3. $\alpha(A+B) \le \alpha(A) + \alpha(B)$*;*
4. *if* $A \subseteq B$*, then* $\alpha(A) \le \alpha(B)$*;*
5. $\alpha(A \cup B) = \max\{\alpha(A), \alpha(B)\}$*;*
6. $\alpha([0,1]A) = \alpha(A)$*;*
7. $\alpha(\overline{\operatorname{co}}\, A) = \alpha(A)$.

Properties (1)–(6) are straightforward consequences of the definition, while the last one is due to Darbo [9].

Given an open subset Ω of E and a continuous map $f\colon \Omega \to F$, we recall the definition of the following two extended real numbers (see e.g. [12]) associated with the map f:

$$\alpha(f) = \sup\left\{\frac{\alpha(f(A))}{\alpha(A)} : A \subseteq \Omega \text{ bounded}, \alpha(A) > 0\right\},$$
$$\omega(f) = \inf\left\{\frac{\alpha(f(A))}{\alpha(A)} : A \subseteq \Omega \text{ bounded}, \alpha(A) > 0\right\}.$$

We point out that $\alpha(f) = 0$ if and only if f is completely continuous and $\omega(f) > 0$ only if f is proper on bounded closed sets. For a comprehensive list of properties of $\alpha(f)$ and $\omega(f)$ we refer to [12]. Here we recall the following one concerning linear operators.

Proposition 4.2. *Let $L : E \to F$ be a bounded linear operator. Then $\omega(L) > 0$ if and only if* $\operatorname{Im} L$ *is closed and* $\dim \operatorname{Ker} L < +\infty$.

As a consequence of Proposition 4.2 one gets that a bounded linear operator L is Fredholm if and only if $\omega(L) > 0$ and $\omega(L^*) > 0$, where L^* is the adjoint of L.

Let $p \in \Omega$ be fixed. We recall the definitions of $\alpha_p(f)$ and $\omega_p(f)$ given in [7] (see also [8]). Roughly speaking, these numbers are the local analogues of $\alpha(f)$ and $\omega(f)$.

Let $B(p,s)$ denote the open ball in E centered at p with radius $s > 0$. Suppose that $B(p,s) \subseteq \Omega$ and consider the number

$$\alpha(f|_{B(p,s)}) = \sup\left\{\frac{\alpha(f(A))}{\alpha(A)} : A \subseteq B(p,s), \alpha(A) > 0\right\},$$

which is nondecreasing as a function of s. Hence, we can define

$$\alpha_p(f) = \lim_{s\to 0} \alpha(f|_{B(p,s)}).$$

Clearly $\alpha_p(f) \leq \alpha(f)$. Analogously, define

$$\omega_p(f) = \lim_{s\to 0} \omega(f|_{B(p,s)}).$$

Obviously, $\omega_p(f) \geq \omega(f)$.

With only minor changes, it is easy to show that the main properties of α and ω hold for α_p and ω_p as well. In fact, the following proposition holds (see [7]).

Proposition 4.3. *Let $f : \Omega \to F$ be continuous and $p \in \Omega$. Then*

1. $\alpha_p(cf) = |c|\alpha_p(f)$ *and* $\omega_p(cf) = |c|\omega_p(f)$, *for any* $c \in \mathbb{R}$;
2. $\omega_p(f) \leq \alpha_p(f)$;
3. $|\alpha_p(f) - \alpha_p(g)| \leq \alpha_p(f+g) \leq \alpha_p(f) + \alpha_p(g)$;
4. $\omega_p(f) - \alpha_p(g) \leq \omega_p(f+g) \leq \omega_p(f) + \alpha_p(g)$;
5. *if f is locally compact,* $\alpha_p(f) = 0$;
6. *if* $\omega_p(f) > 0$, *f is locally proper at p;*
7. *if f is a local homeomorphism and* $\omega_p(f) > 0$, *then* $\alpha_q(f^{-1})\omega_p(f) = 1$, *where* $q = f(p)$.

Clearly, for a bounded linear operator $L : E \to F$, the numbers $\alpha_p(L)$ and $\omega_p(L)$ do not depend on the point p and coincide, respectively, with $\alpha(L)$ and $\omega(L)$. Furthermore, for the C^1 case the following result holds.

Proposition 4.4 ([7])**.** *Let $f : \Omega \to F$ be of class C^1. Then, for any $p \in \Omega$ we have $\alpha_p(f) = \alpha(f'(p))$ and $\omega_p(f) = \omega(f'(p))$.*

If $f : \Omega \to F$ is a Fredholm map, as a straightforward consequence of Propositions 4.2 and 4.4, we obtain $\omega_p(f) > 0$ for any $p \in \Omega$.

The next property of bounded linear operators is useful for a direct computation of α and ω.

Proposition 4.5. *Let $L : E \to F$ be a bounded linear operator, and let $P : E \to E$ and $Q : F \to F$ be two projectors onto finite codimensional subspaces. Then*

$$\alpha(L) = \alpha(QLP) \quad \textit{and} \quad \omega(L) = \omega(QLP).$$

Proof. We have for instance $L = QL + (I-Q)L$. Observe that the operator $(I-Q)L$ is compact since its image is finite dimensional. Thus, $\alpha((I-Q)L) = 0$ and, consequently, $\alpha(L) = \alpha(QL)$. In an analogous way one can easily check that $\alpha(L) = \alpha(QLP)$ and $\omega(L) = \omega(QLP)$. □

The next proposition, which will be used in the sequel, is a sort of nonlinear analogue of Proposition 4.5. The easy proof is omitted.

Proposition 4.6. *Let $f : \Omega \to F$ be continuous and $p \in \Omega$. Let $Q : F \to F$ be a projector onto a finite codimensional subspace. Then*

$$\alpha_p(f) = \alpha_p(Qf) \quad \textit{and} \quad \omega_p(f) = \omega_p(Qf).$$

The following proposition extends to the continuous case an analogous result shown in [7] for C^1 maps.

Proposition 4.7. *Let $g : \Omega \to F$ and $\sigma : \Omega \to \mathbb{R}$ be continuous. Consider the product map $f : \Omega \to F$ defined by $f(x) = \sigma(x)g(x)$. Then, for any $p \in \Omega$ we have $\alpha_p(f) = |\sigma(p)|\alpha_p(g)$ and $\omega_p(f) = |\sigma(p)|\omega_p(g)$.*

Proof. Let us show first that $\sigma(p) = 0$ implies $\alpha_p(f) = 0$. Indeed, fix $\varepsilon > 0$. As σ is continuous, there exists $\overline{s}$ such that for any $0 < s < \overline{s}$ and any $x \in B(p, s)$ one has $|\sigma(x)| \le \varepsilon$. Given a positive $s < \overline{s}$, since $f(x) \in [-\varepsilon, \varepsilon]g(x)$ for any $x \in B(p, s)$, one has $f(A) \subseteq [-\varepsilon, \varepsilon]g(A)$ for any $A \subseteq B(p, s)$. Hence, $\alpha(f(A)) \le \varepsilon\alpha(g(A))$ for any $A \subseteq B(p, s)$ and, consequently, $\alpha\big(f|_{B(p,s)}\big) \le \varepsilon\alpha\big(g|_{B(p,s)}\big)$. Taking the limit as $s \to 0$ we have $\alpha_p(f) \le \varepsilon\alpha_p(g)$. Since ε is arbitrary, we conclude that $\alpha_p(f) = 0$.

In the general case, write

$$f(x) = \sigma(p)g(x) + \widetilde{\sigma}(x)g(x),$$

where $\widetilde{\sigma}(x) = \sigma(x) - \sigma(p)$. As $\widetilde{\sigma}(p) = 0$, we have $\alpha_p(\widetilde{\sigma}g) = 0$. Thus, $\alpha_p(f) = \alpha_p(\sigma(p)g) = |\sigma(p)|\alpha_p(g)$, as claimed. The case of $\omega_p(f)$ is analogous. □

With an argument analogous to that used in [7], by means of Proposition 4.7 one can easily find examples of continuous maps f such that $\alpha(f) = \infty$ and $\alpha_p(f) < \infty$ for any p, and examples of continuous maps f with $\omega(f) = 0$ and $\omega_p(f) > 0$ for any p. Moreover, in [7] there is an example of a map f such that $\alpha(f) > 0$ and $\alpha_p(f) = 0$ for any p.

In the sequel we will consider also maps G defined on the product space $E \times \mathbb{R}$. In order to define $\alpha_{(p,\lambda)}(G)$, we consider the norm

$$||(p, \lambda)|| = \max\{||p||, |\lambda|\}.$$

The natural projection of $E \times \mathbb{R}$ onto the first factor will be denoted by π_1.

Remark 4.8. With the above norm, π_1 is nonexpansive. Therefore $\alpha(\pi_1(X)) \le \alpha(X)$ for any subset X of $E \times \mathbb{R}$. More precisely, since $\mathbb{R}$ is finite dimensional, if $X \subseteq E \times \mathbb{R}$ is bounded, we have $\alpha(\pi_1(X)) = \alpha(X)$.

We conclude the section with the following technical result, which is a straightforward consequence of Proposition 4.7 and which will be useful in the sequel.

Corollary 4.9. *Given a continuous map $\varphi : \Omega \to F$, consider the map*

$$\Phi : \Omega \times [0, 1] \to F, \quad \Phi(x, \lambda) = \lambda\varphi(x).$$

Then, for any fixed pair $(p, \lambda) \in \Omega \times [0, 1]$ we have

$$\alpha_{(p,\lambda)}(\Phi) = \lambda\alpha_p(\varphi).$$

5. Degree for α-Fredholm maps

In this section we sketch the construction of the degree for α-Fredholm maps introduced in [1]. These maps are special noncompact perturbations of Fredholm maps, defined in terms of the numbers α_p and ω_p. Precisely, an *α-Fredholm map* $f : \Omega \to F$ is of the form $f = g - k$, where g is a Fredholm map of index zero, k is continuous and $\alpha_p(k) < \omega_p(g)$ for every p.

The degree is given as an integer valued map defined on a class of triples that we shall call *admissible α-Fredholm triples*. This class is recalled in the following two definitions.

Definition 5.1. Let $g : \Omega \to F$ be a Fredholm map of index zero, $k : \Omega \to F$ a continuous map, and U an open subset of Ω. The triple (g, U, k) is said to be *α-Fredholm* if for any $p \in U$ we have

$$\alpha_p(k) < \omega_p(g).$$

Definition 5.2. An α-Fredholm triple (g, U, k) is said to be *admissible* if

i) g is oriented;

ii) the *solution set* $S = \{x \in U : g(x) = k(x)\}$ is compact.

Definition 5.3. Let (g, U, k) be an admissible α-Fredholm triple and consider a finite covering $\mathcal{V} = \{V_1, \dots, V_N\}$ of open balls of its solution set S. We say that $\mathcal{V}$ is an *α-covering* of S (relative to (g, U, k)) if for any $i \in \{1, \dots, N\}$ the following properties hold:

i) the ball $\widetilde{V}_i$ of double radius and same center as V_i is contained in U;

ii) $\alpha(k|_{\widetilde{V}_i}) < \omega(g|_{\widetilde{V}_i})$.

Let (g, U, k) be an admissible α-Fredholm triple and $\mathcal{V} = \{V_1, \dots, V_N\}$ an α-covering of the solution set S. We define the following sequence $\{C_n\}$ of convex closed subsets of E:

$$C_1 = \overline{\mathrm{co}} \left(\bigcup_{i=1}^{N} \{x \in V_i : g(x) \in k(\widetilde{V}_i)\} \right)$$

and, inductively,

$$C_n = \overline{\mathrm{co}} \left(\bigcup_{i=1}^{N} \{x \in V_i : g(x) \in k(\widetilde{V}_i \cap C_{n-1})\} \right), \quad n \geq 2.$$

Observe that, by induction, $C_{n+1} \subseteq C_n$ and $S \subseteq C_n$ for any $n \geq 1$. Then the set

$$C_\infty = \bigcap_{n \geq 1} C_n$$

turns out to be closed, convex, and containing S. Consequently, if S is nonempty, so is C_∞. In addition C_∞ verifies the following two properties (see [1] for the proof):

(1) $\{x \in V_i : g(x) \in k(\widetilde{V}_i \cap C_\infty)\} \subseteq C_\infty$, for any $i = 1, \dots, N$;

(2) C_∞ is compact.

Definition 5.4. Let (g, U, k) be an admissible α-Fredholm triple, $\mathcal{V} = \{V_1, \dots, V_N\}$ an α-covering of the solution set S, and C a compact convex set. We say that $(\mathcal{V}, C)$ is an *α-pair* (relative to (g, U, k)) if the following properties hold:

(1) $U \cap C \neq \emptyset$;

(2) $C_\infty \subseteq C$;

(3) $\{x \in V_i : g(x) \in k(\widetilde{V}_i \cap C)\} \subseteq C$ for any $i = 1, \dots, N$.

In [1] it is shown that given any admissible α-Fredholm triple (g, U, k) it is always possible to find an α-pair $(\mathcal{V}, C)$ relative to (g, U, k).

Let (g, U, k) be an admissible α-Fredholm triple and let $(\mathcal{V}, C)$ be an α-pair. Consider a retraction $r : E \to C$, whose existence is ensured by Dugundji's Extension Theorem [11]. Denote $V = \bigcup_{i=1}^N V_i$, where $\{V_1, \dots, V_N\} = \mathcal{V}$, and let W be a (possibly empty) open subset of V containing S such that, for any i, $x \in W \cap V_i$ implies $r(x) \in \widetilde{V}_i$. For example, if ρ denotes the minimum of the radii of the balls V_i, one may take as W the set $\{x \in V : \|x - r(x)\| < \rho\}$.

Observe that property (3) in Definition 5.4 implies that the two equations $g(x) = k(x)$ and $g(x) = k(r(x))$ have the same solution set in W (notice that the composition kr is defined in the open set $r^{-1}(U)$ containing W). The map kr is locally compact (even if not necessarily compact), hence the triple $(g - kr, W, 0)$ is qF-admissible (recall Definition 3.7). We define the degree of the triple (g, U, k), $\deg(g, U, k)$ in symbols, as follows:

$$\deg(g, U, k) = \deg_{qF}(g - kr, W, 0),$$

where the right hand side is the degree defined in Section 3.

The following definition summarizes the above construction.

Definition 5.5. Let (g, U, k) be an admissible α-Fredholm triple and $(\mathcal{V}, C)$ an α-pair. Consider a retraction $r : E \to C$. Denote $V = \bigcup_{i=1}^{N} V_i$, where $\{V_1, \dots, V_N\} = \mathcal{V}$. Let W be an open subset of V containing S such that, for any i, $x \in W \cap V_i$ implies $r(x) \in \widetilde{V}_i$. We set

$$\deg(g, U, k) = \deg_{qF}(g - kr, W, 0). \tag{5.1}$$

As proved in [1], the above definition is well posed since the right hand side of formula (5.1) is independent of the choice of the α-pair $(\mathcal{V}, C)$, of the retraction r and of the open set W.

Below we state the most important properties of the degree. Actually, in [1] only the fundamental properties (that is, normalization, additivity and homotopy invariance) were stated and proved. The excision and existence properties are easy consequences of the additivity.

Let us introduce the following concept of *α-Fredholm homotopy.*

Definition 5.6. Let W be an open subset of $E \times [0, 1]$ and $H : W \to F$ a continuous map of the form

$$H(x, \lambda) = G(x, \lambda) - K(x, \lambda).$$

We say that H is an *α-Fredholm homotopy* if the following conditions hold:

i) G is C^1;
ii) for any $\lambda \in [0, 1]$ the partial map G_λ is Fredholm of index zero on the section $W_\lambda = \{x \in E : (x, \lambda) \in W\}$;
iii) for any pair $(p, \lambda) \in W$ we have $\alpha_{(p,\lambda)}(K) < \omega_{(p,\lambda)}(G)$.

Theorem 5.7. *The following properties hold:*

1. *(Normalization) Let the identity I of E be naturally oriented. Then*
$$\deg(I, E, 0) = 1.$$
2. *(Additivity) Given an admissible α-Fredholm triple (g, U, k) and two disjoint open subsets U_1, U_2 of U, assume that $S = \{x \in U : g(x) = k(x)\}$ is contained in $U_1 \cup U_2$. Then*
$$\deg(g, U, k) = \deg(g, U_1, k) + \deg(g, U_2, k).$$
3. *(Excision) Given an admissible α-Fredholm triple (g, U, k) and an open subset U_1 of U, assume that S is contained in U_1. Then*
$$\deg(g, U, k) = \deg(g, U_1, k).$$
4. *(Existence) Given an admissible α-Fredholm triple (g, U, k), if*
$$\deg(g, U, k) \neq 0,$$
then the equation $g(x) = k(x)$ has a solution in U.
5. *(Homotopy invariance) Let W be an open subset of $E \times [0, 1]$ and $H : W \to F$ an α-Fredholm homotopy of the form $H(x, \lambda) = G(x, \lambda) - K(x, \lambda)$. Assume that G is oriented and that the set $H^{-1}(0)$ is compact. Then $\deg(G_\lambda, W_\lambda, K_\lambda)$ is well defined and does not depend on $\lambda \in [0, 1]$.*

6. Nonlinear bifurcation results

In this section we consider the semilinear operator equation

$$Lx + \lambda(h(x) + k(x)) = 0 \tag{6.1}$$

in $\Omega \times [0, +\infty)$, where $L : E \to F$ is a linear Fredholm operator of index zero between real Banach spaces, the maps $h : \Omega \to F$ and $k : \Omega \to F$ are C^1 and continuous, respectively.

Equation (6.1) can be equivalently written as

$$H(x, \lambda) = 0,$$

where

$$H : \Omega \times [0, +\infty) \to F, \quad H(x, \lambda) = Lx + \lambda(h(x) + k(x)).$$

This map is of the form

$$H(x, \lambda) = G(x, \lambda) - K(x, \lambda), \tag{6.2}$$

where $G(x, \lambda) = Lx + \lambda h(x)$ is of class C^1 and $K(x, \lambda) = -\lambda k(x)$. We will suppose that the following conditions hold:

H1) for any $\lambda \geq 0$ the partial map G_λ is Fredholm of index zero;

H2) for any pair $(p, \lambda) \in \Omega \times [0, +\infty)$ we have $\alpha_{(p,\lambda)}(K) < \omega_{(p,\lambda)}(G)$.

Thus, the map H is an α-Fredholm homotopy (see Definition 5.6).

By a solution of (6.1) we mean a pair $(x, \lambda) \in H^{-1}(0)$ and we regard the distinguished subset $(\Omega \cap \operatorname{Ker} L) \times \{0\}$ of $H^{-1}(0)$ as the set of *trivial solutions* of (6.1).

A problem related to equation (6.1) is that of the existence of an (*atypical*) *bifurcation point* (in the terminology of Prodi–Ambrosetti in [17]), that is, a point p in $\Omega \cap \operatorname{Ker} L$ such that $(p, 0)$ lies in the closure of the set of nontrivial solutions (i.e., of the pairs $(x, \lambda) \in H^{-1}(0)$ with $\lambda \neq 0$).

In a recent paper, Benevieri, Furi, Martelli and Pera (see [6]) obtained a global bifurcation result for equation (6.1) in the particular case when $k = 0$. Afterward, the result in [6] was extended by the first two authors (see [5]) by introducing a locally compact perturbation k. In that case the map H as in formula (6.2) is a homotopy of quasi-Fredholm maps (see Definition 3.4). Theorem 6.3 below is a further extension of the result in [5], by considering a not necessarily compact perturbation k. The compactness assumption of k is replaced by condition *H2)* above (which is clearly satisfied when k, and thus K, is locally compact). The proof follows some ideas in [6]. Let us stress that our argument is based on the degree for α-Fredholm maps.

Let F_1 be any fixed (finite dimensional) direct summand of $\operatorname{Im} L$ in F. We consider the decomposition $F = \operatorname{Im} L \oplus F_1$, and we denote by R and π the associated projections onto $\operatorname{Im} L$ and F_1, respectively.

Equation (6.1) is clearly equivalent to the system

$$\begin{cases} Lx + \lambda(Rh(x) + Rk(x)) = 0 \\ \lambda(\pi h(x) + \pi k(x)) = 0. \end{cases} \tag{6.3}$$

In order to investigate the set of nontrivial solutions of (6.3) it is convenient to consider the system

$$\begin{cases} Lx + \lambda(Rh(x) + Rk(x)) = 0 \\ \pi h(x) + \pi k(x) = 0 \end{cases} \tag{6.4}$$

which is equivalent to (6.3) for $\lambda \neq 0$.

The next result provides a necessary condition for $p \in \Omega \cap \operatorname{Ker} L$ to be a bifurcation point. The easy proof, which is based on a simple continuity argument, is given for completeness.

Theorem 6.1. *Assume that p is a bifurcation point for the equation* (6.1)*. Then $h(p) + k(p) \in \operatorname{Im} L$ or, equivalently, $\pi h(p) + \pi k(p) = 0$.*

Proof. Since p is a bifurcation point, there exists a sequence $\{(\lambda_n, x_n)\}$ of nontrivial solutions of (6.1) converging to $(0, p)$. Hence, (λ_n, x_n) is a solution of the system (6.4) for any n, and the result follows from the continuity of the maps πh and πk. □

Our main result, Theorem 6.3 below, is the analogue of Theorem 3.2 in [6] and provides a sufficient condition for the existence of a bifurcation point. The fundamental tools for proving Theorem 6.3 are the homotopy invariance property of the degree for α-Fredholm maps (Theorem 5.7), together with the following crucial lemma, whose proof can be found in [14].

Lemma 6.2. *Let Z be a compact subset of a locally compact metric space X. Assume that any compact subset of X containing Z has nonempty boundary. Then $X \setminus Z$ contains a connected set whose closure is not compact and intersects Z.*

We are now ready to state our main result. The statement involves the Brouwer degree of a map between $\operatorname{Ker} L$ and F_1. Therefore these spaces should be oriented. However, the result is independent of the chosen orientations.

As in Section 5, given an open subset W of $\Omega \times [0, +\infty)$, by W_λ we denote the section $\{x \in \Omega : (x, \lambda) \in W\}$.

Theorem 6.3. *Let $H : \Omega \times [0, +\infty) \to F$ be defined by $H(x, \lambda) = Lx + \lambda(h(x) + k(x))$ and suppose that conditions H1) and H2) above hold, so that H is an α-Fredholm homotopy of the form $H(x, \lambda) = G(x, \lambda) - K(x, \lambda)$. Assume in addition that G is oriented.*

Let $v : \Omega \cap \operatorname{Ker} L \to F_1$ be defined by $v(p) = \pi h(p) + \pi k(p)$. Let W be an open subset of $\Omega \times [0, +\infty)$ and suppose that the Brouwer degree $\deg_B(v, W_0 \cap \operatorname{Ker} L, 0)$ is well defined and nonzero. Then, there exists in W a connected set of nontrivial solutions of (6.1) *whose closure in W is not compact and intersects* $\operatorname{Ker} L \times \{0\}$.

Proof. Let $\widehat{H} : W \to F = \operatorname{Im} L \oplus F_1$ be defined by

$$\widehat{H}(x, \lambda) = Lx + \lambda(Rh(x) + Rk(x)) + \pi h(x) + \pi k(x). \tag{6.5}$$

This map is clearly an α-Fredholm homotopy which can be written as $\widehat{H} = \widehat{G} - \widehat{K}$, where

$$\widehat{G}(x, \lambda) = Lx + \lambda Rh(x) + \pi h(x)$$

is of class C^1 and oriented (with orientation induced by G according to Theorem 2.8), and $\widehat{K}(x, \lambda) = -\lambda Rk(x) - \pi k(x)$. In fact, since $RG = R\widehat{G}$ and $RK = R\widehat{K}$, by Proposition 4.6 we get

$$\alpha_{(p,\lambda)}(\widehat{K}) = \alpha_{(p,\lambda)}(K) \quad \text{and} \quad \omega_{(p,\lambda)}(\widehat{G}) = \omega_{(p,\lambda)}(G)$$

for any pair $(p, \lambda) \in W$. Thus, $\alpha_{(p,\lambda)}(\widehat{K}) < \omega_{(p,\lambda)}(\widehat{G})$ for any $(p, \lambda) \in W$.

Let now

$$Y = \{(x, \lambda) \in W : \widehat{H}(x, \lambda) = 0\}.$$

Notice that the set Y is locally compact. Indeed, the map $\widehat{H}$ is locally proper at any $(p, \lambda) \in W$ since $\alpha_{(p,\lambda)}(\widehat{K}) < \omega_{(p,\lambda)}(\widehat{G})$. Moreover, $Y_0 = v^{-1}(0) \cap W_0$ is compact because we assumed that $\deg_B(v, W_0 \cap \operatorname{Ker} L, 0)$ is well defined.

We apply Lemma 6.2 with $Y_0 \times \{0\}$ in place of Z and with Y in place of X. Assume, by contradiction, that there exists a compact set $Y' \subseteq Y$ containing $Y_0 \times \{0\}$ and with empty boundary in Y. Thus, Y' is also an open subset of Y. Hence, there exists a bounded open subset U of W such that $Y' = U \cap Y$. Since Y' is compact, the homotopy invariance property of the degree (Theorem 5.7) implies that $\deg(\widehat{G}_\lambda, U_\lambda, \widehat{K}_\lambda)$ does not depend on $\lambda \geq 0$. Moreover, the slice $Y'_\lambda = U_\lambda \cap Y_\lambda$ is empty for some positive λ. This implies that $\deg(\widehat{G}_\lambda, U_\lambda, \widehat{K}_\lambda) = 0$ for any $\lambda \in [0, +\infty)$ and, in particular, $\deg(\widehat{G}_0, U_0, \widehat{K}_0) = 0$. The inclusions $v^{-1}(0) \cap W_0 \subseteq U_0 \subseteq W_0$ imply, using the excision property of the degree, that $\deg(\widehat{G}_0, W_0, \widehat{K}_0) = 0$.

Now, observe that the map $\widehat{H}_0 = \widehat{G}_0 - \widehat{K}_0$, which is given by

$$\widehat{H}_0(x) = Lx + \pi h(x) + \pi k(x),$$

is actually an oriented quasi-Fredholm map (being $\widehat{G}$, and thus $\widehat{G}_0$, oriented). Consequently, we get

$$0 = \deg(\widehat{G}_0, W_0, \widehat{K}_0) = \deg_{qF}(\widehat{H}_0, W_0, 0). \tag{6.6}$$

The subspace F_1 contains the image of $\widehat{K}_0$ and is transverse to $\widehat{G}_0$ being transverse to L. Moreover, $\widehat{H}_0^{-1}(F_1) = \widehat{G}_0^{-1}(F_1) = W_0 \cap \operatorname{Ker} L$. Suppose F_1 oriented and, without loss of generality, $W_0 \cap \operatorname{Ker} L$

oriented in such a way that it becomes the oriented $\widehat{G}_0$-preimage of F_1. Hence, by definition of degree for quasi-Fredholm maps (see formula (3.2)), we obtain

$$\deg_{qF}(\widehat{H}_0, W_0, 0) = \deg_B(v, W_0 \cap \operatorname{Ker} L, 0) \neq 0,$$

which contradicts equality (6.6).

Therefore, because of Lemma 6.2, there exists a connected subset of Y whose closure in Y intersects $Y_0 \times \{0\}$ and is not compact. This completes the proof. □

The next consequences of Theorem 6.3, Corollaries 6.4 and 6.5 below, extend analogous results in [6]. The proofs are given for the reader's convenience.

Corollary 6.4. *Let the assumptions of Theorem 6.3 be satisfied. Suppose, moreover, that the map H is proper on bounded and closed subsets of W. Then, equation* (6.1) *admits a connected set Γ of nontrivial solutions such that its closure in $E \times [0, +\infty)$ intersects $\operatorname{Ker} L \times \{0\}$ and is either unbounded or reaches the boundary of W. If, in particular, $\Omega = E$ and $W = E \times [0, +\infty)$, then Γ is unbounded.*

Proof. Let $\overline{\Gamma}$ denote the closure in $E \times [0, +\infty)$ of a connected branch Γ as in Theorem 6.3. Suppose that $\Gamma \cap \partial W = \emptyset$. Thus, the closure of Γ in W coincides with $\overline{\Gamma}$. Hence, $\overline{\Gamma}$ cannot be bounded, since the properness of H on bounded closed subsets of W implies that the map $\widehat{H}$ as in the proof of Theorem 6.3 (see formula (6.5)) has the same property. □

Corollary 6.5. *Let W and v be as in Theorem 6.3. Suppose, moreover, that the map H is proper on bounded and closed subsets of W. Let $p \in W_0 \cap \operatorname{Ker} L$ be such that $v(p) = 0$ and $v'(p)$ is invertible. Then, equation* (6.1) *admits a connected set Γ of nontrivial solutions such that its closure contains p and satisfies at least one of the following three conditions:*

i) *is unbounded;*
ii) *contains a point $q \in W_0 \cap \operatorname{Ker} L$, $q \neq p$;*
iii) *intersects ∂W.*

Proof. The assumptions $v(p) = 0$ and $v'(p)$ invertible imply the existence of an open neighborhood $\widetilde{W}_0$ of p in W_0 such that $v^{-1}(0) \cap \widetilde{W}_0 = \{p\}$ and $\deg(v, \widetilde{W}_0, 0) = \pm 1$. Now apply Corollary 6.4 with the set $\widetilde{W} = \Big(\widetilde{W}_0 \times \{0\}\Big) \cup \{(x, \lambda) \in W : \lambda \neq 0\}$ in place of W. Observe that $\widetilde{W}$ is open, being obtained from W by removing the closed subset $\{(x, 0) \in W : x \notin \widetilde{W}_0\}$, and that the boundary of $\widetilde{W}$ (in $E \times [0, +\infty)$) coincides with the boundary of $\widetilde{W}$ as a subset of $E \times \mathbb{R}$ except $\widetilde{W}_0 \times \{0\}$. □

7. Applications

In this section we provide an application of the bifurcation results obtained in Section 6 to the following boundary value problem depending on a parameter $\lambda \geq 0$:

$$\begin{cases} x'(t) + \lambda\phi(t, x(t), x'(t)) + \lambda\psi(t, x(t), x'(t)) = 0 \\ x(0) = x(T), \end{cases} \tag{7.1}$$

where $\phi : \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ is C^1 and $\psi : \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ is continuous. We suppose that ϕ and ψ are T-periodic with respect to the first variable. Under additional assumptions, to be specified in the sequel, we obtain a global bifurcation result for T-periodic solutions of problem (7.1).

Our first step consists in presenting an example of an α-Fredholm homotopy. Let us fix some notation. We denote by $\mathcal{C}^0$ the Banach space $C([0, T], \mathbb{R}^n)$ endowed with the usual supremum norm

$$\|x\|_0 = \max_{t \in [0,T]} |x(t)|, \quad x \in \mathcal{C}^0,$$

and by $\mathcal{C}^1$ the space $C^1([0,T],\mathbb{R}^n)$ endowed with the norm

$$\|x\|_1 = \max\{\|x'\|_0, |x(0)|\}, \quad x \in \mathcal{C}^1.$$

We endow the product space $\mathcal{C}^0 \times \mathbb{R}^n$ with the norm

$$\|(y,r)\| = \max\{\|y\|_0, |r|\}, \quad (y,r) \in \mathcal{C}^0 \times \mathbb{R}^n.$$

Given a $n \times n$ matrix M, we denote its norm by $\|M\|$.

For simplicity we shall consider ϕ and ψ defined just on $[0,T] \times \mathbb{R}^n \times \mathbb{R}^n$.

Define

$$\begin{array}{ll} \Phi : \mathcal{C}^1 \to \mathcal{C}^0, & \Phi(x)(t) = \phi(t, x(t), x'(t)), \quad t \in [0,T], \\ \Psi : \mathcal{C}^1 \to \mathcal{C}^0, & \Psi(x)(t) = \psi(t, x(t), x'(t)), \quad t \in [0,T], \end{array} \tag{7.2}$$

and set

$$\begin{array}{ll} G : \mathcal{C}^1 \times [0,+\infty) \to \mathcal{C}^0 \times \mathbb{R}^n, & G(x,\lambda) = (x' + \lambda\Phi(x), x(0) - x(T)), \\ K : \mathcal{C}^1 \times [0,+\infty) \to \mathcal{C}^0 \times \mathbb{R}^n, & K(x,\lambda) = -\lambda(\Psi(x), 0). \end{array} \tag{7.3}$$

It is convenient to write $G(x,\lambda) = (\widetilde{G}(x,\lambda), x(0) - x(T))$, where

$$\widetilde{G}(x,\lambda) = x' + \lambda\Phi(x).$$

That is,

$$\widetilde{G}(x,\lambda)(t) = x'(t) + \lambda\phi(t, x(t), x'(t)), \quad t \in [0,T].$$

The map G is C^1 (since so is ϕ) and the Fréchet derivative $G'_\lambda(x) : \mathcal{C}^1 \to \mathcal{C}^0 \times \mathbb{R}^n$ of any partial map G_λ at any $x \in \mathcal{C}^1$ is given by

$$G'_\lambda(x)q = \left(\widetilde{G}'_\lambda(x)q, q(0) - q(T)\right),$$

where

$$\big(\widetilde{G}'_\lambda(x)q\big)(t) = q'(t) + \lambda\partial_2\phi(t, x(t), x'(t))q(t) + \lambda\partial_3\phi(t, x(t), x'(t))q'(t), \quad t \in [0,T]$$

for any $q \in \mathcal{C}^1$. Here $\partial_2\phi$ and $\partial_3\phi$ denote the Jacobian matrices of ϕ with respect to the second and third variable, respectively. In particular, the derivative at any x of $\widetilde{G}_\lambda$ can be written as

$$\big(\widetilde{G}'_\lambda(x)q\big)(t) = (I + \lambda M_x(t))q'(t) + \lambda N_x(t)q(t), \quad t \in [0,T]$$

where, given $x \in \mathcal{C}^1$, M_x and N_x are $n \times n$ matrices of continuous real functions defined in $[0,T]$ by

$$M_x(t) = \partial_3\phi(t, x(t), x'(t)) \quad \text{and} \quad N_x(t) = \partial_2\phi(t, x(t), x'(t)).$$

If x and λ are such that

$$\det(I + \lambda M_x(t)) \neq 0, \quad \text{for any } t \in [0,T], \tag{7.4}$$

then $G'_\lambda(x) : \mathcal{C}^1 \to \mathcal{C}^0 \times \mathbb{R}^n$ is a Fredholm operator of index zero. Indeed, it is the sum of the two compact linear operators $q \mapsto (0, -q(T))$ (having finite dimensional image) and $q \mapsto (\lambda N_x(\cdot)q(\cdot), 0)$ (which is compact since so is the inclusion $\mathcal{C}^1 \hookrightarrow \mathcal{C}^0 \times \mathbb{R}^n$) with the isomorphism

$$\mathcal{C}^1 \to \mathcal{C}^0 \times \mathbb{R}^n, \quad q \mapsto ((I + \lambda M_x(\cdot))q'(\cdot), q(0)).$$

Let us stress that condition (7.4) holds for any pair (x,λ) if we assume that, for every $(t,a,b) \in [0,T] \times \mathbb{R}^n \times \mathbb{R}^n$, the Jacobian matrix $\partial_3\phi(t,a,b)$ has no negative eigenvalues.

Let us now estimate the local measure of noncompactness of the maps G and K. In particular, we look for conditions under which a given pair $(x,\lambda) \in \mathcal{C}^1 \times [0,+\infty)$ verifies the inequality

$$\alpha_{(x,\lambda)}(K) < \omega_{(x,\lambda)}(G).$$

Lemma 7.1. *Suppose that ψ is Lipschitz continuous with respect to the third variable; that is, there exists some $c > 0$ such that*

$$|\psi(t, a, b_1) - \psi(t, a, b_2)| \leq c|b_1 - b_2| \tag{7.5}$$

for any $t \in [0, T]$ and any $a, b_1, b_2 \in \mathbb{R}^n$. Then

$$\alpha_{(x,\lambda)}(K) \leq \lambda c$$

for any pair $(x, \lambda) \in \mathcal{C}^1 \times [0, +\infty)$.

Proof. Let $(x, \lambda) \in \mathcal{C}^1 \times [0, +\infty)$ be fixed. Since $K(x, \lambda) = -\lambda(\Psi(x), 0)$, by Corollary 4.9 we get $\alpha_{(x,\lambda)}(K) = \lambda\alpha_x(\Psi)$. Moreover, the map Ψ is Lipschitz with constant c. Indeed, given $x_1, x_2 \in \mathcal{C}^1$ and $t \in [0, T]$, we have

$$|\Psi(x_1)(t) - \Psi(x_2)(t)| = |\psi(t, x_1(t), x_1'(t)) - \psi(t, x_2(t), x_2'(t))| \leq c|x_1'(t) - x_2'(t)|,$$

and thus

$$\|\Psi(x_1) - \Psi(x_2)\|_0 \leq c\|x_1' - x_2'\|_0 \leq c\|x_1 - x_2\|_1.$$

It follows that $\alpha_x(\Psi) \leq \alpha(\Psi) \leq c$ and, consequently, $\alpha_{(x,\lambda)}(K) \leq \lambda c$. □

Remark 7.2. The assertion of Lemma 7.1 is still valid when

$$\psi(t, a, b) = \psi_1(t, a, b) + \psi_2(t, a),$$

with ψ_1 satisfying condition (7.5) and ψ_2 independent of the third variable. In fact, in this case one can easily check that the map $\widetilde{\Psi} : \mathcal{C}^1 \to \mathcal{C}^0$, defined by

$$\widetilde{\Psi}(x)(t) = \psi_1(t, x(t), x'(t)) + \psi_2(t, x(t)), \quad t \in [0, T],$$

is α-Lipschitz with constant c, being the sum of an α-Lipschitz map with constant c and a completely continuous map.

Lemma 7.3. *Assume that for any $(t, a, b) \in [0, T] \times \mathbb{R}^n \times \mathbb{R}^n$ the Jacobian matrix $\partial_3\phi(t, a, b)$ has no negative eigenvalues. Set*

$$\gamma(\lambda) = \sup_{(t,a,b)} \|(I + \lambda\partial_3\phi(t, a, b))^{-1}\|.$$

Then

$$\omega_{(x,\lambda)}(G) \geq \frac{1}{\gamma(\lambda)}$$

for any pair $(x, \lambda) \in \mathcal{C}^1 \times [0, +\infty)$.

Proof. Let $(x, \lambda) \in \mathcal{C}^1 \times [0, +\infty)$ be fixed. First of all observe that, since G is of class C^1, by Proposition 4.4 we have $\omega_{(x,\lambda)}(G) = \omega\big(G'(x, \lambda)\big)$ and, by Proposition 4.5, $\omega\big(G'(x, \lambda)\big) = \omega\big(G_\lambda'(x)\big)$. Hence,

$$\omega_{(x,\lambda)}(G) = \omega\big(G_\lambda'(x)\big).$$

As we already pointed out, the assumption on the Jacobian matrix $\partial_3\phi(t, a, b)$ implies that condition (7.4) holds for any pair $(x, \lambda) \in \mathcal{C}^1 \times [0, +\infty)$. Consequently, $G_\lambda'(x)$ is a Fredholm operator of index zero.

Now, define the linear operator $\Gamma : \mathcal{C}^1 \to \mathcal{C}^0 \times \mathbb{R}^n$ by $\Gamma q = (\Gamma_1 q, q(0))$, where

$$\Gamma_1 q(t) = (I + \lambda M_x(t))q'(t), \quad t \in [0, T].$$

Since the maps $q \mapsto (0, -q(T))$ and $q \mapsto (\lambda N_x(\cdot)q(\cdot), 0)$ are compact, by (4) and (5) in Proposition 4.3 we have $\omega\big(G_\lambda'(x)\big) = \omega\big(\Gamma\big)$. Moreover, condition (7.4) implies that the linear operator Γ is invertible. Thus, by (7) in Proposition 4.3, we get

$$\omega(\Gamma) = \frac{1}{\alpha(\Gamma^{-1})}.$$

Let us estimate $\alpha(\Gamma^{-1})$. For this purpose, let $P : \mathcal{C}^0 \times \mathbb{R}^n \to \mathcal{C}^0 \times \mathbb{R}^n$ be the natural projection onto $\mathcal{C}^0 \times \{0\}$, defined by $(y,r) \mapsto (y,0)$. By Proposition 4.5, we have $\alpha(\Gamma^{-1}) = \alpha(\Gamma^{-1}P)$. Now, fix $(y,r) \in \mathcal{C}^0 \times \mathbb{R}^n$ and let $q \in \mathcal{C}^1$ be such that $q = \Gamma^{-1}P(y,r)$; that is, q is the solution of the linear problem

$$\begin{cases} q'(t) = (I + \lambda M_x(t))^{-1} y(t) \\ q(0) = 0. \end{cases}$$

We have $|q'(t)| \leq \|(I + \lambda M_x(t))^{-1}\| \, |y(t)|$ for any t, and thus

$$\|q'\|_0 \leq \max\nolimits_{t\in[0,T]} \|(I + \lambda M_x(t))^{-1}\| \, \|y\|_0 \leq$$
$$\sup\nolimits_{(t,a,b)} \|(I + \lambda \partial_3 \phi(t,a,b))^{-1}\| \, \|(y,r)\| = \gamma(\lambda) \, \|(y,r)\|.$$

Consequently, $\|q\|_1 \leq \gamma(\lambda) \, \|(y,r)\|$. It follows

$$\alpha(\Gamma^{-1}) = \alpha(\Gamma^{-1}P) \leq \gamma(\lambda).$$

Hence,

$$\omega_{(x,\lambda)}(G) = \omega(\Gamma) \geq \frac{1}{\gamma(\lambda)}.$$

□

The next proposition summarizes the above two lemmas. The statement involves the map

$$H : \mathcal{C}^1 \times [0,+\infty) \to \mathcal{C}^0 \times \mathbb{R}^n, \quad H(x,\lambda) = G(x,\lambda) - K(x,\lambda), \tag{7.6}$$

where G and K are as in (7.3).

Proposition 7.4. *Let $\phi : [0,T] \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ be C^1 and $\psi : [0,T] \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ be continuous. Assume that the following conditions hold:*

i) *the map ψ is Lipschitz continuous with respect to the third variable with constant $c > 0$;*
ii) *for any $(t,a,b) \in [0,T] \times \mathbb{R}^n \times \mathbb{R}^n$, the Jacobian matrix $\partial_3\phi(t,a,b)$ has no negative eigenvalues;*
iii) *the constant c is such that*

$$\lambda c < \frac{1}{\gamma(\lambda)} \qquad \text{for any } \lambda \in [0,+\infty),$$

where $\gamma(\lambda) = \sup_{(t,a,b)} \|(I + \lambda \partial_3\phi(t,a,b))^{-1}\|$.

Then the map H as in (7.6) *is an α-Fredholm homotopy.*

As an example illustrating condition iii) in Proposition 7.4, consider the case in which there exists $\delta > 0$ such that $\partial_3\phi(t,a,b) = \delta I$ for any (t,a,b). Thus, $\gamma(\lambda) = \dfrac{1}{1+\lambda\delta}$, and condition iii) is clearly satisfied if the Lipschitz constant c of the map ψ is smaller than δ.

Let us come back to our study of problem (7.1). For technical reasons, define

$$\begin{aligned} L &: \mathcal{C}^1 \to \mathcal{C}^0 \times \mathbb{R}^n, & Lx &= (x',\, x(0) - x(T)), \\ h &: \mathcal{C}^1 \to \mathcal{C}^0 \times \mathbb{R}^n, & h(x) &= (\Phi(x),\, 0), \\ k &: \mathcal{C}^1 \to \mathcal{C}^0 \times \mathbb{R}^n, & k(x) &= (\Psi(x),\, 0), \end{aligned}$$

with Φ and Ψ as in (7.2). Then, problem (7.1) is equivalent to the semilinear operator equation

$$Lx + \lambda(h(x) + k(x)) = 0 \tag{7.7}$$

in $\mathcal{C}^1 \times [0,+\infty)$. Observe that equation (7.7) can be equivalently written as

$$H(x,\lambda) = 0$$

where the map

$$H : \mathcal{C}^1 \times [0,+\infty) \to \mathcal{C}^0 \times \mathbb{R}^n, \quad H(x,\lambda) = Lx + \lambda(h(x) + k(x))$$

is the same as in (7.6), being $G(x,\lambda) = Lx + \lambda h(x)$ and $K(x,\lambda) = -\lambda k(x)$.

Now, suppose that conditions i)–iii) in Proposition 7.4 hold. Hence, by Proposition 7.4, the map H is an α-Fredholm homotopy. Therefore, we can apply the results of Section 6 to equation (7.7) obtaining a global bifurcation result (see Theorem 7.5 below).

As we already pointed out, Benevieri, Furi, Martelli and Pera in [6] obtained a global bifurcation result for equation (7.7) in the absence of the perturbation k. That is, in [6] they studied a problem analogous to (7.1) with ψ identically zero. Their result was extended by the first two authors in [5] in the case when ψ is (nonzero and) independent of the third variable. Theorem 7.5 below extends these results, by assuming ψ to be Lipschitz continuous with respect to the third variable, with suitably small Lipschitz constant.

Before stating Theorem 7.5 we need some preliminary remarks. First, to avoid cumbersome notation, any point $p \in \mathbb{R}^n$ is identified with the constant function $t \mapsto p$, so that $\mathbb{R}^n$ can be regarded as the set of trivial solutions of problem (7.1).

Now, it is not difficult to show that the operator L is Fredholm of index zero, with $\operatorname{Ker} L = \mathbb{R}^n$ and

$$\operatorname{Im} L = \left\{ (y,r) \in \mathcal{C}^0 \times \mathbb{R}^n : r = -\textstyle\int_0^T y(t)\,dt \right\}.$$

The reader can easily verify that $\mathcal{C}^0 \times \mathbb{R}^n = \operatorname{Im} L \oplus F_1$, where F_1 is an n-dimensional subspace of $\mathcal{C}^0 \times \mathbb{R}^n$ which can be identified with $\mathbb{R}^n$. In fact, observe that any pair $(y,r) \in \mathcal{C}^0 \times \mathbb{R}^n$ can be uniquely decomposed as

$$(y,r) = \left(y, -\textstyle\int_0^T y(t)\,dt\right) + \left(0, r + \textstyle\int_0^T y(t)\,dt\right).$$

Moreover, the projection π of $\mathcal{C}^0 \times \mathbb{R}^n$ onto $F_1 = \mathbb{R}^n$ can be written as

$$\pi(y,r) = r + \int_0^T y(t)\,dt.$$

Thus, the vector field $v : \mathbb{R}^n \to \mathbb{R}^n$, defined by $v(p) = \pi h(p) + \pi k(p)$, can be written as

$$v(p) = \int_0^T \big(\phi(t,p,0) + \psi(t,p,0)\big)\,dt.$$

We are now ready to state the main result of this section. The statement involves, instead of v, the *mean value vector field* $w : \mathbb{R}^n \to \mathbb{R}^n$ defined by

$$w(p) = \frac{1}{T}\int_0^T \big(\phi(t,p,0) + \psi(t,p,0)\big)\,dt. \tag{7.8}$$

Theorem 7.5. *Let $\phi : [0,T] \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ be C^1 and $\psi : [0,T] \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ be continuous and suppose that conditions* i)–iii) *in Proposition 7.4 hold.*

Let $w : \mathbb{R}^n \to \mathbb{R}^n$ be the mean value vector field defined in (7.8)*. Let W be an open subset of $\mathcal{C}^1 \times [0,+\infty)$ and denote $\widetilde{W}_0 = \{p \in \mathbb{R}^n : (p,0) \in W\}$. Assume that the Brouwer degree $\deg_B(w, \widetilde{W}_0, 0)$ is defined and different from zero. Then, W contains a connected set of nontrivial solutions of problem* (7.1)*, whose closure in W is not compact and intersects $\operatorname{Ker} L \times \{0\} \cong \mathbb{R}^n$ in the compact set $w^{-1}(0) \cap \widetilde{W}_0$.*

Proof. Clearly, $\deg_B(w, \widetilde{W}_0, 0)$ is defined and different from zero if and only if the same is true for $\deg_B(v, \widetilde{W}_0, 0)$. To apply Theorem 6.3 we need the orientability of the map G defined in (7.3). This is a consequence of the fact, proved in [4], that any Fredholm map defined in a simply connected open set (the whole space in this case) is orientable. Thus, the assertion follows from Theorem 6.3. □

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