Bifurcation diagrams for singularly perturbed system.

Matteo Franca*

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Abstract

We consider a singularly perturbed system where the fast dynamic of the unperturbed problem exhibits a trajectory homoclinic to a critical point. We assume that the slow time system is 1-dimensional and it admits a unique critical point, which undergoes to a bifurcation as a second parameter varies: transcritical, saddle-node, or pitchfork. In this setting Battelli and Palmer proved the existence of a unique trajectory $(\breve{x}(t,\varepsilon,\lambda),\breve{y}(t,\varepsilon,\lambda))$ homoclinic to the slow manifold. The purpose of this paper is to construct curves which divide the 2-dimensional parameters space in different areas where $(\breve{x}(t,\varepsilon,\lambda),\breve{y}(t,\varepsilon,\lambda))$ is either homoclinic, heteroclinic, or unbounded. We derive explicit formulas for the tangents of these curves. The results are illustrated by some examples.

Keywords. Singular perturbation, homoclinic trajectory, transcritical bifurcation saddle-node bifurcation.
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1 Introduction

In this paper we consider the following singularly perturbed system:

$$\begin{cases} \dot{x} = \varepsilon f(x,y,\varepsilon,\lambda) \\ \dot{y} = g(x,y,\varepsilon,\lambda) \end{cases} \tag{1.1}$$ n1.1

*Dipartimento di Scienze Matematiche, Università Politecnica delle Marche, Via Brecce Bianche 1, 60131 Ancona - Italy. Partially supported by G.N.A.M.P.A. - INdAM (Italy) and MURST (Italy)

where $x \in \mathbb{R}$, $y \in \mathbb{R}^n$ and $(x, y) \in \Omega$, $\Omega \subset \mathbb{R}^{1+n}$ is open, λ and ε are small real parameters and $f(x, y, \varepsilon, \lambda)$, $g(x, y, \varepsilon, \lambda)$ are C^r–functions in their arguments bounded with their derivatives, $r \geq 2$. We suppose that the following conditions hold:

(i) for any $x \in \mathbb{R}$, we have
$$g(x, 0, 0, 0) = 0,$$

(ii) the infimum over $x \in \mathbb{R}$ of the moduli of the real parts of the eigenvalues of the jacobian matrix $\frac{\partial g}{\partial y}(x, 0, 0, 0)$ is greater than a positive number Λ^g.

(iii) the equation
$$\dot{y} = g(0, y, 0, 0)$$
has a solution $h(t)$ homoclinic to the origin $0 \in \mathbb{R}^n$

(iv) $\dot{h}(t)$ is the unique bounded solution of the linear variational system:
$$\dot{y} = \frac{\partial g}{\partial y}(0, h(t), 0, 0)y \tag{1.2}$$
n1.3

up to a scalar multiple.

According to condition (ii), for any $x \in \mathbb{R}$, the linear system $\dot{y} = \frac{\partial g}{\partial y}(x, 0, 0, 0)y$ has exponential dichotomy on $\mathbb{R}$ with exponent $\Lambda^g > 0$ and projections, say, $P^0(x)$. For simplicity we set $P^0(0) = P^0$. Let $\mathrm{rank}[P^0(x)] = p$, p being the number of eigenvalues of $\frac{\partial g}{\partial y}(x, 0, 0, 0)$ with positive real parts: we stress that p is constant for $|x|$ small enough. From the assumptions (ii) and (iii) it follows that the linear system (1.2) and its adjoint
$$\dot{y} = -\big[\frac{\partial g}{\partial y}(0, h(t), 0, 0)\big]^* y \tag{1.3}$$
n1.4

have exponential dichotomies on both $\mathbb{R}_+$ and $\mathbb{R}_-$; i.e. there are projections $P^\pm$ and $k > 0$ such that
$$\begin{array}{ll} \|Y(t)P^-Y^{-1}(s)\| \leq k\mathrm{e}^{-\Lambda^g(t-s)} & \text{if } s \leq t \leq 0 \\ \|Y(t)(\mathbf{I} - P^-)Y^{-1}(s)\| \leq k\mathrm{e}^{-\Lambda^g(s-t)} & \text{if } t \leq s \leq 0 \\ \|Y(t)P^+Y^{-1}(s)\| \leq k\mathrm{e}^{-\Lambda^g(t-s)} & \text{if } 0 \leq s \leq t \\ \|Y(t)(\mathbf{I} - P^+)Y^{-1}(s)\| \leq k\mathrm{e}^{-\Lambda^g(s-t)} & \text{if } 0 \leq t \leq s \end{array} \tag{1.4}$$
dichot

where $Y(t)$ is the fundamental matrix of (1.2), and the analogous estimate hold for (1.3). Here and later we use the shorthand notation $\pm$ to represent both the $+$ and $-$ equations and functions. Observe that $\mathrm{rank}(P^+) =$

$\operatorname{rank}(P^-) = p$ and the projections of the dichotomy of (1.3) on $\mathbb{R}_\pm$ are $\mathbf{I} - [P^\pm]^*$. Moreover from (i)–(iv) it follows that (1.3) has a unique bounded solution on $\mathbb{R}$, up to a multiplicative constant. We denote one of these solutions by $\psi(t)$. Note that $\psi := \psi(0)$ satisfies $\mathcal{N}[P^+]^* \cap \mathcal{R}[P^-]^* = \operatorname{span}(\psi) = [\mathcal{R}P^+ \cap \mathcal{N}P^-]^\perp$; we assume w.l.o.g. that $|\psi(0)| = 1$. As a second remark we observe that condition (i) implies the existence of a function $v(x, \varepsilon, \lambda)$ defined for x, ε, λ small enough, such that $v(x, 0, 0) \equiv 0$ and the manifold $y = v(x, \varepsilon, \lambda)$ is invariant for the flow of (1.1) (see for example [2, 11]) and satisfies the following: if $(x(t, \varepsilon, \lambda), y(t, \varepsilon, \lambda))$ is the solution of (1.1) such that $(x(0, \varepsilon, \lambda), y(0, \varepsilon, \lambda)) = (\bar{x}, v(\bar{x}, \varepsilon, \lambda))$, then $\|\dot{y}(0, \varepsilon, \lambda)\| \leq C\varepsilon$ for a certain $C > 0$. Moreover $v(x, \varepsilon, \lambda)$ is C^r and bounded with its derivatives. Using the flow of (1.1) we can pass from the local manifold $y = v(x, \varepsilon, \lambda)$ to a global slow manifolds for system (1.1) which will be denoted by $\mathcal{M}^c = \mathcal{M}^c(\varepsilon, \lambda)$.

Let $x_c(t, \xi, \varepsilon, \lambda)$ be the solution of the scalar ODE:

$$\dot{x} = f(x, v(x, \varepsilon, \lambda), \varepsilon, \lambda) \qquad x(0) = \xi \tag{1.5}$$ bl

So $(x_c(t, \xi, \varepsilon, \lambda), v(x_c(t, \xi, \varepsilon, \lambda), \varepsilon, \lambda))$ describes the flow on the slow manifold $\mathcal{M}^c$, and (1.5) is the so called "slow time" system.

The behavior of homoclinic and heteroclinic trajectories subject to singular perturbation has been studied in several papers, see e.g. [1, 2, 4, 5, 6, 11, 12]. In particular in [6] the authors built up a theory to prove the existence of solutions homoclinic to $\mathcal{M}^c$, for the perturbed problem (1.1) assuming conditions (i)–(iv) and giving transversality conditions of several different types. They refine previous results obtained in [4].

This paper is thought as a sequel of [6]. Here we assume that the "slow time" system (1.5) is one-dimensional so that there is a unique solution $(\breve{x}(t, \varepsilon, \lambda), \breve{y}(t, \varepsilon, \lambda))$ homoclinic to $\mathcal{M}^c$. Moreover we assume that (1.5) undergoes to a bifurcation for $\varepsilon = 0$ as λ changes sign. We mainly focus on the transcritical and saddle-node case, i.e. we assume f has one of the following form:

$$f(x, 0, \varepsilon, \lambda) = x^2 - b(\varepsilon)\lambda^2 + O(x^3) \tag{1.6}$$

$$f(x, 0, \varepsilon, \lambda) = x^2 - a(\varepsilon)\lambda + O(x^3) \tag{1.7}$$

where $a(\varepsilon)$ and $b(\varepsilon)$ are positive C^r functions and the terms contained in $O(x^3)$ are C^r in x and ε and C^{r-1} in λ. The aim of this paper is to derive further Melnikov conditions which enable us to divide the ε, λ space in different sets in which $(\breve{x}(t, \varepsilon, \lambda), \breve{y}(t, \varepsilon, \lambda))$ has different behavior: it is homoclinic, heteroclinic or it does not converge to critical points either in the past or in the future. We stress that we have explicit formulas for the derivatives of the curves defining the border of these sets. This is the content of Theorems 3.3, 3.6 which regard respectively the case where (1.5) undergoes to a transcritical or to a saddle-node bifurcation.

We emphasize that the assumptions (1.6) and (1.7) on f are generic. In fact assume $f(0,0,0,0) = 0$, $\frac{\partial f}{\partial x}(0,0,0,0) = 0$ and $\frac{\partial^2 f}{\partial x^2}(0,0,0,0) \neq 0$. Following subsection 11.2 of [**?**], we see that when $\frac{\partial f}{\partial \lambda}(0,0,0,0) \neq 0$ we can find a new parameter $\bar{\lambda} = \bar{\lambda}(\varepsilon, \lambda)$ with C^{r-1} dependence on ε and λ and a C^r change of variables $\bar{x} = \bar{x}(x, \varepsilon, \lambda)$, so that (1.5) takes the form

$$\dot{\bar{x}} = -\bar{\lambda}(\varepsilon, \lambda) + c(\varepsilon)\bar{x}^2 + O(\bar{x}^3),$$

where $c(\varepsilon) > 0$ is C^r (possibly reversing time, i.e. passing from t to $-t$). Hence we reduce to the case where f has the form (1.7), and (1.5) undergoes to a saddle-node bifurcation. When $\frac{\partial f}{\partial \lambda}(0,0,0,0) = 0$, , e.g. if for some physical reasons the origin of the system (1.5) is forced to be a critical point of the system for any λ, generically we have a transcritical bifurcation. In such a case, up to a C^{r-1} change of parameters and a C^r change of variables we can pass from (1.5) to

$$\dot{\bar{x}} = -\bar{x}\bar{\lambda}(\varepsilon, \lambda) + c(\varepsilon)\bar{x}^2 + O(\bar{x}^3),$$

see again subsection 11.2 of [**?**]. Then passing from $\bar{x}$ to $\tilde{x} = \bar{x} - \frac{\bar{\lambda}}{2c(\varepsilon)}$, we reduce to the case where f has the form (1.6), and (1.5) undergoes to a transcritical bifurcation. We emphasize that in all the change of parameters we can and will leave unchanges the singular parameter ε.

Let $u(\varepsilon, \lambda)$, $s(\varepsilon, \lambda)$ be the zeroes of $f(x, v(x, \varepsilon, \lambda), \varepsilon, \lambda) = 0$, and denote by $U(\varepsilon, \lambda) = (u(\varepsilon, \lambda), v(u(\varepsilon, \lambda), \varepsilon, \lambda))$, $S(\varepsilon, \lambda) = (s(\varepsilon, \lambda), v(s(\varepsilon, \lambda), \varepsilon, \lambda))$ the critical points of (1.1) when they exist. When f is either of the form (1.6) or (1.7) (1.5) admits two critical points for $\lambda > 0 = \varepsilon$, i.e. $u(0, \lambda), s(0, \lambda) \in \mathbb{R}$: u is unstable while s is stable.
When f does not depend on ε, the solutions of $u(\varepsilon, \lambda) = s(\varepsilon, \lambda)$ is given by $\lambda = 0$, but when f depends on ε there is a function $q(\lambda)$ such that $u(q(\lambda), \lambda) = s(q(\lambda), \lambda)$. Hence if f is as in (1.6), the critical points $u(\varepsilon, \lambda)$ and $s(\varepsilon, \lambda)$ of (1.5) reverse their stability properties as we pass from $\varepsilon > q(\lambda)$ to $\varepsilon < q(\lambda)$, while if it is as in (1.7) then $u(\varepsilon, \lambda)$ and $s(\varepsilon, \lambda)$ are distinct for $\varepsilon > q(\lambda)$, they coincide for $\varepsilon = q(\lambda)$ and they do not exist for $\varepsilon < q(\lambda)$.
Note that q is a smooth function of λ for (1.6) while it is a smooth function of $\nu = \text{sign}(\lambda)\sqrt{\lambda}$ for (1.7).

Our purpose is to find trajectories of (1.1) which are close for any $t \in \mathbb{R}$ to the homoclinic trajectory $(0, h(t))$ of the unperturbed system. We use the implicit function theorem to construct Melnikov conditions which ensure the existence of such trajectories, and which allow to say if they are homoclinic, heteroclinic or unbounded. The techniques can be applied also to bifurcations of higher order, i.e. when the first nonzero term of the expansion of f_0 in x_0 has degree 3 or more: we discuss shortly this setting in subsection 3.3 focusing in particular on the pitchfork bifurcation. The assumptions used in the main Theorems are the following:

(v)
$$\int_{-\infty}^{\infty} \psi^*(t) \frac{\partial g}{\partial x}(0, h(t), 0, 0) dt \neq 0$$

(vi)
$$\frac{\partial u}{\partial \lambda}(0,0) \neq \frac{\int_{-\infty}^{\infty} \psi^*(t) \frac{\partial g}{\partial \lambda}(0, h(t), 0, 0) dt}{\int_{-\infty}^{\infty} \psi^*(t) \frac{\partial g}{\partial x}(0, h(t), 0, 0) dt} \neq \frac{\partial s}{\partial \lambda}(0,0)\,,$$

where the computable constant B_0 is given in (2.26).

With our techniques we may also consider more degenerate bifurcations, i.e. the first non-zero term in the expansion of $f(x, 0, 0, 0)$ has degree three or larger. However in such a case to obtain a complete unfolding of the singularity more parameters are needed. In fact we just sketch the case of pitchfork bifurcation, which however appears frequently when f is odd in x for any ε and λ for some physical reasons. Again, following subsection 11.2 of [?], we see that, up to changes in variables and parameters, we may reduce to f of the form

$$f(x, 0, \varepsilon, \lambda) = [x^2 - b(\varepsilon)\lambda^2][x - a(\varepsilon)\lambda] + O(x^4) \tag{1.8}$$ pitch

where $a(\varepsilon)$ and $b(\varepsilon)$ are C^r positive functions and the $O(x^4)$ has C^r dependence on ε and λ.

The paper is divided as follows. In section 2 we briefly review some facts, proved in [6]: we construct the solutions asymptotic to the slow manifold $\mathcal{M}^c$, then we match them via implicit function theorem, to construct a solution $(\breve{x}(t, \varepsilon, \lambda), \breve{y}(t, \varepsilon, \lambda))$ homoclinic to $\mathcal{M}^c$. In section 3 we show which is the behavior of $(\breve{x}(t, \varepsilon, \lambda), \breve{y}(t, \varepsilon, \lambda))$ as ε and λ varies, in the transcritical and in the saddle-node case (subsections 3.1 and 3.2 respectively). So we give sufficient conditions in order to have homoclinic, heteroclinic or no bounded solutions close to $(0, h(t))$, as the parameters vary: this is the content of. Theorems 3.3 and 3.6. Finally we explain how the same methods can be extended to describe pitchfork and higher degree bifurcations in subsection 3.3. We illustrate our results drawing some bifurcation diagrams. Finally in section 4 we construct examples for which we can explicitly compute the derivatives of the bifurcation curves appearing in the diagrams.

2 The centre-stable and centre-unstable manifolds

In this section we define the local centre-stable and centre-unstable manifolds and we recall their smoothness properties. These manifolds are (locally) invariant manifolds of solutions that approach the slow manifolds $y = v(x, \varepsilon, \lambda)$ at an exponential rate. In [5, 6] the following result has been proved.

T1 **2.1 Theorem.** *[6] Let f and g be bounded C^r functions, $r \geq 2$, with bounded derivatives, satisfying conditions (i)-(iv) of the Introduction and let the numbers β and σ satisfy $0 < r\sigma < \beta < \Lambda^g$. Then, given suitably small positive numbers μ_1 and μ_2, there exist positive numbers α_0, λ_0, $\varepsilon_0 (< 2\sigma/N$, where N is a bound for the derivatives of $f(x, 0, 0, 0))$, such that for $|\varepsilon| \leq \varepsilon_0$, $|\lambda| \leq \lambda_0$, $|\xi^{\pm}| \leq \alpha_0$, $\zeta^+ \in \mathcal{R}(P^+)$, $|\zeta^+| \leq \mu_1$, $\zeta^- \in \mathcal{N}(P^-)$, $|\zeta^-| \leq \mu_2$, there exists a unique solution*

$$(x^{\pm}(t), y^{\pm}(t)) = (x^{\pm}(t, \xi^{\pm}, \zeta^{\pm}, \lambda), y^{\pm}(t, \xi^{\pm}, \zeta^{\pm}, \varepsilon, \lambda))$$

of (1.1) defined respectively for $t \geq 0$ and for $t \leq 0$ such that

$$e^{|\beta t|}|x^+(t) - x_c(\varepsilon t, \xi^+, \varepsilon, \lambda)| \leq \mu_1, \quad e^{|\beta t|}|y^+(t) - v(x^+(t), \varepsilon, \lambda)| \leq \mu_1 \tag{2.1}$$ n2.5

for $t \geq 0$, and

$$e^{|\beta t|}|x^-(t) - x_c(\varepsilon t, \xi^-, \varepsilon, \lambda)| \leq \mu_2, \quad e^{|\beta t|}|y^-(t) - v(x^-(t), \varepsilon, \lambda)| \leq \mu_2 \tag{2.2}$$ n2.2

for $t \leq 0$, and

$$P^+[y^+(0) - v(x^+(0), \varepsilon, \lambda)] = \zeta^+ , \quad (\mathbf{I} - P^-)[y^-(0) - v(x^-(0), \varepsilon, \lambda)] = \zeta^- \tag{2.3}$$ n2.3

Moreover $y^{\pm}(t, \xi^{\pm}, \zeta^{\pm}, \varepsilon, \lambda) - v(x^{\pm}(t, \xi^{\pm}, \zeta^{\pm}, \varepsilon, \lambda), \varepsilon, \lambda)$ and $x^{\pm}(t, \xi^{\pm}, \zeta^{\pm}, \varepsilon, \lambda) - x_c(\varepsilon t, \xi^{\pm}, \varepsilon, \lambda)$ are C^{r-1} in the parameters $(\xi^{\pm}, \zeta^{\pm}, \varepsilon, \lambda)$ and for $k = 1, \ldots, r - 1$ their k^{th} derivatives also satisfy the estimate (2.2) with β replaced by $\beta - k\sigma$ and μ_1 and μ_2 replaced by possibly larger constants. Also there is a constant N_1 such that for $t \leq 0$

$$\begin{aligned} &e^{|\beta t|}|x^-(t, \xi^-, \zeta^-, \varepsilon, \lambda) - x_c(\varepsilon t, \xi^-, \varepsilon, \lambda)| \leq N_1 |\varepsilon| |\zeta_-|, \\ &e^{|\beta t|}|y^-(t, \xi^-, \zeta^-, \varepsilon, \lambda) - v(x^-(t, \xi^-, \zeta^-, \varepsilon, \lambda), \varepsilon, \lambda))| \leq N_1 |\zeta_-|. \end{aligned} \tag{2.4}$$ addFla1

and for $t \geq 0$

$$\begin{aligned} &e^{|\beta t|}|x^+(t, \xi^+, \zeta^+, \varepsilon, \lambda) - x_c(\varepsilon t, \xi^+, \varepsilon, \lambda)| \leq N_1 |\varepsilon| |\zeta_+|, \\ &e^{|\beta t|}|y^+(t, \xi^+, \zeta^+, \varepsilon, \lambda) - v(x^+(t, \xi^+, \zeta^+, \varepsilon, \lambda), \varepsilon, \lambda))| \leq N_1 |\zeta_+|. \end{aligned} \tag{2.5}$$ addFla2

Following section 2.1 in [6], using Theorem 2.1 we define the local centre-unstable and centre-stable manifolds near the origin in $\mathbb{R}$ as follows

$$\begin{aligned}\mathcal{M}^{cu}_{loc} &:= \{(x^-(0,\xi^-,\zeta^-,\varepsilon,\lambda), y^-(0,\xi^-,\zeta^-,\varepsilon,\lambda)) \ : \ |\zeta^-|<\mu_0,\ |\xi^-|<\alpha_0\},\\ \mathcal{M}^{cs}_{loc} &:= \{(x^+(0,\xi^+,\zeta^+,\varepsilon,\lambda), y^+(0,\xi^+,\zeta^+,\varepsilon,\lambda)) \ : \ |\zeta^+|<\mu_0,\ |\xi^+|<\alpha_0\}.\end{aligned}$$

In [6] it has been proved that $\mathcal{M}^{cu}_{loc}$ and $\mathcal{M}^{cs}_{loc}$ are respectively negatively and positively invariant for (1.1). Thus, going respectively forward and backward in t, we can construct from $\mathcal{M}^{cu}_{loc}$ and $\mathcal{M}^{cs}_{loc}$ the global manifold $\mathcal{M}^{cu}$ and $\mathcal{M}^{cs}$, see Lemma 2.3 in section 2.2 in [6]. Therefore $\mathcal{M}^{cu}$ and $\mathcal{M}^{cs}$ are respectively $p1$ and $n-p+1$ dimensional immersed manifolds of $\mathbb{R}^{n+1}$, made up by the trajectories asymptotic to $\mathcal{M}^c$ resp. in the past and in the future.

Following the discussion after Theorems 2.1 and 2.2 in [6], we see that the k^{th} derivatives of $x^+(t,\xi,\zeta^+,\varepsilon,\lambda)$ and of $x^-(t,\xi,\zeta^-,\varepsilon,\lambda)$ with respect to $(\xi,\zeta^\pm,\varepsilon,\lambda)$ are bounded above in absolute value by $C_k e^{(k+1)\sigma|t|}$ for $t\in\mathbb{R}$ respectively, where C_k is a constant and $\sigma > N\varepsilon_0$ is a positive number that satisfies $0<r\sigma<\beta<\Lambda^g$. Finally, because of uniqueness of $(x^\pm(t,\xi^\pm,\zeta^\pm,\varepsilon,\lambda), y^\pm(t,\xi,\zeta^\pm,\varepsilon,\lambda))$, we see that the following properties hold:

$$\begin{aligned}x^\pm(t,\xi^\pm,v(\xi^\pm,\varepsilon,\lambda),\varepsilon,\lambda) &= x_c(\varepsilon t,\xi^\pm,\varepsilon,\lambda),\\ y^\pm(t,\xi^\pm,v(\xi^\pm,\varepsilon,\lambda),\varepsilon,\lambda) &= v(x_c(\varepsilon t,\xi^\pm,\varepsilon,\lambda),\varepsilon,\lambda)\end{aligned} \tag{2.6}$$ n2.7

and

$$x^\pm(t,\xi^\pm,\zeta^\pm,0,\lambda)=\xi^\pm \tag{2.7}$$ n2.8

see [6]. Since $x_c(0,\xi,\varepsilon,\lambda)=\xi$, we see that the slow manifold $\mathcal{M}^c$ defined by $y=v(\xi,\varepsilon,\lambda)$ is contained in the intersection between $\mathcal{M}^{cu}$ and $\mathcal{M}^{cs}$.

Exploiting section 2.3 in [6] we can define a foliation of $\mathcal{M}^{cu}_{loc}$ and $\mathcal{M}^{cs}_{loc}$ as follows. Let $|\xi|$ be sufficiently small, we set

$$\begin{aligned}\mathcal{M}^{cu}(\xi) &:= \{(x^-(t,\xi,\zeta^-,\varepsilon,\lambda), y^-(t,\xi,\zeta^-,\varepsilon,\lambda)) \mid |\zeta^-|<\mu_0,\ \zeta^-\in\mathcal{N}P^-, t\in\mathbb{R}\}\\ \mathcal{M}^{cs}(\xi) &:= \{(x^+(t,\xi,\zeta^+,\varepsilon,\lambda), y^+(t,\xi,\zeta^+,\varepsilon,\lambda)) \mid |\zeta^+|<\mu_0,\ \zeta^+\in\mathcal{R}P^+, t\in\mathbb{R}\}.\end{aligned}$$

Using the flow of (1.1) we can remove the smallness assumption on ξ (but we get $\mu_1=\mu_1(|\xi|)$, $\mu_2=\mu_2(|\xi|)$, $N_1=N_1(|\xi|)$ in the estimates (2.2), (2.1), (2.4), (2.5)).

From section 2.3 in [6] we see that that $\mathcal{M}^{cu}(\xi)$ and $\mathcal{M}^{cs}(\xi)$ are p and $n-p$ manifolds for any $\xi\in\mathbb{R}$, and that $\mathcal{M}^{cu}=\cup_{\xi\in\mathbb{R}}\mathcal{M}^{cu}(\xi)$, $\mathcal{M}^{cs}=\cup_{\xi\in\mathbb{R}}\mathcal{M}^{cs}(\xi)$, are the global centre-unstable and centre-stable manifolds defined above. Moreover given $\bar\xi,\tilde\xi\in\mathbb{R}$ then either $\mathcal{M}^{cu}(\bar\xi)$ and $\mathcal{M}^{cu}(\tilde\xi)$ coincide or they do not intersect; similarly either $\mathcal{M}^{cs}(\bar\xi)$ and $\mathcal{M}^{cs}(\tilde\xi)$ coincide or they do not intersect: thus $\mathcal{M}^{cu}(\xi)$ and $\mathcal{M}^{cs}(\xi)$ define indeed foliations for $\mathcal{M}^{cu}$ and $\mathcal{M}^{cs}$.

We denote by $B(\xi,\rho)$ the ball with centre $\xi \in \mathbb{R}$ and radius $\rho > 0$. Let $A \subset \mathbb{R}$ be a set, we define $\mathrm{dist}(\xi, A) = \inf\{|\xi - \eta| \mid \eta \in A\}$. We borrow from [6] a theorem which ensures the existence of a solution of (1.1) homoclinic to $\mathcal{M}^c$.

vecchio **2.2 Theorem.** *[6] Let f and g be bounded C^r functions, $r \geq 2$, with bounded derivatives, satisfying conditions (i)–(v) of the Introduction. Then there exist positive numbers λ_0, ε_0 such that for any $|\varepsilon| < \varepsilon_0$, $|\lambda| < \lambda_0$ there is a solution $(\tilde{x}(t,\xi_{\hat{0}},\varepsilon,\lambda), \tilde{y}(t,\xi_{\hat{0}},\varepsilon,\lambda)) \in (\mathcal{M}^{cs} \cap \mathcal{M}^{cu}) \setminus \mathcal{M}^c$ satisfying*

$$\lim_{|t|\to\infty} dist\big((\tilde{x}(\varepsilon,\lambda,t), \tilde{y}(\varepsilon,\lambda,t)), \mathcal{M}^c\big) = 0$$

Moreover there is a neighborhood Ω^0 of $(0, h(0))$ such that, if $(x(t), y(t)) \in (\mathcal{M}^{cs} \cap \mathcal{M}^{cu})$ and $(x(0), y(0)) \in \Omega^0$, then $(x(t), y(t)) \equiv (\tilde{x}(t,\varepsilon,\lambda), \tilde{y}(t,\varepsilon,\lambda))$, so local uniqueness is ensured.

We sketch the proof since some details will be useful later on. To prove theorem 2.2 Battelli and Palmer in [6] look for a bifurcation function whose zeroes correspond to solutions of the system

$$\begin{aligned} y^+(-T,\xi^+,\zeta^+,\varepsilon,\lambda) &= y^-(T,\xi^-,\zeta^-,\varepsilon,\lambda) && (2.8)\\ x^+(-T,\xi^+,\zeta^+,\varepsilon,\lambda) &= x^-(T,\xi^-,\zeta^-,\varepsilon,\lambda) && (2.9) \end{aligned}$$

where $T > 0$, and $|\xi^\pm| < \rho_0$. They begin from (2.8), so they rewrite it as

$$\begin{cases} x^+(-T,\xi^+,\zeta^+,\varepsilon,\lambda) - \xi = 0 \\ x^-(T,\xi^-,\zeta^-,\varepsilon,\lambda) - \xi = 0 \end{cases} \tag{2.10}$$

n3.2

Since $x^+(-T,\xi^+,\zeta^+,0,\lambda) = \xi^+$ and $x^-(T,\xi^-,\zeta^-,0,\lambda) = \xi^-$, using the implicit function theorem they find unique C^r functions $\bar{\xi}^\pm = \bar{\xi}^\pm(\xi,\zeta^\pm,\varepsilon,\lambda)$, such that $\bar{\xi}^\pm(\xi,\zeta^\pm,0,\lambda) = \xi$ and

$$x^+(-T,\bar{\xi}^+(\xi,\zeta^+,\varepsilon,\lambda),\zeta^+,\varepsilon,\lambda) = \xi = x^-(T,\bar{\xi}^-(\xi,\zeta^-,\varepsilon,\lambda),\zeta^-,\varepsilon,\lambda). \tag{2.11}$$

n3.2a

for any $(\xi,\zeta^+,\zeta^-,\varepsilon)$ such that $|\zeta^\pm| \leq \mu_0$, $|\xi| \leq \rho_0$, $|\varepsilon| \leq \varepsilon_0$ and any λ. Set

$$\begin{aligned} \bar{x}^\pm(t,\xi,\zeta^\pm,\varepsilon,\lambda) &= x^\pm(t,\bar{\xi}^\pm(\xi,\zeta^\pm,\varepsilon,\lambda),\zeta^\pm,\varepsilon,\lambda)\\ \bar{y}^\pm(t,\xi,\zeta^\pm,\varepsilon,\lambda) &= y^\pm(t,\bar{\xi}^\pm(\xi,\zeta^\pm,\varepsilon,\lambda),\zeta^\pm,\varepsilon,\lambda) \end{aligned} \tag{2.12}$$

n3.2.5

and note that $\frac{\partial\bar{\xi}^\pm}{\partial\xi}(\xi,\zeta^\pm,0,\lambda) = \mathbf{I}$, while $\frac{\partial\bar{\xi}^\pm}{\partial\zeta^\pm}(\xi,\zeta^\pm,0,\lambda) = 0$, $\frac{\partial\bar{\xi}^\pm}{\partial\lambda}(\xi,\zeta^\pm,0,\lambda) = 0$. Following [6] we see that

$$\begin{aligned} \frac{\partial\bar{\xi}^\pm}{\partial\varepsilon}(\xi,\zeta^\pm,\varepsilon,\lambda)\lfloor_{(0,0,0,0)} &= \int_0^{\pm\infty} f(0,h(s),0,0)ds\\ \frac{\partial\bar{x}^\pm}{\partial\varepsilon}(t\mp T,0,\zeta^\pm,0,0) &= \int_0^t f(0,h(s),0,0)ds \end{aligned} \tag{2.13}$$

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Observe further that

$$\dot{\bar{y}}^{\pm}(t,\xi,\zeta^{\pm},\varepsilon,\lambda) = g(\bar{x}^{\pm}(t,\xi,\zeta^{\pm},\varepsilon,\lambda),\bar{y}^{\pm}(t,\xi,\zeta^{\pm},\varepsilon,\lambda),\varepsilon,\lambda) \qquad (2.14) \quad \boxed{\texttt{basey}}$$

Following section 3.1 in [6], by the uniqueness in Theorem 2.1 we see that

$$\bar{y}^{+}(t,0,\zeta_h^{+},0,0) = h(t+T), \qquad \bar{y}^{-}(t,0,\zeta_h^{-},0,0) = h(t-T)$$

where

$$\zeta_h^{+} = P^{+}h(T) \quad \text{and} \quad \zeta_h^{-} = (\mathbf{I}-P^{-})h(-T)\,.$$

Since $(\bar{x}^{\pm},\bar{y}^{\pm})(t,0,\zeta_h^{\pm},0,0) = (0,h(t\pm T))$ it follows that

$$\dot{\bar{y}}^{\pm}(t,0,\zeta_h^{\pm},0,0) = \dot{y}_h(t\pm T) = g(t\pm T,0,h(t\pm T),0,0)\,. \qquad (2.15) \quad \boxed{\texttt{newstar}}$$

Using (2.15) we can differentiate (2.14) to evaluate the derivatives of $\bar{y}^{\pm}$ with respect to all the variables. Recalling that $\frac{\partial \bar{x}^{\pm}}{\partial \xi}(\mp T,\xi,\zeta^{\pm},\varepsilon,\lambda) = \mathbf{I}$ we find

$$\psi^{*}\frac{\partial \bar{y}^{\pm}}{\partial \xi_j}(\mp T,0,\zeta_h^{\pm},0,0) = \int_{\pm\infty}^{0}\psi^{*}(t)\frac{\partial g}{\partial x_j}(0,h(t),0,0)\mathrm{d}t \qquad (2.16) \quad \boxed{\texttt{Hxi-}}$$

see [6] for details. Let us set $\frac{\partial \bar{y}^{+}}{\partial \varepsilon}(t,0,\zeta_h^{+},0,0) = Y(t)$, and observe that $Y(t)$ solves

$$\begin{cases} \dot{Y}(t) = \frac{\partial g}{\partial y}(0,h(t+T),0,0)Y(t) + h^{+}(t+T) \\ P^{+}(Y(0)) = P^{+}\frac{\partial v}{\partial \varepsilon}(0,0,0) \end{cases} \qquad (2.17) \quad \boxed{\texttt{tre}}$$

where $h^{+}(t) := \frac{\partial g}{\partial x}(0,h(t),0,0)\frac{\partial \bar{x}^{+}}{\partial \varepsilon}(t-T,0,\zeta_h^{+},0,0) + \frac{\partial g}{\partial \varepsilon}(0,h(t),0,0)$, and we have used (2.3). Using the variation of constants formula and the fact that $\psi \in (\mathcal{R}P^{+}\cap NP^{-})^{\perp}$, and repeating the argument for $\bar{y}^{-}$, we find the following:

$$\psi^{*}\frac{\partial \bar{y}^{\pm}}{\partial \varepsilon}(\mp T,0,\zeta_h^{\pm},0,0) = \int_{\pm\infty}^{0}\psi^{*}(s)h^{\pm}(s)ds \qquad (2.18) \quad \boxed{\texttt{quattroa}}$$

where $h^{\pm}(t) := \frac{\partial g}{\partial x}(0,h(t),0,0)\frac{\partial \bar{x}^{\pm}}{\partial \varepsilon}(t\mp T,0,\zeta_h^{\pm},0,0) + \frac{\partial g}{\partial \varepsilon}(0,h(t),0,0)$, see also [6]. Let us denote by

$$K(\xi,\zeta^{+},\zeta^{-},\varepsilon,\lambda) = \bar{y}^{+}(-T,\xi,\zeta^{+},\varepsilon,\lambda) - \bar{y}^{-}(T,\xi,\zeta^{-},\varepsilon,\lambda)\;, \qquad (2.19) \quad \boxed{\texttt{defH}}$$

It is easy to see that $K(0,\zeta_h^{+},\zeta_h^{-},0,0) = 0$. Following pages 449-450 in [6] we evaluate the derivatives with respect to (ζ^{+},ζ^{-}) to apply the implicit function theorem. Note that $Z^{\pm}(t) = \frac{\partial \bar{y}^{\pm}}{\partial \zeta^{\pm}}(t,0,\zeta_h^{\pm},0,0)$ solve

$$\dot{Z}^{\pm}(t) = \frac{\partial g}{\partial y}(0,h(t\pm T),0,0)Z^{\pm}(t). \qquad (2.20) \quad \boxed{\texttt{n3.9}}$$

and are such that $e^{(\beta-\sigma)t}|Z^+(t)|$ and $e^{(\beta-\sigma)|t|}|Z^-(t)|$ are bounded respectively for $t \geq 0$ and for $t \leq 0$; moreover $Z^+(0)\zeta^+ = \zeta^+$ and $Z^-(0)\zeta^- = \zeta^-$ whenever $\zeta^+ \in \mathcal{R}P^+$ and $\zeta^- \in \mathcal{N}P^-$. We stress that (2.20) is a $\pm T$-translate of (1.2), so it admits exponential dichotomy. We introduce the continuous family of projections $P^\pm(t) = Y(t)P^\pm Y^{-1}(t)$ for (1.2). Observe that $\lim_{t\to\pm\infty} P^\pm(t) = P^0$ and that $\mathcal{R}P^+(T)$ and $\mathcal{N}P^-(-T)$ are assigned but we can and will choose $\mathcal{N}P^+(T) = \mathcal{N}P^+$ and $\mathcal{R}P^-(-T) = \mathcal{R}P^-$, see [4] and [10]. Then

$$\begin{aligned}
\frac{\partial \bar{y}^+}{\partial \zeta^+}(t,0,\zeta_h^+,0,0) =& Y(t+T)Y^{-1}(T)P^+(T) = Y(t+T)P^+Y^{-1}(T) \\
\frac{\partial \bar{y}^-}{\partial \zeta^-}(t,0,\zeta_h^-,0,0) =& Y(t-T)(\mathbf{I}-P^-)Y^{-1}(-T)\,.
\end{aligned}$$

Therefore

$$\frac{\partial K}{\partial \zeta^+}(0,\zeta_h^-,\zeta_h^+,0,0) = P^+Y^{-1}(T)\,, \qquad \frac{\partial K}{\partial \zeta^-}(0,\zeta_h^-,\zeta_h^+,0,0) = -(\mathbf{I}-P^-)Y^{-1}(-T) \tag{2.21}$$

n3.11c

If $(v_+,v_-) \in (\mathcal{R}P^+ \times \mathcal{N}P^-)$ is in the kernel of $\frac{\partial K}{\partial(\zeta^+,\zeta^-)}(0,\zeta_h^-,\zeta_h^+,0,0)$, then

$$P^+Y^{-1}(T)v^+ = (\mathbf{I}-P^-)Y^{-1}(-T)v^- = w \in (\mathcal{R}P^+ \cap \mathcal{N}P^-)\,.$$

From (iv) we find $w = c\dot{y}_h(0)$ for a certain $c \in \mathbb{R}$. It follows that $P^+(T)v^+ = c\dot{y}_h(T)$, $(\mathbf{I}-P^-(-T))v^- = c\dot{y}_h(-T)$; so $v^+ = cP^+\dot{y}_h(T)$ and similarly $v^- = c(\mathbf{I}-P^-)\dot{y}_h(-T)$. Thus the kernel of $\frac{\partial K}{\partial(\zeta^+,\zeta^-)}(0,\zeta_h^-,\zeta_h^+,0,0)$ is the one dimensional space spanned by $(P^+\dot{y}_h(T),(\mathbf{I}-P^-)\dot{y}_h(-T))$, see page 450 of [6] for more details. The range of $\frac{\partial H}{\partial(\zeta^+,\zeta^-)}(0,\zeta_h^+,\zeta_h^-,0,0)$ is a subspace of codimension one in $\mathbb{R}^n$ and it is contained in $\mathcal{R}P^+ + \mathcal{N}P^- = \psi^\perp$, so it coincides with the whole $\psi^\perp$. We apply the Lijapunov–Schmidt reduction and consider, instead of (2.8) the system:

$$\begin{aligned}
\tilde{G}(\xi,\zeta^+,\zeta^-,\varepsilon,\lambda) :=& K(\xi,\zeta^+,\zeta^-,\varepsilon,\lambda) - \left[\psi^* K(\xi,\zeta^+,\zeta^-,\varepsilon,\lambda)\right]\psi = 0 \\
& \psi^* K(\xi,\zeta^+,\zeta^-,\varepsilon,\lambda) = 0
\end{aligned} \tag{2.22}$$

n3.12

together with the anchor condition

$$q(\xi,\zeta^+,\varepsilon,\lambda) := [\bar{y}^+(-T,\xi,\zeta^+,\varepsilon,\lambda) - h(0)]^*\dot{y}_h(0) = 0\,,$$

which has to be added to ensure uniqueness.

Following [6] page 450 we see that $\frac{\partial(\tilde{G},q)}{\partial(\zeta^+,\zeta^-)}(0,\zeta_h^-,\zeta_h^+,0,0)$ is invertible. So, from the Implicit Function Theorem we find unique C^r-functions $\check{\zeta}^\pm = \check{\zeta}^\pm(\xi,\varepsilon,\lambda)$ for $|\xi|$, $|\lambda|$ and $|\varepsilon|$ sufficiently small which solve $\tilde{G}(\xi,\check{\zeta}^+(\xi,\varepsilon,\lambda),$

$\check{\zeta}^-(\xi,\varepsilon,\lambda),\varepsilon,\lambda)=0$ and $q(\xi,\check{\zeta}^+(\xi,\varepsilon,\lambda),\varepsilon,\lambda)=0$. Because of uniqueness we get $\check{\zeta}^{\pm}(0,0,0)=\zeta_h^{\pm}$. Hence we are left with solving the bifurcation equation:

$$G(\xi,\varepsilon,\lambda)=\psi^*[\check{y}^+(-T,\xi,\varepsilon,\lambda)-\check{y}^-(T,\xi,\varepsilon,\lambda)]=0 \tag{2.23}$$ n3.14

where we have set $\check{y}^{\pm}(t,\xi,\varepsilon,\lambda):=\bar{y}^{\pm}(t,\xi,\check{\zeta}^{\pm}(\xi,\varepsilon,\lambda),\varepsilon,\lambda)$; so $G(\xi,\varepsilon,\lambda)$ is the bifurcation function we looked for. Since $\psi\in(\mathcal{R}P^+\cap\mathcal{N}P^-)^{\perp}$, from (2.21) it follows that $\psi^*\frac{\partial\bar{y}^{\pm}}{\partial\zeta^{\pm}}(\mp T,\xi,\zeta_h^{\pm},0,0)=0$, so

$$\begin{gathered}\frac{\partial G}{\partial\xi}(0,0,0)=\psi^*\frac{\partial K}{\partial\xi}(0,\zeta_h^+,\zeta_h^-,0,0)\,,\quad \frac{\partial G}{\partial\lambda}(0,0,0)=\psi^*\frac{\partial K}{\partial\lambda}(0,\zeta_h^+,\zeta_h^-,0,0)\,,\\ \frac{\partial G}{\partial\varepsilon}(0,0,0)=\psi^*\frac{\partial K}{\partial\varepsilon}(0,\zeta_h^+,\zeta_h^-,0,0)\,.\end{gathered} \tag{2.24}$$ attento

From (2.8) we see that $G(0,0,0)=0$; we need to evaluate all the derivatives of G. From (2.16), (2.24), (2.19) and (v) we see that

$$\frac{\partial}{\partial\xi}G(0,0,0)=-\int_{-\infty}^{+\infty}\psi^*(t)\frac{\partial g}{\partial x}(0,h(t),0,0)dt\neq 0\,.$$

Reasoning in the same way we see that

$$\frac{\partial}{\partial\lambda}G(0,0,0)=-\int_{-\infty}^{+\infty}\psi^*(t)\frac{\partial g}{\partial\lambda}(0,h(t),0,0)\mathrm{d}t. \tag{2.25}$$ cizero

Therefore, from the Implicit Function Theorem we obtain the following

$$\frac{\partial}{\partial\lambda}\tilde{\xi}(\varepsilon,\lambda)\lfloor_{(0,0)}=B_0:=-\frac{\int_{-\infty}^{\infty}\psi^*(t)\frac{\partial g}{\partial\lambda}(0,h(t),0,0)dt}{\int_{-\infty}^{\infty}\psi^*(t)\frac{\partial g}{\partial x}(0,h(t),0,0)dt} \tag{2.26}$$ sei

Similarly, using (2.18), and (2.24) we find

$$\frac{\partial G}{\partial\varepsilon}(0,0,0)=-\int_{-\infty}^{\infty}\psi^*(s)\left[\frac{\partial g}{\partial\varepsilon}(s)+\frac{\partial g}{\partial x}(s)\left(\int_0^s f(t)dt\right)\right]ds \tag{2.27}$$ newderep

where $g(s)$ stands for $g(0,h(s),0,0)$, $f(s)$ for $f(0,h(s),0,0)$ and similarly for their derivatives; thus

$$\begin{gathered}\frac{\partial\tilde{\xi}}{\partial\varepsilon}(0,0)=-\frac{\frac{\partial G}{\partial\varepsilon}}{\frac{\partial G}{\partial\xi}}=\mathfrak{A}\,,\qquad\text{where}\\ \mathfrak{A}:=-\frac{\int_{-\infty}^{\infty}\psi^*(s)\frac{\partial g}{\partial\varepsilon}(s)ds+\int_{-\infty}^{\infty}(\psi^*(s)\frac{\partial g}{\partial x}(s)\int_0^s f(t)dt)ds}{\int_{-\infty}^{\infty}\psi^*(s)\frac{\partial g}{\partial x}(s)ds}\,.\end{gathered} \tag{2.28}$$ quattroc

So, if (v) holds, for any $(\xi_{\hat{0}}, \varepsilon, \lambda) \in \mathbb{R}^{m+2}$ small enough, there is a unique solution of (1.1) which is homoclinic to the slow manifold $\mathcal{M}^c$, i.e.:

$$\begin{aligned} \tilde{x}(t, \xi_{\hat{0}}, \varepsilon, \lambda) &= \begin{cases} \breve{x}^{+}(t - T, \tilde{\xi}_0(\xi_{\hat{0}}, \varepsilon, \lambda), \xi_{\hat{0}}, \varepsilon, \lambda) & t \geq 0, \\ \breve{x}^{-}(t + T, \tilde{\xi}_0(\xi_{\hat{0}}, \varepsilon, \lambda), \xi_{\hat{0}}, \varepsilon, \lambda) & t \leq 0. \end{cases} \\ \tilde{y}(t, \xi_{\hat{0}}, \varepsilon, \lambda) &= \begin{cases} \breve{y}^{+}(t - T, \tilde{\xi}_0(\xi_{\hat{0}}, \varepsilon, \lambda), \xi_{\hat{0}}, \varepsilon, \lambda) & t \geq 0, \\ \breve{y}^{-}(t + T, \tilde{\xi}_0(\xi_{\hat{0}}, \varepsilon, \lambda), \xi_{\hat{0}}, \varepsilon, \lambda) & t \leq 0. \end{cases} \end{aligned} \tag{2.29}$$ omo

This concludes the proof of Theorem 2.2. We stress that in [6] the authors just require $\frac{\partial G}{\partial \xi}(0,0,0) \neq 0$ (i.e. $\frac{\partial G}{\partial \xi_j}(0,0,0) \neq 0$ for a certain $j = 0, 1, \ldots, m$) and use such a condition and the implicit function theorem to construct the solution defined in (2.29). Our request is slightly more restrictive: we need $\frac{\partial G}{\partial \xi_0}(0,0,0) \neq 0$ (i.e. we ask the j-coordinate to be the 0 one).

3 Existence of Homoclinic and Heteroclinic solutions.

In this section we state and prove our main results. Let Ω_h be a neighborhood of the graph of the unperturbed homoclinic, i.e. $\Omega_h \supset \{(0, h(t)) \mid t \in \mathbb{R}\}$; we stress that, if Ω_h is small enough, each solution $(x(t), y(t))$ of (1.1) which is contained in Ω_h, is in fact contained in $\mathcal{M}^{cu} \cap \mathcal{M}^{cs}$ or it is in $\mathcal{M}^c$. The latter case is trivial, i.e. $(x(t), y(t))$ coincides with one of the critical points $U(\varepsilon, \lambda) = (u(\varepsilon, \lambda), v(u(\varepsilon, \lambda), \varepsilon, \lambda))$, $S(\varepsilon, \lambda) = (s(\varepsilon, \lambda), v(s(\varepsilon, \lambda), \varepsilon, \lambda))$, or it is a heteroclinic connection between them lying on the slow manifold. So we are interested in the former case, for which we have $\sup_{t \in \mathbb{R}} \|(x(t, \varepsilon, \lambda), y(t, \varepsilon, \lambda) - h(t))\| = O(|\lambda| + |\varepsilon|)$.

unidim **3.1 Theorem.** *Let f and g be $\mathcal{C}^r$ functions, $r \geq 2$, bounded with their derivatives, satisfying conditions (i)–(v) of the Introduction. Then there are $\varepsilon_0 > 0$ and $\lambda_0 > 0$ such that for any $0 < |\varepsilon| < \varepsilon_0$ and $|\lambda| < \lambda_0$ there exists a trajectory $(\breve{x}(t, \varepsilon, \lambda), \breve{y}(t, \varepsilon, \lambda))$ homoclinic to $\mathcal{M}^c$.*

We look for a trajectory contained in $(\mathcal{M}^{cu}(C(\varepsilon, \lambda)) \cap \mathcal{M}^{cs}(C(\varepsilon, \lambda))$. Since $(\breve{x}(t, \varepsilon, \lambda), \breve{y}(t, \varepsilon, \lambda))$ is constructed via implicit function theorem we have local uniqueness, see the explanation just after theorem 2.2. Our purpose is to divide the parameters space in different subsets in which the solution $(\breve{x}(t, \varepsilon, \lambda), \breve{y}(t, \varepsilon, \lambda))$ constructed via Theorem 3.1 has a different asymptotic behavior: for this purpose we need to evaluate all the derivatives of $\tilde{\xi}_0^{\pm}$.

...

We observe that, if $v(x, \varepsilon, \lambda) \equiv 0$, (**??**) holds and f does not depend on ε, the stability properties of the critical points change when the parameters cross the coordinate axes ε, λ. E. g. in the **(s.n.)** case u and s do not exist for $\lambda < 0$, they coincide for $\lambda = 0$, and they split for $\lambda > 0$. In the general case, when (**??**) holds, there is a smooth function $\lambda = q(\varepsilon)$, defined for ε small, such that $u_0(\varepsilon, \lambda) = s_0(\varepsilon, \lambda)$ if and only if $\lambda = q(\varepsilon)$. Then it follows that $U(\varepsilon, q(\varepsilon)) = S(\varepsilon, q(\varepsilon))$. We can assume w.l.o.g. that $u_0(\varepsilon, \lambda) > s_0(\varepsilon, \lambda)$ for $\lambda > q(\varepsilon)$; when $\lambda < q(\varepsilon)$ if **(tr.)** holds we have $u_0(\varepsilon, \lambda) < s_0(\varepsilon, \lambda)$, while when **(s.n.)** holds there are no critical points of (1.5) in a neighborhood of the origin. So the critical points $U(\varepsilon, \lambda)$ and $S(\varepsilon, \lambda)$ change their stability properties when the parameters cross either the line $\varepsilon = 0$ or $\lambda = q(\varepsilon)$. Thus we need to argue separately in the 4 different quadrants in which these lines divide a neighborhood of the origin. Note that

$$q'(0) = -\frac{\frac{\partial u_0}{\partial \varepsilon}(0,0) - \frac{\partial s_0}{\partial \varepsilon}(0,0)}{\frac{\partial u_0}{\partial \lambda}(0,0) - \frac{\partial s_0}{\partial \lambda}(0,0)}\,. \tag{3.1}$$

derivoq

In fact we could reduce the general case to the case where $q(\varepsilon) = 0$, simply by making a change of parameters from (ε, λ) to $(\varepsilon, \bar{\lambda})$, where $\bar{\lambda} = \lambda - q(\varepsilon)$. However the expression of $q(\varepsilon)$ is a priori unknown, while we can compute explicitly the derivative $q'(\varepsilon)$.
At this point we need to distinguish between f satisfying **(tr.)** and **(s.n.)**: we start from the former.

3.1 Transcritical bifurcation.

We argue separately in each quadrant, so we start from $\varepsilon > 0$ and $\lambda \geq q(\varepsilon)$. The key point to understand the behavior in the future is to establish the mutual positions of $\breve{\xi}^+(\varepsilon, \lambda)$ and $u(\varepsilon, \lambda)$, while to understand the behavior in the past we need to know the positions of $\breve{\xi}^-(\varepsilon, \lambda)$ with respect to $s(\varepsilon, \lambda)$. So we define the functions $J_1^{\pm} : [-\varepsilon_0, \varepsilon_0] \times [-\lambda_0, \lambda_0] \to \mathbb{R}$ as follows

$$\begin{aligned} J_1^+(\varepsilon, \lambda) &= \breve{\xi}^+(\varepsilon, \lambda) - u)\varepsilon, \lambda)\,, \\ J_1^-(\varepsilon, \lambda) &= \breve{\xi}^-(\varepsilon, \lambda) - s(\varepsilon, \lambda) \end{aligned} \tag{3.2}$$

acca2

We want to construct via implicit function theorem two curves, $\lambda_1^+(\varepsilon)$ and $\lambda_1^-(\varepsilon)$, satisfying $\lambda_1^{\pm}(0) = 0$, and such that $J_1^{\pm}(\varepsilon, \lambda_1^{\pm}(\varepsilon)) = 0$. Then $(\breve{x}(t, \varepsilon, \lambda_1^+(\varepsilon)),$ $\breve{y}(t, \varepsilon, \lambda_1^+(\varepsilon)))$ converges to $U(\varepsilon, \lambda_1^+(\varepsilon))$ as $t \to +\infty$, while $(\breve{x}(t, \varepsilon, \lambda_1^-(\varepsilon)), \breve{y}(t, \varepsilon, \lambda_1^-(\varepsilon)))$ converges to $S(\varepsilon, \lambda_1^-(\varepsilon))$ as $t \to -\infty$. Since

$$\frac{\partial \breve{\xi}^{\pm}}{\partial \lambda}(0,0) = \frac{\partial \tilde{\xi}_0^{\pm}}{\partial \xi_{\hat{0}}}(0,0,0)\frac{\partial \breve{\xi}_{\hat{0}}}{\partial \lambda}(0,0) + \frac{\partial \tilde{\xi}_0^{\pm}}{\partial \lambda}(0,0,0)\,,$$

using (**??**), (2.26) and (**??**) we find $\frac{\partial \breve\xi^\pm}{\partial\lambda}(0,0) = B_0$ where B_0 is defined in (2.26). Hence in particular $\frac{\partial \breve\xi^-}{\partial\lambda}(0,0) = \frac{\partial \breve\xi^+}{\partial\lambda}(0,0)$. Similarly we find [1]

$$\frac{\partial \breve\xi^\pm}{\partial\varepsilon}(0,0) = \frac{\partial \tilde\xi_0^\pm}{\partial\varepsilon}(0,0,0)\,.$$

Hence using (**??**), (**??**), (**??**) we find $\frac{\partial \breve\xi^\pm}{\partial\varepsilon}(0,0) = A_0^\pm$. Using (**??**) and (**??**) we find

$$\begin{array}{l} \frac{\partial J_1^+}{\partial\lambda}(0,0) = \frac{\partial \breve\xi^+}{\partial\lambda}(0,0) - \frac{\partial h_b}{\partial\lambda}(0,0,0) = B_0 - \frac{\partial u_0}{\partial\lambda}(0,0) \\ \frac{\partial J_1^-}{\partial\lambda}(0,0) = \frac{\partial \breve\xi^-}{\partial\lambda}(0,0) - \frac{\partial h_a}{\partial\lambda}(0,0,0) = B_0 - \frac{\partial s_0}{\partial\lambda}(0,0) \end{array}\;; \qquad (3.3)$$ derivo1

so, if (vi) holds we can apply the implicit function theorem and construct the curves $\lambda_1^\pm(\varepsilon)$ (defined for $0 \le \varepsilon \le \varepsilon_0$) such that $J_1^\pm(\varepsilon, \lambda_1^\pm(\varepsilon)) = 0$. Moreover

$$\begin{aligned} \frac{d}{d\varepsilon}\lambda_1^+(0) &= -\frac{\frac{\partial}{\partial\varepsilon}\breve\xi^+(0,0) - \frac{\partial u}{\partial\varepsilon}(0,0)}{\frac{\partial}{\partial\lambda}\breve\xi^+(0,0) - \frac{\partial u_0}{\partial\lambda}(0,0)} = -\frac{A_0^+\frac{\partial u}{\partial\varepsilon}(0,0)}{B_0 - \frac{\partial u}{\partial\lambda}(0,0)} \\ \frac{d}{d\varepsilon}\lambda_1^-(0) &= -\frac{A_0^- - \frac{\partial s}{\partial\varepsilon}(0,0)}{B_0 - \frac{\partial s}{\partial\lambda}(0,0)} \end{aligned} \qquad (3.4)$$ diff6

nb *3.2 Remark.* The curves $\lambda_1^+(\varepsilon)$ and $\lambda_1^-(\varepsilon)$ may not intersect the open set $Q_1 = \{(\varepsilon,\lambda) \mid \lambda - q(\varepsilon) > 0 \text{ and } \varepsilon > 0\}$. If this is the case for any $(\varepsilon,\lambda) \in Q_1$ the trajectory $(\breve x(t,\varepsilon,\lambda), \breve y(t,\varepsilon,\lambda))$ does not converge respectively to U in the future neither to S in the past.

Let Ω be a sufficiently small neighborhood of the origin in $\mathbb{R}$, independent of ε and λ, and denote by

$$\begin{array}{ll} \mathfrak{A}^+ = \{x < u(\varepsilon,\lambda) \mid x \in \Omega\} & \mathfrak{B}^+ = \{x > u(\varepsilon,\lambda) \mid x \in \Omega\} \\ \mathfrak{A}^- = \{x < s(\varepsilon,\lambda) \mid x \in \Omega\} & \mathfrak{B}^- = \{x > s(\varepsilon,\lambda) \mid x \in \Omega\} \end{array}$$

By construction $s(\varepsilon,\lambda) \in A^+$ and $u(\varepsilon,\lambda) \in B^-$; hence if $\breve\xi^+(\varepsilon,\lambda) \in \mathfrak{A}^+$ then the trajectory $x_c(t, \breve\xi^+(\varepsilon,\lambda), \varepsilon, \lambda)$ of (1.5) converges to $s(\varepsilon,\lambda)$, while if $\breve\xi^+(\varepsilon,\lambda) \in \mathfrak{B}^+$ there is $T > 0$ such that $x_c(T, \breve\xi^+(\varepsilon,\lambda),\varepsilon,\lambda) \notin \Omega_x$. So, if $\breve\xi^+(\varepsilon,\lambda) \in \mathfrak{A}^+$ then $(\breve x(t,\varepsilon,\lambda), \breve y(t,\varepsilon,\lambda)) \to S(\varepsilon,\lambda)$ as $t \to +\infty$, while if $\breve\xi^+(\varepsilon,\lambda) \in \mathfrak{B}^+$ then there is $T > 0$ such that $(\breve x(T,\varepsilon,\lambda), \breve y(T,\varepsilon,\lambda))$ is not close to the homoclinic trajectory of the unperturbed problem $(0, h(t))$ (obviously $\breve\xi^+(\varepsilon, \lambda_1^+(\varepsilon)) \in W^s(u(\varepsilon, \lambda_1^+(\varepsilon)))$ so $(\breve x(t,\varepsilon,\lambda), \breve y(t,\varepsilon,\lambda)) \to U(\varepsilon,\lambda)$ as $t \to +\infty$). Furthermore

$$\begin{aligned} J_1^+(\varepsilon,\lambda) &= J_1^+(\varepsilon,\lambda_1^+(\varepsilon)) + \frac{\partial J_1^+}{\partial\lambda}(\varepsilon,\lambda_1^+(\varepsilon))(\lambda - \lambda_1^+(\varepsilon)) + O((\lambda-\lambda_1^+(\varepsilon))^2) \\ J_1^-(\varepsilon,\lambda) &= J_1^-(\varepsilon,\lambda_1^-(\varepsilon)) + \frac{\partial J_1^-}{\partial\lambda}(\varepsilon,\lambda_1^-(\varepsilon))(\lambda - \lambda_1^-(\varepsilon)) + O((\lambda-\lambda_1^-(\varepsilon))^2) \end{aligned} \qquad (3.5)$$ tay

[1] check

From (3.3) we know the signs of $\frac{\partial}{\partial\lambda}J_1^{\pm}(\varepsilon,\lambda_1^{\pm}(\varepsilon))$; thus, exploiting these two elementary observations we deduce for which values of the nonnegative parameters ε,λ the point $\breve{\xi}^{+}(\varepsilon,\lambda)$ belongs to $\mathfrak{A}^{+}$, $\mathfrak{B}^{+}$ or $W^{s}(u(\varepsilon,\lambda))$, and we obtain a detailed bifurcation diagram (we give some examples in figures 1, 2, 3).

To complete the picture we need to repeat the analysis for ε and $\lambda-q(\varepsilon)$ negative. When $\lambda-q(\varepsilon)\le 0<\varepsilon$ the critical points $u(\varepsilon,\lambda)$ and $s(\varepsilon,\lambda)$ change their stability properties; hence u is stable and s is unstable with respect to the flow of (1.5). So we define

$$\begin{aligned} J_4^{+}(\varepsilon,\lambda)&=\breve{\xi}^{+}(\varepsilon,\lambda)-s(\varepsilon,\lambda)\,,\\ J_4^{-}(\varepsilon,\lambda)&=\breve{\xi}^{-}(\varepsilon,\lambda)-u(\varepsilon,\lambda) \end{aligned} \tag{3.6}$$ acca2bis

Thus, if (vi) holds, we can apply the implicit function theorem and construct the curves $\lambda_4^{\pm}(\varepsilon)$ such that $J_4^{\pm}(\varepsilon,\lambda_4^{\pm}(\varepsilon))$. Moreover we find

$$\begin{aligned} \frac{d}{d\varepsilon}\lambda_4^{+}(0)&=-\frac{\frac{\partial}{\partial\varepsilon}J_4^{+}(0,0)}{\frac{\partial}{\partial\lambda}J_4^{+}(0,0)}=-\frac{A_0^{+}-\frac{\partial s_0}{\partial\varepsilon}(0,0)}{B_0-\frac{\partial s_0}{\partial\lambda}(0,0)}\\ \frac{d}{d\varepsilon}\lambda_4^{-}(0)&=-\frac{\frac{\partial}{\partial\varepsilon}J_4^{-}(0,0)}{\frac{\partial}{\partial\lambda}J_4^{-}(0,0)}=-\frac{A_0^{-}-\frac{\partial u_0}{\partial\varepsilon}(0,0)}{B_0-\frac{\partial u_0}{\partial\lambda}(0,0)} \end{aligned} \tag{3.7}$$ diff6bis

Obviously Remark 3.2 holds also in this setting with trivial modifications (and when $\varepsilon<0$ as well, see below). Moreover reasoning as above and using a Taylor expansion analogous to (3.5), we can draw a detailed bifurcation diagram (we give some examples in figures 1, 2, 3).

When $\varepsilon<0$ we have an inversion in the stability properties of the critical points of (1.1) with respect to the stability properties of (1.5). Therefore if $\breve{\xi}^{+}(\varepsilon,\lambda)=u(\varepsilon,\lambda)$ then $\breve{x}(t,\varepsilon,\lambda)$ converges to $u(\varepsilon,\lambda)$ as $t\to+\infty$, while if $\breve{\xi}^{-}(\varepsilon,\lambda)=s(\varepsilon,\lambda)$ then $\breve{x}(t,\varepsilon,\lambda)$ converges to $s(\varepsilon,\lambda)$ as $t\to-\infty$. Once again we assume (vi) and we distinguish between negative and positive values of $\lambda-q(\varepsilon)$. When $\lambda-q(\varepsilon)>0$ we use again the functions $J_1^{\pm}$ defined in (3.2) and we extend the curves $\lambda_1^{\pm}(\varepsilon)$ to $\varepsilon<0$; similarly for $\lambda-q(\varepsilon)<0$ we use $J_4^{\pm}$ defined in (3.6) and we extend the curves $\lambda_4^{\pm}(\varepsilon)$. Note that also in these cases the derivatives of $\lambda_1^{\pm}$ and $\lambda_4^{\pm}$ are the ones given in (3.4) and in (3.7) so the curves are C^1 in the origin.

The bifurcation diagram changes according to the signs of the nonzero computable constants $\frac{\partial J_i^{\pm}}{\partial\lambda}(0,0)$ and of the following computable constants which may be zero

$$\frac{d}{d\varepsilon}\lambda_i^{+}(0)-q'(0),\qquad \frac{d}{d\varepsilon}\lambda_i^{-}(0)-q'(0),\qquad \frac{d}{d\varepsilon}\lambda_i^{+}(0)-\frac{d}{d\varepsilon}\lambda_i^{-}(0) \tag{3.8}$$ constant

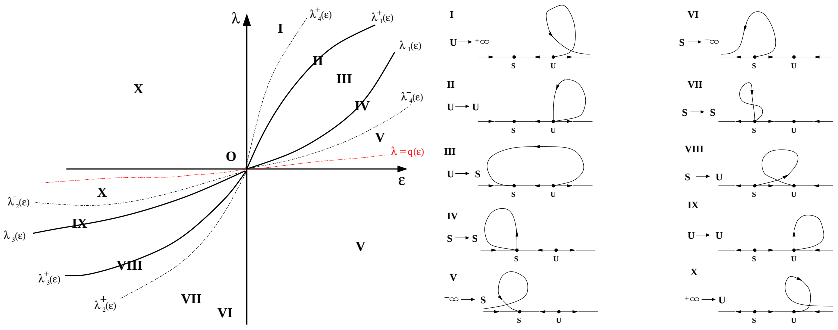

Figure 1: Bifurcation diagrams in the transcritical case: when $\frac{\partial s_0}{\partial \lambda} < 0$. Here we assume $\frac{\partial J_1^{\pm}}{\partial \lambda} > 0$, and $\frac{d\lambda_i^+}{d\varepsilon} > \frac{d\lambda_i^-}{d\varepsilon} > q'(0)$ for $i = 1, 2, 3, 4$. transcrit1

for $i = 1, 4$. To illustrate the meaning of Theorem 3.3 we draw some pictures for specific nonzero values of the constants given in (3.8), the other possibilities can be obtained similarly (not all the combinations are effectively possible). In section 4 we construct a differential equation for which the values of these constants are explicitly computed.

picture **3.3 Theorem.** *Assume that Hypotheses (i)–(vi) of the Introduction hold and that f satisfies (**??**) and **(Tr.)**. Then we can draw the bifurcation diagram for system (1.1), see figures 1, 2, 3)*

asympt *3.4 Remark.* We think it is worthwhile to observe that generically, when $(\breve{x}(t,\varepsilon,\lambda), \breve{y}(t,\varepsilon,\lambda))$ tends to a critical point it has a slow rate of convergence. Namely if it converges to $S(\varepsilon,\lambda)$ as $t \to +\infty$ there is $C_1 > 0$ (independent of ε and λ), such that $\|(\breve{x}(t,\varepsilon,\lambda), \breve{y}(t,\varepsilon,\lambda)) - S(\varepsilon,\lambda)\| \exp(C_1|\varepsilon\lambda|t)$ is bounded for $t > 0$. However when $\lambda = \lambda_1^+(\varepsilon)$ and λ and ε are both positive, we have faster convergence i. e. there is $C_2 > 0$ (independent of ε and λ), such that $\|(\breve{x}(t,\varepsilon,\lambda), \breve{y}(t,\varepsilon,\lambda)) - U(\varepsilon, \lambda_1^+(\varepsilon))\| \exp(\varepsilon C_2 t)$ is uniformly positive for $t > 0$.

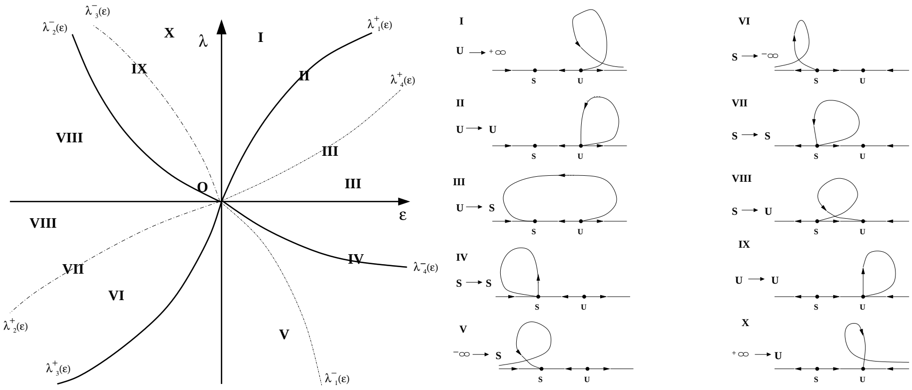

Figure 2: Bifurcation diagram in the transcritical case: $\frac{\partial J_1^{\pm}}{\partial \lambda} > 0$, $\frac{d}{d\varepsilon}\lambda_i^- < 0 < \frac{d}{d\varepsilon}\lambda_i^+ < 0$ for $i = 1, 2, 3, 4$ (and $\frac{\partial s_0}{\partial \lambda} < 0$). `transcrit2`

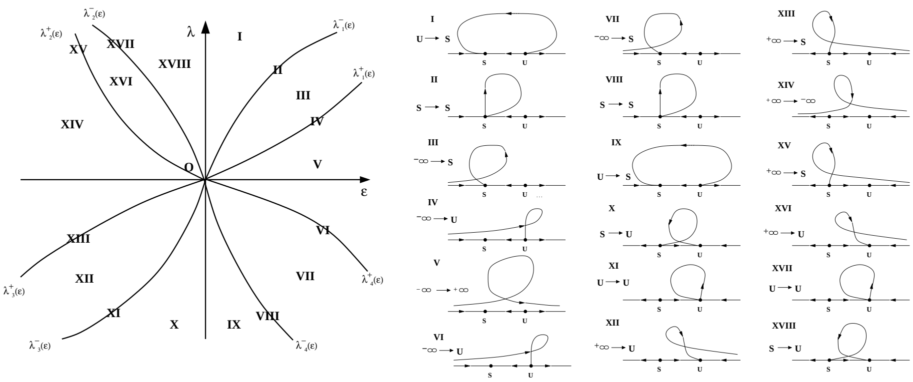

Figure 3: Bifurcation diagram in the transcritical case: $\frac{\partial J^+}{\partial \lambda} < 0 < \frac{\partial J^-}{\partial \lambda}$, $q'(0) < \frac{\partial \lambda_i^+}{\partial \varepsilon} < \frac{\partial \lambda_i^-}{\partial \varepsilon}$, and $\frac{\partial \lambda_j^-}{\partial \varepsilon} < \frac{\partial \lambda_j^+}{\partial \varepsilon} < 0$, for $i = 1, 3,\ j = 2, 4$ (and $\frac{\partial s_0}{\partial \lambda} < 0$). `transcritica5a`

For completeness we observe that, when $\varepsilon = 0$ (1.1) reduces to

$$\dot{y} = g(\xi, y, 0, \lambda) \qquad y(\xi, t)\lfloor_{t=0} = \zeta \in \mathbb{R}^n \qquad \xi \in \mathbb{R} \tag{3.9}$$ red

From Hypothesis (v) it follows that there is $\delta > 0$ such and a unique $\bar{\xi} \in (-\delta, \delta)$ (which is not necessarily a critical point for (1.5)) such that $y(\bar{\xi}, t)$ is a homoclinic trajectory.

When the computable constants given in (3.8) are null we cannot draw the bifurcation diagram in all details. However, using the expansion (3.5), we obtain the asymptotic behavior of $(\breve{x}(t, \varepsilon, \lambda), \breve{y}(t, \varepsilon, \lambda))$, far from the $\lambda = q(\varepsilon)$ axis. When $\frac{d\lambda_i^+}{d\varepsilon}(0) = \frac{d\lambda_i^-}{d\varepsilon}(0)$ for either $i = 1, 2, 3, 4$, we cannot exactly determine the behavior of $(\breve{x}(t, \varepsilon, \lambda), \breve{y}(t, \varepsilon, \lambda))$ for (λ, ε) close to the curves $\lambda = \lambda_i^{\pm}(\varepsilon)$.

Similarly when either $\frac{d\lambda_i^+}{d\varepsilon}(0) - \frac{dq}{d\varepsilon}(0) = 0$ or $\frac{d\lambda_i^-}{d\varepsilon} - \frac{dq}{d\varepsilon}(0) = 0$ for $i = 1, 2, 3, 4$, we cannot say wether the curves $\lambda_i^{\pm}$ are above or below the line $\lambda = q(\varepsilon)$. We think it is worth observing that in the former case a new scenario may arise. In fact a priori we could have uncountably many intersections between λ_i^+ and λ_i^-. These intersections would correspond to heteroclinic trajectories with fast convergence and following the unusual direction: when ε and λ are positive the trajectory tend to S in the past and to U in the future. So $(\breve{x}(t, \varepsilon, \lambda), \breve{y}(t, \varepsilon, \lambda))$, together with the heteroclinic connection between U and S contained in $\mathcal{M}^c(\varepsilon, \lambda)$, form a heteroclinic cycle.

3.5 Remark. Observe that the classification result can be developed also when Hyp. (vi) is not satisfied. In such a case we should replace condition (vi) with the following, which is more difficult to handle:

(vi$_\varepsilon$) $\frac{\partial \breve{\xi}^+}{\partial \varepsilon}(0, 0) \neq \frac{du_0}{d\varepsilon}(0)$ and $\frac{\partial \breve{\xi}^-}{\partial \varepsilon}(0, 0) \neq \frac{ds_0}{d\varepsilon}(0)$,

where the formulas for $\frac{d\breve{\xi}_0^{\pm}}{d\varepsilon}(0, 0)$ are given in (**??**) and (**??**).

In fact we may use the implicit function Theorem to prove the existence of curves $\varepsilon_i^{\pm}(\lambda)$ in the i^{th} quadrant, such that the trajectory $(\breve{x}(t, \varepsilon_i^{\pm}(\lambda), \lambda), \breve{y}(t, \varepsilon_i^{\pm}(\lambda), \lambda))$ converges respectively to the unstable point of $\mathcal{M}(C(\varepsilon, \lambda))$ as $t \to +\infty$ and to the stable point of $\mathcal{M}(C(\varepsilon, \lambda))$ as $t \to -\infty$. However if $\frac{\partial \breve{\xi}^+}{\partial \lambda}(0, 0) = \frac{\partial u_0}{\partial \lambda}(0, 0)$ the curve $\varepsilon_i^+(\varepsilon, \lambda)$ would be tangent to the $\varepsilon = 0$ axis, so once again we could not decide the behavior of the trajectory $(\breve{x}(t, \varepsilon, \lambda), \breve{y}(t, \varepsilon, \lambda))$ for (ε, λ) close to the $\varepsilon = 0$ axis.

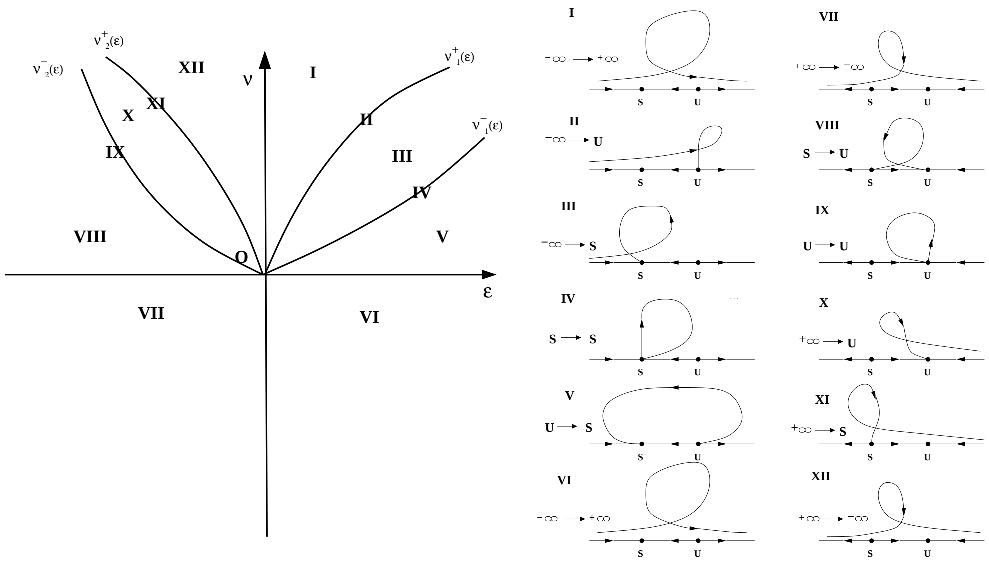

Figure 4: Bifurcation diagram in the saddle node case. We have assumed $q(\varepsilon) \equiv 0$, $\frac{\partial \check{\xi}^+}{\partial \varepsilon}(0,0) < 0 < \frac{\partial \check{\xi}^-}{\partial \varepsilon}(0,0)$, $\frac{\partial \check{\xi}^+}{\partial \varepsilon}(0,0) < 0 < \frac{\partial \check{\xi}^-}{\partial \varepsilon}(0,0)$ (and $\frac{\partial s_0}{\partial \nu}(0) < 0$).

sadnode

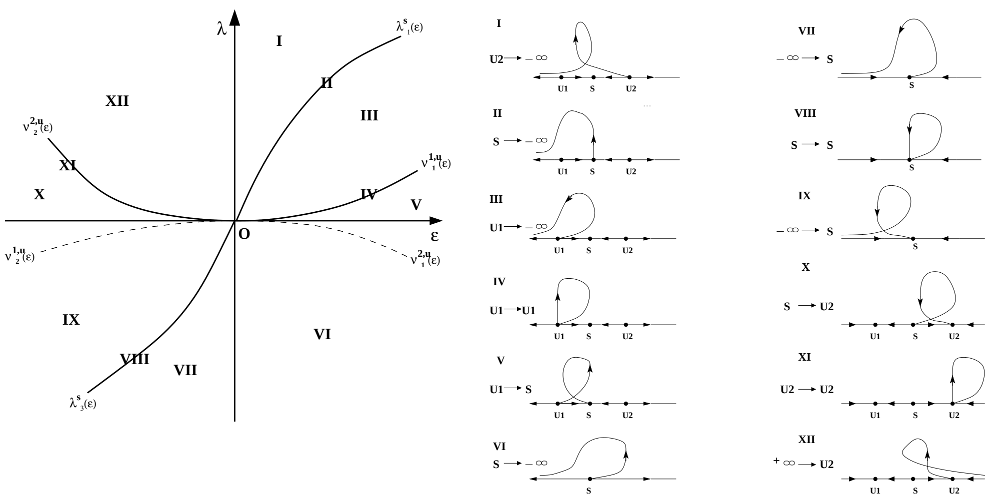

Figure 5: An example of bifurcation diagram in the pitchfork case. We have assumed $q(\varepsilon) \equiv 0$, $A_0^{\pm} + A_m + C_m > 0$, $A_0^{\pm} + A_m + \tilde{C}_m > 0$, $\frac{\partial u_0^1}{\partial \nu} > 0$ and $B_0 + B_m - \frac{\partial s_0}{\partial \lambda} > 0$.

pitchf

3.2 Saddle-node bifurcation.

We briefly consider the case where f satisfies **(s.n.)** so that the origin undergoes to a saddle-node bifurcation. We need to introduce the auxiliary variable $\nu = \text{sign}(\lambda)\sqrt{|\lambda|}$ and we observe that $u(\varepsilon, \nu|\nu|)$ and $s(\varepsilon, \nu|\nu|)$ are smooth functions (while they are just holder functions of λ). Via implicit function theorem we define the smooth curve $\nu = Q(\varepsilon)$ such that $u(\varepsilon, Q(\varepsilon)|Q(\varepsilon)|) = s(\varepsilon, Q(\varepsilon)|Q(\varepsilon)|)$; we find $Q(0) = 0$ and

$$\frac{dQ}{d\varepsilon}(0) = -\frac{\frac{d(u_0-s_0)}{d\varepsilon}(0,0)}{\frac{d(u_0-s_0)}{d\nu}(0,0)} = \frac{\frac{d(u_0-s_0)}{d\varepsilon}(0,0)}{2\frac{ds_0}{d\nu}(0,0)}\,.$$

Set $q(\varepsilon) = Q(\varepsilon)|Q(\varepsilon)|$, then both (1.5) and (1.1) admit two critical points in a neighborhood of the origin for $\lambda > q(\varepsilon)$ and no critical points for $\lambda < q(\varepsilon)$. Note that $\frac{dq}{d\varepsilon}(0) = 0$.

Theorem 3.3 works also in this setting, with some minor changes, but condition (vi) is not needed anymore. Once again we have to argue separately in each quadrant of the parameters plane; we start from ε and $\lambda - q(\varepsilon)$ positive, and we define

$$\begin{aligned} \tilde{J}_1^+(\varepsilon,\nu) &= \breve{\xi}^+(\varepsilon,\nu|\nu|) - u(\varepsilon,\nu|\nu|) \qquad \text{and} \\ \tilde{J}_1^-(\varepsilon,\nu) &= \breve{\xi}^-(\varepsilon,\nu|\nu|) - s(\varepsilon,\nu|\nu|) \end{aligned}$$

and we repeat the analysis made in the previous subsection. So the solution defined by (**??**) converges to U as $t \to +\infty$ if $\tilde{J}_1^+(\varepsilon,\nu) = 0$ and to S as $t \to -\infty$ if $\tilde{J}_1^-(\varepsilon,\nu) = 0$. We stress that $\frac{\partial \breve{\xi}_0^\pm}{\partial \nu}(0,0) = 0$ since $\frac{\partial \lambda}{\partial \nu}(0) = 0$, therefore

$$\frac{\partial \tilde{J}_1^-}{\partial \nu}(0,0) = -\frac{\partial s_0}{\partial \nu}(0,0) = -\frac{\partial \tilde{J}_1^+}{\partial \nu}(0,0)\,.$$

So we can apply the implicit function Theorem and construct smooth curves $\nu_1^\pm(\varepsilon)$ such that $\nu_1^\pm(0) = 0$, $\tilde{J}_1^+(\varepsilon, \nu_1^+(\varepsilon)) = 0$ and $\tilde{J}_1^-(\varepsilon, \nu_1^-(\varepsilon)) = 0$ respectively. Furthermore

$$\frac{d}{d\varepsilon}\nu_1^+(0) = -\frac{A_0^+ - \frac{\partial}{\partial\varepsilon}u_0(0,0)}{\frac{\partial}{\partial\nu}s_0(0,0)}\,, \quad \frac{d}{d\varepsilon}\nu_1^-(0,0) = \frac{A_0^- - \frac{\partial}{\partial\varepsilon}s_0(0,0)}{\frac{\partial}{\partial\nu}s_0(0,0)} \qquad (3.10) \quad \boxed{\texttt{diff7}}$$

When $\varepsilon < 0 \le \lambda - q(\varepsilon)$ we define

$$\begin{aligned} \tilde{J}_2^+(\varepsilon,\nu) &= \breve{\xi}^+(\varepsilon,\nu|\nu|) - s(\varepsilon,\nu|\nu|)\,, \\ \tilde{J}_2^-(\varepsilon,\nu) &= \breve{\xi}^-(\varepsilon,\nu|\nu|) - u(\varepsilon,\nu|\nu|) \end{aligned}$$

and we find again curves $\nu_2^{\pm}(\varepsilon)$ such that $\nu_2^{\pm}(0) = 0$, $\tilde{J}_2^{\pm}(\varepsilon, \nu_2^{\pm}(\varepsilon)) = 0$ respectively, and

$$\frac{d}{d\varepsilon}\nu_2^{+}(0) = \frac{A_0^{+} - \frac{\partial}{\partial\varepsilon}s_0(0,0)}{\frac{\partial}{\partial\nu}s_0(0,0)}, \quad \frac{d}{d\varepsilon}\nu_2^{-}(0,0) = -\frac{A_{-}^{0} - \frac{\partial}{\partial\varepsilon}u_0(0,0)}{\frac{\partial}{\partial\nu}s_0(0,0)} \tag{3.11}$$ diff7,5

As usual the solution defined by (**??**) converges to S as $t \to +\infty$ if $\tilde{J}_2^{+}(\varepsilon, \nu) = 0$ and to U as $t \to -\infty$ if $\tilde{J}_2^{-}(\varepsilon, \nu) = 0$.

Obviously in both the cases for $\lambda < q(\varepsilon)$ there are no critical points and hence no bounded trajectories. Arguing as in the previous subsection we obtain a result analogous to Theorem 3.3.

picture2 **3.6 Theorem.** *Assume that Hypotheses (i)–(v) of the Introduction hold and that f satisfies (**??**) and **(s.n.)**. Then we can draw the bifurcation diagram for system (1.1).*

If $\frac{\partial \tilde{J}_1^{+}}{\partial\nu}(0,0) = -\frac{\partial s_0}{\partial\nu}(0,0) > 0$, i.e. $\frac{\partial^2 f_0}{(\partial x_0)^2}(0,0,0,0) > 0$, the bifurcation diagram of (1.1) described in Theorem 3.6 depends on the signs of the following computable constants:

$$\frac{d\nu_i^{+}}{d\varepsilon}(0), \qquad \frac{d\nu_i^{-}}{d\varepsilon}(0), \qquad \frac{d}{d\varepsilon}\nu_i^{+}(0) - \frac{d}{d\varepsilon}\nu_i^{-}(0). \tag{3.12}$$ constant2

We give again one example for illustrative purposes, see figure 4.

3.3 Degree 3 or more.

In this subsection we show briefly how our methods can be applied to unfold more degenerate singularities of (1.1). We just sketch the case where (1.5) undergoes to a pitchfork bifurcation, stressing that the construction can be easily generalized to describe singularities of higher order. So we assume that (**??**) holds but $\frac{\partial^2 f_0}{(\partial x_0)^2}(0,0,0,0) = 0$ and $\frac{\partial^3 f_0}{(\partial x_0)^3}(0,0,0,0) \neq 0$ and we consider the following assumption.

(pitch) The equation $f(x,0,0,\lambda) = 0$ admits three solutions for $\lambda > 0$, say $u^1(0,\lambda) < s(0,\lambda) < u^2(0,\lambda)$, and one solution $s(0,\lambda)$ for $\lambda \leq 0$.

We denote by u_0^1, s_0 and u_0^2 the x_0 coordinates of the critical points, and we set $U^1(\varepsilon,\lambda) = (u^1(\varepsilon,\lambda), v(u^1(\varepsilon,\lambda),\varepsilon,\lambda))$, $U^2(\varepsilon,\lambda) = (u^2(\varepsilon,\lambda), v(u^2(\varepsilon,\lambda),\varepsilon,\lambda))$, $S(\varepsilon,\lambda) = (s(\varepsilon,\lambda), v(s(\varepsilon,\lambda),\varepsilon,\lambda)))$. We think it is worthwhile to stress that to achieve a complete unfolding of the singularity one more parameter is needed. Similarly to the previous subsections, we can construct smooth curves $\nu^i = Q^i(\varepsilon)$ for $i = 0, 1, 2$, and $\lambda^i = q^i(\varepsilon) = Q^i(\varepsilon)|Q^i(\varepsilon)|$ such that $q^i(0) = Q^i(0) = 0$, and

$$u^1(\varepsilon, q^0(\varepsilon)) = u^2(\varepsilon, q^0(\varepsilon)) \,, \quad s(\varepsilon, q^1(\varepsilon)) = u^1(\varepsilon, q^1(\varepsilon)) \,, \quad s(\varepsilon, q^2(\varepsilon)) = u^2(\varepsilon, q^2(\varepsilon)) \,;$$

moreover $\frac{dq^i}{d\varepsilon}(0) = 0$ for $i = 0, 1, 2$. In fact

$$\frac{dQ^0}{d\varepsilon} = \frac{\frac{d(u_0^1 - u_0^2)}{d\varepsilon}(0,0)}{2\frac{du_0^2}{d\nu}(0,0)} \,, \quad \frac{dQ^1}{d\varepsilon} = \frac{\frac{d(s_0 - u_0^1)}{d\varepsilon}(0,0)}{\frac{du_0^1}{d\nu}(0,0)} \,, \quad \frac{dQ^2}{d\varepsilon} = -\frac{\frac{d(s_0 - u_0^2)}{d\varepsilon}(0,0)}{\frac{du_0^1}{d\nu}(0,0)} \,. \tag{3.13}$$

chepalle

We assume for simplicity that $u^1(\varepsilon, 0) = u^2(\varepsilon, 0)$, i.e. $q^0(\varepsilon) \equiv 0$. We assume w.l.o.g. that $u^1(\varepsilon, \lambda)$ and $u^2(\varepsilon, \lambda)$ are unstable for the restriction of system (1.5) to $C(\varepsilon, \lambda)$ (when $\lambda > 0$, they do not exist for $\lambda < 0$) while $s(\varepsilon, \lambda)$ is stable for $\lambda > 0$ and unstable for $\lambda < 0$.

Theorem 3.1 holds in this case too, so using the function H defined in (**??**) for $\varepsilon > 0$, and the function $\tilde{H}$ defined in (**??**) for $\varepsilon < 0$, via implicit function theorem we construct the smooth function $\breve{\xi}_{\hat{0}}(\varepsilon, \lambda)$ such that the solution defined by (**??**) is homoclinic to $\mathcal{M}^c(C(\varepsilon, \lambda))$.

Similarly to the saddle-node case the functions $u_0^1(\varepsilon, \lambda)$ and $u_0^2(\varepsilon, \lambda)$ are not smooth in the origin, so we need to introduce the parameter $\nu = \sqrt{\lambda}$. On the other hand the function $s_0(\varepsilon, \lambda)$ is smooth and its derivative with respect to ν is null; so, in order to apply the implicit function theorem, we have to work with $u_0^1(\varepsilon, \nu^2)$, $u_0^2(\nu^2)$ and $s_0(\varepsilon, \lambda)$.

Let us start assuming $\lambda \geq q(\varepsilon) \equiv 0$ and $\varepsilon > 0$, in analogy to the previous subsection we define the functions $\bar{h}_b^1, \bar{h}_b^2 : A_b \times [-\varepsilon_0, \varepsilon_0] \times [-\lambda_0, \lambda_0] \to \mathbb{R}$, $\bar{h}_a : A_a \times [-\varepsilon_0, \varepsilon_0] \times [-\lambda_0, \lambda_0] \to \mathbb{R}$ such that $\breve{\xi} \in W^s(u^i(\varepsilon, \lambda)) \cap A$ if and only if $\breve{\xi}_0 = \bar{h}_b^i(\breve{\xi}_b, \varepsilon, \nu)$ and $\breve{\xi}_a = h_{(0,b)}(\breve{\xi}_0, \breve{\xi}_b, \varepsilon, \nu^2)$ for $i = 1, 2$, while $\breve{\xi} \in W^u(s(\varepsilon, \lambda)) \cap A$ if and only if $\breve{\xi}_0 = \bar{h}_a(\breve{\xi}_a, \varepsilon, \lambda)$ and $\breve{\xi}_b = h_{(0,a)}(\breve{\xi}_0, \breve{\xi}_a, \varepsilon, \lambda)$. Again the derivatives of $\bar{h}_a$ and $\bar{h}_b^j$ in $(0, 0, 0)$ with respect to ε and x_i are null, and

$$\frac{\partial \bar{h}_b^1}{\partial \nu}(0,0,0) = \frac{\partial u_0^1}{\partial \nu}(0,0) = -\frac{\partial \bar{h}_b^2}{\partial \nu}(0,0,0) \,, \qquad \frac{\partial \bar{h}_a}{\partial \lambda}(0,0) = \frac{\partial s_0}{\partial \lambda}(0,0) \,.$$

Then we define the functions

$$J_1^{i,u}(\varepsilon, \nu) = \breve{\xi}^+(\varepsilon, \nu^2) - \bar{h}_b^i(\breve{\xi}_b, \varepsilon, \nu) \,, \qquad J_1^s(\varepsilon, \lambda) = \breve{\xi}^-(\varepsilon, \lambda) - \bar{h}_a(\breve{\xi}_a, \varepsilon, \lambda)$$

for $i = 1, 2$; obviously $J_1^{i,u}(0, 0) = 0$ for $i = 1, 2$ and $J_1^s(0, 0) = 0$. We stress that $\frac{\partial}{\partial \nu}\breve{\xi}^+(\varepsilon, \nu^2) = 0$ for $(\varepsilon, \nu) = (0, 0)$. To apply the implicit function theorem we just need to assume

(vi') $\frac{\partial u_0^1}{\partial \nu}(0,0) \neq 0\,,$ and $B_0 + B_m - \frac{\partial}{\partial \lambda} s_0(0,0) \neq 0$

So we prove the existence of curves $\nu_1^{i,u}(\varepsilon)$, $\lambda_1^s(\varepsilon)$ such that $(\breve{x}(t,\varepsilon,\lambda), \breve{y}(t,\varepsilon,\lambda))$ converges to U^i as $t \to +\infty$ when $\lambda = [\nu_1^{i,u}(\varepsilon)]^2$ for $i = 1,2$, and to S as $t \to -\infty$ when $\lambda = \lambda_1^s(\varepsilon)$. Moreover

$$\begin{aligned}
\frac{d}{d\varepsilon}\nu_1^{1,u}(0) &= \frac{\frac{\partial}{\partial\varepsilon}\breve{\xi}^+(0,0) - \frac{\partial u_0^1}{\partial \varepsilon}(0,0)}{\frac{\partial u_0^1}{\partial \nu}(0,0)} = \frac{A_m + B_0^+ + C_m - \frac{\partial u_0^1}{\partial \varepsilon}(0,0)}{\frac{\partial u_0^1}{\partial \nu}(0,0)} \\
\frac{d}{d\varepsilon}\nu_1^{2,u}(0) &= \frac{\frac{\partial}{\partial\varepsilon}\breve{\xi}^+(0,0) - \frac{\partial u_0^2}{\partial \varepsilon}(0,0)}{\frac{\partial u_0^2}{\partial \nu}(0,0)} = -\frac{A_m + B_0^+ + C_m - \frac{\partial u_0^2}{\partial \varepsilon}(0,0)}{\frac{\partial u_0^1}{\partial \nu}(0,0)} \\
\frac{d}{d\varepsilon}\lambda_1^s(0) &= -\frac{\frac{\partial}{\partial\varepsilon}\breve{\xi}^-(0,0) - \frac{\partial s_0}{\partial \varepsilon}(0,0)}{\frac{\partial}{\partial\lambda}\breve{\xi}_0(0,0) - \frac{\partial s_0}{\partial \lambda}(0,0)} = -\frac{A_m + B_0^- + C_m - \frac{\partial s_0}{\partial \varepsilon}(0,0)}{B_0 + B_m - \frac{\partial s_0}{\partial \lambda}(0,0)}\,,
\end{aligned} \tag{3.14}$$

diff6.3p

When $\lambda \leq 0$ the only critical point of (1.5) in a neighborhood of the origin is $s(\varepsilon,\lambda)$, which is unstable in the direction of $C(\varepsilon,\lambda)$. So we define the function $\bar{h}_b$ such that $\xi \in W^s(S) \cap A$ if and only if $\xi_0 = \bar{h}_b(\xi_b,\varepsilon,\lambda)$ and $\xi_a = h_{(0,b)}(\xi_0,\xi_b,\varepsilon,\lambda)$ and

$$J_4^s(\varepsilon,\lambda) = \breve{\xi}^+(\varepsilon,\lambda) - \bar{h}_b(\breve{\xi}_b,\varepsilon,\lambda)\,.$$

Then via implicit function theorem we construct the curve $\lambda_4^s(\varepsilon)$ such that $(x(t,\varepsilon,\lambda_4^s(\varepsilon)), y(t,\varepsilon,\lambda_4^s(\varepsilon)))$ converges to S as $t \to +\infty$; moreover

$$\frac{d}{d\varepsilon}\lambda_4^s(0) = -\frac{\frac{\partial}{\partial\varepsilon}\breve{\xi}^+(0,0) - \frac{\partial s_0}{\partial \varepsilon}(0,0)}{\frac{\partial}{\partial\lambda}\breve{\xi}_0(0,0) - \frac{\partial s_0}{\partial \lambda}(0,0)} = -\frac{A_m + B_0^+ + C_m - \frac{\partial s_0}{\partial \varepsilon}(0,0)}{B_0 + B_m - \frac{\partial s_0}{\partial \lambda}(0,0)}\,. \tag{3.15}$$

diff6.4p

When $\lambda < 0 < \varepsilon$ the trajectory $(\breve{x}(t,\varepsilon,\lambda), \breve{y}(t,\varepsilon,\lambda))$ homoclinic to $\mathcal{M}(C(\varepsilon,\lambda))$ converges to S as $t \to -\infty$.

When $\varepsilon < 0$ as usual the critical points of (1.1) reverse their stability properties, so we have to redefine the auxiliary functions as we did in the previous section. When $\varepsilon < 0 \leq \lambda$ we construct via implicit function theorem the curves $\nu_2^{1,u}(\varepsilon)$, $\nu_2^{2,u}(\varepsilon)$ and $\lambda_2^s(\varepsilon)$ with the following properties: the trajectory defined by (**??**) converges to U^i as $t \to -\infty$ when $\sqrt{\lambda}$ equals $\nu_2^{i,u}(\varepsilon)$ for $i = 1,2$, and to S as $t \to +\infty$ when $\lambda = \lambda_2^s(\varepsilon)$. Moreover

$$\begin{aligned}
\frac{d}{d\varepsilon}\nu_2^{1,u}(0) &= \frac{A_m + B_0^- + \tilde{C}_m - \frac{\partial u_0^1}{\partial \varepsilon}(0,0)}{\frac{\partial u_0^1}{\partial \nu}(0,0)} \\
\frac{d}{d\varepsilon}\nu_2^{2,u}(0) &= -\frac{A_m + B_0^- + \tilde{C}_m - \frac{\partial u_0^2}{\partial \varepsilon}(0,0)}{\frac{\partial u_0^1}{\partial \nu}(0,0)} \\
\frac{d}{d\varepsilon}\lambda_2^s(0) &= -\frac{A_m + B_0^+ + \tilde{C}_m - \frac{\partial s_0}{\partial \varepsilon}(0,0)}{B_0 + B_m - \frac{\partial s_0}{\partial \lambda}(0,0)}\,,
\end{aligned} \tag{3.16}$$

diff6.5p

Similarly when ε and λ are negative, we construct the curve $\lambda_3^s(\varepsilon)$, such that the trajectory defined by (**??**) converges to S as $t \to -\infty$. Moreover

$$\frac{d}{d\varepsilon}\lambda_3^s(0) = -\frac{A_m + B_0^- + \tilde{C}_m - \frac{\partial s_0}{\partial \varepsilon}(0,0)}{B_0 + B_m - \frac{\partial s_0}{\partial \lambda}(0,0)}\,.$$

Furthermore the trajectory $(\breve{x}(t,\varepsilon,\lambda), \breve{y}(t,\varepsilon,\lambda))$ homoclinic to $\mathcal{M}^c(C(\varepsilon,\lambda))$ converges to $S(\varepsilon,\lambda)$ as $t \to +\infty$. Now, similarly to the previous subsections, using a Taylor expansion analogous to (3.5), we can draw the bifurcation diagram for (1.1). Once again the bifurcation diagram depends on the sign of some computable constants, i. e. $B_0 + B_m - \frac{\partial s_0}{\partial \lambda}(0,0)$, $\frac{\partial \nu_i^{1,u}}{\partial \varepsilon}$, for $i = 1, 2$, $\frac{\partial \lambda_i^s}{\partial \varepsilon}$ for $i = 1, 2, 3, 4$, see figure 5.

effort *3.7 Remark.* When f does not depend on λ or anyway ε is the only parameter involved in the bifurcation, we can still perform our analysis, with some trivial (and simplifying) changes. When both f and g do not depend on λ, we cannot unfold completely the singularity. However the behavior of the solution $(\breve{x}(t,\varepsilon), \breve{y}(t,\varepsilon))$ defined by (**??**) is determined in the transcritical case by the sign of the following constants:

$$\begin{aligned}
K^+ &= \frac{\partial \breve{\xi}_0^+}{\partial \varepsilon}(0) - \frac{\partial u_0}{\partial \varepsilon}(0) = A_0^+ + A_m + C_m - \frac{\partial u_0}{\partial \varepsilon}(0)\,, \\
K^- &= \frac{\partial \breve{\xi}_0^-}{\partial \varepsilon}(0) - \frac{\partial s_0}{\partial \varepsilon}(0) = A_0^- + A_m + C_m - \frac{\partial s_0}{\partial \varepsilon}(0)\,, \\
\tilde{K}^+ &= \frac{\partial \breve{\xi}_0^+}{\partial \varepsilon}(0) - \frac{\partial u_0}{\partial \varepsilon}(0) = A_0^+ + A_m + \tilde{C}_m - \frac{\partial u_0}{\partial \varepsilon}(0)\,, \\
\tilde{K}^- &= \frac{\partial \breve{\xi}_0^-}{\partial \varepsilon}(0) - \frac{\partial s_0}{\partial \varepsilon}(0) = A_0^- + A_m + \tilde{C}_m + \frac{\partial s_0}{\partial \varepsilon}(0)\,,
\end{aligned} \tag{3.17}$$

costantiK

see (**??**), (**??**). E.g. if $K^\pm$ are positive and $\frac{\partial s_0}{\partial \varepsilon}(0) < \frac{\partial u_0}{\partial \varepsilon}(0)$, using a Taylor expansion we find that $\breve{\xi}_0^+(\varepsilon) - u_0(\varepsilon)$ and $\breve{\xi}_0^-(\varepsilon) - s_0(\varepsilon)$ are positive for $\varepsilon > 0$; thus $(\breve{x}(t,\varepsilon), \breve{y}(t,\varepsilon))$ converges to $U(\varepsilon)$ as $t \to -\infty$ and gets out from a neighborhood of the origin for t large. Similarly if $\varepsilon < 0$ and $\tilde{K}^\pm$ are both positive we find that $\breve{\xi}_0^+(\varepsilon) - u_0(\varepsilon)$ and $\breve{\xi}_0^-(\varepsilon) - s_0(\varepsilon)$ are both negative, so again $(\breve{x}(t,\varepsilon), \breve{y}(t,\varepsilon))$ converges to $U(\varepsilon)$ as $t \to -\infty$ and gets out from a neighborhood of the origin for t large.

Reasoning in the same way it is easy to see that when (1.5) exhibits a saddle-node bifurcation, then $(x(t,\varepsilon), y(t,\varepsilon))$ is a heteroclinic connection between U and S and converges to the former in the past and to the latter in the future, since $s_0(\varepsilon) < \breve{\xi}_0^\pm(\varepsilon) < u_0(\varepsilon)$ for $\varepsilon > 0$; in fact $\frac{\partial s_0}{\partial \varepsilon}(0) = -\infty$ and $\frac{\partial u_0}{\partial \varepsilon}(0) = +\infty$.

4 Examples.

In this section we construct examples for which the conditions of Theorems 3.1, 3.3, 3.6 are fulfilled and the derivatives of the bifurcation curves can be explicitly computed. Let us consider the following system:

$$\begin{cases} \dot{x}_0 &= \varepsilon\big[x_0^2 - (\sigma_0\lambda)^2 + \alpha y_1 y_2 + \omega_1(x,y,\varepsilon,\lambda)\big] := \varepsilon f_0(x,y,\varepsilon,\lambda) \\ \dot{x}_1 &= \varepsilon\big[x_1 - \sigma_1\lambda + \beta y_1^2 y_2 + \omega_2(x,y,\varepsilon,\lambda)\big] \\ \dot{x}_2 &= \varepsilon\big[-x_2 + \sigma_2\lambda + \gamma y_2 + \omega_3(x,y,\varepsilon,\lambda)\big] \\ \dot{y}_1 &= y_2 + x_0(a'y_1 + a''y_2) + a'''x_1 y_2 + a^{iv}x_2 y_1 + \lambda y_2 k(y_1) + O(\lambda|x|) \\ \dot{y}_2 &= y_1 - (y_1)^3 + x_0(b'y_1 + b''y_2) + x_1 y_1 + \lambda h(y_1) + O(\lambda|x|) \end{cases} \tag{4.1}$$

esempio

where h and k are smooth functions satisfying $h(0) = 0 = k(0)$, $\omega_i(x,y,\varepsilon,\lambda) = O(|y||x|) + o(\varepsilon^2 + \lambda^2 + |x|^2)$ for $i = 1,2,3$.

We stress that the y component of (4.1) is constructed on the unperturbed problem

$$\begin{cases} \dot{y}_1 &= g_1(0,y,0,0) := & y_2 \\ \dot{y}_2 &= g_2(0,y,0,0) := & y_1 - (y_1)^3 \end{cases} \tag{4.2}$$

esempio0

which admits two homoclinic trajectories $\pm(\chi_1(t), \chi_2(t))$ where

$$\chi_1(t) = \frac{2\sqrt{2}}{e^t + e^{-t}}, \quad \chi_2(t) = -2\sqrt{2}\frac{e^t - e^{-t}}{(e^t + e^{-t})^2}$$

and $\chi_1^4/2 - \chi_1^2 + \chi_2^2 = 0$. So $\chi(t) = (0,0,0,\chi_1(t),\chi_2(t))$ and $-\chi(t)$ are homoclinic trajectory for (4.1) for $\varepsilon = \lambda = 0$. Note that the adjoint variational systems $\dot{y} = -[\partial g/\partial y]^*(0,\pm\chi(t),0,0)y$ admits the unique (up to multiplicative constant) solutions $\pm\psi(t) = \pm(\{\chi_1(t) - [\chi_1(t)]^3\}, -\chi_2(t))$.
To simplify matter we have assumed that g does not depend on ε so that the slow manifold $\mathcal{M}^c$ is given simply by $y = v(x,\varepsilon,\lambda) \equiv 0$. From a straightforward computation we find $\frac{\partial u}{\partial\lambda}(0,0) = (\sigma_0,\sigma_1,\sigma_2)$, $\frac{\partial s}{\partial\lambda}(0,0) = (-\sigma_0,\sigma_1,\sigma_2)$, and $\frac{\partial u_0}{\partial\varepsilon}(0,0) = \frac{\partial s_0}{\partial\varepsilon}(0,0) = 0$. It follows that the line $\lambda = q(\varepsilon)$ such that $U(\varepsilon,\lambda) = S(\varepsilon,\lambda)$ satisfies $q'(\varepsilon) = 0$. If we assume further that f does not depend on ε, the line $\lambda = q(\varepsilon)$ is simply defined by $\lambda = q(\varepsilon) \equiv 0$.

From further computations we get $\chi_1(0) = \sqrt{2}$, $\chi_2(0) = 0$, $\int_{\mathbb{R}}\chi_1^4 = \int_{\mathbb{R}}\chi_2^2 = \frac{16}{3}$, $\int_{\mathbb{R}}\chi_1^2 = 4$, $\int_{\mathbb{R}}\chi_1^6 = \frac{128}{15}$, $\int_{\mathbb{R}}\chi_1^2\chi_2^2 = \frac{16}{15}$, $\int_{\mathbb{R}}\chi_1^3 = \pi\sqrt{2}$, $\int_{\mathbb{R}}\chi_1^5 = \frac{3\pi}{\sqrt{2}}$.

$$G_0 = \int_{-\infty}^{\infty} \pm\psi^*(t)\frac{\partial g}{\partial x_0}(\pm\chi(t),0,0)dt = \int_{-\infty}^{\infty}\big[a'(\chi_1^2 - \chi_1^4(t)) - b''\chi_2^2(t)\big]dt =$$

$$= -\frac{4}{3}a' - \frac{16}{3}b'', \qquad G_1 = \int_{-\infty}^{\infty} \pm\psi^*(t)\frac{\partial g}{\partial x_1}(\pm\chi(t),0,0)dt = 0,$$

$$G_2 = \int_{-\infty}^{\infty} \pm\psi^*(t)\frac{\partial g}{\partial x_2}(\pm\chi(t),0,0)dt = -\frac{4}{3}a^{iv}$$

Moreover

$$\int_0^t f_0(\pm\chi(s),0,0)ds = \frac{\alpha}{2}[\chi_1^2(t)-\chi_1^2(0)]\,,$$
$$\int_0^t f_1(\pm\chi(s),0,0)ds = \pm\frac{\beta}{3}[\chi_1^3(t)-\chi_1^3(0)]\,, \qquad \int_0^t f_2(\pm\chi(s),0,0)ds = \pm\gamma[\chi_1(t)-\chi_1(0)]$$
$$F_1^{\pm} = \int_0^{-\infty} f_1(\pm\chi(t),0,0)dt - \frac{\partial s_1}{\partial\varepsilon}(0,0) = \mp\frac{2\sqrt{2}\beta}{3} = \tilde{F}_1^{\pm}$$
$$F_2^{\pm} = \int_0^{+\infty} f_2(\pm\chi(t),0,0)dt - \frac{\partial s_2}{\partial\varepsilon}(0,0) = \mp\gamma\sqrt{2} = \tilde{F}_2^{\pm}$$

and

$$K_0 = \int_{-\infty}^{\infty} \pm\psi^*(t)\frac{\partial g}{\partial x_0}(\pm\chi(t),0,0)\Big[\int_{\pm\infty}^t f_0(\pm\chi(s),0,0)ds\Big]dt = -\frac{8\alpha}{15}\left(3a'+b''\right)\,,$$
$$\int_{-\infty}^{\infty} \pm\psi^*(t)\frac{\partial g}{\partial\lambda}(\pm\chi(t),0,0)dt = \int_{-\infty}^{\infty} \pm\psi^*(t)\frac{\partial g}{\partial\varepsilon}(\pm\chi(t),0,0)dt = 0$$
$$K_1 = \int_{-\infty}^{\infty} \pm\psi^*(t)\frac{\partial g}{\partial x_1}(\pm\chi(t),0,0)\Big[\int_0^t f_1(\pm\chi(s),0,0)ds\Big]dt = 0$$
$$K_2^{\pm} = \int_{-\infty}^{\infty} \pm\psi^*(t)\frac{\partial g}{\partial x_2}(\pm\chi(t),0,0)\Big[\int_0^t f_2(\pm\chi(s),0,0)ds\Big]dt = \pm a^{iv}\gamma\left(\frac{4\sqrt{2}}{3}-\frac{\pi}{\sqrt{2}}\right).$$

We stress that condition (v) is satisfied whenever $G_0\neq 0$, so it is satisfied for both $\pm\chi$ when $a'\neq -4b''$. Condition (vi) is satisfied whenever $\sigma_1G_1+\sigma_2G_2\neq \pm\sigma_0G_0$.
The values of F_i^+, $\tilde{F}_i^+$ and K_2^+ change to F_i^-, $\tilde{F}_i^-$ and K_2^- passing from χ to $-\chi$, while the other values remain the same. For simplicity from now on we restrict our attention to $+\chi(t)$. So when (v) and (vi) hold, using (**??**), (**??**), (**??**) and (**??**) we find:

$$A_0^{\pm}+A_m = -\frac{K_0+K_1+K_2^+}{G_0}\,,\quad B_0+B_m = -\frac{\sigma_1G_1+\sigma_2G_2}{G_0}\,,$$
$$C_m = \tilde{C}_m = \frac{F_1G_1+F_2G_2}{G_0}\,.$$

Thus, from (3.4), (3.7), (**??**), (**??**) we get the following:

$$\frac{\partial\lambda_1^{\pm}}{\partial\varepsilon}(0) = \frac{\partial\lambda_4^{\mp}}{\partial\varepsilon}(0) = \frac{\partial\lambda_2^{\mp}}{\partial\varepsilon}(0) = \frac{\partial\lambda_3^{\pm}}{\partial\varepsilon}(0) = \frac{F_1G_1+F_2G_2-K_0-K_2^+}{\sigma_1G_1+\sigma_2G_2\pm\sigma_0G_0}\,. \quad (4.3) \quad \boxed{\texttt{contiderivo1}}$$

So we can draw explicitly the bifurcation diagram of (1.1), and our description is accurate at least at the first order.

If we replace f_0 in (4.1) by

$$f_0(x,y,\varepsilon,\lambda) := x_0^2 - (\sigma_0\lambda) + \alpha y_1 y_2 + \omega_{sn}(x,y,\varepsilon,\lambda)$$

where $\omega_{sn}(x,y,\varepsilon,\lambda) = O(|y||x|) + o(\varepsilon^2 + \lambda + |x_0|^2 + |x_{\hat{0}}|)$ and $\sigma_0 > 0$ we have a saddle-node bifurcation. We recall that in the saddle-node case we always have that the line $\lambda = q(\varepsilon)$ satisfies $q'(0) = 0$. Once again condition (v) is satisfied whenever $G_0 \neq 0$, and using (3.10) we find

$$\frac{\partial \nu_1^{\pm}}{\partial \varepsilon}(0) = \mp \frac{F_1G_1 + F_2G_2 - K_0 - K_2^+}{\sqrt{\sigma_0}G_0} = \frac{\partial \nu_2^{\mp}}{\partial \varepsilon}(0) \tag{4.4}$$

contoderivo2

So we can draw the bifurcation diagram of (1.1), also in this case.

If we replace f_0 in (4.1) by

$$f_0(x,y,\varepsilon,\lambda) := (x_0 - \tilde{\sigma}_0\lambda)(x_0^2 - \sigma_0\lambda) + \alpha y_1 y_2 + \omega_p(x,y,\varepsilon,\lambda)$$

where $\omega_p(x,y,\varepsilon,\lambda) = 0(|y||x|) + o(\varepsilon^2 + \lambda^2 + |x_0|^3 + |x_{\hat{0}}|^2)$ and $\sigma_0 > 0$, we have a pitchfork bifurcation. So we find

$$\begin{aligned}
&\frac{\partial \nu_1^{1,u}}{\partial \varepsilon}(0) = \frac{\partial \nu_2^{2,u}}{\partial \varepsilon}(0) = -\frac{K_0 + K_2^+ + \alpha G_0 - F_1G_1 - F_2G_2}{\sqrt{\sigma_0}G_0} = -\frac{\partial \nu_1^{2,u}}{\partial \varepsilon}(0) = -\frac{\partial \nu_2^{1,u}}{\partial \varepsilon}(0)\\
&\frac{\partial \lambda_i^s}{\partial \varepsilon}(0) = -\frac{K_0 + K_2^+ - F_1G_1 - F_2G_2}{\sigma_1G_1 + \sigma_2G_2 + \tilde{\sigma}_0G_0} \qquad \text{for } i = 1,2,3,4
\end{aligned} \tag{4.5}$$

contoderivo3

Thus we obtain the bifurcation diagram of (1.1) in this case, too.

References

B [1] F. Battelli, *Heteroclinic orbits in singular systems: a unifying approach*, J. Dynamics Diff. Eqns., **6** (1994), 147–173.

BFec [2] F. Battelli and M. Fečkan, *Global centre manifolds in singular systems*, NoDEA, **3** (1996), 19–34.

BPD [3] F. Battelli and K.J. Palmer, *Chaos in the Duffing equation*, J. Diff. Eqns., **101** (1993), 276–301.

BP1 [4] F. Battelli and K.J. Palmer, *Transverse intersection of invariant manifolds in singular systems*, J. Diff. Eqns., **177** (2001), 77–120.

BP2 [5] F. Battelli and K.J. Palmer, *Singular perturbations, transversality, and Sil'nikov saddle focus homoclinic orbits*, J. Dynamics Diff. Eqns., **15** (2003), 357–425.

BPnew [6] F. Battelli and K.J. Palmer, *Heteroclinic connections in singularly perturbed systems,* D.C.D.S., **9** n. 3/4 (2008), 431–461.

C [7] W.A. Coppel, "Dichotomies in Stability Theory," Lecture Notes in Math., Vol. 629, Springer Verlag, Berlin, 1978.

KL [8] H. Kurland and M. Levi, *Transversal heteroclinic intersections in slowly varying systems* in "Dynamical Systems Approaches to Nonlinear Problems in Systems and Circuits" (eds. F.A.Salam and M. Levi), SIAM, (1988), 29-38.

P2 [9] K.J. Palmer, *Existence of a transversal homoclinic point in a degenerate case*, Rocky Mt. J. Math., **20** (1990), 1099–1118.

Pbook [10] K.J. Palmer: *Shadowing in Dynnamical systems. Theory and Applications*, **501**, Kluwer Academic Publishers, Dordrecht (2000).

Saka [11] K. Sakamoto, *Invariant manifolds in singular perturbation problems for ordinary differential equations*, Proc. Royal Soc. Edinburgh Sect. A, **116** (1990), 45–78.

SZ [12] P. Szmolyan, *Transversal heteroclinic and homoclinic orbits in singular perturbation problems*, J. Diff. Eqns., **92** (1991), 252–281.