Title: Positive solutions for semilinear elliptic equations: two simple models with several bifurcations.

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POSITIVE SOLUTIONS FOR SEMILINEAR ELLIPTIC EQUATIONS: TWO SIMPLE MODELS WITH SEVERAL BIFURCATIONS.

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Abstract. In this paper we analyze the structure of positive radial solutions for the following semi-linear equations:

$$\Delta u + f(u, |\mathbf{x}|) = 0$$

where $\mathbf{x} \in \mathbb{R}^n$ and f is superlinear. In fact we just consider two very special non-linearities, i.e.

$$f(u, |\mathbf{x}|) = u|u|^{q-2} \max\{|\mathbf{x}|^{\delta^s}, |\mathbf{x}|^{\delta^u}\} \qquad -2 < \delta^u < \lambda^* < \delta^s < \lambda_*, \tag{0.1}$$

i.e. f is supercritical for $|\mathbf{x}|$ small and subcritical for $|\mathbf{x}|$ large, and

$$f(u) = \max\{u|u|^{q^s-2}, u|u|^{q^u-2}\}, \quad 2_* < q^s < 2^* < q^u \tag{0.2}$$

i.e. f is subcritical for u small and supercritical for u large.

We find a surprisingly rich structure for both the non-linearities, similar to the one detected by Bamon, et al. for $f = u^{q^u-1} + u^{q^s-1}$ when $2_* < q^s < 2^* < q^u$. More precisely if we fix q^s and we let q^u vary in (0.2) we find that there are no ground states for q^u large, and an arbitrarily large number of ground states with fast decay as q^u approaches 2^*. We also find the symmetric result when we fix q^u and let q^s vary. We also prove the existence of a further resonance phenomenon which generates small windows with a large number of ground states with fast decay. Similar results hold for (0.1).

MSC (2000): 35J05, 34B18,37B55

Key words: Radial solutions, mixed nonlinearities, ground states, invariant manifolds.

Dedicated to Professor Russell Johnson on the occasion of his sixtieth birthday.

1. Introduction

Our purpose is to shed some light on the structure of positive radial solutions for the following semi-linear elliptic equation:

$$\Delta u(\mathbf{x}) + f(u, |\mathbf{x}|) = 0 \tag{1.1}$$

where $\mathbf{x} \in \mathbb{R}^n$, $n > 2$ and f is supercritical for u large and $|\mathbf{x}|$ small and subcritical for $|u|$ small and $|\mathbf{x}|$ large. In fact in this paper we just consider two very special non-linearities which are particularly suitable to be studied with our methods, i.e.

$$f(u, |\mathbf{x}|) = u|u|^{q-2} \begin{cases} |\mathbf{x}|^{\delta^u} & \text{if } |\mathbf{x}| \le 1 \\ |\mathbf{x}|^{\delta^s} & \text{if } |\mathbf{x}| \ge 1 \end{cases}, \tag{1.2}$$

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where $q > 2$ and $-2 < \delta^u < \lambda^* < \delta^s < \lambda_*$, $\lambda_* := (n-2)[q - 2\frac{n-1}{n-2}] > \lambda^* := \frac{n-2}{2}[q - 2\frac{n}{n-2}]$; and

$$f(u) = \begin{cases} u|u|^{q^s-2} & \text{if } u \le 1 \\ u|u|^{q^u-2} & \text{if } u \ge 1 \end{cases}, \tag{1.3}$$

where $2_* := \frac{2n-2}{n-2} < q^s < 2^* := \frac{2n}{n-2} < q^u$.

We just consider radial solutions and we commit the following abuse of notation: we write $u(r)$ for $u(\mathbf{x})$ where $|\mathbf{x}| = r$. Since we only deal with radial solutions we consider in fact the following singular O.D.E.

$$u'' + \frac{n-1}{r}u' + f(u,r) = 0. \tag{1.4}$$

Here $'$ denotes the derivative with respect to r. We call "regular" positive solutions $u(d,r)$ of (1.4) satisfying $u(0) = d > 0$ and $u'(0) = 0$. We call "singular" positive solutions $u(r)$ which are singular in the origin, that is $\lim_{r\to 0} u(r) = +\infty$. In particular we focus our attention on the problem of existence of ground states (G.S.), of singular ground states (S.G.S.) and of crossing solutions. By G.S. we mean a positive regular solution $u(r)$ defined for any $r = |\mathbf{x}| \ge 0$ such that $\lim_{r\to\infty} u(r) = 0$. A S.G.S. of equation (1.1) is a positive solution $v(r)$ such that $\lim_{r\to 0} v(r) = +\infty$ and $\lim_{r\to+\infty} v(r) = 0$. Crossing solutions are solutions $u(\mathbf{x})$ such that there is $R > 0$ for which $u(\mathbf{x}) > 0$ for any $0 \le |\mathbf{x}| < R$ and $u(\mathbf{x}) = 0$ for $|\mathbf{x}| = R$, so they can be considered as solutions of the Dirichlet problem in the ball of radius R. We are also concerned with the asymptotic behavior of positive solutions: we say that a positive solution $u(r)$ has fast decay (f.d.) if $u(r)r^{n-2}$ has positive finite limit as $r \to \infty$ and that it has slow decay (s.d.) if $\lim_{r\to+\infty} u(r)r^{n-2} = \infty$. When f is either of type (1.2) or of type (1.3) we can give very precise estimates of the asymptotic behavior of singular and s.d. solutions, see below.

In literature there are many results concerning (1.1) in the subcritical case, e.g. when f has the form

$$f(u,r) = k(r)u|u|^{q-2} \tag{1.5}$$

and $k(r)r^{-\lambda^*}$ is non-decreasing and non-constant. Such a case is usually analyzed through variational techniques, or exploiting the Pohozaev identity, which is a clever way to restate Green's formula, see e.g. [25]. In fact these methods allow to study the problem in non-radial domains as well, and in general to look for the existence of non-radial solutions. However, if the domain is radial (e.g. the whole $\mathbb{R}^n$), G.S., S.G.S and crossing solutions inherit this symmetry, under very mild assumptions, see [5, 28]. In particular this is the case for all the equations we discuss in this paper, see [5] for f of type (1.2) and [28] for f of type (1.3), and again [5] for a remarkable counterexample. Here we follow the way paved by Johnson, Pan, Yi [19, 20, 21] and later followed also by Battelli [3], Franca [16, 14, 15], Bamon et al., and Flores [2, 13]. So we introduce a change of variable, known as Fowler transformation, and we turn to consider a two dimensional dynamical system, which is suitable to be studied through dynamical tools, such as invariant manifold theory and phase diagrams. This approach cannot be adapted to study non radial solutions, but it is particularly helpful to analyze the spatial dependent case, singular solutions and to discuss the supercritical case, e.g. when f is of type (1.5) and $k(r)r^{-\lambda^*}$ is non-increasing. In fact in this latter setting variational techniques are difficult to be applied and the analysis is usually restricted to radial solutions, as far as we are aware.

It is well known that in the subcritical case all the regular solutions are crossing solutions, there are uncountably many S.G.S. with f.d. and one S.G.S. with s.d., see e.g. [14] also for a more general definition of subcriticality in this context. In the

supercritical case all the regular solutions are G.S. with s.d., there are uncountably many solutions of the Dirichlet problem in exterior domains, and one S.G.S. with s.d., see [14]. The situation becomes more interesting and challenging when f exhibits both subcritical and supercritical behaviors. A remarkable case in which this happens is given by the scalar curvature equation, i.e. f of type (1.5) and $q = 2^*$. This setting lies on the border between the subcritical and the supercritical ones and is very sensitive to the behavior at $r = 0$ and at $r = \infty$ of the function k, see in particular [5]. In [4, 5, 6, 21, 3, 16] are given rather natural conditions for positive solutions to have structure **Mix** described below and even richer structures, which collect features of both the subcritical and the supercritical case.

Another case, well studied in literature, is the one in which f exhibits supercritical behavior for u small and subcritical for u large. In this setting the solutions of (1.4) typically have the following structure:

Mix: There is $d^* > 0$ such that $u(d, r)$ is a crossing solutions for $d > d^*$, a G.S. with f.d. if $d = d^*$ and a G.S. with s.d. if $0 < d < d^*$. Furthermore there are uncountably many S.G.S. with f.d. and S.G.S. with s.d.

In fact this is a quite general feature, and this structure has been found for a rather large family of spatial dependent nonlinearity see [22, 8, 9, 27, 15], also for the p-Laplace case. However for the context discussed in this paper, these results reduces to the following Theorem.

1.1. **Theorem.** *Consider (1.4) and assume either f of type (1.3) and $2_* < q^u < 2^* < q^s$ or f of type (1.2), $q > 2$ and $-2 < \delta^s < \lambda^* < \delta^u < \lambda_*$. Then positive solutions have a structure of type **Mix**. If f is of type (1.3) and $2 < q^u < 2^* < q^s$ or f of type (1.2), $q > 2$ and $-2 < \delta^s < \lambda^* < \delta^u$ we lose the result concerning singular solutions but regular solutions continue to have structure **Mix**.*

See [8, 9] for the (1.3) case, [22] for the conditions for the uniqueness of the G.S. with f.d., [27] for the (1.2) case, and [15] for both (1.2) and (1.3). We stress that in [15] a unifying approach, similar to the one exploited here, has been used which allows to insert f of type (1.2) and (1.3) in a larger family and to discuss singular solutions, too.

In this paper we consider the opposite situations: we assume that f is subcritical for u small and supercritical for u large. This setting seems to be less understood: this is probably due to the fact that the structure of positive solutions undergoes to at least two different types of bifurcations, as we will see below.

As far as we are aware in literature it is discussed just the case where $f(u) = u|u|^{q^s-2} + u|u|^{q^u-2}$, where $2_* < q^s < 2^* < q^u$, and results appeared rather recently. In [2] Bamon et al. proved the following very interesting result.

1.2. **Theorem.** *Consider equation (1.4) where $f(u) = u|u|^{q^s-2} + u|u|^{q^u-2}$. Fix $q^s \in (2_*, 2^*)$, then for any $k \in \mathbb{N}$ there is $\bar{\epsilon}_k(q^s) > 0$ such that (1.4) admits at least k G.S. with f.d. for any $q^u \in (2^*, 2^* + \bar{\epsilon}_k]$.*

Analogously fix $q^u > 2^*$, then for any $k \in \mathbb{N}$ there is $\bar{\epsilon}_k(q^u) > 0$ such that (1.4) admits at least k G.S. with f.d. for any $q^s \in (2^* - \bar{\epsilon}_k, 2^*)$.

The proof is achieved through a dynamical approach similar to the one used in this paper. In fact Campos in [7] gave a different proof of the same result using a variational argument on the equation obtained via Fowler transformation. He also showed that the G.S. with f.d. found through Theorem 1.2 can be approximated by a finite sum of translates of the G.S. obtained in the critical case (for which there is an explicit formula).

We recall that in fact $\bar{\epsilon}_k$ cannot be too large. In fact in [2] the authors also proved the following:

1.3. **Theorem.** *Consider equation (1.4) where* $f(u) = u|u|^{q^s-2} + u|u|^{q^u-2}$. *Fix* $q^u > 2^*$, *then there is* $\bar{\epsilon}_0(q^u) > 0$ *such that (1.4) admits no G.S. with f.d. for any* $q^s \in [2_*, 2_* + \bar{\epsilon}_0(q^u))$.

We stress that these results have perturbative nature and give no clue to detect "how large" the $\bar{\epsilon}_k$ and $\bar{\epsilon}_0$ are. The first result for this type of non-linearity is due to Ni who proved (by explicit calculation) the existence of a G.S. with s.d. when $q^u = 2(q^s - 1)$, see [24]. As it is observed in [13] such a result is non generic, even if it appears for particular values of the parameters also for f of type (1.3) and (1.2). However the existence of this "rare" solutions gains more relevance by a further resonance phenomenon explained by Flores in [13], which makes the situation more complex. Let us denote by $\sigma^* := 2\frac{n-2\sqrt{n-1}-2}{n-2\sqrt{n-1}-4}$ if $n > 10$ and $+\infty$ if $n \leq 10$ and by $\sigma_* := 2\frac{n+2\sqrt{n-1}-2}{n+2\sqrt{n-1}-4}$ (the origin of these numbers will be explained in section 2). Observe that $2_* < \sigma_* < 2^* < \sigma^*$.

1.4. **Theorem.** *Consider equation (1.4) where* $f(u) = u|u|^{q^s-2} + u|u|^{q^u-2}$.

- **(a):** *Assume (1.4) admits a G.S. with s.d. for* $2_* < \bar{q}^s < 2^* < \bar{q}^u < \sigma^*$. *Then there are infinitely many G.S. with f.d.*
- **(b):** *Assume (1.4) admits a S.G.S. with f.d. for* $\sigma_* < \bar{q}^s < 2^* < \bar{q}^u$. *Then there are infinitely many G.S. with f.d.*
- **(c):** *If* $\bar{q}^s < \bar{q}^u$ *satisfy either* ***(a)*** *or* ***(b)*** *then for any* $k \in \mathbb{N}$, $k \geq 1$, *there is* $\eta_k > 0$ *such that (1.4) admits at least k G.S. with f.d. whenever* $|q^u - \bar{q}^u| + |q^s - \bar{q}^s| < \eta_k$.

We will see that for f of type (1.3) and (1.2) we can produce results analogous to Theorem 1.2, 1.3, 1.4, and to complete the symmetry of the non-existence result 1.3 by fixing q^s and letting q^u tend to $+\infty$. In fact we are also able to give conditions for the existence of the "rare" G.S. with s.d. (similarly to Ni's result) and S.G.S. with f.d. for our f. More precisely we prove the following results for f of type (1.2).

1.5. **Theorem.** *Consider f of type (1.2). For any* $\delta^s \in (\lambda^*, \lambda_*)$ *we can find* $\epsilon_1(\delta^s) > 0$ *such that (1.4) admits at least a G.S. with f.d. whenever* $\delta^u \in (\lambda^* - \epsilon_1(\delta^s), \lambda^*)$. *Analogously for any* $-2 < \delta^u < \lambda^*$ *we can find* $\epsilon_1(\delta^u) > 0$ *such that (1.4) admits at least a G.S. with f.d. whenever* $\delta^s \in (\lambda^*, \lambda^* + \epsilon_1(\delta^u))$.

1.6. **Theorem.** *Consider f of type (1.2). For any* $\delta^s \in (\lambda^*, \lambda_*)$ *we can find* $\epsilon_k(\delta^s) > 0$ *such that (1.4) admits at least k G.S. with f.d. whenever* $\delta^u \in (\lambda^* - \epsilon_k(\delta^s), \lambda^*)$. *Analogously for any* $-2 < \delta^u < \lambda^*$ *we can find* $\epsilon_k(\delta^u) > 0$ *such that (1.4) admits at least k G.S. with f.d. whenever* $\delta^s \in (\lambda^*, \lambda^* + \epsilon_k(\delta^u))$.

1.7. **Theorem.** *Consider f of type (1.2). Fix* $\lambda^* < \delta^s < \lambda_*$, *we can find* $N_0(\delta^s) > 0$ *such that (1.4) admits no positive solutions either regular or singular whenever* $\delta^u \in (-2, -2 + N_0(\delta^s))$.

Analogously fix $-2 < \delta^u < \lambda^*$, *we can find* $\epsilon_0(\delta^u) > 0$ *such that (1.4) admits no positive solutions either regular or singular whenever* $\lambda_* - \epsilon_0(\delta^u) < \delta^s < \lambda_*$.

We introduce now two further critical values: $\Sigma_* := \frac{2}{\sigma_*-2}(q - \sigma_*)$ and $\Sigma^* := \frac{2}{\sigma^*-2}(q - \sigma^*)$ (we set $\Sigma^* = -2$ when $\sigma^* = +\infty$, i.e. $n \leq 10$).

1.8. **Theorem.** *Consider equation (1.4) with f of type (1.2). Fix* $\delta^u \in (-2, \lambda^*)$; *we can find a decreasing sequence of values* $r_j(\delta^u) \in (\lambda^*, \Sigma_*)$, $r_j(\delta^u) \to \lambda^*$ *as* $j \to \infty$, *such that (1.4) with* $\delta^s = r_j(\delta^u)$ *admits one S.G.S. with f.d.*

Analogously fix $\lambda^* < \delta^s < \lambda_*$, *then we can find an increasing sequence of values* $r_j(\delta^u) \in (\Sigma^*, \lambda^*)$, $r_j(\delta^u) \to \lambda^*$ *as* $j \to \infty$, *such that (1.4) with* $\delta^u = r_j(\delta^s)$ *admits one G.S. with s.d.*
Moreover for any fixed $j \in \mathbb{N}$, $r_j(\delta^u) \to \lambda^*$ *as* $\delta^u \to -2$ *and* $r_j(\delta^s) \to \lambda^*$ *as* $\delta^s \to \lambda_*$.

1.9. **Corollary.** *Consider equation (1.4) with f of type (1.2) for $\delta^s = \hat{\delta}^u$ and $\delta^u = \hat{\delta}^u$. Assume that (1.4) either admits a G.S with s.d. and $\hat{\delta}^s \in (\lambda^*, \Sigma_*)$ or a S.G.S with f.d. and $\hat{\delta}^u \in (\Sigma^*, \lambda^*)$. Then (1.4) admits infinitely many G.S. with f.d. too. Furthermore for any integer $k > 0$ there is $\eta(k) > 0$ small enough so that (1.4) admits at least k G.S. with f.d. whenever $|\delta^s - \hat{\delta}^u| + |\delta^u - \hat{\delta}^s| < \eta(k)$.*

Moreover we prove the following results for f of type (1.3).

1.10. **Theorem.** *Consider f of type (1.3). Fix $q^u > 2^*$, then there is $\epsilon_1(q^u) > 0$ such that (1.4) admits at least one G.S. with f.d. whenever $q^s \in (2^* - \epsilon_1(q^u), 2^*)$. Analogously fix $q^s \in (2_*, 2^*)$, then there is $\epsilon_1(q^s) > 0$ such that (1.4) admits at least one G.S. with f.d. whenever $q^u \in (2^*, 2^* + \epsilon_1(q^s))$.*

1.11. **Theorem.** *Consider f of type (1.3). For any $q^u > 2^*$ we can find $\varepsilon_k(q^u) > 0$ such that (1.4) admits at least k G.S. with f.d. whenever $q^s \in (2^* - \varepsilon_k(q^u), 2^*)$. Analogously for any $q^s < 2^*$ we can find $\varepsilon_k(q^s) > 0$ such that (1.4) admits at least k G.S. with f.d. whenever $q^u \in (2^*, 2^* + \varepsilon_k(q^s))$.*

1.12. **Theorem.** *Consider f of type (1.3). Fix $q^u > 2^*$, then there is $\epsilon_0(q^u) > 0$ such that (1.4) admits no solutions $u(r)$ positive for any $r > 0$, either regular or singular, for any $q^s \in (2_*, 2_* + \epsilon_0(q^u))$. Analogously fix $2_* < q^s < 2^*$, then there is $N_0(q^s) > 2^*$ such that (1.4) admits no solutions $u(r)$ positive for any $r > 0$, either regular or singular for any $q^u > N_0(q^s)$.*

1.13. **Theorem.** *Consider f of type (1.3). Fix $q^s \in (2_*, 2^*)$; we can find a decreasing sequence of values $r^j(q^s) \in (2^*, \sigma^*)$, $r^j(q^s) \to 2^*$ as $j \to \infty$, such that (1.4) with $q^u = r^j(q^s)$ admits one G.S. with s.d.*

Analogously fix $q^u > 2^*$, then we can find an increasing sequence of values $r^j(q^u) \in (\sigma_*, 2^*)$, $r^j(q^u) \to 2^*$ as $j \to \infty$, such that (1.4) with $q^s = r^j(q^u)$ admits one S.G.S. with f.d.
Moreover for any fixed $j \in \mathbb{N}$, $r^j(q^s) \to 2^*$ as $q^s \to 2_*$ and $r^j(q^u) \to 2^*$ as $q^u \to +\infty$.

1.14. **Corollary.** *Consider f of type (1.3) where $q^u = \hat{q}^u$ and $q^s = \hat{q}^s$ and $2_* < \hat{q}^s < 2^* < \hat{q}^u$. Assume that (1.4) either admits a G.S with s.d. and $\hat{q}^s \in (\sigma_*, 2^*)$ or a S.G.S with f.d. and $\hat{q}^u \in (2^*, \sigma^*)$. Then it admits infinitely many G.S. with f.d. too. Furthermore for any integer $k > 0$ there is $\eta(k) > 0$ small enough so that (1.4) admits at least k G.S. with f.d. whenever $|q^u - \hat{q}^u| + |q^s - \hat{q}^s| < \eta(k)$.*

All the proofs will be performed on the auxiliary systems (2.2), (2.3) and (2.4). In fact the constants ϵ_k, for $k = 0, 1, \ldots$ and N_0, appearing in Theorems concerning (1.2) and in (1.3) are functions of the same constants of these systems and have a precise geometrical meanings. The proof of Theorems 1.5,1.6,1.7,1.10 and 1.12 are developed with sharper constants (ϵ_k and N_0) denoted with $\tilde{}$, but we preferred to give the Theorems in these forms for homogeneity and also because we think that in this form the results might be extended to a larger family of functions f.

We stress that our proofs are quite constructive and this gives a hint to perform a rather easy computer assisted proof, to see how large the values ϵ_1 in Theorems 1.10 and 1.5, and ϵ_0 and N_0 in 1.7 and 1.12 are, see Remark 3.6 for details. Moreover we find analytically values for which the nonexistence result holds, see Corollaries 3.4 and 4.6.

We point out that Theorem 1.5 can be deduced from Theorem 1.6 and in fact they are proved together. However we preferred to distinguish them since, exploiting Remark 3.6, we could give a positive lower bound for the values ϵ_1, while we cannot for ϵ_k when $k \geq 2$. However Theorem 1.10 cannot be deduced from Theorem 1.11. In fact $\epsilon_1(l^s) > \varepsilon_1(l^s)$ and $\epsilon_1(l^u) > \varepsilon_1(l^u)$, and again Remark 3.6 allows to estimate ϵ_1 while it does not work for ε_1.

Note that Theorems 1.6 and 1.11 are the analogous of Theorem 1.2, while Theorems 1.7 and 1.12 are the analogous of Theorem 1.3 (even if they include also its symmetric counterpart), Corollaries 1.9 and 1.14 are the analogous of Theorem 1.4 (and in fact their proof is strongly inspired by the proof of Theorem 1.4). Furthermore Theorems 1.8 and 1.13 are analogous to the existence result for G.S. with s.d. found by Ni (even if they include also its symmetric counterpart: existence of S.G.S. with f.d.).

Also for these similarities we regard this paper as the second step in the comprehension of equation (1.1) with f supercritical for u large and $|\mathbf{x}|$ small and subcritical for u small and $|\mathbf{x}|$ large. In fact we think that this rich structure is a general feature for f with this characteristic.

After this paper was submitted we knew about two interesting papers [1, 10] related to the topic studied here. They both consider $f(u,r) = k(r)u|u|^{q-2}$, where $k(r) = r^{\delta^s} + r^{\delta^u}$. In [1] the authors prove the analogous of theorem 1.6 for this equations, using variational methods (so they also have an estimate of G.S. with f.d. in terms of translates of the solutions of the critical case, but they cannot estimate the smallness of the parameter ϵ_k involved). They also conjectured the existence of the same result for f of type (1.3), which is proved in Theorem 1.11 in this paper. In [10] the authors manage to prove the coexistence of G.S. with s.d. and of S.G.S. with f.d. for particular values of the parameters and special functions $k(r)$. As a consequence they also find two different sequences of G.S. with f.d. So they prove a result related to both the existence parts of Theorem 1.8 (in fact they let both the parameters vary together and found the most topologically complex situation: a double intersection between stable and unstable manifold). They exploit topological methods starting from a new idea: they let the so called "natural dimension" change values.

For completeness we recall that (1.4) has a subcritical behavior (i.e. all the regular solutions are crossing solutions, there are uncountably many S.G.S. with f.d. and one S.G.S. with s.d.) whenever f is of type (1.2) and $\lambda^* \le \delta^u < \delta^s < \lambda_*$, or f is of type (1.3) and $2_* < q^s < q^u \le 2^*$; while it has a supercritical behavior (i.e. all the regular solutions are G.S. with s.d., there is one S.G.S. with s.d.) whenever f is of type (1.2) and $-2 < \delta^u < \delta^s \le \lambda^*$, or f is of type (1.3) and $2^* \le q^s < q^u$.

The paper is divided as follows. In section 2 we introduce the Fowler transformation, and we review some known results concerning the autonomous case, such as the existence of unstable and stable manifold M^u and M^s. We also introduce some of the tools used in the following sections, such as Kelvin transformation, which in our setting works as an inversion of the time "variable" t. In section 3 we discuss (1.4) when f is of type (1.2) and we prove Theorems 1.5, 1.6, 1.7, 1.8, and Corollary 1.9. In section 4 we discuss the case where f is of type (1.3) and we prove Theorems 1.10, 1.11, 1.12, 1.13, and Corollary 1.14. At the end of sections 3 and 4 we also derive non-existence, existence and multiplicity results for the corresponding Dirichlet problem in balls of radius R.

2. Fowler transformation and dynamical interpretation of classical results

We begin this section by introducing the Fowler transformation for the Laplace operator, which changes equation (1.4) in a two dimensional dynamical system. Let us set

$$\alpha_l = \tfrac{2}{l-2}, \qquad \beta_l = \tfrac{l}{l-2}, \qquad \gamma_l = \beta_l - (n-1), \qquad l > 2$$
$$x_l = u(r)r^{\alpha_l} \quad y_l = u'(r)r^{\beta_l} \quad r = e^t \tag{2.1}$$

then (1.4) with $f(u,r) = r^\delta u|u|^{q-2}$, and $\delta > -2$ becomes

$$\begin{pmatrix} \dot{x}_l \\ \dot{y}_l \end{pmatrix} = \begin{pmatrix} \alpha_l & 1 \\ 0 & \gamma_l \end{pmatrix} \begin{pmatrix} x_l \\ y_l \end{pmatrix} + \begin{pmatrix} 0 \\ -e^{[\delta+2(l-q)/(l-2)]t} x_l|x_l|^{q-2} \end{pmatrix} \tag{2.2}$$

Thus, setting $l = l(\delta) := 2\frac{\delta+q}{\delta+2}$, we obtain the following autonomous system

$$\begin{pmatrix} \dot{x}_l \\ \dot{y}_l \end{pmatrix} = \begin{pmatrix} \alpha_l & 1 \\ 0 & \gamma_l \end{pmatrix} \begin{pmatrix} x_l \\ y_l \end{pmatrix} + \begin{pmatrix} 0 \\ -x_l|x_l|^{q-2} \end{pmatrix} \tag{2.3}$$

Observe that when $\delta = 0$ we can simply set $l = q$ to obtain

$$\begin{pmatrix} \dot{x}_q \\ \dot{y}_q \end{pmatrix} = \begin{pmatrix} \alpha_q & 1 \\ 0 & \gamma_q \end{pmatrix} \begin{pmatrix} x_q \\ y_q \end{pmatrix} + \begin{pmatrix} 0 \\ -x_q|x_q|^{q-2} \end{pmatrix} \tag{2.4}$$

Note that $\alpha_l > 0$ whenever $l > 2$ and $\gamma_l < 0$ whenever $l > 2_*$. Moreover $\alpha_l + \gamma_l$ has the same sign as $2^* - l$ and it is null if and only if $l = 2^*$. We wish to recall that in [17] the authors introduced a change of variables which allows to change solutions $u(r)$ of eq. (1.4) with $f(u) = u|u|^{q-2}$ into solutions $v(r)$ of eq. (1.4) with $f(v) = r^\delta v|v|^{q-2}$, where the dimension n of the variable x has been changed into the so called "natural dimension" $N := 2\frac{n+\delta}{2+\delta}$. The change of variables of [17], which works more in general for spatial dependent p-Laplace equations, is in fact equivalent to the change of parameter from q to l in (2.1), in this context. Clearly this change in the dimension n affects the critical values 2_* and 2^*, so we can either evaluate them replacing n by N, or we can maintain their values and compare them with l instead of q (in this paper we follow this second idea).

The following notation will be in force throughout all the paper. We use bold letters for vectors and normal letters for scalars; we write $\mathbf{x_{\bar{l}}}(t,\tau;\mathbf{Q})$ for a trajectory of (2.2) or (2.3) where $l = \bar{l}$, departing from $\mathbf{Q} \in \mathbb{R}^2$ at $t = \tau$. The following sets will be often used in the whole paper.

$$\mathbb{R}^2_+ := \{(x_l, y_l) \mid x_l \geq 0\} \qquad \mathbb{R}^2_\pm := \{(x_l, y_l) \mid y_l < 0 < x_l\}$$

$$A^0_l := \{(x_l, y_l) \in \mathbb{R}^2_\pm \mid \alpha_l x_l + y_l = 0\}$$

$$A^+_l := \{(x_l, y_l) \in \mathbb{R}^2_\pm \mid \alpha_l x_l + y_l > 0\} \qquad A^-_l := \{(x_l, y_l) \in \mathbb{R}^2_\pm \mid \alpha_l x_l + y_l < 0\}$$

System (2.3) admits three critical points for $q > 2$ and $l > 2_*$: the origin $O = (0,0)$, $\mathbf{P}(l) = (P_x(l), P_y(l))$, where $P_y(l) < 0 < P_x(l)$, and $-\mathbf{P}(l)$, where $P_x(l) = (-\gamma_l\alpha_l)^{\frac{1}{q-2}}$ and $P_y(l) = -[-\gamma_l(\alpha_l)^{q-1}]^{\frac{1}{q-2}}$. The origin is a saddle point and it admits a one-dimensional stable manifold $M^s_l(q)$ and a one-dimensional unstable manifold $M^u_l(q)$, which in the origin are tangent respectively to the x axis and the line $y = -(n-2)x$, corresponding respectively to A^0_∞ and to $A^0_{2_*}$. Since we are just interested in positive solutions $u(r)$ of (1.4), we will commit the following abuse of notation: we call stable and unstable manifold $M^s_l(q)$ and $M^u_l(q)$ the branch which departs from the origin and get into $\mathbb{R}^2_\pm$ deprived of the origin.

The critical point $\mathbf{P}(l)$ is asymptotically stable if $l > 2^*$, asymptotically unstable if $2_* < l < 2^*$ and a center if $l = 2^*$, for any $q > 2$. Let us denote by $\sigma_* < \sigma^*$ the roots of $(\alpha_l + \gamma_l)^2 + 4\alpha_l\gamma_l(q-1)$. It is easy to show that $\mathbf{P}(l)$ is a focus if $\sigma_* < l < \sigma^*$, and it is an unstable node if $2_* < l \leq \sigma_*$ and a stable node if $l \geq \sigma^*$; we stress that the values σ_*, σ^* are the ones defined just before the statement of Theorem 1.4. See figure 1 for a sketch of the phase portrait.

From some asymptotic estimates we deduce the following useful result, see [12] for the proof.

2.1. *Remark.* Regular solutions $u(r)$ of Eq. (1.4) correspond to trajectories $\mathbf{x_l}(t)$ of system (2.3) departing from points in $M^u_l(q)$ and viceversa. Positive solutions $u(r)$ with f.d. correspond to trajectories $\mathbf{x_l}(t)$ of system (2.3) departing from points in $M^s_l(q)$ and viceversa.

Moreover singular solutions $v(r)$ of (1.4) and slow decay solutions $u(r)$ correspond to the trajectories $\mathbf{x_l}(t) \equiv \mathbf{P}(l)$; thus $\lim_{r\to 0} v(r)r^{\alpha_l} = P_x(l)$ and $\lim_{r\to+\infty} v(r)r^{\alpha_l} = P_x(l)$.

Observe that system (2.3) is invariant for translations in t. Therefore if $X(t)$ is a solution, $X^\tau(t) = X(t+\tau)$ is a solution as well. Equivalently if $u(r)$ is a solution of (1.4), then $v(r) = u(re^\tau)e^{\alpha\tau}$ is a solution as well.

We recall now two useful basic results.

2.2. *Remark.* Consider system (2.3); then $\mathbf{x_l}(t;\mathbf{Q})$ is well defined for any $t \in \mathbb{R}$, and any $\mathbf{Q} \in \mathbb{R}^2$. Moreover if $\lim_{t\to\pm\infty} \|\mathbf{x_l}(t;\mathbf{Q})\| = +\infty$, then $\mathbf{x_l}(t;\mathbf{Q})$ crosses the coordinate axes infinitely many times rotating clockwise.

2.3. *Remark.* Consider system (2.3) where $l = \bar{l} \in (2_*, 2^*)$ and choose $L \geq \bar{l}$. Then for any $\mathbf{Q} \in A_L^+$ there is $T(\mathbf{Q}) > 0$ such that $\mathbf{x_{\bar{l}}}(t,0;\mathbf{Q}) \in A_L^+$ whenever $0 < t < T(\mathbf{Q})$ and it crosses A_L^0 transversally in a point $\bar{\mathbf{Q}} = (X,Y)$ such that $x > P_x(L)$ at $t = T(\mathbf{Q})$.
Analogously let $l = \bar{l} > 2^*$, $L \leq \bar{l}$ and $\mathbf{Q} \in A_L^-$; then there is $T(\mathbf{Q}) < 0$ such that $\mathbf{x_{\bar{l}}}(t,0;\mathbf{Q}) \in A_L^-$ for any $T(\mathbf{Q}) < t < 0$ and it crosses A_L^0 transversally in a point $\bar{\mathbf{Q}} = (X,Y)$ such that $x > P_x(L)$ at $t = T(\mathbf{Q})$.

Proof. Remark 2.2 follows from the super-linearity of f and it is easily proved passing to polar coordinates, see Lemma 2.5 in [14] for a detailed proof in a more general context. We prove Remark 2.3. Assume first $L = \bar{l} \in (2_*, 2^*)$ and let $\mathbf{Q} \in A_L^+$; if $\lim_{t\to+\infty} |\mathbf{x_{\bar{l}}}(t;\mathbf{Q})| = +\infty$, then $\mathbf{x_{\bar{l}}}(t;\mathbf{Q})$ crosses the coordinate axes indefinitely so Remark 2.3 follows from Remark 2.2. Assume that $\mathbf{x_{\bar{l}}}(t;\mathbf{Q}) \in A_L^+$ for any $t \in [0,T)$; if $T = +\infty$, it follows that $x_{\bar{l}}(t;\mathbf{Q})$ is positive and increasing, so admits a limit: we find easily that $\lim_{t\to+\infty} \mathbf{x_{\bar{l}}}(t;\mathbf{Q}) = \mathbf{P}(\bar{l})$. But if $\bar{l} < 2^*$, $\mathbf{P}$ is a repulser so we have found a contradiction. It follows that $T < \infty$ and the flow of (2.3) on $\mathbf{Q} = \mathbf{x_{\bar{l}}}(T;\mathbf{Q})$ is transversal to A_L^0.

Now fix $L > \bar{l}$: from the previous argument for any $\mathbf{Q} \in A_L^+ \subset A_{\bar{l}}^+$ we find $T > 0$ such that $\mathbf{x_{\bar{l}}}(t;\mathbf{Q})$ crosses $A_{\bar{l}}^0$ at $t = T$. From a continuity argument then we find $T(L) \in (0,T)$ such that $\mathbf{x_{\bar{l}}}(t;\mathbf{Q})$ crosses A_L^0 at $t = T(L)$. We prove that such a crossing is transversal. We denote by $m(\mathbf{x_{\bar{l}}}) := \dot{y}_{\bar{l}}/\dot{x}_{\bar{l}}$ evaluated along A_L^0, i.e.

$$m(x, \alpha_L x) = \frac{\alpha_L \gamma_{\bar{l}} + x^{q-2}}{\alpha_L - \alpha_{\bar{l}}}\,. \tag{2.5}$$

Since $m(x,\alpha_L x)$ is monotone in x and the flow of (2.3) on A_L^0 points towards A_L^+ for $x > 0$ small enough, and towards A_L^- for x large enough, there is a unique point of A_L^0 in which the flow of (2.3) is tangent to A_L^0; from a straightforward computation we find that the tangency point is in fact $\mathbf{P}(L)$. From a simple analysis of the phase portrait of (2.3) we find that $\mathbf{x_{\bar{l}}}(T(L);\mathbf{Q}) = (C_x, C_y)$ where $C_x > P_x(L)$ so the crossing is transversal. The proof when $\bar{l} > 2^*$ is completely analogous and will be omitted. □

Using the previous argument and the fact that (2.3) admits no critical points in A_L^0 for $L \neq l$ we get the following.

2.4. *Remark.* Consider system (2.3) where $l = l_u > 2^*$ and choose $L^u > l_u$. Then for any $\mathbf{Q} \in A_{L^u}^+$ there is $T(\mathbf{Q}) > 0$ such that $\mathbf{x_{l_u}}(t,0;\mathbf{Q}) \in A_{L^u}^+$ whenever $0 < t < T(\mathbf{Q})$ and it crosses $A_{L^u}^0$ transversally in a point $\bar{\mathbf{Q}} = (X,Y)$ such that $x > P_x(L^u)$ at $t = T(\mathbf{Q})$.
Analogously let $l = l_s \in (2_*, 2^*)$, $L^s \in (2_*, l_s)$ and $\mathbf{Q} \in A_{L^s}^-$; then there is $T(\mathbf{Q}) < 0$ such that $\mathbf{x_{l_s}}(t,0;\mathbf{Q}) \in A_{L^s}^-$ for any $T(\mathbf{Q}) < t < 0$ and it crosses $A_{L^s}^0$ transversally in a point $\bar{\mathbf{Q}} = (X,Y)$ such that $x > P_x(L^s)$ at $t = T(\mathbf{Q})$.

Using the t- invariance property of (2.3) we easily get the following.

2.5. *Remark.* Let $u(d,r)$ be a regular solution of (1.4) and $\mathbf{x_l}(t,\tau;\mathbf{Q^u})$ the corresponding trajectory of (2.3), so that $\mathbf{Q^u}\in M_l^u$. Then d is a continuous monotone function of τ such that $d(\tau)\to+\infty$ as $\tau\to-\infty$ and $d(\tau)\to 0$ as $\tau\to+\infty$, and viceversa. Furthermore if we fix τ, $d(\mathbf{Q})\to 0$ as $\mathbf{Q}\to(0,0)$ and viceversa, and if $q<2^*$ then $d(\mathbf{Q})\to+\infty$ as $\mathbf{Q}$ tends to the critical point $\mathbf{P}$.

Analogously let $v(L,r)$ be a f.d. solution of (1.4) such that $\lim_{r\to+\infty}v(L,r)r^{n-2}=L>0$, and $\mathbf{x_l}(t,\tau;\mathbf{Q^s})$ the corresponding trajectory of (2.2) such that $\mathbf{Q^s}\in M_l^s$. Then L is a smooth monotone function of τ such that $L(\tau)\to+\infty$ as $\tau\to-\infty$ and $L(\tau)\to 0$ as $\tau\to+\infty$, and viceversa. Furthermore if we fix τ, $L(\mathbf{Q})\to 0$ as $\mathbf{Q}\to(0,0)$ and viceversa, and if $q>2^*$ then $L(\mathbf{Q})\to+\infty$ as $\mathbf{Q}$ tends to the critical point $\mathbf{P}$.

Now we see what happens if we switch between different values of l in (2.2). Let $u(r)$ be a solution of (1.4) and $\mathbf{x_{l_1}}(t,\tau;\mathbf{Q^1})$ and $\mathbf{x_{l_2}}(t,\tau;\mathbf{Q^2})$ be the corresponding trajectories of (2.2) with $l=l_1$ and $l=l_2$. Then we denote by $\aleph^t_{l_2,l_1}(\mathbf{x})$ the smooth family of linear maps such that $\aleph^t_{l_2,l_1}\big(\mathbf{x_{l_1}}(t,\tau;\mathbf{Q^1})\big)=\mathbf{x_{l_2}}(t,\tau;\mathbf{Q^2})$, that is

$$\aleph^t_{l_2,l_1}(x,y)=\big(x,y\big)\exp[(\alpha_{l_2}-\alpha_{l_1})t]\,. \tag{2.6}$$

Observe that the sets A_l^-, A_l^0 and A_l^+ are invariant for this maps, for any l.

A key tool for the analysis of equations of type (1.1) is the so called Pohozaev identity, which is a clever way to restate Green's formula. Fix $q>2$ and consider a solution $u(r)$ of (1.4) where $f(u,r)=r^\delta u|u|^{q-2}$, and the corresponding trajectories $\mathbf{x_l}(t)$ and $\mathbf{x_{2^*}}(t)$ of (2.2). We introduce the Pohozaev function

$$P(u,u',r)=\frac{n-2}{2}r^{n-1}uu'+\frac{r^n|u'|^2}{2}+\frac{|u|^q}{q}r^{n+\delta},$$

and its translation for this dynamical context:

$$H(\mathbf{x},t):=\frac{n-2}{2}xy+\frac{y^2}{2}+e^{(\delta-\lambda^*)t}\frac{|x|^q}{q}\,.$$

In this context the Pohozaev identity can be restated as follows:

$$\frac{dH}{dt}(x_{2^*}(t),y_{2^*}(t),t)=(\delta-\lambda^*)\,e^{(\delta-\lambda^*)t}\frac{|x_{2^*}(t)|^q}{q}\,. \tag{2.7}$$

Therefore if $\mathbf{x_{2^*}}(t)$ solves (2.2), then $H(\mathbf{x_{2^*}}(t),t)$ is monotone decreasing when $\delta<\lambda^*$, constant when $\delta=\lambda^*$ and increasing when $\delta>\lambda^*$. Note that if $q=2^*$ then $\lambda^*=0$ and if $\delta=0$, then $H(\mathbf{x_{2^*}}(t),t)$ is decreasing if $q>2^*$, constant if $q=2^*$ and increasing if $q<2^*$.

We also need the following functions

$$H_*(\mathbf{x}):=\frac{n-2}{2}xy+\frac{y^2}{2}+\frac{|x|^q}{q}\,.$$

$$\hat H_l(\mathbf{x},t):=\frac{n-2}{2}xy+\frac{y^2}{2}+e^{[\delta+\alpha_l(l-q)]t}\frac{|x|^q}{q}\,.$$

Observe that $H(\mathbf{x},t)=\hat H_{2^*}(\mathbf{x},t)$ and $\hat H_L(\mathbf{x},t)$ reduces to $H_*(\mathbf{x})$ when (2.2) reduces to (2.3) for $l=L$; furthermore if $u(r)$ solves (1.4) and $\mathbf{x_{2^*}}(t)$ and $\mathbf{x_L}(t)$ are the corresponding trajectories of (2.2) with $l=2^*$ and with $l=L$ respectively, we find

$$P(u(r),u'(r),r)=H(\mathbf{x_{2^*}}(t),t)=\hat H_L(\mathbf{x_L}(t),t)\mathrm{e}^{-(\alpha_L+\gamma_L)t} \tag{2.8}$$

Let us consider a regular solution $u(r)$ and a f.d. solution $v(r)$ and the corresponding trajectories $\mathbf{x^u_{2^*}}(t)$ and $\mathbf{x^v_{2^*}}(t)$. From (2.7) we easily get that $H(\mathbf{x^v_{2^*}}(t),t)<0<H(\mathbf{x^u_{2^*}}(t),t)$ for any $t\in\mathbb{R}$ if $\delta>\lambda^*$ (subcritical case), $H(\mathbf{x^u_{2^*}}(t),t)<0<$

$H(\mathbf{x}^{\mathbf{v}}_{\mathbf{2}^*}(t),t)$ for any $t \in \mathbb{R}$ if $\delta < \lambda^*$ (supercritical case) and $H(\mathbf{x}^{\mathbf{v}}_{\mathbf{2}^*}(t),t) = 0 = H(\mathbf{x}^{\mathbf{u}}_{\mathbf{2}^*}(t),t)$ if $\delta = \lambda^*$ (critical case). From (2.8) we also find the following.

2.6. *Remark.* Let $\mathbf{x}_u(t)$ and $\mathbf{x}_s(t)$ be the trajectories of (2.3) with $l = u$ and $l = s$ respectively, corresponding to a given solution $u(r)$ of (1.4). Then $H_*(\mathbf{x}_u(t))$ has the same sign as $H_*(\mathbf{x}_s(t))$ and they become null for the same values of t.

Using this simple information and noticing that the level sets $H_*(x,y) = 0$ are 8-shaped and bounded, we can draw a picture of the stable and the unstable manifold $M^s_l(q)$ and $M^u_l(q)$ for (2.3), and we obtain figures 1, see [12, 14] and Remarks 2.8 and 2.9 below for details.

Another change of variables which is very useful in the context of equation of type (1.1), is known in literature as "Kelvin transformation". Let us set

$$s = r^{-1} \quad \tilde{u}(s) = s^{2-n}u(1/s) \quad \tilde{f}(\tilde{u},s) = f(\tilde{u},1/s) \tag{2.9}$$

Then (1.4) is transformed into

$$\tilde{u}_{ss}(s) + \frac{n-1}{s}\tilde{u}_s(s) + \tilde{f}(\tilde{u},s) = 0\,. \tag{2.10}$$

In particular if $f(u,r) = k(r)u|u|^{q-2}$, then $\tilde{f}(\tilde{u},s) = k(1/s)s^{2\lambda^*}\tilde{u}|\tilde{u}|^{q-2}$. Moreover if $q = 2^*$ then $\lambda^* = 0$ so transformation (2.9) simply acts as a reversion of time in (2.2). Combining (2.9) and (2.1) we obtain the following.

2.7. *Remark.* Let us consider (1.4) where $f(u,r) = r^\delta u|u|^{q-2}$ and the corresponding autonomous system (2.3). Let $u(r)$ and $v(r)$ be a regular and a f.d. solution of (1.4) respectively, and let $\mathbf{x}^{\mathbf{u}}_{\mathbf{l}}(t)$ and $\mathbf{x}^{\mathbf{v}}_{\mathbf{l}}(t)$ be the corresponding trajectories of (2.3). Let $\tilde{u}(s)$ and $\tilde{v}(s)$ be the solutions of (2.10) obtained from $u(r)$ and $v(r)$ through (2.9): then $\tilde{v}(s)$ is a regular solution and $\tilde{u}(s)$ has f.d.

Moreover if we apply (2.1) with $l_k = 2\frac{(n-1)l-2n}{(n-2)l-2n+2}$ to (2.10) we obtain the autonomous system (2.3); we emphasize that $\alpha_{l_k} = -\gamma_l$ and $\gamma_{l_k} = -\alpha_l$. It follows that a subcritical system is changed into a supercritical one and viceversa. Note further that if $l = \sigma^*$ then $l_k = \sigma_*$ and viceversa, $l_k \to 2_*$ as $l \to +\infty$ and $l_k \to +\infty$ as $l \to 2_*$.

So the phase portrait of $M^u_l(q)$ and $M^s_l(q)$ of a subcritical system (2.3) is changed by the Kelvin inversion into a supercritical one and viceversa.

Now we give a key result which is summed up in figure 1.

2.8. *Remark.* Let $q > 2$ be fixed, and $\mathbf{Q^u} \in M^u_l(q)$, $\mathbf{Q^s} \in M^s_l(q)$.
If $l > 2^*$ then $M^s_l(q)$ crosses the coordinate axis indefinitely rotating counter-clockwise. Moreover $H_*(\mathbf{Q^u}) < 0 < H_*(\mathbf{Q^s})$. If $l \geq \sigma^*$ then there is a negative decreasing function $h^u : [0, P_x(l)] \to \mathbb{R}$ such that $h^u(0) = 0$, $h^u(P_x(l)) = P_y(l)$ and $M^u_l(q) := \{(x, h^u(x)) \,|\, x \in (0, P_x(l))\}$. Furthermore $M^u_l(q) \cap A^-_l = \emptyset$.
If $2^* < l < \sigma^*$ then $M^u_l(q)$ is a spiral which joins the origin and $\mathbf{P}(l)$. There exists $U(l) \in (2_*, l)$ such that $M^u_l(q)$ is tangent to A^0_U in the point $\mathbf{P}(U)$, and $M^u_l(q) \cap A^-_U = \emptyset$.

If $l = 2^*$ then $M^u_l(q) = M^s_l(q) \subset \mathbb{R}^2_\pm$ and they are the graph of a homoclinic trajectory. Moreover $H_*(\mathbf{Q^u}) = 0 = H_*(\mathbf{Q^s})$ and the interior of the bounded set enclosed by $M^u_l(q)$ is filled by periodic trajectories which are the negative level sets of the function H_*; $H_*(\mathbf{P}(2^*)) < 0$ is the minimum for H_*.

If $2_* < l < 2^*$ then $M^u_l(q)$ crosses the coordinate axis indefinitely rotating clockwise, while $M^s_l(q) \subset \mathbb{R}^2_\pm$; $H_*(\mathbf{Q^u}) > 0 > H_*(\mathbf{Q^s})$.
If $\sigma_* < l < 2^*$ then $M^s_l(q)$ is a spiral which joins the origin and $\mathbf{P}(l)$. There exists $S(l) > l$ such that $M^s_l(q)$ is tangent to A^0_S in the point $\mathbf{P}(S)$, and $M^s_l(q) \cap A^+_S = \emptyset$.
If $2_* < l \leq \sigma_*$ there is a negative decreasing function $h^s : [0, P_x(l)] \to \mathbb{R}$ such

that $h^s(0)=0$, $h^s(P_x(l))=P_y(l)$ and $M_l^s(q):=\{(x,h^s(x))\,|\,x\in(0,P_x(l))\}$. Furthermore $M_l^s(q)\cap A_l^+=\emptyset$. Moreover $U(l)\to 2_*$ as $l\to 2^*$ and $S(l)\to+\infty$ as $l\to 2^*$.

Proof. Most of the results of this Remark are proved in a more general context in [12, 14]. The mutual position of $M_l^u(q)$ and $M_l^s(q)$ is easily obtained evaluating the function H_*, whose sign easily follows from (2.8) and (2.7). The uniqueness of the tangent points between $M_l^u(q)$ and A_U^0, and $M_l^s(q)$ and A_S^0, as well as the fact that these points are in fact $\mathbf{P}(U)$ and $\mathbf{P}(S)$ follows from Remark 2.3, and (2.5).

The really new part concern the shape of $M_l^u(q)$ and $M_l^s(q)$ when $l\geq\sigma^*$ and $l\leq\sigma_*$ respectively. Assume first $l=q\geq\sigma^*$: Wang in [26] proved that if $d_2>d_1$, then $u(d_2,r)>u(d_1,r)$ for any $r\geq 0$ (this monotony property of regular solutions does not hold for $2^*<q<\sigma^*$, since $\mathbf{P}(q)$ is a focus).

Combining this property with the translation in t invariance of system (2.4) we easily deduce that the trajectory $\mathbf{x_q}(t)$ corresponding to $u(1,r)$ is in A_q^+ for any $t\in\mathbb{R}$. Then we easily deduce that $M_q^u(q)$ is a graph on the x axis. Moreover, as we have already observed, system (2.3) corresponds to equation (1.4) where the dimension n is replaced by the natural dimension $N=2\frac{\delta+n}{\delta+2}$. So the modified equation (1.4) is endowed with the monotony property proved in [26], and it follows that the manifold $M_l^u(q)$ is a graph on A_l^0, when $l\neq q$ too. The monotonicity of h^u (which by the way is not relevant in this paper), follows again from the uniqueness of the point in which the flow of (2.3) is tangent to $A_{\bar l}^0$ for $\bar l>2_*$, see (2.5) and Remark 2.3.

Now set $l\leq\sigma_*$ and consider $\mathbf{Q^s}\in M_l^s(q)$, the trajectory $\mathbf{x_l}(t,0;\mathbf{Q^s})$ and the corresponding fast decay solution $v(L,r)$ where $L=\lim_{r\to+\infty}v(L,r)r^{n-2}$. Let us apply the Kelvin inversion (2.9) so that $v(L,r)$ becomes the regular solution $u(L,s)$ of (2.10) with $f(u,s)=s^\eta u|u|^{q-2}$, where $\eta=(n-2)(q-2^*)-\delta$. Then we apply the change of variables (2.1) with $l=l_k$ and we obtain system (2.3) with $l_k\geq\sigma^*$. So $M_l^s(q)$ with $2_*<l<\sigma_*$ is changed into $M_{l_k}^u(q)$ with $l_k>\sigma^*$, see Remark 2.7. Using the fact that regular solutions are such that $u(d_2,s)>u(d_1,s)$ for any $s\geq 0$, whenever $d_2\geq d_1$, we find that the f.d. solutions $v(L_2,r)$ and $v(L_1,r)$ inherit the same property: $v(L_2,r)>v(L_1,r)$ whenever $L_2>L_1$. Combining this order preserving characteristic with the t-invariance we easily get that $M_l^s(q)$ is a monotone graph on the x positive semi-axis. □

By construction the functions $U(l):(2^*,+\infty)\to(2_*,+\infty)$ and $S(l):(2_*,2^*)\to(2_*,+\infty)$ defined in Remark 2.8 are such that $U(l)=l$ for $l\geq\sigma^*$ while $U(l)<l$ for $2_*<l<\sigma^*$, and $S(l)=l$ for $2_*<l\leq\sigma_*$ and $S(l)>l$ for $l>\sigma_*$. Since $\mathbf{P}$ is a focus, then $M_l^u(q)$ intersects A_l^0 indefinitely whenever $2^*<l<\sigma^*$. Follow $M_l^u(q)$ from the origin towards $\mathbf{P}(l)$; we denote by $\tilde{\mathbf{Q}}_\mathbf{l}^\mathbf{u}(1)$ the first intersection met between $M_l^u(q)$ and A_l^0, and by $\tilde{\mathbf{Q}}_\mathbf{l}^\mathbf{u}(k)$ the kth; we set $\tilde{\mathbf{Q}}_\mathbf{l}^\mathbf{u}(0)=(0,0)$. We denote by $\tilde M_l^u(k)$ the branch of M_l^u between $\tilde{\mathbf{Q}}_\mathbf{l}^\mathbf{u}(k-1)$ and $\tilde{\mathbf{Q}}_\mathbf{l}^\mathbf{u}(k)$. Observe that $\lim_{k\to\infty}\tilde{\mathbf{Q}}_\mathbf{l}^\mathbf{u}(k)=\mathbf{P}(l)$ and that $\tilde M_l^u(2k)\cap A_l^+=\emptyset$, while $\tilde M_l^u(2k+1)\cap A_l^-=\emptyset$, for any $k\in\mathbb{N}$.

Analogously we follow $M_l^s(q)$ from the origin towards $\mathbf{P}(l)$ when $\sigma_*<l<2^*$; we denote by $\tilde{\mathbf{Q}}_\mathbf{l}^\mathbf{s}(k)$ the kth intersection met between $M_l^s(q)$ and A_l^0; we set $\tilde{\mathbf{Q}}_\mathbf{l}^\mathbf{s}(0)=(0,0)$. We denote by $\tilde M_l^s(k)$ the branch of M_l^s between $\tilde{\mathbf{Q}}_\mathbf{l}^\mathbf{s}(k-1)$ and $\tilde{\mathbf{Q}}_\mathbf{l}^\mathbf{s}(k)$. Again $\lim_{k\to\infty}\tilde{\mathbf{Q}}_\mathbf{l}^\mathbf{s}(k)=\mathbf{P}(l)$ and $\tilde M_l^s(2k)\cap A_l^-=\emptyset$, while $\tilde M_l^s(2k+1)\cap A_l^+=\emptyset$, for any $k\in\mathbb{N}$. Observe that the crossings $\tilde{\mathbf{Q}}_\mathbf{l}^\mathbf{u}(k)$ and $\tilde{\mathbf{Q}}_\mathbf{l}^\mathbf{s}(k)$ between A_l^0 and M_l^u and M_l^s respectively, are transversal, for any $k\in\mathbb{N}$, since the flow on A_l^0 is vertical.

2.9. *Remark.* The functions $U(l):(2^*,+\infty)\to(2_*,+\infty)$ and $S(l):(2_*,2^*)\to(2_*,+\infty)$, $\tilde{\mathbf{Q}}_\mathbf{l}^\mathbf{u}(k):(2^*,\sigma^*)\to A_l^0$, $\tilde{\mathbf{Q}}_\mathbf{l}^\mathbf{s}(k):(\sigma_*,2^*)\to A_l^0$ for $k\in\mathbb{N}$, depend continuously on l. Moreover for any $k\in\mathbb{N}$, $\tilde{\mathbf{Q}}_\mathbf{l}^\mathbf{u}(k)\to\mathbf{P}(\sigma^*)$ as $l\to\sigma^*$ and

$\tilde{\mathbf{Q}}^{\mathbf{s}}_{\mathbf{l}}(k) \to \mathbf{P}(\sigma_*)$ as $l \to \sigma_*$. Furthermore for any fixed $k \in \mathbb{N}$, $\tilde{\mathbf{Q}}^{\mathbf{u}}_{\mathbf{l}}(2k+1) \to \tilde{\mathbf{Q}}^{\mathbf{u}}_{\mathbf{2^*}}(1)$, $\tilde{\mathbf{Q}}^{\mathbf{u}}_{\mathbf{l}}(2k) \to (0,0)$, $\tilde{\mathbf{Q}}^{\mathbf{s}}_{\mathbf{l}}(2k+1) \to \tilde{\mathbf{Q}}^{\mathbf{s}}_{\mathbf{2^*}}(1) = \tilde{\mathbf{Q}}^{\mathbf{u}}_{\mathbf{2^*}}(1)$ and $\tilde{\mathbf{Q}}^{\mathbf{u}}_{\mathbf{l}}(2k) \to (0,0)$, $U(l) \to 2_*$ and $S(l) \to +\infty$ as $l \to 2^*$.

Proof. The continuity of $\tilde{\mathbf{Q}}^{\mathbf{u}}_{\mathbf{l}}(k)$ with respect to l for $l \in (2^*, \sigma^*)$, follows from the continuous dependence of the flow of (2.3) on parameters, and the transversality of the crossings. From Remark 2.3 we know that M^u_l is tangent to $A^0_{U(l)}$ in $\mathbf{P}(U(l))$. Observe further that $\mathbf{P}(U(l)) \in \tilde{M}^u_l(2)$ since M^u_l cannot have self-intersections; consider the line L orthogonal to $A^0_{U(l)}$ passing through $\mathbf{P}(U(l))$: by construction $\tilde{M}^u_l(2)$ crosses L transversally in $\mathbf{P}(U(l))$. Using again the continuous dependence of the flow of (2.3) on the parameters we find that $\mathbf{P}(U(l))$ depends continuously on l. The continuity of $U(l)$ then follows from the continuity of $\mathbf{P}(U(l))$.

Let $B(\mathbf{Q}, R)$ denote the ball of radius R with center in $\mathbf{Q}$; observe that for any $\epsilon > 0$ we can find $\delta > 0$ such that $\tilde{M}^u_l(1) \cap B(\mathbf{P}(\sigma^*), \epsilon/2) \neq \emptyset$ whenever $0 < l - \sigma^* < \delta$. Let us choose $\mathbf{Q} \in \tilde{M}^u_l(1) \cap B(\mathbf{P}(\sigma^*), \epsilon/2)$ and consider $\mathbf{x}_l(t, 0; \mathbf{Q})$. Using the fact that $\mathbf{P}(l)$ is attractive and that we can choose $\delta > 0$ such that $|\mathbf{P}(\sigma^*) - \mathbf{P}(l)| < \epsilon/2$, we see that $\mathbf{x}_l(t, 0; \mathbf{Q}) \in B(\mathbf{P}(\sigma^*), \epsilon)$ for any $t > 0$. Moreover, since $\mathbf{P}(l)$ is a focus there is a sequence $\tau_k \to +\infty$ such that $\mathbf{x}_l(\tau_k, 0; \mathbf{Q}) = \tilde{\mathbf{Q}}^{\mathbf{u}}_{\mathbf{l}}(k)$ for $k \in \mathbb{N}$. For the arbitrariness of $\epsilon > 0$ it follows that $\tilde{\mathbf{Q}}^{\mathbf{u}}_{\mathbf{l}}(k) \to \mathbf{P}(\sigma^*)$ as $l \to \sigma^*$, for any $k \in \mathbb{N}$. Since $\mathbf{P}(U(l)) \in \tilde{M}^u_l(2)$ we find $\mathbf{P}(U(l)) \to \mathbf{P}(\sigma^*)$ as $l \to \sigma^*$, too. Therefore we also have $U(l) \to \sigma^*$ as $l \to \sigma^*$.

Now we prove that $\tilde{\mathbf{Q}}^{\mathbf{u}}_{\mathbf{l}}(2k+1) \to \tilde{\mathbf{Q}}^{\mathbf{u}}_{\mathbf{2^*}}(1)$ as $l \to 2^*$, for any fixed $k \in \mathbb{N}$. Let us fix $l \in [2^*, \sigma^*)$ and denote by a^d_l the segment of A^0_l between $\mathbf{P}(l)$ and $\tilde{\mathbf{Q}}^{\mathbf{u}}_{\mathbf{l}}(1)$ and by a^u_l the segment of A^0_l between the origin and $\mathbf{P}(l)$. Note that for any $\mathbf{Q} \in a^d_l \setminus \{\mathbf{P}(l)\}$ there is $T(\mathbf{Q}) > 0$ such that the trajectory $\mathbf{x_l}(t, 0; \mathbf{Q}) \in A^-_l$ for any $0 < t < T(\mathbf{Q})$ and $\mathbf{x_l}(T(\mathbf{Q}), 0; \mathbf{Q}) \in a^u_l$. Analogously for any $\mathbf{Q} \in a^u_l \setminus \{\mathbf{P}(l), (0,0)\}$ there is $T(\mathbf{Q}) > 0$ such that the trajectory $\mathbf{x_l}(t, 0; \mathbf{Q}) \in A^+_l$ for any $0 < t < T(\mathbf{Q})$ and $\mathbf{x_l}(T(\mathbf{Q}), 0; \mathbf{Q}) \in a^d_l$. So we can define the function $\Upsilon^d_l(\mathbf{Q}) : a^d_l \to a^u_l$ as $\Upsilon^d_l(\mathbf{P}(l)) = \mathbf{P}(l)$ and $\Upsilon^d_l(\mathbf{Q}) = \mathbf{x_l}(T(\mathbf{Q}), 0; \mathbf{Q})$ otherwise; for continuity we set $\Upsilon^d_{2^*}(\tilde{\mathbf{Q}}^{\mathbf{u}}_{\mathbf{2^*}}(1)) = (0,0)$. Analogously we can define $\Upsilon^u_l(\mathbf{Q}) : a^u_l \to a^d_l$ as $\Upsilon^u_l(0,0) = \tilde{\mathbf{Q}}^{\mathbf{u}}_{\mathbf{l}}(1)$, $\Upsilon^u_l(\mathbf{P}(l)) = \mathbf{P}(l)$ and $\Upsilon^u_l(\mathbf{Q}) = \mathbf{x_l}(T(\mathbf{Q}), 0; \mathbf{Q})$ otherwise. From continuous dependence on initial data and parameters it follows easily that Υ^d_l and Υ^s_l are continuous in $\mathbf{Q}$ and that they depend continuously on l (observe that $a^d_{\bar{l}} \to a^d_l$ and $a^u_{\bar{l}} \to a^u_l$ with respect to the Hausdorff distance between sets, as $l \to \bar{l}$). Let us define the continuous function $\Lambda^1_l(\mathbf{Q}) : a^d_l \to a^d_l$ as $\Lambda^1_l(\mathbf{Q}) = \Upsilon^u_l\big(\Upsilon^d_l(\mathbf{Q})\big)$, and denote by $\Lambda^2_l = \Lambda^1_l \circ \Lambda^1_l$, and $\Lambda^k_l = \Lambda^1_l \circ \Lambda^{k-1}_l$, for $k \geq 2$. Observe that $\Lambda^k_l(\tilde{\mathbf{Q}}^{\mathbf{u}}_{\mathbf{l}}(1)) = \tilde{\mathbf{Q}}^{\mathbf{u}}_{\mathbf{l}}(2k+1)$. By construction for any $k \in \mathbb{N}$, Λ^k_l is continuous and depends continuously on l as well, and $\Lambda^k_{2^*}(\mathbf{Q}) = \mathbf{Q}$ for any $\mathbf{Q} \in a^d(2^*)$. Thus it follows that for any $k \in \mathbb{N}$, $\tilde{\mathbf{Q}}^{\mathbf{u}}_{\mathbf{l}}(2k+1) \to \tilde{\mathbf{Q}}^{\mathbf{u}}_{\mathbf{2^*}}(1)$ as $l \to 2^*$. Using the function $\Upsilon^d_l\big(\Upsilon^u_l(\mathbf{Q})\big)$ and reasoning as above we can prove that $\tilde{\mathbf{Q}}^{\mathbf{u}}_{\mathbf{l}}(2k) \to (0,0)$ as $l \to 2^*$ for any $k \in \mathbb{N}$.
Using the fact that $\tilde{\mathbf{Q}}^{\mathbf{u}}_{\mathbf{l}}(1) \to \tilde{\mathbf{Q}}^{\mathbf{u}}_{\mathbf{2^*}}(1)$ as $l \to 2^*$ and continuous dependence on parameters, we get that $U(l) \to U(2^*) = 2_*$ as $l \to 2^*$.

The proof concerning $S(l)$, and $\tilde{\mathbf{Q}}^{\mathbf{s}}_{\mathbf{l}}(k)$ is completely analogous so will be omitted. □

Let $2_* < l_s < 2^* < l_u$; we set

$$
\begin{aligned}
&\epsilon_1(l_s) := \sup\{l - 2^* > 0 \,|\, U(L) < l_s \text{ for any } 2^* < L < l\} \\
&\epsilon_1(l_u) := \sup\{|l - 2^*| \,|\, S(L) > l_u \text{ for any } l < L < 2^*\} \\
&N_0(l_s) := \inf\{l \,|\, U(L) > l_s \text{ for any } L > l\} \\
&\epsilon_0(l_u) := \sup\{l - 2_* > 0 \,|\, S(l) < l_u \text{ for any } 2_* < L < l.\}
\end{aligned} \tag{2.11}
$$

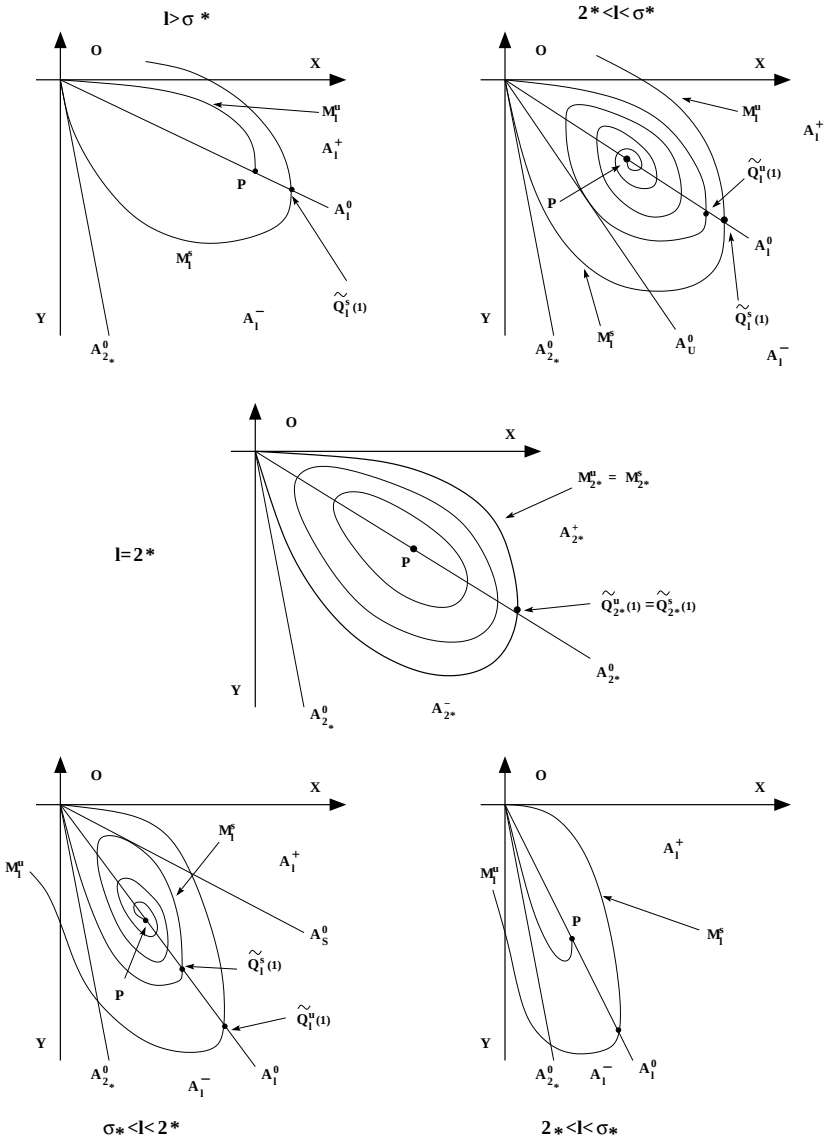

FIGURE 1. Sketch of the phase portrait of (2.3), for $q > 2$ fixed.

The values defined in (2.11) are meaningful (nonzero and bounded), thanks to Remark 2.9. In particular $N_0(l_s) < \max\{S(l_s), \sigma^*\}$ and $\epsilon_0(l_u) > \min\{U(l_u), \sigma_*\} - 2_*$.

2.10. *Remark.* Fix $l_s \in (2_*, 2^*)$; for any $l_u \in [2^*, 2^* + \epsilon_1(l_s))$ we have $[A^0_{l_s} \cap M^u_{l_u}(q)] \neq \emptyset$, while for any $l_u > N_0(l_s)$ we have $M^u_{l_u}(q) \subset A^+_{l_s}$. Analogously fix $l_u > 2^*$; for any $l_s \in (2^* - \epsilon_1(l_u), 2^*)$ we have $[A^0_{l_u} \cap M^s_{l_s}(q)] \neq \emptyset$, while for any $l_s \in (2_*, 2_* + \epsilon_0(l_u))$ we have $M^s_{l_s}(q) \subset A^-_{l_u}$.

Let us define

$$
\begin{aligned}
&\tilde{\epsilon}_1(l_s) = \sup\{l - 2^* \,|\, U(l) < S(l_s) \text{ for any } 2^* < L < l\} \\
&\tilde{\epsilon}_1(l_u) = \sup\{|l - 2^*| \,|\, S(L) > U(l_u) \text{ for any } l < L < 2^*\} \\
&\tilde{N}_0(l_s) = \inf\{l \,|\, U(l) > S(l_s) \text{ for any } L > l\} \\
&\tilde{\epsilon}_0(l_u) = \sup\{l - 2_* \,|\, S(l) < U(l_u) \text{ for any } 2_* < L < l\}
\end{aligned}
\tag{2.12}
$$

We stress the values in (2.11) are the ones that appear in Theorems 1.10 and 1.12 and that the values $\epsilon_1(\delta^u)$, $\epsilon_1(\delta^s)$, $\epsilon_0(\delta^u)$, $N_0(\delta^s)$ which appear in Theorems 1.5 and 1.7 are obtained simply setting $\epsilon_1(\delta^u) := \epsilon_1(\delta(l_u))$, $\epsilon_1(\delta^s) := \epsilon_1(\delta(l_s))$ and so on.

Observe further that if the functions $U(l)$ and $S(l)$ are monotone as we conjecture than $2^* - \tilde{\epsilon}_1(l_u) = 2_* + \tilde{\epsilon}_0(l_u)$ and $2^* + \tilde{\epsilon}_1(l_s) = \tilde{N}_0(l_s)$, and the definitions of the functions ϵ, $\tilde{\epsilon}$, N_0 and $\tilde{N}_0$ simplify. We stress that by construction $\tilde{\epsilon}_1(l_u) > \epsilon_1(l_u)$, $\tilde{\epsilon}_1(l_s) > \epsilon_1(l_s)$, $\tilde{\epsilon}_0(l_u) > \epsilon_0(l_u)$ and $\tilde{N}_0(l_s) < N_0(l_s)$. The values ϵ_1, ϵ_0 and N_0 in Theorems 1.5, 1.7, 1.10, 1.12 might be replaced by the better constants $\tilde{\epsilon}_1$, $\tilde{\epsilon}_0$ and $\tilde{N}_0$ defined in (2.12). However the values defined in (2.11) can be easily estimated by a computer assisted proof. This way we may obtain precise values for which we have either existence or non-existence of G.S. with f.d., see Remark 3.6.

In next sections we need also this Lemma.

2.11. **Lemma.** *Fix $l_s \in (2_*, 2^*)$; for any $k \in \mathbb{N}$ we can find $\epsilon_k(l_s) > 0$ such that $M^u_{l_u}(q)$ crosses $A^0_{l_s}$ at least $2k$ times, whenever $l_u \in [2^*, 2^* + \epsilon_k(l_s))$. Analogously fix $l_u > 2^*$; for any $k \in \mathbb{N}$ we can find $\epsilon_k(l_u) > 0$ such that $M^s_{l_s}(q)$ crosses $A^0_{l_u}$ at least $2k$ times, whenever $l_s \in (2^* - \epsilon_k(l_u), 2^*)$.*

Proof. From Remark 2.9 we know that $\tilde{\mathbf{Q}}^{\mathbf{u}}_{\mathbf{l}}(2k+1) \to \tilde{\mathbf{Q}}^{\mathbf{u}}_{\mathbf{2^*}}(1)$ and $\tilde{\mathbf{Q}}^{\mathbf{u}}_{\mathbf{l}}(2k) \to (0,0)$ as $l \to 2^*$. Using continuous dependence on initial data we easily find that $\tilde{M}^u_l(2k+1) \to \tilde{M}^u_{2^*}(1)$ and $\tilde{M}^u_l(2k) \to \tilde{M}^u_{2^*}(2)$ as $l \to 2^*$ with respect to the Hausdorff distance between sets. So the existence of $\epsilon_k(l_s)$ is guaranteed by a continuity argument.

Reasoning as above we see that $\tilde{M}^s_l(2k+1) \to \tilde{M}^s_{2^*}(1)$ and $\tilde{M}^s_l(2k) \to \tilde{M}^s_{2^*}(2)$ as well, so the existence of $\epsilon_k(l_u)$ follows. □

Reasoning as in Remark 2.8, for any $k \in \mathbb{N}$ we can define the functions $U_k : [2^*, \sigma^*] \to [2_*, \sigma^*]$ as follows: $U_k(2^*) = 2_*$, $U_k(\sigma^*) = \sigma^*$ and $U_k(l) = L$, where $L \in (2_*, \sigma^*)$ is the unique value such that $\tilde{M}^u_l(2k)$ crosses A^0_u for any $u > L$ and it is tangent to A^0_L. Analogously we define $S_k : [\sigma_*, 2^*) \to [\sigma_*, +\infty)$ as $S_k(\sigma_*) = \sigma_*$, and $S_k(l) = L$, where $L > \sigma_*$ is the unique value such that $\tilde{M}^s_l(2k)$ crosses A^0_s for any $s < L$ and it is tangent to A^0_L.

From Remark 2.3 and reasoning as in Remarks 2.8 and 2.9 we find the following.

2.12. *Remark.* For any $k \in \mathbb{N}$ the functions $U_k : [2^*, \sigma^*] \to [2_*, \sigma^*]$ and $S_k : [\sigma_*, 2^*) \to [\sigma_*, +\infty)$ are continuous and $S_k(l) \to +\infty$ as $l \to 2^*$. Moreover $\tilde{M}^u_l(2k)$ is tangent to $A^0_{U_k(l)}$ in $\mathbf{P}(U_k(l))$ and $\tilde{M}^s_l(2k)$ is tangent to $A^0_{S_k(l)}$ in $\mathbf{P}(S_k(l))$.

Let us denote by

$$
\begin{aligned}
&\epsilon_k(l_s) := \sup\{l - 2^* \,|\, U_k(L) < l_s \text{ for any } 2^* < L < l\}\\
&\epsilon_k(l_u) := \sup\{|l - 2^*| \,|\, S_k(L) > l_u \text{ for any } l < L < 2^*\}\\
&\tilde{\epsilon}_k(l_s) := \sup\{l - 2^* \,|\, U_k(L) < S(l_s) \text{ for any } 2^* < L < l\}\\
&\tilde{\epsilon}_k(l_u) := \sup\{|l - 2^*| \,|\, S_k(L) > U(l_u) \text{ for any } l < L < 2^*\}
\end{aligned}
\tag{2.13}
$$

Observe that $\tilde{\epsilon}_{k-1}(l_u) > \tilde{\epsilon}_k(l_u) > \epsilon_k(l_u)$, $\tilde{\epsilon}_{k-1}(l_s) > \tilde{\epsilon}_k(l_s) > \epsilon_k(l_s)$ whenever $k \geq 2$. We stress that the values $\epsilon_k(\delta^u)$ and $\epsilon_k(\delta^s)$ of Theorem 1.6 can be obtained simply setting $\epsilon_k(\delta^u) = \epsilon_k(\delta(l_u))$ and $\epsilon_k(\delta^s) = \epsilon_k(\delta(l_s))$.

For any $l_u \in (2^*, \sigma^*)$ we denote by $\tilde{B}^u_{l_u}(j)$ the bounded set enclosed by $\tilde{M}^u_{l_u}(j)$ and the segment of $A^0_{l_u}$ between $\tilde{\mathbf{Q}}^{\mathbf{u}}_{\mathbf{l_u}}(j-1)$ and $\tilde{\mathbf{Q}}^{\mathbf{u}}_{\mathbf{l_u}}(j)$. We have $\tilde{B}^u_{l_u}(2j) \subset \tilde{B}^u_{l_u}(2(j-1)) \subset (A^0_{l_u} \cup A^-_{l_u})$ and $\tilde{B}^u_{l_u}(2j+1) \subset \tilde{B}^u_{l_u}(2j-1) \subset (A^0_{l_u} \cup A^+_{l_u})$ for any $j \in \mathbb{N}$.

Analogously for any $l_s \in (\sigma_*, 2^*)$ we denote by $\tilde{B}^s_{l_s}(j)$ the bounded set enclosed by $\tilde{M}^s_{l_s}(j)$ and the segment of $A^0_{l_s}$ between $\tilde{\mathbf{Q}}^{\mathbf{s}}_{\mathbf{l_s}}(j-1)$ and $\tilde{\mathbf{Q}}^{\mathbf{s}}_{\mathbf{l_s}}(j)$. We have $\tilde{B}^s_{l_s}(2j) \subset$

$\tilde{B}^s_{l_s}(2(j-1)) \subset (A^0_{l_s} \cup A^+_{l_s})$ and $\tilde{B}^s_{l_s}(2j+1) \subset \tilde{B}^s_{l_s}(2j-1) \subset (A^0_{l_s} \cup A^-_{l_s})$ for any $j \in \mathbb{N}$.

Let $l_u \in (2^*, 2^* + \epsilon_k(l_s))$ and follow $M^u_{l_u}(q)$ from the origin towards $\mathbb{R}^2_\pm$: it crosses $A^0_{l_s}$ in at least $2k$ distinct points. We denote by $\breve{\mathbf{Q}}^{\mathbf{u}}_{\mathbf{l_u}}(j; l_s) = (\breve{X}^u(j), \breve{Y}^u(j))$ the jth intersections, for any $j = 1, \dots, 2k$ and we set $\breve{\mathbf{Q}}^{\mathbf{u}}_{\mathbf{l_u}}(0; l_s) = (0, 0)$; it follows that

$$\breve{X}^u(2) < \breve{X}^u(2i) < \breve{X}^u(2(i+1)) < P_x(l_s) < \breve{X}^u(2j+1) < \breve{X}^u(2j-1) < \breve{X}^u_1 \ ,$$

for any $i, j = 2, \dots, k-1$. We denote by $\breve{M}^u_{l_u}(j; l_s)$ the branch of $M^u_{l_u}$ between $\breve{\mathbf{Q}}^{\mathbf{u}}(j-1)$ and $\breve{\mathbf{Q}}^{\mathbf{u}}(j)$, for $j = 1, \dots, 2k$, and by $\breve{B}^u_{l_u}(j; l_s)$ the bounded set enclosed by $\breve{M}^u_{l_u}(j)$ and the segment of $A^0_{l_s}$ between $\breve{\mathbf{Q}}^{\mathbf{u}}(j-1)$ and $\breve{\mathbf{Q}}^{\mathbf{u}}(j)$. It is easy to check that $\breve{B}^u_{l_u}(2j; l_s) \subset \breve{B}^u_{l_u}(2(j-1); l_s) \subset (A^0_{l_s} \cup A^-_{l_s})$ and $\tilde{B}^u_{l_u}(2j+1) \subset \breve{B}^u_{l_u}(2j+1; l_s) \subset \breve{B}^u_{l_u}(2j-1); l_s) \subset (A^0_{l_s} \cup A^+_{l_s})$ for any $j = 1, \dots, k-1$.

Analogously fix $l_u > 2^*$ and choose $l_s \in (2^* - \epsilon_k(l_u), 2^*)$ so that $M^s_{l_s}(q)$ crosses $A^0_{l_u}$ in at least $2k$ distinct points. Follow $M^s_{l_s}(q)$ from the origin towards $\mathbb{R}^2_\pm$; we denote by $\breve{\mathbf{Q}}^{\mathbf{s}}_{\mathbf{l_s}}(j; l_u) = (\breve{X}^s(j), \breve{Y}^s(j))$ the jth intersections met, for any $j = 1, \dots, 2k$ and we set $\breve{\mathbf{Q}}^{\mathbf{s}}_{\mathbf{l_s}}(0; l_u) = (0, 0)$; again we have

$$\breve{X}^s(2) < \breve{X}^s(2i) < \breve{X}^s(2(i+1)) < P_x(l_u) < \breve{X}^s(2j+1) < \breve{X}^s(2j-1) < \breve{X}^s_1 \ ,$$

for any $i, j = 2, \dots, k-1$. We denote by $\breve{M}^s_{l_s}(j; l_u)$ the branch of $M^s_{l_s}$ between $\breve{\mathbf{Q}}^{\mathbf{s}}_{\mathbf{l_s}}(j-1; l_u)$ and $\breve{\mathbf{Q}}^{\mathbf{s}}_{\mathbf{l_s}}(j; l_u)$, for $j = 1, \dots, 2k$, and by $\breve{B}^s_{l_s}(j; l_u)$ the bounded set enclosed by $\breve{M}^s_{l_s}(j; l_u)$ and the segment of $A^0_{l_u}$ between $\breve{\mathbf{Q}}^{\mathbf{s}}_{\mathbf{l_s}}(j-1; l_u)$ and $\breve{\mathbf{Q}}^{\mathbf{s}}_{\mathbf{l_s}}(j; l_u)$. It is easy to check that $\breve{B}^s_{l_s}(2j; l_u) \subset \breve{B}^s_{l_s}(2(j-1); l_u) \subset (A^0_{l_u} \cup A^+_{l_u})$ and $\tilde{B}^s_{l_s}(2j+1) \subset \breve{B}^s_{l_s}(2j+1; l_u) \subset \breve{B}^s_{l_s}(2j-1); l_u) \subset (A^0_{l_u} \cup A^-_{l_u})$ for any $j = 1, \dots, k-1$.
These constructions will be useful in next sections.

Denote by $\mathbf{Q^0}(L) = (X^0(L), Y^0(L))$ the unique intersection between A^0_L and the level set $H_*(\mathbf{x}) = 0$, such that $X^0(L) > 0$ for $L > 2_*$. Consider system (2.3) where $2_* < l < 2^*$ and follow M^u_l from the origin towards $\mathbb{R}^2_\pm$: using again (2.5), (2.8), and the fact that M^u_l crosses the y negative semi-axis, we see that it crosses A^0_L once in a point $\mathbf{C^u_l}(L)$ such that $H_*(\mathbf{C^u_l}(L)) > 0$, for any $L \geq 2_*$. Therefore there is $T^u < 0$ such that $\mathbf{x}_l(t, 0; \mathbf{Q^0}(L)) \in A^+_L$ for $T^u < t < 0$ and $\mathbf{x}_l(T^u, 0; \mathbf{Q^0}(L)) = \mathbf{R^u_l}(L) \in A^0_L$. Denote by $D^u_l(L) := \{\mathbf{x}_l(t, 0; \mathbf{Q^0}(L)) \,|\, T^u \leq t \leq 0\}$, by $N^u_l(L)$ the compact set enclosed by $D^u_l(L)$ and A^0_L, and by $E^u_l(L) := A^+_L \setminus N^u_l(L)$ see figure 2.

Analogously consider system (2.3) where $l > 2^*$; and follow M^s_l from the origin towards $\mathbb{R}^2_\pm$: reasoning as above we see that it crosses A^0_L once in a point $\mathbf{C^s_l}(L)$ such that $H_*(\mathbf{C^s_l}(L)) > 0$, for any $L \geq 2_*$. There is $T^s > 0$ such that $\mathbf{x}_l(t, 0; \mathbf{Q^0}(L)) \in A^-_L$ for $0 < t < T^s$ and $\mathbf{x}_l(T^s, 0; \mathbf{Q^0}(L)) = \mathbf{R^s_l}(L) \in A^0_L$. Denote by $D^s_l(L) := \{\mathbf{x}_l(t, 0; \mathbf{Q^0}(L)) \,|\, 0 \leq t \leq T^s\}$, by $N^s_l(L)$ the compact set enclosed by $D^s_l(L)$, and A^0_L, and by $E^s_l(L) := A^-_L \setminus N^s_l(L)$. From an elementary analysis on the phase portrait we get the following.

2.13. *Remark.* Consider system (2.3) where $2_* < l < 2^*$, and choose $\mathbf{Q}$ in the interior of $E^u_l(L)$, where $L > 2_*$. Then there is $\tilde{T}^u(\mathbf{Q}, L) > 0$ such that $\mathbf{x_l}(t, 0; \mathbf{Q}) \in A^+_L$ for any $0 < t < \tilde{T}^u(\mathbf{Q}, L)$, $\mathbf{x_l}(\tilde{T}^u(\mathbf{Q}, L), 0; \mathbf{Q}) = \mathbf{C^u} \in A^0_L$ and $H_*(\mathbf{C^u}) > 0$. Moreover if $\mathbf{R^u_l}(L) = (\Delta^u_l(L), -\alpha_L \Delta^u_l(L))$ then $\Delta^u_l(L)$ is positive and continuous for $2_* < l < 2^*$ and tends to 0 as $l \to 2^*$.

Analogously set $l > 2^*$ and choose $\mathbf{Q}$ in the interior of $E^s_l(L)$, where $L > 2_*$. Then there is $\tilde{T}^s(\mathbf{Q}, L) < 0$ such that $\mathbf{x_l}(t, 0; \mathbf{Q}) \in A^-_l$ for any $\tilde{T}^s(\mathbf{Q}, L) < t < 0$, $\mathbf{x_l}(\tilde{T}^s(\mathbf{Q}, L), 0; \mathbf{Q}) = \mathbf{C^s} \in A^0_l$ and $H_*(\mathbf{C^s}) > 0$. Moreover if $\mathbf{R^s_l}(L) = (\Delta^s_l(L), -\alpha_L \Delta^s_l(L))$ then $\Delta^s_l(L) > 0$ is positive and continuous for $l > 2^*$ and tends to 0 as $l \to 2^*$.

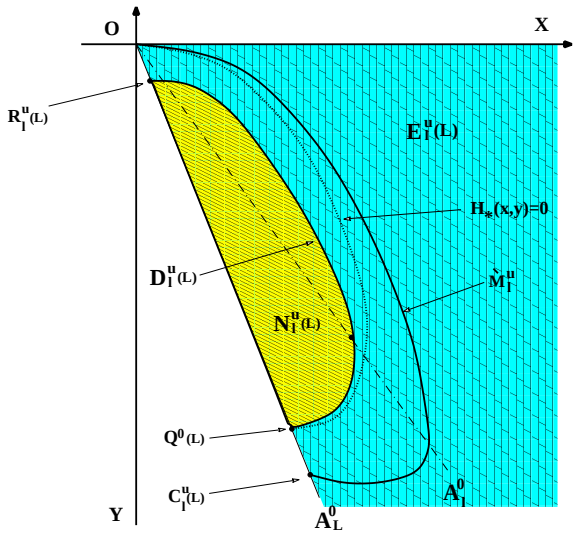

FIGURE 2. Construction of $E_l^u(L)$.

3. f OF TYPE (1.2)

In this section we analyze (1.4) assuming that f has the form (1.2). This particular non-linearity is suitable for a rather easy discussion, performed gluing together two 2 dimensional autonomous systems. In this case the problem is simple enough to detect several bifurcations (in fact we think all of them). However we believe that the bifurcation diagram developed in Remarks 3.10 and 3.11, which sums up the results of Theorems 1.6, 1.7, 1.8 and Corollary 1.9, should be typical for a much larger class of nonlinearities f exhibiting supercritical behavior for u large and r small, and supercritical behavior for u small and r large (or at least part of the diagram). So we regard this simplest case as a prototype.

In the whole section we consider $q > 2$ fixed so we leave the dependence on this parameter unsaid. Consider (1.4) where $f(u,r) = r^\delta u|u|^{q-2}$; if we set $l = l_u := l(\delta^u) = 2(\delta^u + q)/(\delta^u + 2)$ we find that (2.2) reduces to the autonomous system (2.3) when $\delta = \delta^u$, and if we set $l = l_s$ it reduces to (2.3) when $\delta = \delta^s$. We develop all the proofs setting $l_s < 2^* < l_u$ for homogeneity with the next section. So we obtain values of the parameters ϵ_k, $\tilde{\epsilon}_k$, N_0 and $\tilde{N}_0$, giving bounds for the validity of the Theorems, depending on l_u and l_s. The original values which appear in the statements of the Theorems have the form $\epsilon_k(\delta^u) := \epsilon_k(\delta(l_u))$, $\tilde{\epsilon}_k(\delta^u) := \tilde{\epsilon}_k(\delta(l_u))$, and so on.

We modify the change of variables (2.6) as follows: we set

$$\alpha = \alpha(t) = \begin{cases} \alpha_{l_u}, & \text{if } t \le 0 \\ \alpha_{l_s}, & \text{if } t > 0. \end{cases} \quad \text{and} \quad \begin{array}{l} \beta = \beta(t) = \alpha(t) + 1 \\ \gamma = \gamma(t) = \alpha(t) + 2 - n \end{array} \tag{3.1}$$

Then, setting $x_*(t) = u(e^t)e^{\alpha(t)t}$ and $y_*(t) = u'(e^t)e^{\beta(t)t}$, we obtain:

$$\begin{pmatrix} \dot{x}_* \\ \dot{y}_* \end{pmatrix} = \begin{pmatrix} \alpha(t) & 1 \\ 0 & \gamma(t) \end{pmatrix} \begin{pmatrix} x_* \\ y_* \end{pmatrix} + \begin{pmatrix} 0 \\ -x_*|x_*|^{q-2} \end{pmatrix} \tag{3.2}$$

This way we have introduced a discontinuity at $t = 0$, however the trajectories obtained are C^1 for t positive and for t negative and locally Lipschitz for any $t \in \mathbb{R}$. A solution $\mathbf{x}_*(t)$ of (3.2) will be a continuous and piecewise C^1 function such that $\lim_{t\to 0^\pm} \dot{\mathbf{x}}_*(t) = \lim_{t\to 0^\pm} \mathbf{F}(\mathbf{x}_*(t), t)$, where $\mathbf{F}(\mathbf{x}_*(t), t)$ is the right hand side of (3.2).

Observe that for $\tau \leq 0$ the origin admits a 1-dimensional unstable manifold, denoted by $W^u(\tau)$, which is in fact constant, equal to $M^u_{l_u}(q)$, and invariant for the flow whenever $t < 0$. Here and later we continue to abuse the notation and for unstable manifold we mean just the branch which departs from the origin in the direction of $\mathbb{R}^2_\pm$, since we are interested just in positive solutions. Using the flow of (3.2) we can define the unstable manifold for $\tau > 0$ as follows:

$$W^u(\tau) := \{\mathbf{x}_{l_s}(\tau, 0; \mathbf{Q}) \mid \mathbf{Q} \in M^u_{l_u}\}.$$

It is easy to check that $W^u(\tau)$ varies continuously with respect to τ but it is not constant for $\tau > 0$ and it is not invariant for the flow for $t > 0$. Furthermore observe that for any $\tau \in \mathbb{R}$ we have

$$W^u(\tau) = \{\mathbf{Q} \neq (0,0) \mid \lim_{t\to-\infty} \mathbf{x}_*(t, \tau; \mathbf{Q}) = (0,0)\}.$$

Thus, using Remark 2.1, we see that the trajectories $\mathbf{x}_*(t, \tau; \mathbf{Q})$ such that $\mathbf{Q} \in W^u(\tau)$ correspond to regular solutions $u(r)$ of (1.4) and viceversa.

Analogously we define the stable manifold $W^s(\tau) \equiv M^s_{l_s}$ for $\tau > 0$ and

$$W^s(\tau) := \{\mathbf{x}_{l_u}(\tau, 0; \mathbf{Q}) \mid \mathbf{Q} \in M^s_{l_s}\},$$

for $\tau \leq 0$. Again $W^s(\tau)$ depends continuously on τ, therefore $W^u(0) = M^s_{l_s}$; moreover we have

$$W^s(\tau) = \{\mathbf{Q} \neq (0,0) \mid \lim_{t\to+\infty} \mathbf{x}_*(t, \tau; \mathbf{Q}) = (0,0)\},$$

so trajectories $\mathbf{x}_*(t, \tau; \mathbf{Q})$ such that $\mathbf{Q} \in W^s(\tau)$ correspond to f.d. solutions $u(r)$ of (1.4) and viceversa.

From the previous construction, Remark 2.1, and the stability properties of the critical points $\mathbf{P}(l_u)$ and $\mathbf{P}(l_s)$ we get the following.

3.1. *Remark.* Eq. (1.4) with f of type (1.2) admits a unique positive singular solution, say $v(r)$, and $v(r) \equiv P_x(l_u) r^{-\alpha_{l_u}}$ for $r \leq 1$, and a unique s.d. solution $w(r)$, and $w(r) \equiv P_x(l_s) r^{-\alpha_{l_s}}$ for $r \geq 1$

Since $\mathbf{P}(l_u) \neq \mathbf{P}(l_s)$ whenever $l_u \neq l_s$, we get the following.

3.2. *Remark.* Equation (1.4) admits no S.G.S. with s.d. whenever f is of type (1.2) and $\lambda_* < \delta^u < \lambda^* < \delta^s$.

Now we are ready to prove Theorem 1.5 and 1.6. We stress that both the results are obtained directly working with the constants $\tilde{\epsilon}_k$ defined in (2.11) and (2.13) which are larger than the ones used in the statement of the Theorems.

Proof of Theorems 1.5 and 1.6. Fix $k \in \mathbb{N}$, $\delta^s \in (\lambda^*, \lambda_*)$ and correspondingly $l_s \in (2_*, 2^*)$; choose $l_u \in (2^*, 2^* + \tilde{\epsilon}_k(l_s))$ and correspondingly δ^u. From the previous section we know that $W^u(0) \equiv M^u_{l_u}$ intersects $A^0_{l_s}$ transversally in $\breve{\mathbf{Q}}^{\mathbf{u}}_{\mathbf{l_u}}(j, l_s)$ for any $j = 1, \ldots, 2k$. For any $j = 1, \ldots, k$ we denote by $\breve{B}^u(j) = \breve{B}^u_{l_u}(2j-1, l_s) \cup \breve{B}^u_{l_u}(2j, l_s)$. Since $\mathbf{P}(l_s)$ is in the interior of the segment of $A^0_{l_s}$ between $\breve{\mathbf{Q}}^{\mathbf{u}}_{\mathbf{l_u}}(2k-1, l_s)$ and $\breve{\mathbf{Q}}^{\mathbf{u}}_{\mathbf{l_u}}(2k, l_s)$, it follows that $\mathbf{P}(l_s)$ is in the interior of $\breve{B}^u(k) \subset \breve{B}^u(j)$ for $j = 1, \ldots, k$, see Remark 2.3. Observe that $M^s_{l_s}$ is a continuous path that joins the origin and $\mathbf{P}(l_s)$. Moreover the flow of (2.3) where $l = l_s$ on the segment of $A^0_{l_s}$ between $\breve{\mathbf{Q}}^{\mathbf{u}}_{\mathbf{l_u}}(2j-2, l_s)$ and $\breve{\mathbf{Q}}^{\mathbf{u}}_{\mathbf{l_u}}(2j, l_s)$ points towards $\breve{B}^u(j)$ for any $j = 1, \ldots, k$ (remember that $\breve{\mathbf{Q}}^{\mathbf{u}}_{\mathbf{l_u}}(0, l_s) = (0,0)$). So if we follow $M^s_{l_s}$ from the origin towards $\mathbf{P}(l_s)$ we find $\mathbf{Q}^*(j) \in \big(\breve{M}^u_{l_u}(2j-1; l_s) \cup \breve{M}^u_{l_u}(2j; l_s)\big) \cap M^s_{l_s}$. Now consider the trajectory $\mathbf{x}_*(t, 0; \mathbf{Q}^*(j))$ of (3.2); since $\mathbf{Q}^*(j) \in W^u_{l_u}(0) \cap W^s_{l_s}(0)$ it follows that $\mathbf{x}_*(t, 0; \mathbf{Q}^*(j)) \in M^u_{l_u} \subset \mathbb{R}^2_\pm$ for any $t \leq 0$, and $\mathbf{x}_*(t, 0; \mathbf{Q}^*(j)) \in M^s_{l_s} \subset \mathbb{R}^2_\pm$ for any $t \geq 0$ and it is homoclinic to the origin. So the corresponding solution $u^j(r)$ of (1.4)

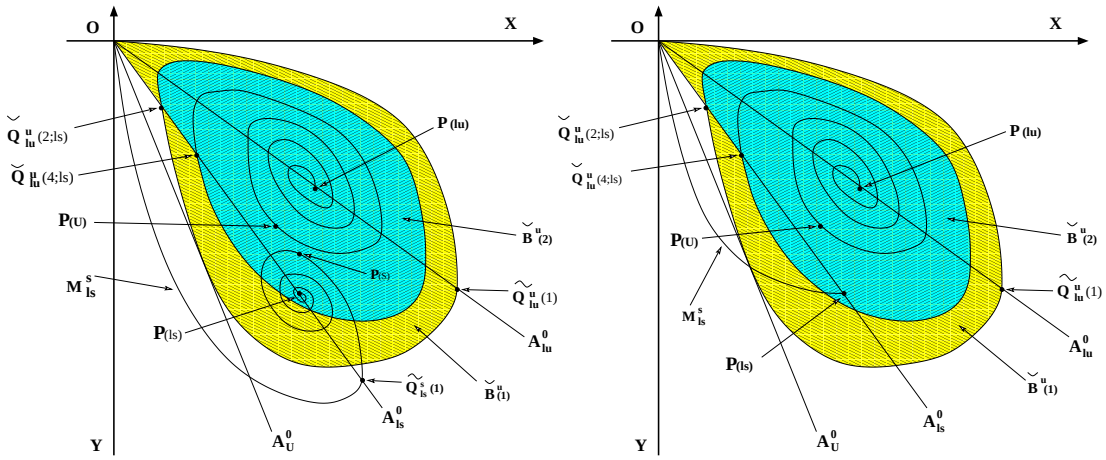

FIGURE 3. Sketch of $M^u_{l_u}$ and $M^s_{l_s}$ for $l_s \in (2_*, 2^*)$ and $l_u \in (2^*, 2^* + \epsilon_2(l_s))$. To the left the case $2_* < l_s \leq \sigma^*$ to the right the case $\sigma^* < l_s < 2^*$

is a monotone decreasing G.S. with f.d. for $j = 1, \dots, k$. We also stress that $\mathbf{Q}^*(i) \neq \mathbf{Q}^*(j)$ for $i \neq j$ since there is no intersection between $\breve{M}^u_{l_u}(2i-1; l_s) \cup \breve{M}^u_{l_u}(2i; l_s)$ and $\breve{M}^u_{l_u}(2j-1; l_s) \cup \breve{M}^u_{l_u}(2j; l_s)$; thus $u^i(r) \not\equiv u^j(r)$ for $i \neq j$.

The proof in the case $-2 < \delta^u < \lambda^*$ and $\delta^s \in (\lambda^*, \lambda^* + \tilde{\epsilon}_k(\delta^u))$, i. e. $l_u > 2^*$ and $l_s \in (2^* - \tilde{\epsilon}_k(l_u), 2^*)$ is completely analogous and will be omitted.

In fact we could work out a proof in this latter case also using the Kelvin inversion as follows. Let us choose δ^u and correspondingly $l_u > 2^*$: we look for a value $l_s \in (2_*, 2^*)$ such that $M^u_{l_u} \cap M^s_{l_s}$ contains at least k distinct points. Then we can conclude the existence of k decreasing G.S. with f.d. reasoning as above. Let us apply the Kelvin inversion: f.d. solutions are changed into regular solutions and viceversa. Moreover using Remark 2.7 we see that, applying (3.1) where $L_u(l_s) := 2\frac{(n-1)l_s - 2n}{(n-2)l_s - 2n + 2}$ and $L_s(l_u) := 2\frac{(n-1)l_u - 2n}{(n-2)l_u - 2n + 2}$ to the equation (2.10) obtained through (2.9), we obtain again a system of the form (3.2). Note that $L_u > 2^*$ and $L_s \in (2_*, 2^*)$. So applying the result just proved we see that there is $\tilde{\epsilon}_k(L_s)$ such that $M^u_{L_u} \cap M^s_{L_s}$ contains at least k distinct points for any $L_u \in (2^*, 2^* + \tilde{\epsilon}_k(L_s))$ and we find k G.S. with f.d. $\tilde{u}^j(s)$ for (2.10). Thus the function $u^j(r) = \tilde{u}^j(1/r)r^{2-n}$ obtained inverting (2.9) solves (1.4) and has fast decay (because $\tilde{u}^j(s)$ is a regular solution), and it is regular (because $\tilde{u}^j(s)$ has f.d.) and it is always positive: so it is a G.S. with f.d. Rewriting $L_u(l_s) < 2^* + \tilde{\epsilon}_k[L_s(l_u)]$ as $l_s > 2^* - \bar{\epsilon}_k(l_u)$, we get

$$\bar{\epsilon}_k(l_u) = \frac{2\tilde{\epsilon}_k(L_s(l_u))}{1 + (n-2)\tilde{\epsilon}_k(L_s(l_u))}$$

Thus we find that the original equation (1.4) with f of type (1.2) has k decreasing G.S. with f.d. whenever $l_u > 2^*$ and $l_s \in (2^* - \bar{\epsilon}_k(l_u), 2^*)$, i.e. $\lambda_* < \delta^u < \lambda^*$ and $\delta^s \in (\lambda^*, \lambda^* + \bar{\epsilon}_k(\delta^u))$. $\square$

3.3. *Remark.* From the argument at the end of the proof of Theorem 1.5 we get a further symmetric relationship between $\epsilon_k(l_u)$ and $\epsilon_k(l_s)$ for any k, i.e.:

$$\tilde{\epsilon}_k(l_u) = \frac{2\tilde{\epsilon}_k(L_s(l_u))}{1 + (n-2)\tilde{\epsilon}_k(L_s(l_u))} \qquad \tilde{\epsilon}_k(l_s) = \frac{2\tilde{\epsilon}_k(L_u(l_s))}{1 + (n-2)\tilde{\epsilon}_k(L_u(l_s))} \tag{3.3}$$

Now we prove the non-existence counterpart of the previous Theorem. Once again it is developed with the better constants $\tilde{\epsilon}_0$ and $\tilde{N}_0$.

Proof of Theorem 1.7. Fix $-2 < \delta^u < \lambda^*$ and correspondingly $l_u > 2^*$. From the

definition of $\tilde{\epsilon}_0(l_u)$, see (2.12), we get that $M^s_{l_s} \subset A^-_{l_u}$ for any $l_s < 2_* + \tilde{\epsilon}_0(l_u)$ and $M^s_{l_s} \cap M^u_{l_u} = \emptyset$. So consider (3.2) with these values for l_u and l_s: it follows that $W^u(0) \cap W^s(0) = \emptyset$. Therefore there are no G.S. with f.d. Moreover, since neither $\mathbf{P}(l_u) \in M^s_{l_s}$, nor $\mathbf{P}(l_s) \in M^u_{l_u}$, there are no S.G.S with f.d., and no G.S. with s.d. Moreover no S.G.S. with s.d. might exist, see Remark 3.2.

Analogously fix $\delta^s > 0$ and correspondingly $l_s \in (2_*, 2^*)$. Reasoning as above we conclude that (1.4) admits no G.S. neither S.G.S. (either with fast or s.d.), whenever $l_u > \tilde{N}_0(l_s)$. Once again we could prove this second non-existence result using the Kelvin inversion (2.9) and Remark 2.7 as in the proof of Theorem 1.5, obtaining a further symmetric relationship analogous to (3.3). □

From the previous proof and Remark 2.9 we get also the following Corollary.

3.4. **Corollary.** *Theorems 1.5, 1.6, and 1.7 hold also if we replace ϵ_k, ϵ_0, N_0 by the constants $\tilde{\epsilon}_k \geq \epsilon_k$, $\tilde{\epsilon}_0 \geq \epsilon_0$ and $\tilde{N}_0 \leq N_0$ defined in (2.12).*
Moreover $2_* + \tilde{\epsilon}_0(l_u) < \sigma_*$ for any $l_u \geq \sigma^*$ and $N_0(l_s) < \sigma^*$ for any $l_s \in (2_*, \sigma_*]$. In particular (1.4) admits no positive solutions either regular or singular, whenever $l_u \geq \sigma^*$ and $l_s \leq \sigma_*$.

3.5. *Remark.* Let us denote by $\omega^s_* := \sup\{l \,|\, S(L) < 2^* \text{ for any } 2_* < L < l\}$ and by $\omega^u_* := \inf\{l \,|\, U(L) > 2^* \text{ for any } L > l\}$. Then (1.4) admits no positive solutions either regular or singular, whenever $l_s < \omega^s_* < \omega^u_* < l_u$.

Note that $\sigma_* < \omega^s_* < 2^* < \omega^s_* < \sigma^*$.

3.6. *Remark.* We explain now, how a computer assisted proof could be worked out in order to evaluate the values of $\epsilon_1(l_u), \epsilon_1(l_s), \epsilon_0(l_u), N_0(l_s)$ used in Theorems 1.5, 1.10, 1.7, 1.12.

Using a software capable to evaluate an approximated trajectory of an O.D.E. taking into account the errors, we can draw an approximated unstable manifold $M^u_{l_u}$ for $l_u > 2^*$ as follows. We choose $\mathbf{Q} \in A^0_{l_u}$ where $|\mathbf{Q}|$ is small and we consider the real trajectory $\mathbf{x}_{l_u}(t, 0; \mathbf{Q})$. There exists a value $\acute{U}$ such that $\mathbf{x}_{l_u}(t, 0; \mathbf{Q})$ crosses A^0_l for any $l > \acute{U}$ and it is tangent to $A^0_{\acute{U}}$. Analogously we choose $\mathbf{D} = (0, \delta)$ where $\delta > 0$ is small and we consider the real trajectory $\mathbf{x}_{l_u}(t, 0; \mathbf{D})$; if $\delta > 0$ is small enough there exists a value $\grave{U}$ such that $\mathbf{x}_{l_u}(t, 0; \mathbf{Q})$ crosses A^0_l for any $l > \grave{U}$ and it is tangent to $A^0_{\grave{U}}$. It is clear from the uniqueness of the solutions of (2.3) that $\grave{U} < U(l_u) < \acute{U}$. Using an appropriate software we can replace the values $\grave{U}$ and $\acute{U}$ by two approximating intervals, say $[\grave{U}_a, \grave{U}_b]$ and $[\acute{U}_a, \acute{U}_b]$. More precisely we evaluate intervals such that surely $\grave{U} \in [\grave{U}_a, \grave{U}_b]$ and $\acute{U} \in [\acute{U}_a, \acute{U}_b]$. Then, setting $U_A := \min\{\grave{U}_a, \acute{U}_a\}$, $U_B := \max\{\grave{U}_b, \acute{U}_b\}$, we are sure that the real value $U(l_u) \in [U_A, U_B]$. Note that we can assume $2_* \leq U_A < U_B \leq l_u$. Then it follows that no positive solutions exist for any $l_s < U_A$ and there is at least a G.S. with f.d. whenever $l_s > U_B$ (this corresponds to giving an estimate of the value $\epsilon_0(l_u)$ and $\epsilon_1(l_u)$).

Reasoning in the same way we obtain an estimate of the value $S(l_s)$ i.e. we find two values $l_s \leq S_A < S_B$ such that surely $S(l_s) \in [S_A, S_B]$; then we find values l_u for which we have either non-existence of positive solutions or existence of G.S. with f.d. for prescribed values of l_u and l_s.
In fact we can have better results through a construction analogous to the one of $\tilde{\epsilon}$ and $\tilde{N}_0$, but we need more computational power: (1.4) admits at least a G.S. with f.d. whenever $l_u > l_s$ have been chosen so that $U_B < S_A$, and no positive solution if $S_B < U_A$.

Clearly if the function $U(l)$ and $S(l)$ are monotone as we conjecture this argument can be improved further.

From the proof of Theorems 1.5 and 1.6 we can easily infer also Theorem 1.8.
Proof of Theorem 1.8. Let us fix $\lambda^* < \delta^s < \lambda_*$ and correspondingly $l_s \in (2_*, 2^*)$. Fix $k \in \mathbb{N}$; from Remark 2.12 we know that the function $U_k(l) : [\sigma^*, 2^*] \to [2_*, \sigma^*]$ is surjective. Thus there is $r_k(l_s)$ such that $\tilde{M}^u_{r_k(l_s)}(2k)$ is tangent to $A^0_{l_s}$ in $\mathbf{P}(l_s)$. Consider system (3.2) where $l_s = l_s$ and $l_u = r_k(l_s)$, and the trajectory $\mathbf{x}_*(t, 0; \mathbf{P}(l_s))$; let $u^k_*(r)$ be the corresponding solution of (1.4) with f of type (1.2) with $\delta^u = r_k(\delta^s)$ and $\delta^s = \delta(l_s)$ where (abusing the notation) we set $r_k(\delta^s) = \delta(r_k(l_s))$. Then $\mathbf{x}_*(t, 0; \mathbf{P}(l_s)) \in M^u_{r_k(l_s)} \subset \mathbb{R}^2_\pm$ for any $t \le 0$ and $\mathbf{x}_*(t, 0; \mathbf{P}(l_s)) \equiv \mathbf{P}(l_s)$ for any $t \ge 0$: so $u^k_*(r)$ is a G.S. with s.d.
Since $\tilde{M}^u_{r_i(l_s)}(2i) \cap \tilde{M}^u_{r_j(l_s)}(2j) = \emptyset$ for $i \ne j$ it follows that $r_i(l_s) \ne r_j(l_s)$. Moreover we can assume $r_k(l_s) \to 2^*$ as $l_s \to 2_*$ for any fixed k, since $U_k(2^*) = 2_*$. We observe that in fact we could have two values $\acute{r}_k(l_s)$ and $\grave{r}_k(l_s)$ giving a G.S. with s.d., since U_k is not a priori monotone, however we can always choose $r_k(l_s)$ monotone decreasing in k. Furthermore by construction $r_k(l_s) \to 2^*$ as $k \to +\infty$ for any fixed l_s.

Now fix $\delta^u \in (-2, \lambda^*)$ in (1.2) and correspondingly $l_u > 2^*$. Reasoning as above, for any $k \in \mathbb{N}$ we can find $r_k(l_u) \in (\sigma_*, 2^*)$ such that $\mathbf{P}(l_u) \in \tilde{M}^s_{r_k}(2k)$. So it follows that $\mathbf{x}_*(t, 0, \mathbf{P}(l_u)) \equiv \mathbf{P}(l_u)$ for $t \le 0$ and $\mathbf{x}_*(t, 0, \mathbf{P}(l_u)) \in M^s_{r_k}$ for $t \ge 0$, and the solution $u(r)$ of (1.4) corresponding to $\mathbf{x}_*(t, 0, \mathbf{P}(l_u))$ is a decreasing S.G.S. with f.d. Then, reasoning as above, we get the monotonicity and the asymptotic properties of $r_k(l_u)$. An alternative proof can be obtained using Kelvin inversion (2.9) and reasoning as in the proof of Theorems 1.5 and 1.6. □
With the same argument we obtain easily the following counterpart result. Set $\Omega^s_* := \inf\{l \,|\, S(L) > 2^* \text{ for any } l < L < 2^*\}$ and by $\Omega^u_* := \sup\{l \,|\, U(L) < 2^* \text{ for any } 2^* < L < l\}$.

3.7. **Corollary.** *Consider equation (1.4) with f of type (1.2). Fix $\delta^u \in (\delta(\Omega^u_*), \lambda^*)$, then there is at least a value $\delta^s = R(\delta^u) \in (\lambda^*, \lambda_*)$ such that (1.4) admits a G.S. with s.d.*

Analogously fix $\delta^s \in (\lambda^*, \delta(\Omega^s_*))$, then there is at least a value $\delta^u = R(\delta^s) \in (-2, \lambda^*)$ such that (1.4) admits a S.G.S. with f.d.

Moreover observe that if $\delta^u \in (-2, \delta(\omega^u_*))$ (1.4) admits no G.S. with s.d. whenever $\delta^s \in (\lambda^*, \lambda_*)$, and if $\delta^s \in (\delta(\omega^s_*), \lambda_*)$ (1.4) admits no S.G.S. with f.d. whenever $\delta^u \in (-2, \lambda^*)$

Proof. Fix l_u corresponding to δ^u and observe that if $l_u \in (2^*, \sigma^*)$ then $M^u_{l_u}$ is tangent to $A^0_{U_k(l_u)}$ in $\mathbf{P}(U_k(l_u))$ for any $k \in \mathbb{N}$. So if $U_k(l_u) < 2^*$ we can set $l_s = U_k(l_u)$, and reasoning as in the proof of Theorem 1.8 we get a G.S. with s.d. If $l_u \in (2^*, \Omega^u_*)$ then $U_1(l_u) < 2^*$ by construction, so the claim concerning the existence of G.S. with s.d. follows.
Moreover if $l_u > \omega^u_*$ then $M^u_{l_u} \cap A^-_{2^*} = \emptyset$. Since $\mathbf{P}(l_s) \in A^-_{2^*}$ for any $l_s \in (2_*, 2^*)$ we find that there cannot be G.S. with s.d. The claim concerning S.G.S. with f.d. can be proved in the same way. □

The existence of G.S. with s.d. and of S.G.S. with f.d. for the equations discussed in this paper, should be a rare phenomenon (non-generic) as pointed out in [2], and as we will see in the next section. However, besides its intrinsic importance, it is relevant because it indicates the appearance of the resonance phenomenon described in Corollary 1.9, which was first detected by Flores in [11] for a similar problem.

To prove this result we need the following Lemma strongly inspired by Lemma 4.1 in [11]. Let $\mathbf{P} \in \mathbb{R}^2$ and consider the curves

$$\mathbf{S}(s) = \mathbf{P} + \rho(s)e^{i\theta(s)}\,; \qquad U(s) = \mathbf{P} + R(s)e^{iw(s)}\,, \tag{3.4}$$

both defined for $s \in (0, +\infty)$. Assume that ρ, θ, R, w are continuous; assume further that $\rho(s) > 0$ and $R(s) > 0$ for any $s > 0$, $\rho(0^+) = R(0^+) = 0$, and that the limits $\lim_{s\to 0} w(s) = w(0)$, $\lim_{s\to 0} \theta(s) = \theta(0)$, $\lim_{s\to+\infty} w(s) = w(\infty)$ and $\lim_{s\to+\infty} \theta(s) = \theta(\infty)$ are well defined but possibly infinite. Furthermore assume that $S(s)$ and $U(s)$ have no self-intersections. Then we have the following.

3.8. **Lemma.** *Assume $\theta(0) = +\infty > \theta(+\infty)$ (possibly $\theta(+\infty) = -\infty$), and that $\rho(+\infty) \le R(+\infty)$. Assume further that either $w(0)$ and $w(+\infty)$ are both finite or $w(0) < w(+\infty)$; then the curves $U(s)$ and $S(s)$ intersects infinitely many times.*

Moreover there exists $\epsilon_k > 0$ and $M_k > 0$ such that for any couple of continuous curves $U_1, S_1 : (0, M_k) \to \mathbb{R}^2$ satisfying $|U(s) - U_1(s)| + |S(s) - S_1(s)| < \epsilon_k$ for any $s \in (0, M_k)$, there are at least k intersections between U_1 and S_1.

Proof. We lift U and S to the universal covering of $\mathbb{R}^2 \setminus \{\mathbf{P}\}$, hence the lifting of U is $\tilde{U}(s) = (R(s), w(s))$ and the lifting of S is $\tilde{S}(s) = (\rho(s), \theta(s))$. Our assumptions implies that both $\tilde{U}$ and $\tilde{S}$ lie in the strip $[0, R(\infty)] \times \mathbb{R}$, which is divided by the graph of $\tilde{U}$ into two components A^- and A^+, to the "left" and to the "right" of $\tilde{U}$ respectively. Consider the family of translates $\tilde{S}_j(s) = (\rho(s), \theta(s) - 2j\pi)$ which are lifting of S as well. Observe that choosing $k > 0$ large enough we find $\tilde{S}_j(0) \in A^+$ and $\tilde{S}_j(+\infty) \in A^-$ for any $j > k$, hence there is an intersection (ρ_j, θ_j) between $\tilde{S}_j$ and $\tilde{U}$. In fact given $n > 0$ we can find j such that $\rho_j < 1/n$. For the arbitrariness of n the original curves S and U inherits infinitely many distinct intersections.

Now we consider the perturbed curves U_1 and S_1. For any integer k we can find $\delta(k) > 0$ such that there are at least k intersections (ρ_j, θ_j), for $j = 1, \dots, k$ and $\rho_j \in (\delta(k), \rho(+\infty)/4)$. Let $S_1(0) = \mathbf{P_1}$ and $U_1(0) = \mathbf{Q_1}$; we introduce again the universal covering of $\mathbb{R}^2 \setminus \{\mathbf{P_1}\}$ and we choose ϵ_k small enough so that the lifting $\tilde{U}_1(s) = (R_1(s), w_1(s))$ has $R_1(0) < \delta(k)/2$ and $R_1(+\infty) > \rho(+\infty)/2$. Using a continuity argument (possibly choosing a smaller $\epsilon_k > 0$) we find that the k intersections persist. □

In the next section we need to apply this Lemma in a more general framework.

3.9. *Remark.* Lemma 3.8 works with no changes in the proof also if we replace the hypotheses on the limit $\lim_{t\to 0^+} w(s)$ by the following:

$$\text{both } \liminf_{t\to 0^+} w(s) \text{ and } \limsup_{t\to 0^+} w(s) \text{ are bounded.}$$

Now we are ready to prove Corollary 1.9.

Proof of Corollary 1.9. Assume that (1.4) admits a G.S. with s.d. $u(r)$ and let $\mathbf{x}_*(t, 0; \mathbf{Q})$ be the corresponding trajectory of (3.2). Let $\hat{l}_s$ and $\hat{l}_u$ be the values corresponding to $\hat{\delta}^s$ and $\hat{\delta}^u$ respectively. Then $\mathbf{Q} = \mathbf{P}(\hat{l}_s)$ and $\mathbf{x}_*(t, 0; \mathbf{Q}) \equiv \mathbf{P}(\hat{l}_s)$ for $t \ge 0$ since $u(r)$ has s.d., and $\mathbf{x}_*(t, 0; \mathbf{Q}) \in M^u_{\hat{l}_u}$ for $t \le 0$ since $u(r)$ is a regular solution. Therefore $\mathbf{P}(\hat{l}_s) \in M^u_{\hat{l}_u}$; denote by $\hat{M}^u_{\hat{l}_u}$ the branch of $M^u_{\hat{l}_u}$ between the origin and $\mathbf{P}(\hat{l}_s)$. Since $\hat{l}_s \in (\sigma_*, 2^*)$, $\mathbf{P}(\hat{l}_s)$ is an unstable focus, therefore $M^s_{\hat{l}_s}$ winds around $\mathbf{P}(\hat{l}_s)$ indefinitely.

Let $\mathbf{F}(\mathbf{x}, \hat{l}_u)$ denote the right hand side of (2.3) where $l = \hat{l}_u$; then $\mathbf{F}(\mathbf{P}(\hat{l}_s), \hat{l}_u) \neq (0, 0)$. So $\hat{M}^u_{\hat{l}_u}$ has a definite tangent in $\mathbf{P}(\hat{l}_s)$; hence we can apply Lemma 3.8, where $\mathbf{P}(\hat{l}_s) = \mathbf{P}$, $S(t)$ and $U(t)$ are parameterizations of $M^s_{\hat{l}_s}$ and $\hat{M}^u_{\hat{l}_u}$ respectively, $\rho(\infty) = R(\infty) = \|\mathbf{P}(\hat{l}_s)\|$, $\theta(\infty) = w(+\infty)$ are finite, and $\theta(0) = +\infty$ while $w(0)$ is finite.

Hence in an arbitrarily small neighborhood of $\mathbf{P}(\hat{l}_s)$ we can find infinitely many points $\mathbf{P^j} \in M^u_{\hat{l}_u} \cap M^s_{\hat{l}_s}$.

Consider the trajectories $\mathbf{x}_*(t,0;\mathbf{P^j})$ and the corresponding solutions $u^j(r)$ of (1.4). It follows that $\mathbf{x}_*(t,0;\mathbf{P^j}) \in M^u_{\hat{l}_u} \subset \mathbb{R}^2_\pm$ for any $t \le 0$, and $\mathbf{x}_*(t,0;\mathbf{P^j}) \in M^s_{\hat{l}_s} \subset \mathbb{R}^2_\pm$ for any $t \ge 0$, so it is homoclinic to the origin; hence $u^j(r)$ is a monotone decreasing G.S. with f.d., for any $j \in \mathbb{N}$.

From Lemma 3.8, we also infer the existence of $\eta(k) > 0$ such that there are k points $\mathbf{Q^1},\dots,\mathbf{Q^k} \in M^s_{l^s} \cap M^u_{l^u}$ whenever $|l^u - \hat{l}_u| + |l^s - \hat{l}_s| < \eta(k)$. So arguing as above we see that the solutions $v^j(r)$ of (1.4) corresponding to $\mathbf{x}_*(t,0;\mathbf{Q^j})$ are G.S. with f.d. The proof when a S.G.S. with f.d. exists is completely analogous.□

3.10. *Remark.* Consider equation (1.4) with f of type (1.2) for $\delta^u = \delta(l_u)$ and $\delta^s = \delta(l_s)$ where $l_s \in (2_*, \sigma_*]$. Summing up the previous results we find that (1.4) admits no positive solutions either regular or singular whenever $l_u > \tilde{N}_0(l_s)$. Then we find a decreasing sequence $r_1(l_s) > \dots > r_k(l_s) \to 2^*$ such that (1.4) admits a G.S. with s.d. whenever $l_u = r_k(l_s)$ and at least k G.S. with f.d. whenever $l_u \in (r_{k+1}(l_s), r_k(l_s))$ for any $k \ge 1$.

Analogously assume $l_u \ge \sigma^*$; then (1.4) admits no positive solutions either regular or singular whenever $2_* < l_s < 2_* + \tilde{\epsilon}_0(l_u)$. Moreover there is an increasing sequence $r_1(l_u) < \dots < r_k(l_u) \to 2^*$ such that (1.4) admits a S.G.S. with f.d. whenever $l_s = r_k(l_u)$ and at least k G.S. with f.d. whenever $l_s \in (r_k(l_u), r_{k+1}(l_u))$ for any $k \ge 1$.

We stress that if the functions $U(l)$ and $S(l)$ are monotone we also have $\tilde{N}_0(l_s) = r_1(l_s)$ and $2_* + \tilde{\epsilon}_0(l_u) = r_1(l_u)$.

This Remark follows putting together Theorems 1.7, 1.6, 1.8, and observing that, by construction, we can choose $\tilde{\epsilon}_k(l_s) + 2^* = r_k(l_s)$ and $2^* - \tilde{\epsilon}_k(l_u) = r_k(l_u)$.

3.11. *Remark.* Consider equation (1.4) with f of type (1.2) for $\delta^u = \delta(l_u)$ and $\delta^s = \delta(l_s)$ where $l_s \in (\sigma_*, 2^*)$. Then (1.4) admits no positive solutions either regular or singular, whenever $l_u > \tilde{N}_0(l_s)$. Then we find a decreasing sequence of values $r_j(l_s) \to 2^*$ such that (1.4) admits a G.S. with s.d. and infinitely many G.S. with f.d. whenever $l_u = r_j(l_s)$. Close to these values there are small windows of amplitude $\eta_k(r_j) > 0$ such that if $|l_u - r_j(l_s)| < \eta_k(r_j)$, then (1.4) admits at least k G.S. with f.d. for any integer $k > 0$. We can assume w.l.o.g. that $\eta_k(r_j) < \min\{|r_j - r_{j-1}|; |r_{j+1} - r_j|\}$. Moreover whenever $l_u \in [r_{k+1}(l_s), r_k(l_s)]$, there are at least k G.S. with f.d.

Analogously assume $2^* < l_u < \sigma^*$; then (1.4) admits no positive solutions either regular or singular, whenever $l_s < \tilde{\epsilon}_0(l_u) + 2_*$. Moreover there is an increasing sequence $r_j(l_u) \to 2^*$ such that (1.4) admits a S.G.S. with f.d. and infinitely many G.S. with f.d. whenever $l_s = r_j(l_u)$. Close to these values there are small windows of amplitude $\eta_k(r_j) > 0$ such that if $|l_s - r_j(l_u)| < \eta_k(r_j)$, then (1.4) admits at least k G.S. with f.d. for any integer $k > 0$ (again $\eta_k(r_j) < \min\{|r_j - r_{j-1}|; |r_{j+1} - r_j|\}$). Furthermore whenever $l_s \in [r_k(l_u), r_{k+1}(l_u)]$, there are at least k G.S. with f.d.

We see now briefly which are the consequences of our analysis for the Dirichlet problem in the ball. Let us denote by $u(d,r)$ the regular solution of (1.4) with $u(d,0) = d$. From the proof of Theorem 1.5 and Remark 2.5 it follows easily that there is $d_* > 0$ such that $u(d,r)$ is a crossing solution for any $0 < d < d_*$. Moreover, using Remark 2.5 and continuous dependence on initial data of (3.2) we can check easily that the set

$$C := \{d > 0 \,|\, u(d,r) \text{ is a crossing solution } \} \tag{3.5}$$

is open. Denote by $R(d)$ the first zero of $u(d,r)$: using again Remark 2.5 and continuous dependence on initial data we find that $R(d)$ is continuous on C. Furthermore from Remark 2.5 we get $R(d) \to +\infty$ as $d \to 0$. Let $v(r)$ be the unique

singular solution and R^* its first zero (we set $R^* = +\infty$ if v is a S.G.S.), then $R(d) \to R^*$ as $d \to +\infty$: this follows using Remark 2.5 and continuous dependence of (3.2) from initial data.

Using these observations and the fact that if $u(d^*, r)$ is a G.S. then $R(d) \to +\infty$ as $d \to d^*$, we find the following.

3.12. **Proposition.** *Consider (1.4) with f of type (1.2); there are $\rho_2 \geq \rho_1 > 1$ such that the Dirichlet problem in the ball of radius R admits no solutions whenever $0 < R < \rho_1$, at least two solutions for $R \in (\rho_1, \rho_2)$ and at least one for $R \geq \rho_2$. Moreover assume that there are exactly k G.S. with f.d. (or infinitely many of them). Then there are $1 < \rho_0 \leq \rho_1 \leq \rho_2 \leq \ldots \leq \rho_k < +\infty$ (respectively an increasing sequence $\rho_k \to \infty$), such that the Dirichlet problem in the ball of radius R admits no solutions for $0 < R < \rho_0$, at least $2j+1$ solutions for any $R \geq \rho_j$ for $j = 0, \ldots, k$ (respectively no solutions for $0 < R < \rho_0$, at least $2j+1$ solutions for any $R \geq \rho_j$ for $j \in \mathbb{N}$).*

Using Remark 2.5 and analyzing the proof of Corollary 1.9 we get the following.

3.13. **Proposition.** *Consider (1.4) with f of type (1.2) and assume that $\delta^s \in (\lambda^*, \Sigma_*)$ while $\delta^u = r_j(\delta^s)$, so that there is a G.S. with s.d. $u(\bar{d}, r)$, see Theorem 1.8 and Corollary 1.9. Then there is a sequence $d_j \to \bar{d}$ such that $u(d_j, r)$ is a G.S. with f.d.*

Analogously assume that $\delta^u \in (\Sigma^*, \lambda^*)$ and $\delta^s = r_j(\delta^u)$, so that there is a S.G.S. with f.d., see Theorem 1.8 and Corollary 1.9. Then there is a sequence $d_j \to +\infty$ such that $u(d_j, r)$ is a G.S. with f.d.

The proof easily follows observing that the points $\mathbf{P^j}$ defined in the proof of Corollary 1.9 are such that $\mathbf{P^j} \to \mathbf{P}(l_s)$ as $j \to +\infty$ when a G.S. with s.d. exists, and $\mathbf{P^j} \to \mathbf{P}(l_u)$ as $j \to +\infty$ when a S.G.S. with f.d. exists.
We stress that our analysis can be extended to slightly more general non-linearities f. If $u(r)$ is a solution of (1.4) with f of type (1.2), then $\tilde{u}(r) = u(r)K^{-1/(q-2)}$ with $K > 0$ solves (1.4) where f is replaced by $\tilde{f} = Kf$. So all the Theorems continue to hold.

Furthermore, thanks to the t-invariance property of the autonomous system (2.3), if $\mathbf{x}(t)$ solves (3.2), then $\mathbf{x}(t-t_0)$ solves the system (3.2) where $(\alpha(t), \beta(t), \gamma(t)) = (\alpha_{l_u}, \beta_{l_u}, \gamma_{l_u})$ if $t \leq t_0$ and $(\alpha(t), \beta(t), \gamma(t)) = (\alpha_{l_s}, \beta_{l_s}, \gamma_{l_s})$ if $t \geq t_0$. So we have the following.

3.14. *Remark.* All the results proved for (1.4) with f of type (1.2) can be trivially extended to functions f of the form

$$f(u, r) = Ku|u|^{q-2} \begin{cases} (r/r_0)^{\delta^u} & \text{if } r \leq r_0 \\ (r/r_0)^{\delta^s} & \text{if } r \geq r_0 \end{cases}, \tag{3.6}$$

4. f of type (1.3)

In this section we discuss (1.4) when f is of type (1.3) using methods similar to the one exploited in the previous section. First of all we introduce the following non-autonomous system, applying (2.1) with $l = q^u$, so that we pass from (1.4) to the following:

$$\begin{pmatrix} \dot{x}_\top \\ \dot{y}_\top \end{pmatrix} = \begin{pmatrix} \alpha_{q^u} & 1 \\ 0 & \gamma_{q^u} \end{pmatrix} \begin{pmatrix} x_\top \\ y_\top \end{pmatrix} + \begin{pmatrix} 0 \\ -h^u(x_\top, t) \end{pmatrix} \tag{4.1}$$

where

$$h^u(x,t) := \begin{cases} x|x|^{q^u-2} & \text{if } x \geq \exp(\alpha_{q^u} t) \\ x|x|^{q^s-2} e^{\delta^u t} & \text{if } x \leq \exp(\alpha_{q^u} t) \end{cases}, \qquad \delta^u = \frac{2(q^u - q^s)}{q^u - 2} > 0\,.$$

Observe that $h^u(x,t)$ is continuous and locally Lipschitz in the x variable, uniformly with respect to t for $t \le \tau$ for any given $\tau \in \mathbb{R}$. It follows that local uniqueness of the solutions of (3.2) is still ensured, and this allows us to establish the existence of the unstable manifold. In [18, 21] it is proved that a non-autonomous system such as (3.2) admits a local unstable Lipschitz manifolds, denoted by $W^u_{q^u,\mathrm{loc}}(\tau)$, whenever the equation is Lipschitz in the (x,y) variables and uniformly continuous in the t-variable. More precisely the sets

$$W^u_{q^u,\mathrm{loc}}(\tau) := \{\mathbf{Q} \in \mathfrak{O} \mid \lim_{t\to-\infty} \mathbf{x}(t,\tau;\mathbf{Q}) = 0\}$$

are topological 1-dimensional manifolds and their intersections with a transversal vary continuously in τ (they inherit the smoothness of (3.2)), if $\mathfrak{O}$ is a sufficiently small neighborhood of the origin, see [18]. In fact the sets $W^u_{q^u,\mathrm{loc}}(\tau)$ are graph of locally Lipschitz functions on the x axis (however for our purposes continuity is enough), see [18].

Then using the flow it is possible to construct global manifolds, denoted by $W^u_{q^u}(\tau)$, as follows:

$$W^u_{q^u}(\tau) := \{\mathbf{Q} \in \mathbb{R}^2 \mid \exists\, T \le \tau \,\text{such that}\, \mathbf{x}_\top(T,\tau;\mathbf{Q}) \in W^u_{q^u,\mathrm{loc}}(T)\} \tag{4.2}$$

The existence of these unstable sets can be proved also in a more standard way, followed in [2, 11]. Let us add to (4.1) the extra variable $z(t) = \mathrm{e}^{\delta^u t}$, in order to obtain an autonomous system. The system obtained is locally Lipschitz and the origin of the 3-dimensional system admits a 2-dimensional Lipschitz unstable manifold, say $\mathbf{W^u_{q^u}}$ (again for Lipschitz manifold we mean a set which is locally the graph of a Lipschitz function of 2 variables).

If we set $W^u_{q^u}(\tau) := \{(x,y) \mid (x,y,e^{\delta^u\tau}) \in \mathbf{W^u_{q^u}}\}$ we obtain again the manifolds defined through (4.2). This gives a simple explanation of the smooth dependence on τ of these sets. In order to construct the stable manifold we need to set $l = q^s$ in (2.1) to obtain

$$\begin{pmatrix} \dot{x}_\perp \\ \dot{y}_\perp \end{pmatrix} = \begin{pmatrix} \alpha_{q^s} & 1 \\ 0 & \gamma_{q^s} \end{pmatrix} \begin{pmatrix} x_\perp \\ y_\perp \end{pmatrix} + \begin{pmatrix} 0 \\ -h^s(x_\perp,t) \end{pmatrix} \tag{4.3}$$

where

$$h^s(x,t) := \begin{cases} x|x|^{q^u-2}e^{\delta^s t} & \text{if } x \ge \exp(\alpha_{q^s}t) \\ x|x|^{q^s-2} & \text{if } x \le \exp(\alpha_{q^s}t) \end{cases}, \qquad \delta^s = \frac{2(q^s-q^u)}{q^s-2} < 0\,.$$

Again $h^s(x,t)$ is continuous and locally Lipschitz in the x variable, uniformly with respect to t for $t \ge \tau$ for any given $\tau \in \mathbb{R}$. Arguing as above we construct the sets

$$W^s_{q^s,\mathrm{loc}}(\tau) := \{\mathbf{Q} \in \mathfrak{O} \mid \lim_{t\to+\infty} \mathbf{x}(t,\tau;\mathbf{Q}) = 0\}$$

which are graph of locally Lipschitz functions from $A^0_{2_*}$ to $\mathbb{R}^2$, see [18]. Then we define global stable sets as follows:

$$W^s_{q^s}(\tau) := \{\mathbf{Q} \in \mathbb{R}^2 \mid \exists\, T \ge \tau \text{ such that } \mathbf{x}_\perp(T,\tau;\mathbf{Q}) \in W^s_{q^s,\mathrm{loc}}(T)\}$$

Again we might define $W^s_{q^s}(\tau)$ by introducing a three dimensional autonomous system, where we have added the extra variable $z = \exp(\delta^s t)$, and proceeding as above.

Then, using the diffeomorphism $\aleph$, we can change variables from $\mathbf{x}_\top$ to $\mathbf{x}_\perp$ and viceversa and define a stable manifold for (4.1) and an unstable manifold for (4.3). More precisely we set

$$\begin{aligned} W^s_{q^u}(\tau) &:= \aleph^\tau_{q^u,q^s} W^s_{q^s}(\tau) = \{\mathbf{Q}\exp[-(\alpha_{q^s}-\alpha_{q^u})\tau] \in \mathbb{R}^2 \mid \mathbf{Q} \in W^s_{q^s}(\tau)\} \\ W^u_{q^s}(\tau) &:= \aleph^\tau_{q^s,q^u} W^u_{q^u}(\tau) = \{\mathbf{Q}\exp[(\alpha_{q^s}-\alpha_{q^u})\tau] \in \mathbb{R}^2 \mid \mathbf{Q} \in W^u_{q^u}(\tau)\} \end{aligned} \tag{4.4}$$

Observe that $W^s_{q^u}(\tau)$ and $W^u_{q^s}(\tau)$ are bounded for any $\tau \in \mathbb{R}$ but the former becomes unbounded as $\tau \to -\infty$ while the latter becomes unbounded as $\tau \to +\infty$.

It is straightforward to check that systems (4.1) and (4.3) inherit from (2.4) the property described in Remark 2.1.

4.1. *Remark.* Trajectories $\mathbf{x}_\top(t,\tau;\mathbf{Q})$ of (4.1) correspond to regular solutions $u(r)$ of (1.4) if and only if $\mathbf{Q} \in W^u_{q^u}(\tau)$, and to fast decay solutions if and only if $\mathbf{Q} \in W^s_{q^u}(\tau)$.

Analogously trajectories $\mathbf{x}_\perp(t,\tau;\mathbf{Q})$ of (4.3) correspond to regular solutions of (1.4) if and only if $\mathbf{Q} \in W^u_{q^s}(\tau)$, and to fast decay solutions if and only if $\mathbf{Q} \in W^s_{q^s}(\tau)$. Moreover

$$\begin{aligned} W^u_{q^s}(\tau) &\subset \overline{W^u_{q^s}(\tau)} := \{\mathbf{Q} \,|\, \lim_{t\to-\infty}\mathbf{x}_\perp(t,\tau;\mathbf{Q}) = (0,0)\} \\ W^s_{q^u}(\tau) &\subset \overline{W^s_{q^u}(\tau)} := \{\mathbf{Q} \,|\, \lim_{t\to+\infty}\mathbf{x}_\top(t,\tau;\mathbf{Q}) = (0,0)\}\,, \end{aligned} \tag{4.5}$$

and from simple asymptotic estimates we see that $\overline{W^u_{q^s}(\tau)} = W^u_{q^s}(\tau) \cup \{\mathbf{S}\}$ where $\mathbf{x}_\perp(t,\tau;\mathbf{S})$ corresponds to the unique singular solution of (1.4), while $\overline{W^s_{q^u}(\tau)} = W^s_{q^u}(\tau) \cup \{\mathbf{T}\}$ where $\mathbf{x}_\top(t,\tau;\mathbf{T})$ corresponds to the unique slow decay solution of (1.4).

Also the analysis of this section is heavily based on the knowledge of the autonomous system (2.4) and of its stable and unstable manifolds. In order to deal with less cumbersome notation we set $M^u_q := M^u_q(q)$ and $M^s_q := M^s_q(q)$. We set $\tilde{\mathbf{Q}}^{\mathbf{u}}_{\mathbf{q^u}}(2) = (\tilde{X}^u(2), \tilde{Y}^u(2))$, and $\tau^u := \ln(\tilde{X}^u(2)/2)/\alpha_{q^u}$, if $2^* < q^u < \sigma^*$, and $\tau^u := \ln(P_x(q^u)/2)/\alpha_{q^u}$ if $q^u \geq \sigma^*$ (recall that from Remark 2.12 we know that $\tilde{X}^u(k) \to P_x(\sigma^*)$ as $q^u \to \sigma^*$, for any $k \in \mathbb{N}$).

Let $2^* < q^u < \sigma^*$ and $\mathbf{Q} = (Q_x, Q_y) \in M^u_{q^u}$ where $Q_x > \tilde{X}^u(2)/2$, and consider the trajectory $\mathbf{x}_{\mathbf{q^u}}(t,\tau;\mathbf{Q})$ of (2.4) and the corresponding solution $u(r)$ of (1.4). Since $u(e^t)$ is decreasing as long as $\mathbf{x}_{\mathbf{q^u}}(t,\tau;\mathbf{Q}) \in \mathbb{R}^2_\pm$ it is easy to check that $u(r)$ is a regular solution and $u(r) > 1$ for $r < e^\tau$ and $\tau < \tau^u$. Hence $\mathbf{x}_{\mathbf{q^u}}(t,\tau;\mathbf{Q})$ solves (4.1), for any $t \leq \tau \leq \tau^u$. It follows that $\mathbf{Q} \in W^u_{q^u}(\tau^u)$ as well. Thus

$$\mathfrak{M}^u = \{\mathbf{Q} = (Q_x, Q_y) \,|\, \mathbf{Q} \in M^u_{q^u} \quad \& \quad Q_x > \tilde{X}^u(2)/2\} \subset W^u_{q^u}(\tau^u)\,. \tag{4.6}$$

Denote by $Y = \max\{y \,|\, (x,y) \in \mathfrak{M}^u\}$ and by $\mathbf{R^u} = (\tilde{X}^u(2)/2, Y)$: it follows that $\mathfrak{M}^u$ is a smooth path that joins $\mathbf{R^u}$ with $\mathbf{P}(q^u)$. Follow $W^u_{q^u}(\tau^u)$ from the origin towards $\mathbb{R}^2_\pm$; we denote by $\mathfrak{N}^u$ the branch of $W^u_{q^u}(\tau^u)$ between the origin and $\mathbf{R^u}$. Using the ideas of subsection 3.1 in [14] it can be proved that $\mathfrak{N}^u \in A^+_{q^s}$; furthermore using the fact that $h^u(x,t) \geq x|x|^{q^u-2}$ for any x, t we can show that $\mathfrak{N}^u$ lies below $M^u_{q^u}$, see again [14]. In fact roughly speaking $W^u_{q^u}(\tau^u)$ is just a slight deformation of $M^u_{q^u}$ where the first small branch between the origin and $\mathbf{R^u}$ has been pushed downwards leaving unchanged the endpoints.

Now assume $q^u \geq \sigma^*$: it is easy to check that the argument still goes through if we replace $\tilde{X}^u(2)/2$ by $P_x(q^u)/2$. The only difference is that $\mathfrak{M}^u$ is not a spiral but a graph on the x axis (see Remark 2.8).

Similarly set $\tilde{\mathbf{Q}}^{\mathbf{s}}_{\mathbf{q^s}}(1) = (\tilde{X}^s(1), \tilde{Y}^s(1))$ and $\tau^s := \ln(\tilde{X}^s(1))/\alpha_{q^s}$ if $\sigma_* < q^s < 2^*$ and $\tau^s := \ln(P_x(q^s))/\alpha_{q^s}$ if $2_* < q^s \leq \sigma_*$. Note that for any $\mathbf{Q} = (Q_x, Q_y) \in M^s_{q^s}$ we have $Q_x \leq \tilde{X}^s(1)$ when $\sigma_* < q^s < 2^*$, and $Q_x < P_x(q^s)$ when $2_* < q^s \leq \sigma_*$. Consider the trajectory $\mathbf{x}_{\mathbf{q^s}}(t,\tau;\mathbf{Q})$ of (2.4) and the corresponding solution $v(r)$ of (1.4): $v(\exp(t)) \leq 1$ for any $t \geq \tau \geq \tau^s$, so $\mathbf{x}_{\mathbf{q^s}}(t,\tau;\mathbf{Q})$ solves (4.3) for $t \geq \tau \geq \tau^s$ as well. Therefore $W^s_{q^s}(\tau) \equiv M^s_{q^s}$; for any $\tau \in \mathbb{R}$ and $W^s_{q^s}(\tau) \equiv \{\mathbf{x}_\perp(\tau,\tau^s;\mathbf{Q}) \mid \mathbf{Q} \in M^s_{q^s}\ \}$ for any $\tau \leq \tau^s$. This in fact could be a simpler definition of $W^s_{q^s}(\tau)$, however we cannot do the same for $W^u_{q^u}(\tau)$, so the non-autonomous invariant manifold theory developed in [18] is in fact needed.

Follow $W^u_{q^u}(\tau^u)$ and $W^s_{q^s}(\tau^s)$ from the origin towards $\mathbb{R}^2_\pm$; we denote by $\tilde{W}^u_{q^u}(\tau^u)$ the branch of $W^u_{q^u}(\tau^u)$ between the origin and $\tilde{\mathbf{Q}}^{\mathbf{u}}_{\mathbf{q^u}}(1)$, and by $\tilde{W}^s_{q^s}(\tau^s) = \tilde{M}^s_{q^s}(1)$ the branch of $W^s_{q^s}(\tau^s)$ between the origin and $\tilde{\mathbf{Q}}^{\mathbf{s}}_{\mathbf{q^s}}(1)$.

Now fix $q^s \in (2_*, 2^*)$ and $q^u \in (2^*, 2^* + \epsilon_k(q^s))$, for $k \in \mathbb{N}$, so that $M^u_{q^u}$ crosses $2k$ times $A^0_{q^s}$. From the definition of ϵ_k we see that we can choose $L^s < q^s$ such that $M^u_{q^u}$ crosses $2k$ times $A^0_{L^s}$ too. Then we denote by $\breve{W}^u_{q^u}(1, \tau^u; L^s)$ the branch of $W^u_{q^u}(\tau^u)$ between the origin and $\breve{\mathbf{Q}}^{\mathbf{u}}_{\mathbf{q^u}}(1; L^s)$. In general we denote by $\breve{W}^u_{q^u}(j, \tau^u; L^s)$ the branch of $W^u_{q^u}(\tau^u)$ between $\breve{\mathbf{Q}}^{\mathbf{u}}_{\mathbf{q^u}}(j-1; L^s)$ (excluded) and $\breve{\mathbf{Q}}^{\mathbf{u}}_{\mathbf{q^u}}(j; L^s)$ (included) whenever $j = 1, \dots, 2k$ (recall that $\breve{\mathbf{Q}}^{\mathbf{u}}_{\mathbf{q^u}}(0; L^s) = (0,0) = \tilde{\mathbf{Q}}^{\mathbf{s}}_{\mathbf{q^s}}(0)$).

Analogously fix $q^u > 2^*$ and $q^s \in (2^* - \epsilon_k(q^u), 2^*)$; by definition of ϵ_k we can find $L^u > q^u$ such that $M^s_{q^s}$ crosses $2k$ times $A^0_{L^u}$. We denote by $\breve{W}^s_{q^s}(j, \tau^s; L^u) = \breve{M}^s_{q^s}(j; L^u)$ the branch of $W^s_{q^s}(\tau^s)$ between $\tilde{\mathbf{Q}}^{\mathbf{s}}_{\mathbf{q^s}}(j-1)$ (excluded) and $\tilde{\mathbf{Q}}^{\mathbf{s}}_{\mathbf{q^s}}(j)$ (included) for $j = 1, \dots, 2k$. We stress that by construction we have $\breve{W}^u_{q^u}(j, \tau^u; L^s) = \breve{M}^u_{q^u}(j; L^s)$, whenever $j = 2, \dots, 2k$, and $\breve{W}^s_{q^s}(j, \tau^s; L^u) = \breve{M}^s_{q^s}(j; L^u)$ whenever $j = 1, \dots, 2k$.

Let $\mathbf{Q^u} \in W^u_{q^u}(\tau^u)$, $\mathbf{Q^s} \in W^s_{q^s}(\tau^s)$ and denote by

$$\breve{T}^u(\mathbf{Q^u}) := \sup\{\tau \in \mathbb{R} \,|\, \mathbf{x}_\top(t, \tau^u; \mathbf{Q^u}) \in A^+_{L^s} \text{ for any } t < \tau\}$$
$$\breve{T}^s(\mathbf{Q^s}) := \inf\{\tau \in \mathbb{R} \,|\, \mathbf{x}_\perp(t, \tau^s; \mathbf{Q^s}) \in A^-_{L^u} \text{ for any } t > \tau\}$$

In order to prove Theorem 1.10 we need the following Lemma.

4.2. **Lemma.** *Fix $q^s \in (2_*, 2^*)$ and consider (4.1) where $l = q^u \in (2^*, 2^* + \epsilon_1(q^s))$. Choose $L^s < q^s$ and $q^s - L^s$ small enough so that $q^u \in (2^*, 2^* + \epsilon_1(L^s))$. Then $\breve{T}^u(\mathbf{Q})$ is finite whenever $\mathbf{Q} \in W^u_{q^u}(\tau^u)$, and $\breve{T}^u(\mathbf{Q}) \geq \tau^u$ for $\mathbf{Q} \in \breve{W}^u_{q^u}(1, \tau^u; L^s)$. Moreover the function $\breve{T}^u : W^u_{q^u}(\tau^u) \to \mathbb{R}$ is continuous and $\breve{T}^u(\mathbf{Q}) \to -\infty$ as $\mathbf{Q} \to \mathbf{P}(q^u)$ and $\breve{T}^u(\mathbf{Q}) \to +\infty$ as $\mathbf{Q} \to (0,0)$.*

Analogously fix $q^u > 2^*$ and consider (4.3) where $l = q^s \in (2^* - \epsilon_1(q^u), 2^*)$. Choose $L^u > q^u$ and $L^u - q^u$ small enough so that $q^s \in (2^* - \epsilon_1(L^u), 2^*)$. Then $\breve{T}^s(\mathbf{Q})$ is finite whenever $\mathbf{Q} \in W^s_{q^s}(\tau^s)$, and $\breve{T}^s(\mathbf{Q}) \leq \tau^s$ for $\mathbf{Q} \in \breve{W}^s_{q^s}(1, \tau^s; L^u)$. Moreover the function $\breve{T}^s : W^s_{q^s}(\tau^s) \to \mathbb{R}$ is continuous and $\breve{T}^s(\mathbf{Q}) \to +\infty$ as $\mathbf{Q} \to \mathbf{P}(q^s)$ and $\breve{T}^s(\mathbf{Q}) \to -\infty$ as $\mathbf{Q} \to (0,0)$.

Proof. Fix $q^s \in (2_*, 2^*)$ and $q^u \in (2^*, 2^* + \epsilon_1(q^s))$. Let $\mathbf{Q} \in M^u_{q^u}$ and consider the trajectory $\mathbf{x_{q^u}}(t, \tau^u; \mathbf{Q})$ of (2.4) and the corresponding regular solution $u(d, r)$ of (1.4). Observe that there is $\bar{T}^u_1(\mathbf{Q})$ such that $\mathbf{x_{q^u}}(t, \tau^u; \mathbf{Q}) \in A^+_{L^s}$ for $t < \bar{T}^u_1(\mathbf{Q})$ and $\mathbf{x_{q^u}}(\bar{T}^u_1(\mathbf{Q}), \tau^u; \mathbf{Q}) \in A^0_{L^s}$. Moreover $\bar{T}^u_1(\mathbf{Q}) \geq \tau^u$ if and only if $\mathbf{Q} \in \breve{M}^u_{q^u}(1; L^s)$.

Assume first $d > 1$; there is a unique value $\bar{T}^u_0(\mathbf{Q})$ such that $x_{q^u}(\bar{T}^u_0(\mathbf{Q}), \tau^u; \mathbf{Q})$ $\exp[-\alpha_{q^u}\bar{T}^u_0(\mathbf{Q})] = u(\exp[\bar{T}^u_0(\mathbf{Q})]) = 1$. It is easy to check that $\bar{T}^u_0(\mathbf{Q})$ is continuous.

If $\bar{T}^u_0(\mathbf{Q}) \geq \bar{T}^u_1(\mathbf{Q})$, i.e. $u(e^t) > 1$ for any $t < \exp[\bar{T}^u_1(\mathbf{Q})]$, then $\mathbf{x_{q^u}}(t, \tau^u; \mathbf{Q})$ solves (4.1) too, and $\mathbf{x_{q^u}}(t, \tau^u; \mathbf{Q}) \equiv \mathbf{x}_\top(t, \tau^u; \mathbf{Q})$ for any $t \leq \bar{T}^u_1(\mathbf{Q})$ and $\bar{T}^u_1(\mathbf{Q}) = \breve{T}^u(\mathbf{Q})$. Note that

$$\left[M^u_{q^u} \setminus \breve{M}^u_{q^u}(1, L^s)\right] \subset \{\mathbf{Q} \in M^u_{q^u} \,|\, \bar{T}^u_0(\mathbf{Q}) \geq \bar{T}^u_1(\mathbf{Q})\} \subset W^u_{q^u}(\tau^u)\,.$$

Now assume $d \leq 1$ so that in particular $\mathbf{Q} \in \mathfrak{N}^u = W^u_{q^u}(\tau^u) \setminus \mathfrak{W}^u$. Set $\mathbf{Q^s} = \mathbf{Q}\exp[(\alpha_{q^s} - \alpha_{q^u})\tau^u]$, then the solution $\mathbf{x_{q^s}}(t, \tau^u; \mathbf{Q^s})$ of (2.4) with $q = q^s$ solves (4.3) as well, since $u(d, r) \leq 1$ until it becomes null, and $\mathbf{Q^s} \in \tilde{M}^u_{q^s}$. Therefore from Remark 2.8 and (2.5), we find $\bar{T}^u_3(\mathbf{Q})$ such that $\mathbf{x_{q^s}}(t, \tau^u; \mathbf{Q}) \in A^+_{L^s}$ for any $t < \bar{T}^u_3(\mathbf{Q})$ and it crosses $A^0_{L^s}$ transversally at $t = \bar{T}^u_3(\mathbf{Q})$. Hence $\mathbf{x}_\top(t, \tau^u; \mathbf{Q}) = \mathbf{x_{q^s}}(t, \tau^u; \mathbf{Q^s})\exp[(\alpha_{q^u} - \alpha_{q^s})t]$ whenever $t \leq \bar{T}^u_3(\mathbf{Q})$, and $\breve{T}^u(\mathbf{Q}) = \bar{T}^u_3(\mathbf{Q})$.

From the t-invariance property of (2.4) where $q = q^u$ and $q = q^s$ it follows that $\breve{T}^u(\mathbf{Q})$ is continuous respectively when $\breve{T}^u(\mathbf{Q}) = \bar{T}^u_1(\mathbf{Q})$ and $\breve{T}^u(\mathbf{Q}) = \bar{T}^u_3(\mathbf{Q})$.

Now we go back to the case $d > 1$ but we assume $\bar{T}^u_0(\mathbf{Q}) < \bar{T}^u_1(\mathbf{Q})$, so that $\mathbf{x}_{\mathbf{q^u}}(t, \tau^u; \mathbf{Q})$ solves (4.1) for $t \le \bar{T}^u_0(\mathbf{Q})$. Let us denote by $\bar{\mathbf{Q}}^{\mathbf{u}} = \mathbf{x}_{\mathbf{q^u}}(\bar{T}^u_0(\mathbf{Q}), \tau^u; \mathbf{Q})$ and by $\bar{\mathbf{Q}}^{\mathbf{s}} := \aleph^{\bar{T}^u_0(\mathbf{Q})}_{q^s,q^u}\big(\bar{\mathbf{Q}}^{\mathbf{u}}\big) = \bar{\mathbf{Q}}^{\mathbf{u}} \exp[(\alpha_{q^s} - \alpha_{q^u})\bar{T}^u_0(\mathbf{Q})]$: observe that $\bar{\mathbf{Q}}^{\mathbf{s}} \in A^+_{q^s} \cap W^u_{q^s}(\bar{T}^u_0(\mathbf{Q}))$. Consider the trajectory $\mathbf{x}_{\mathbf{q^s}}(t, \bar{T}^u_0(\mathbf{Q}); \bar{\mathbf{Q}}^{\mathbf{s}})$ of (2.4) where $q = q^s$; from Remark 2.3 we find that there is $\tilde{T}^u_2(\mathbf{Q})$ such that $\mathbf{x}_{\mathbf{q^s}}(t, \bar{T}^u_0(\mathbf{Q}); \bar{\mathbf{Q}}^{\mathbf{s}}) \in A^+_{q^s}$ for $t < \tilde{T}^u_2(\mathbf{Q})$ and it crosses $A^0_{q^s}$ at $t = \tilde{T}^u_2(\mathbf{Q})$, and such a crossing is transversal.

Denote by $N_{q^u}(\tau^u)$ the set of all the $\mathbf{Q} \in W^u_{q^u}(\tau^u)$ such that $\bar{T}^u_0(\mathbf{Q}) < \bar{T}^u_1(\mathbf{Q})$ and $d > 1$; then we set $N_{q^s}(\tau^u) := N_{q^u}(\tau^u) \exp[(\alpha_{q^s} - \alpha_{q^u})\tau^u] \subset W^u_{q^s}(\tau^u)$. It is easy to check that the closure of $N_{q^s}(\tau^u)$ is contained in $A^+_{q^s}$, and that there is $\delta > 0$ such that $|\mathbf{Q}^{\mathbf{s}}| + |\mathbf{Q}^{\mathbf{s}} - \mathbf{P}(q^s)| > \delta$ for any $\mathbf{Q}^{\mathbf{s}} \in N_{q^s}(\tau^u)$. So, using a continuity argument, we can find L^s close enough to q^s so that all the trajectories $\mathbf{x}_{\mathbf{q^s}}(t, \tau^u; \mathbf{Q}^{\mathbf{s}})$ cross $A^0_{L^s}$ transversally too, whenever $\mathbf{Q}^{\mathbf{s}} \in N_{q^s}(\tau^u)$. Thus for any $\mathbf{Q} \in N_{q^u}(\tau^u)$ there is $\bar{T}^u_2(\mathbf{Q})$ slightly larger than $\tilde{T}^u_2(\mathbf{Q})$ such that $\mathbf{x}_{\mathbf{q^s}}(t, \bar{T}^u_0(\mathbf{Q}); \bar{\mathbf{Q}}^{\mathbf{s}}) \in A^+_{L^s}$ for $t < \tilde{T}^u_2(\mathbf{Q})$ and it crosses $A^0_{L^s}$ at $t = \tilde{T}^u_2(\mathbf{Q})$, and such a crossing is transversal.

Using this property and the fact $\bar{T}^u_0$ is a continuous function of $\mathbf{Q}$, we find that $\bar{T}^u_2(\mathbf{Q})$ is continuous too. Observe that by construction

$$\mathbf{x}_\top(t, \tau^u; \mathbf{Q}) = \begin{cases} \mathbf{x}_{\mathbf{q^u}}(t, \tau^u; \mathbf{Q}), & \text{if } t \le \bar{T}^u_0(\mathbf{Q}). \\ \mathbf{x}_{\mathbf{q^s}}(t, \bar{T}^u_0(\mathbf{Q}); \bar{\mathbf{Q}}^{\mathbf{s}}) \exp[(\alpha_{q^u} - \alpha_{q^s})t], & \text{if } \bar{T}^u_0(\mathbf{Q}) < t < \bar{T}^u_2(\mathbf{Q}). \end{cases}$$

Therefore $\mathbf{x}_\top(\bar{T}^u_2(\mathbf{Q}), \tau^u; \mathbf{Q}) \in A^0_{L^s}$ and $\breve{T}^u(\mathbf{Q}) = \bar{T}^u_2(\mathbf{Q})$; so $\breve{T}^u(\mathbf{Q})$ inherits the continuity of $\bar{T}^u_2(\mathbf{Q})$. So we have proved that $\breve{T}^u(\mathbf{Q})$ is finite and continuous whenever $\mathbf{Q} \in W^u_{q^u}(\tau^u)$.

From the previous argument it follows also that $\breve{T}^u(\mathbf{Q}) = \bar{T}^u_1(\mathbf{Q}) \to -\infty$ as $\mathbf{Q} \to \mathbf{P}(q^u)$ and $\breve{T}^u(\mathbf{Q}) = \bar{T}^u_3(\mathbf{Q}) \to +\infty$ as $\mathbf{Q} \to (0, 0)$.

The proof of claim concerning $\breve{T}^s(\mathbf{Q})$ is easily obtained by repeating the argument developed for $\breve{T}^u(\mathbf{Q})$ (in fact just the cases $\breve{T}^u(\mathbf{Q}) = \bar{T}^u_1(\mathbf{Q})$ and $\breve{T}^u(\mathbf{Q}) = \bar{T}^u_2(\mathbf{Q})$). □

Let us define the functions $\breve{\psi}^u : W^u_{q^u}(\tau^u) \to A^0_{L^s}$ and $\breve{\psi}^s : W^s_{q^s}(\tau^s) \to A^0_{L^u}$, as $\breve{\psi}^u(\mathbf{Q}) = \mathbf{x}_\top(\breve{T}^u(\mathbf{Q}), \tau^u; \mathbf{Q})$ and $\breve{\psi}^s(\mathbf{Q}) = \mathbf{x}_\perp(\breve{T}^s(\mathbf{Q}), \tau^s; \mathbf{Q})$.

Observe that from the continuity of $\breve{T}^u(\mathbf{Q})$ and $\breve{T}^s(\mathbf{Q})$ it follows that $\breve{\psi}^u$ is well defined and continuous whenever $q^s \in (2_*, 2^*)$ and $q^u \in (2^*, 2^* + \epsilon_1(L^s))$, while $\breve{\psi}^s$ is well defined and continuous whenever $q^u > 2^*$ and $q^s \in (2^* - \epsilon_1(L^u), 2^*)$.

Furthermore, reasoning as in Lemma 4.2 we see that for any $q^u > 2^*$ and $2_* < q^s < 2^*$ we can define the functions $\tilde{T}^u : W^u_{q^u}(\tau^u) \to \mathbb{R}$ and $\tilde{T}^s : W^s_{q^s}(\tau^s) \to \mathbb{R}$ as

$$\tilde{T}^u(\mathbf{Q^u}) := \sup\{\tau \in \mathbb{R} \,|\, \mathbf{x}_\top(t, \tau^u; \mathbf{Q^u}) \in A^+_{L^u} \text{ for any } t < \tau\},$$

$$\tilde{T}^s(\mathbf{Q^s}) := \inf\{\tau \in \mathbb{R} \,|\, \mathbf{x}_\perp(t, \tau^u; \mathbf{Q^s}) \in A^-_{L^s} \text{ for any } t > \tau\}.$$

If $q^s \in (\sigma_*, 2^*)$ we could also choose $L^s = q^s$; but if $q^s \le \sigma_*$ we need to choose L^s slightly smaller than q^s, so that there is a unique point of intersection between $M^s_{q^s} = \tilde{M}^s_{q^s}$ and $A^0_{L^s}$, say $\tilde{\mathbf{Q}}^{\mathbf{s}}_{\mathbf{q^s}}(L^s)$. Analogously if $q^u \in (2^*, \sigma^*)$ we could also choose $L^u = q^u$; but if $q^u \ge \sigma^*$ we choose $L^u > q^u$, so that $M^u_{q^u} = \tilde{M}^u_{q^u}$ intersects $A^0_{L^u}$ in a unique point denoted by $\tilde{\mathbf{Q}}^{\mathbf{u}}_{\mathbf{q^u}}(L^u)$.

Reasoning as in Lemma 4.2 and using Remarks 2.3 and 2.4 we see that $\mathbf{x}_\top(t, \tau^u; \mathbf{Q^u})$ crosses transversally $A^0_{L^u}$ and $\mathbf{x}_\perp(t, \tau^s; \mathbf{Q^s})$ crosses transversally $A^0_{L^s}$; so the continuity of $\tilde{T}^u$ and $\tilde{T}^s$ follows. Thus the functions $\tilde{\psi}^u : W^u_{q^u}(\tau^u) \to A^0_{L^u}$, $\tilde{\psi}^s : W^s_{q^s}(\tau^s) \to A^0_{L^s}$, defined by $\tilde{\psi}^u(\mathbf{Q}) = \mathbf{x}_\perp(\tilde{T}^u(\mathbf{Q}), \tau^u; \mathbf{Q} \exp[(\alpha_{q^s} - \alpha_{q^u})\tau^u])$, $\tilde{\psi}^s(\mathbf{Q}) = \mathbf{x}_\top(\tilde{T}^s(\mathbf{Q}), \tau^s; \mathbf{Q} \exp[(\alpha_{q^u} - \alpha_{q^s})\tau^s])$, are continuous too.

Let us consider the 3-dimensional autonomous system obtained adding to (4.1) the extra variable $\tau = t$. We introduce the following sets

$$\breve{\mathbf{\Xi}}^{\mathbf{u}} := \{(\breve{\psi}^u(\mathbf{Q}), \breve{T}^u(\mathbf{Q})) \mid \mathbf{Q} \in W^u_{q^u}(\tau^u)\} = \breve{\mathbf{W}}^{\mathbf{u}}(1) \cap \mathbf{A}^{\mathbf{0}}_{\mathbf{L^s}}$$
$$\tilde{\mathbf{\Xi}}^{\mathbf{s}} := \{(\tilde{\psi}^s(\mathbf{Q}), \tilde{T}^s(\mathbf{Q})) \mid \mathbf{Q} \in W^s_{q^s}(\tau^s)\} := \tilde{\mathbf{W}}^{\mathbf{s}}(1) \cap \mathbf{A}^{\mathbf{0}}_{\mathbf{L^s}}$$

where $\mathbf{A}^{\mathbf{0}}_{\mathbf{L^s}} := A^0_{L^s} \times \mathbb{R}$, $\breve{\mathbf{W}}^{\mathbf{u}}(1) = \{(\mathbf{Q}, \tau) \mid \mathbf{Q} \in \breve{W}^u_{q^u}(1, \tau; L^s)\}$ and $\tilde{\mathbf{W}}^{\mathbf{s}}(1) = \{(\mathbf{Q}, \tau) \mid \mathbf{Q} \in \breve{W}^s_{q^s}(1, \tau; L^s)\}$. Note that by construction $\breve{\mathbf{\Xi}}^{\mathbf{u}}$ and $\tilde{\mathbf{\Xi}}^{\mathbf{s}}$ are connected (they are images of continuous functions).

We observe that the maps $\tilde{T}^u, \breve{T}^u$ are surjective but a priori they might not be injective. Such a phenomenon correspond to the appearance of tangencies between $A^0_{L^s}$ and the manifold $W^u(T)$ for $T > 0$, which is not a priori excluded. A similar conclusion holds for $\tilde{T}^s, \breve{T}^s$. Thus we cannot exclude the existence of certain $T \in \mathbb{R}$ such that $\breve{\mathbf{\Xi}}^{\mathbf{u}}$ and $\tilde{\Xi}^s$ intersect the plane $\tau = T$ in more than one point or in whole segments.

We stress that by construction if $(\breve{\mathbf{Q}}^{\mathbf{u}}(\tau), \tau) \in \breve{\mathbf{\Xi}}^{\mathbf{u}}$ and $(\tilde{\mathbf{Q}}^{\mathbf{s}}(\tau), \tau) \in \tilde{\Xi}^s$ we have $\lim_{t\to-\infty} \mathbf{x}_\top(t, \tau; \breve{\mathbf{Q}}^{\mathbf{u}}(\tau)) = (0, 0) = \lim_{t\to+\infty} \mathbf{x}_\top(t, \tau; \tilde{\mathbf{Q}}^{\mathbf{s}}(\tau))$. Analogously we denote by

$$\breve{\mathbf{\Xi}}^{\mathbf{s}} := \{(\breve{\psi}^s(\mathbf{Q}), \breve{T}^s(\mathbf{Q})) \mid \mathbf{Q} \in W^s_{q^s}(\tau^s)\} = \breve{\mathbf{W}}^{\mathbf{s}}(1) \cap \mathbf{A}^{\mathbf{0}}_{\mathbf{L^u}}$$
$$\tilde{\mathbf{\Xi}}^{\mathbf{u}} := \{(\tilde{\psi}^u(\mathbf{Q}), \tilde{T}^u(\mathbf{Q})) \mid \mathbf{Q} \in W^u_{q^u}(\tau^u)\} = \tilde{\mathbf{W}}^{\mathbf{u}}(1) \cap \mathbf{A}^{\mathbf{0}}_{\mathbf{L^u}}$$

where $\mathbf{A}^{\mathbf{0}}_{\mathbf{L^u}} := A^0_{L^u} \times \mathbb{R}$, $\breve{\mathbf{W}}^{\mathbf{s}}(1) = \{(\mathbf{Q}, \tau) \mid \mathbf{Q} \in \breve{W}^s_{q^s}(1, \tau; L^u)\}$ and $\tilde{\mathbf{W}}^{\mathbf{u}}(1) = \{(\mathbf{Q}, \tau) \mid \mathbf{Q} \in \breve{W}^u_{q^u}(1, \tau; L^u)\}$. Again we have that $\lim_{t\to-\infty} \mathbf{x}_\perp(t, \tau; \tilde{\mathbf{Q}}^{\mathbf{u}}(\tau)) = (0, 0) = \lim_{t\to+\infty} \mathbf{x}_\perp(t, \tau; \breve{\mathbf{Q}}^{\mathbf{s}}(\tau))$ whenever $(\tilde{\mathbf{Q}}^{\mathbf{u}}(\tau), \tau) \in \tilde{\mathbf{\Xi}}^{\mathbf{u}}$ and $(\breve{\mathbf{Q}}^{\mathbf{s}}(\tau), \tau) \in \breve{\mathbf{\Xi}}^{\mathbf{s}}$.

We are ready to state the following Lemma.

4.3. **Lemma.** *Fix $2_* < q^s < 2^*$, $q^u \in (2^*, 2^* + \epsilon_1(q^s))$ and choose $L^s < q^s$ such that $q^u \in (2^*, 2^* + \epsilon_1(L^s))$, too. Then for any $\tau \in \mathbb{R}$ we can find $\breve{\mathbf{Q}}^{\mathbf{u}}(\tau)$ and $\tilde{\mathbf{Q}}^{\mathbf{s}}(\tau)$ such that $(\breve{\mathbf{Q}}^{\mathbf{u}}(\tau), \tau) \in \breve{\mathbf{\Xi}}^{\mathbf{u}}$ and $(\tilde{\mathbf{Q}}^{\mathbf{s}}(\tau), \tau) \in \tilde{\Xi}^s$. Moreover there are $T^- < T^+$ such that $H_*(\breve{\mathbf{Q}}^{\mathbf{u}}(\tau)) < 0 < H_*(\tilde{\mathbf{Q}}^{\mathbf{s}}(\tau))$ for any $\tau \le T^-$ and $H_*(\tilde{\mathbf{Q}}^{\mathbf{s}}(\tau)) < 0 < H_*(\breve{\mathbf{Q}}^{\mathbf{u}}(\tau))$ for any $\tau \ge T^+$.*

Fix $q^u > 2^*$, $q^s \in (2^* - \epsilon_1(q^u), 2^*)$ and choose $L^u > q^u$ such that $q^s \in (2^* - \epsilon_1(L^u), 2^*)$, too. Then for any $\tau \in \mathbb{R}$ we can find $\tilde{\mathbf{Q}}^{\mathbf{u}}(\tau)$ and $\breve{\mathbf{Q}}^{\mathbf{s}}(\tau)$ such that $(\tilde{\mathbf{Q}}^{\mathbf{u}}(\tau), \tau) \in \tilde{\Xi}^u$ and $(\breve{\mathbf{Q}}^{\mathbf{s}}(\tau), \tau) \in \breve{\mathbf{\Xi}}^{\mathbf{s}}$. Moreover there are $T^- < T^+$ such that $H_*(\tilde{\mathbf{Q}}^{\mathbf{u}}(\tau)) < 0 < H_*(\breve{\mathbf{Q}}^{\mathbf{s}}(\tau))$ for any $\tau \le T^-$, $H_*(\breve{\mathbf{Q}}^{\mathbf{s}}(\tau)) < 0 < H_*(\tilde{\mathbf{Q}}^{\mathbf{u}}(\tau))$ for any $\tau \ge T^+$.

Proof. Let $2_* < q^s < 2^*$, $q^u \in (2^*, 2^* + \epsilon_1(q^s))$; we can choose L^s such that $q^u \in (2^*, 2^* + \epsilon_1(L^s))$, see (2.11). Let us set $\breve{\mathbf{Q}}^{\mathbf{u}}_{\mathbf{q^u}}(1; L^s) = (\breve{X}^u(1), \breve{Y}^u(1))$ and $\mathfrak{T}^- := \ln(\breve{X}^u(1))/\alpha_{q^u}$; by construction we find that the set $\{\mathbf{Q} \mid (\mathbf{Q}, \tau) \in \breve{\Xi}^u\}$ reduces to $\{\breve{\mathbf{Q}}^{\mathbf{u}}_{\mathbf{q^u}}(1; L^s)\}$ for $\tau \le \mathfrak{T}^-$ while it changes with τ for $\tau > \mathfrak{T}^-$. In fact $\mathbf{x}_\top(t, \tau; \breve{\mathbf{Q}}^{\mathbf{u}}_{\mathbf{q^u}}(1; L^s))$ is such that the corresponding solution $u(e^t)$ of (1.4) satisfy $u(e^t) \ge 1$ for any $t \le \tau \le \mathfrak{T}^-$. So $\mathbf{x}_\top(t, \tau; \breve{\mathbf{Q}}^{\mathbf{u}}_{\mathbf{q^u}}(1; L^s)) \equiv \mathbf{x}_{\mathbf{q^u}}(t, \tau; \breve{\mathbf{Q}}^{\mathbf{u}}_{\mathbf{q^u}}(1; L^s))$ for $t \le \tau \le \mathfrak{T}^-$, and setting $\breve{\mathbf{Q}}^{\mathbf{u}}(\tau) := \breve{\mathbf{Q}}^{\mathbf{u}}_{\mathbf{q^u}}(1; L^s)$ for $\tau \le \mathfrak{T}^-$, we get $(\breve{\mathbf{Q}}^{\mathbf{u}}(\tau), \tau) \in \breve{\mathbf{\Xi}}^{\mathbf{u}}$ and $H_*(\breve{\mathbf{Q}}^{\mathbf{u}}(\tau)) < 0$ for any $\tau \le \mathfrak{T}^-$, see Remark 2.8.

Analogously set $\breve{\mathbf{Q}}^{\mathbf{s}}_{\mathbf{q^s}}(1; L^s) = (\tilde{X}^s(1), \tilde{Y}^s(1))$ and $\mathfrak{T}^+ := \ln(\tilde{X}^s(1))/\alpha_{q^s}$; then $\mathbf{x}_\perp(t, \tau; \tilde{\mathbf{Q}}^{\mathbf{s}}_{\mathbf{q^s}}(1; L^s)) \equiv \mathbf{x}_{\mathbf{q^s}}(t, \tau; \tilde{\mathbf{Q}}^{\mathbf{s}}_{\mathbf{q^s}}(1; L^s))$ for $t \ge \tau \ge \mathfrak{T}^+$. It follows that

$$\{(\tilde{\mathbf{Q}}^{\mathbf{s}}(\tau), \tau) \mid \tau \ge \mathfrak{T}^+\} = \tilde{\Xi}^s \cap \{(\mathbf{Q}, \tau) \mid \tau \ge \mathfrak{T}^+\},$$

if $\tilde{\mathbf{Q}}^{\mathbf{s}}(\tau) = \tilde{\mathbf{Q}}^{\mathbf{s}}_{\mathbf{q^s}}(1; L^s) \exp[(\alpha_{q^u} - \alpha_{q^s})\tau]$. Furthermore observe that $H_*(\tilde{\mathbf{Q}}^{\mathbf{s}}_{\mathbf{q^s}}(1; L^s)) < 0$; so from (2.8) we find that $H_*(\tilde{\mathbf{Q}}^{\mathbf{s}}(\tau)) < 0$ whenever $\tau \ge \mathfrak{T}^+$, too.

Consider now the solution $u(1,r)$ of (1.4) where $f(u,r) = u|u|^{q^s-2}$ and the corresponding trajectory $\mathbf{x}_{\mathbf{q^s}}(t)$ of (2.4). Then there is τ^+ such that $\mathbf{x}_{\mathbf{q^s}}(\tau^+) = \breve{\mathbf{Q}}^{\mathbf{u}}_{\mathbf{q^s}}(1;L^s) \in A^0_{L^s}$ and $\mathbf{x}_{\mathbf{q^s}}(t) \in A^+_{L^s}$ for any $t < \tau^+$. Consider the trajectories $\mathbf{x}_{\mathbf{q^s}}(t,\tau;\breve{\mathbf{Q}}^{\mathbf{u}}_{\mathbf{q^s}}(1;L^s))$ of (2.4) and the corresponding solutions $u(d(\tau),r)$ of (1.4); using the monotonicity property described in Remark 2.5 we find $d(\tau) < 1$ for any $\tau > \tau^+$. Once again, since $u(d(\tau),r)$ is decreasing in r for $0 \le r \le e^\tau$ we find $u(d(\tau),r) \le 1$ for any $r \in [0,e^\tau]$. It follows that $u(d(\tau),r)$ solves (1.4) with f of type (1.3) too, for any $r \le e^\tau$ and any $\tau \ge \tau^+$; therefore $\mathbf{x}_{\mathbf{q^s}}(t,\tau;\breve{\mathbf{Q}}^{\mathbf{u}}_{\mathbf{q^s}}(1;L^s))$ solves (4.3) for $t \le \tau$ and $\tau \ge \tau^+$. Let us denote by $\breve{\mathbf{Q}}^{\mathbf{u}}(\tau) := \aleph^\tau_{q^u,q^s}\breve{\mathbf{Q}}^{\mathbf{u}}_{\mathbf{q^s}}(1;L^s) = \breve{\mathbf{Q}}^{\mathbf{u}}_{\mathbf{q^s}}(1;L^s)\exp[(\alpha_{q^u}-\alpha_{q^s})\tau]$ for $\tau \ge \tau^+$; then $(\breve{\mathbf{Q}}^{\mathbf{u}}(\tau),\tau) \in \breve{\Xi}^{\mathbf{u}}$ for any $\tau \ge \tau^+$.

Since $q^s < 2^*$ we have $H_*(\breve{\mathbf{Q}}^{\mathbf{u}}_{\mathbf{q^s}}(1;L^s)) > 0$; so from Remark 2.6 it follows that $H_*(\breve{\mathbf{Q}}^{\mathbf{u}}(\tau)) > 0$ for any $\tau \ge \tau^+$, too.

Now choose $\mathbf{Q}^{\mathbf{s}}_{\mathbf{q^s}} = (X^s_{q^s}, Y^s_{q^s}) \in \breve{M}^s_{q^s}(1;L^s)$ and set $\delta = X^s_{q^s}$. Denote by $\tilde{\tau}_0 = \ln(\delta)/\alpha_{q^s}$ and consider the trajectory $\mathbf{x}_{\mathbf{q^s}}(t,\tilde{\tau}_0;\mathbf{Q}^{\mathbf{s}}_{\mathbf{q^s}})$ of (2.4) where $q = q^s$, and the corresponding fast decay solution $v_s(r)$ of (1.4). Observe that $v_s(e^{\tilde{\tau}_0}) = 1$ hence $\mathbf{x}_{\mathbf{q^s}}(t,\tilde{\tau}_0;\mathbf{Q}^{\mathbf{s}}_{\mathbf{q^s}}) = \mathbf{x}_\perp(t,\tilde{\tau}_0;\mathbf{Q}^{\mathbf{s}}_{\mathbf{q^s}})$ solves (4.3) for $t \ge \tilde{\tau}_0$.

Let us denote by $\mathbf{Q}^{\mathbf{s}}_{\mathbf{q^u}} := \mathbf{Q}^{\mathbf{s}}_{\mathbf{q^s}}\exp[(\alpha_{q^u}-\alpha_{q^s})\tilde{\tau}_0] \in W^s_{q^u}(\tilde{\tau}_0)$ and consider the trajectory $\mathbf{x}_{\mathbf{q^u}}(t,\tilde{\tau}_0;\mathbf{Q}^{\mathbf{s}}_{\mathbf{q^u}})$ of (2.4) where $q = q^u$ and the corresponding solution $v_u(r)$ of (1.4). Observe that the x coordinate of $\mathbf{Q}^{\mathbf{s}}_{\mathbf{q^u}}$ is $\delta^{(q^s-2)/(q^u-2)}$ and by construction $\mathbf{Q}^{\mathbf{s}}_{\mathbf{q^u}} \in A^-_{L^s}$. So, possibly choosing a smaller δ so that

$$\delta^{\frac{q^s-2}{q^u-2}} < \delta^{\frac{2_*-2}{q^u-2}} < \Delta^s_{q^u}(L^s)$$

where $\Delta^s_{q^u}(L^s)$ is defined in Remark 2.13, we find that $\mathbf{Q}^{\mathbf{s}}_{\mathbf{q^u}} \in E^s_{q^u}(L^s)$. So there is $\tau = \tilde{T}^s(\mathbf{Q}^{\mathbf{s}}_{\mathbf{q^u}},L^s) < \tilde{\tau}_0$ such that $\mathbf{x}_{\mathbf{q^u}}(\tilde{T}^s(\mathbf{Q}^{\mathbf{s}}_{\mathbf{q^u}},L^s),\tilde{\tau}_0;\mathbf{Q}^{\mathbf{s}}_{\mathbf{q^u}}) := \tilde{\mathbf{Q}}^{\mathbf{s}}(\tau) \in A^0_{L^s}$. Furthermore we also get $H_*(\tilde{\mathbf{Q}}^{\mathbf{s}}(\tau)) > 0$.

Now observe that the function $v(r)$ defined as $v_u(r)$ for $e^\tau \le r \le e^{\tilde{\tau}_0}$ and as $v_s(r)$ for $r \ge \ln(\tilde{\tau}_0)$, solves (1.4) with f of type (1.3). It follows that $(\tilde{\mathbf{Q}}^{\mathbf{s}}(\tau),\tau) \in \tilde{\Xi}^s$. Letting $\delta \to 0$ we have $\tilde{\tau}_0 \to -\infty$, thus $\tau \to -\infty$. Therefore there is τ^- such that $(\tilde{\mathbf{Q}}^{\mathbf{s}}(\tau),\tau) \in \tilde{\Xi}^s$ and $H_*(\tilde{\mathbf{Q}}^{\mathbf{s}}(\tau)) > 0$ for any $\tau < \tau^-$.

From the connectedness of $\breve{\Xi}^u$ and $\tilde{\Xi}^s$ we get that they both intersect the plane $z = \tau$ for any $\tau \in \mathbb{R}$. So, setting $T^- := \min\{\tau^-, \mathfrak{T}^-\}$ and $T^+ := \max\{\tau^+, \mathfrak{T}^+\}$, the first part of the Lemma follows.

The proof in the case $q^u > 2^*$ and $q^s \in (2^* - \epsilon_1(q^u), 2^*)$, is very similar and will be omitted. □

Now we are ready to prove Theorem 1.10.

Proof of Theorem 1.10. Assume $2_* < q^s < 2^*$, $q^u \in (2^*, 2^* + \epsilon_1(q^s))$, and choose $L^s < q^s$ such that $q^u \in (2^*, 2^* + \epsilon_1(L^s))$. From Lemma 4.3 we know that $\tilde{\Xi}^s$ divides the plane $\mathbf{A}^{\mathbf{0}}_{\mathbf{L^s}}$ into two disjoint open sets: $\mathfrak{A}^+$ which contains the subset $\{(x,y,\tau)\,|\,\tau \ge T^+ \;\&\; x > \tilde{X}^s(\tau)\}$ and $\mathfrak{A}^-$ which contains the subset $\{(x,y,\tau)\,|\,\tau \ge T^+ \;\&\; 0 < x < \tilde{X}^s(\tau)\}$, where $\tilde{X}^s(\tau)$ is the x-coordinate of $\tilde{\mathbf{Q}}^{\mathbf{s}}(\tau)$.

From the continuity of H_* and Lemma 4.3, for any $\tau \le T^-$ we can find $\breve{\mathbf{Q}}^{\mathbf{u}}(\tau) \in \mathfrak{A}^- \cap \breve{\Xi}^{\mathbf{u}}$, while for any $\tau \ge T^+$ there is $\breve{\mathbf{Q}}^{\mathbf{u}}(\tau) \in \mathfrak{A}^+ \cap \breve{\Xi}^{\mathbf{u}}$. It follows that there are τ^* and $\mathbf{Q}^*$ such that $(\mathbf{Q}^*,\tau^*) \in \breve{\Xi}^{\mathbf{u}} \cap \tilde{\Xi}^s$. Consider the trajectory $\mathbf{x}_\top(t,\tau^*;\mathbf{Q}^*)$ of (4.1) and the corresponding solution $u(r)$ of (1.4). Since $\mathbf{Q}^* \in \big(\breve{W}^u_{q^u}(1,\tau^*;L^s) \cap W^s_{q^u}(\tau^*)\big)$, then $u(r)$ is a regular solution and it has fast decay. Moreover $\mathbf{x}_\top(t,\tau^*;\mathbf{Q}^*)$ belongs to $A^+_{L^s}$ for $t < \tau^*$, to $A^-_{L^s}$ for $t > \tau^*$, so it is in $\mathbb{R}^2_\pm$ for any $t \in \mathbb{R}$, thus $u(r)$ is a monotone decreasing G.S. with f.d. □

In order to prove Theorem 1.11 we need some further auxiliary functions and sets. Fix $k \in \mathbb{N}$, $q^s \in (2_*, 2^*)$, $q^u \in (2^*, 2^* + \epsilon_k(q^s))$ and choose $L^s < q^s$ such that $q^u \in$

$(2^*, 2^* + \epsilon_k(L^s))$, too; we set $\breve{\mathbf{Q}}^{\mathbf{u}}_{\mathbf{q^u}}(2k-2; L^s) = (\delta^u_k, -\alpha_{L^s}\delta^u_k)$ and $\breve{\tau}^u_k = \ln(\delta^u_k)/\alpha_{q^u}$. We denote by $\mathbf{Z}^{\mathbf{u}}_{\mathbf{q^u}}(j)$ the unique intersection between $x = \delta^u_k$ and the manifold $\breve{M}^u_{q^u}(2j-1; L^s)$ for $j = 2, \dots, k$; observe that $\mathbf{Z}^{\mathbf{u}}_{\mathbf{q^u}}(k) = \breve{\mathbf{Q}}^{\mathbf{u}}_{\mathbf{q^u}}(2k-2; L^s)$. We denote by $\hat{M}^u_{q^u}(2j-1; L^s)$ the branch of $M^u_{q^u}$ between $\mathbf{Z}^{\mathbf{u}}_{\mathbf{q^u}}(j)$ and $\breve{\mathbf{Q}}^{\mathbf{u}}_{\mathbf{q^u}}(2j-1; L^s)$, for $j = 2, \dots, k$, so that $\hat{M}^u_{q^u}(2j-1; L^s) \subset \breve{M}^u_{q^u}(2j-1; L^s)$ for $j = 2, \dots, k-1$, and $\hat{M}^u_{q^u}(2k-1; L^s) = \breve{M}^u_{q^u}(2k-1; L^s)$. Moreover observe that $\hat{M}^u_{q^u}(2j-1; L^s) \subset W^u_{q^u}(\breve{\tau}^u_k)$, for any $j = 2, \dots, k$. Following Lemma 4.2 for any $j = 2, \dots, k$ and any $\mathbf{Q} \in \hat{M}^u_{q^u}(2j-1; L^s)$ we denote by

$$\breve{T}^u_j(\mathbf{Q}) := \sup\{T \,|\, \mathbf{x}_\top(t, \breve{\tau}^u_k; \mathbf{Q}) \in A^+_{L^s} \text{ for } \breve{\tau}^u_k < t < T\}, \tag{4.7}$$

and we set $\breve{T}^u_j(\breve{\mathbf{Q}}^{\mathbf{u}}_{\mathbf{q^u}}(2j-1; L^s)) = \breve{\tau}^u_k$. Observe that the solution $u(d(\mathbf{Q}), r)$ of (1.4) corresponding to $\mathbf{x}_\top(t, \breve{\tau}^u_k; \mathbf{Q})$ is such that $u(d(\mathbf{Q}), r) \geq \delta^u_k \exp[-\alpha_{q^u}\breve{\tau}^u_k] = 1$, whenever $r \leq \ln(\breve{\tau}^u_k)$. So, repeating the reasoning of Lemma 4.2 (the cases where $d > 1$) we see that the function $\breve{T}^u_j : \hat{M}^u_{q^u}(2j-1; L^s) \to [\breve{\tau}^u_k, +\infty)$ defined by (4.7) and the function $\breve{\psi}^{u,j} : \hat{M}^u_{q^u}(2j-1; L^s) \to A^0_{L^s}$ defined by $\breve{\psi}^{u,j}(\mathbf{Q}) = \mathbf{x}_\top(\breve{T}^u_j(\mathbf{Q}), \breve{\tau}^u_k; \mathbf{Q})$ are continuous. We denote by

$$\begin{aligned}
\breve{\boldsymbol{\Xi}}^{\mathbf{u,j}} &:= \{(\breve{\psi}^{u,j}(\mathbf{Q}), \breve{T}^u_j(\mathbf{Q})) \,|\, \mathbf{Q} \in \hat{M}^u_{q^u}(2j-1; L^s)\} \cup \{(\breve{\mathbf{Q}}^{\mathbf{u}}_{\mathbf{q^u}}(2j-1; L^s), \tau) \,|\, \tau \leq \breve{\tau}^u_k\} \\
\breve{\xi}^{\mathbf{u,j}} &:= \{(\mathbf{Q}, \tau) \,|\, \mathbf{Q} \in W^u_{q^u}(\tau) \cap A^0_{L^s}, \mathbf{x}_\top(t, \tau; \mathbf{Q}) \text{ crosses } A^0_{L^s} \; 2j-2 \text{ times for } t < \tau\}.
\end{aligned}$$

for any $j = 2, \dots, k$. By construction $\breve{\boldsymbol{\Xi}}^{\mathbf{u,j}}$ is connected and $\breve{\boldsymbol{\Xi}}^{\mathbf{u,j}} \subset \breve{\xi}^{\mathbf{u,j}}$; moreover $H_*(\breve{\mathbf{Q}}^{\mathbf{u}}_{\mathbf{q^u}}(2j-1; L^s)) < 0$, since $q^u > 2^*$.

4.4. **Lemma.** *Fix $2_* < q^s < 2^*$, $k \in \mathbb{N}$, $k \geq 2$. Then there are $L^s < q^s$ and $\varepsilon_k(q^s) \in (0, \epsilon_k(L^s)]$, $T^-_k < T^+_j$, $\breve{\mathbf{Q}}^{\mathbf{u,j}}(\tau)$ and $\tilde{\mathbf{Q}}^{\mathbf{s}}(\tau)$ such that $(\breve{\mathbf{Q}}^{\mathbf{u,j}}(\tau), \tau) \in \breve{\boldsymbol{\Xi}}^{\mathbf{u,j}}$ and $(\tilde{\mathbf{Q}}^{\mathbf{s}}(\tau), \tau) \in \tilde{\Xi}^s$ for any $T^-_k \leq \tau \leq T^+_j$, whenever $q^u \in (2^*, 2^* + \varepsilon_k(q^s))$, and $j = 2, \dots, k$. Moreover $H_*(\breve{\mathbf{Q}}^{\mathbf{u,j}}(T^-_k)) < 0 < H_*(\tilde{\mathbf{Q}}^{\mathbf{s}}(T^-_k))$ and $H_*(\breve{\mathbf{Q}}^{\mathbf{u,j}}(T^+_j)) > 0 > H_*(\tilde{\mathbf{Q}}^{\mathbf{s}}(T^+_j))$, for any $j = 2, \dots, k$.*

Proof. Fix $2_* < q^s < 2^*$, $k \in \mathbb{N}$, $k \geq 2$ and $q^u \in (2^*, 2^* + \epsilon_k(q^s))$; choose $L^s < q^s$ such that $q^u \in (2^*, 2^* + \epsilon_k(L^s))$, too. Setting $\breve{\mathbf{Q}}^{\mathbf{u,j}}(\tau) := \breve{\mathbf{Q}}^{\mathbf{u}}_{\mathbf{q^u}}(2j-1; L^s)$ for any $\tau \leq \breve{\tau}^u_k$, and $T^-_k = \min\{\breve{\tau}^u_k, T^-\}$ we find $H_*(\breve{\mathbf{Q}}^{\mathbf{u,j}}(\tau)) < 0 < H_*(\tilde{\mathbf{Q}}^{\mathbf{s}}(\tau))$ for any $\tau \leq T^-_k$, and any $j = 2, \dots, k$, see also Lemma 4.3.

Consider the trajectory $\mathbf{x}_\top(t, \breve{\tau}^u_k; \mathbf{Z}^{\mathbf{u}}_{\mathbf{q^u}}(j))$ and the corresponding regular solution $u^j(r)$ of (1.4) with f of type (1.3). Observe that $u^j[\exp(\breve{\tau}^u_k)] = 1$ so $\mathbf{x}_\top(t, \breve{\tau}^u_k; \mathbf{Z}^{\mathbf{u}}_{\mathbf{q^u}}(j)) \equiv \mathbf{x}_{\mathbf{q^u}}(t, \breve{\tau}^u_k; \mathbf{Z}^{\mathbf{u}}_{\mathbf{q^u}}(j))$ for $t \leq \breve{\tau}^u_k$. Denote by $\mathbf{Z}^{\mathbf{u}}_{\mathbf{q^s}}(j) = \mathbf{Z}^{\mathbf{u}}_{\mathbf{q^u}}(j)\exp[(\alpha_{q^s} - \alpha_{q^u})\breve{\tau}^u_k] \in W^u_{q^s}(\breve{\tau}^u_k)$; observe that the x coordinate of $\mathbf{Z}^{\mathbf{u}}_{\mathbf{q^s}}(j)$ is $(\delta^u_k)^{(q^u-2)/(q^s-2)}$ and that $\mathbf{Z}^{\mathbf{u}}_{\mathbf{q^s}}(j) \in A^+_{L^s}$ for $j = 2, \dots, k-1$ and $\mathbf{Z}^{\mathbf{u}}_{\mathbf{q^s}}(k) \in A^0_{L^s}$.

Note that δ^u_k is a continuous function of q^u and that $\delta^u_k \to 0$ as $q^u \to 2^*$, see Remarks 2.8 and 2.9. So we can find $\varepsilon_k(q^s) < \epsilon_k(L^s)$ such that $(\delta^u_k)^{(q^u-2)/(q^s-2)} < \delta^u_k < \Delta^u_{q^s}(L^s)$, where $\Delta^u_{q^s}(L^s)$ is the constant defined in Remark 2.13; thus $\mathbf{Z}^{\mathbf{u}}_{\mathbf{q^s}}(j) \in E^u_{q^s}(L^s)$ for any $j = 2, \dots, k$. So from Remark 2.13 we find that there is $\breve{T}^u_j(\mathbf{Z}^{\mathbf{u}}_{\mathbf{q^s}}(j)) > \breve{\tau}^u_k$ such that the trajectory $\mathbf{x}_{\mathbf{q^s}}(t, \breve{\tau}^u_k; \mathbf{Z}^{\mathbf{u}}_{\mathbf{q^s}}(j))$ of (2.4) where $q = q^s$ is in $A^+_{L^s}$ for $\breve{\tau}^u_k < t < \breve{T}^u_j(\mathbf{Z}^{\mathbf{u}}_{\mathbf{q^s}}(j))$, and it crosses $A^0_{L^s}$ in a point $\mathbf{Q}^{\mathbf{u,j}}$, such that $H_*(\mathbf{Q}^{\mathbf{u,j}}) > 0$. Denote by $v^j(r)$ the solution of (1.4) corresponding to $\mathbf{x}_{\mathbf{q^s}}(t, \breve{\tau}^u_k; \mathbf{Z}^{\mathbf{u}}_{\mathbf{q^s}}(j))$; the function $w^j(r)$ defined as $u^j(r)$ for $r \leq \exp[\breve{\tau}^u_k]$ and as $v^j(r)$ for $\exp[\breve{\tau}^u_k] \leq r \leq \exp[\breve{T}^u_j(\mathbf{Z}^{\mathbf{u}}_{\mathbf{q^s}}(j))]$ is continuous and solves (1.4) with f of type (1.3). Therefore $\mathbf{x}_\top(t, \breve{\tau}^u_k; \mathbf{Z}^{\mathbf{u}}_{\mathbf{q^u}}(j)) \equiv \mathbf{x}_{\mathbf{q^s}}(t, \breve{\tau}^u_k; \mathbf{Z}^{\mathbf{u}}_{\mathbf{q^s}}(j))\exp[(\alpha_{q^u} - \alpha_{q^s})t]$ for any $t \in (\breve{\tau}^u_k, \breve{T}^u_j(\mathbf{Z}^{\mathbf{u}}_{\mathbf{q^s}}(j)))$. Set $\breve{\mathbf{Q}}^{\mathbf{u,j}}(\tau) = \mathbf{Q}^{\mathbf{u,j}}\exp[(\alpha_{q^u} - \alpha_{q^s})\tau]$, where $\tau = \breve{T}^u_j(\mathbf{Z}^{\mathbf{u}}_{\mathbf{q^s}}(j))$; by (2.8) we have $H_*(\breve{\mathbf{Q}}^{\mathbf{u,j}}(\tau)) > 0$. Furthermore note that, for any $j = 2, \dots, k$, $\breve{T}^u_j(\mathbf{Z}^{\mathbf{u}}_{\mathbf{q^s}}(j)) \to$

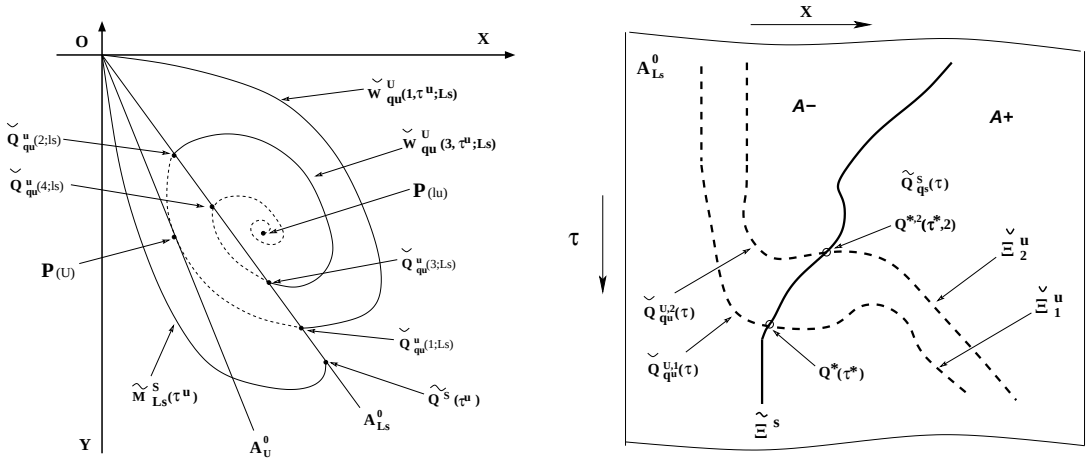

FIGURE 4. Construction of $\breve{\mathbf{\Xi}}^{\mathbf{u,j}}$ for $j = 2, \dots, k$.

$+\infty$ as $\delta_k^u \to 0$, i.e. $q^u \to 2^*$. Hence we can choose δ_k^u small enough so that $T_j^+ := \breve{T}_j^u(\mathbf{Z}_{\mathbf{q^s}}^{\mathbf{u}}(j)) > T^+$ for any $j = 2, \dots, k$, so that $(\breve{\mathbf{Q}}^{\mathbf{u,j}}(T_j^+), T_j^+) \in \breve{\mathbf{\Xi}}_{\mathbf{j}}^{\mathbf{u}}$ and $H_*(\breve{\mathbf{Q}}^{\mathbf{u,j}}(T_j^+)) > 0 > H_*(\tilde{\mathbf{Q}}^{\mathbf{s}}(T_j^+))$. Thanks to the connectedness of $\breve{\mathbf{\Xi}}_{\mathbf{j}}^{\mathbf{u}}$ there is $(\breve{\mathbf{Q}}^{\mathbf{u,j}}(\tau), \tau) \in \breve{\mathbf{\Xi}}_{\mathbf{j}}^{\mathbf{u}}$ for any $\tau \in [T_k^-, T_j^+]$ for any $j = 2, \dots, k$. □

Now fix $q^u > 2^*$, $k \in \mathbb{N}$, $k \geq 2$, $q^s \in (2^* - \epsilon_k(q^u), 2^*)$, and choose $L^u > q^u$ such that $q^s \in (2^* - \epsilon_k(L^u), 2^*)$. Let $\mathbf{W}_{\mathbf{q^s}}^{\mathbf{s}} := \{(\mathbf{Q}, \tau) \mid \mathbf{Q} \in W_{q^s}^s(\tau),\ \tau \in \mathbb{R}\}$ and

$$\breve{\xi}^{\mathbf{s,j}} = \{(\mathbf{Q}, \tau) \in \mathbf{W}_{\mathbf{q^s}}^{\mathbf{s}} \cap \mathbf{A}_{\mathbf{L^u}}^{\mathbf{0}} \mid \mathbf{x}_\perp(t, \tau; \mathbf{Q}) \text{ crosses } A_{L^u}^0 \text{ exactly } 2j-2 \text{ times for } t > \tau\}.$$

Let us set $\breve{\mathbf{Q}}_{\mathbf{q^s}}^{\mathbf{s}}(j; L^u) = (\breve{X}^s(j), -\alpha_{L^u}\breve{X}^s(j))$, $\tau_j^0 := \ln(\breve{X}^s(2j-1))/\alpha_{q^s}$, and $T_j^0 := \ln(\breve{X}^s(2j-2))/\alpha_{q^s}$, so that $T_j^0 < \tau_j^0$. Consider the trajectory $\mathbf{x}_{\mathbf{q^s}}(t, \tau_j^0; \breve{\mathbf{Q}}_{\mathbf{q^s}}^{\mathbf{s}}(2j-1; L^u))$ of (2.4) where $q = q^s$, and the corresponding fast decay solution $v^j(r)$ of (1.4). Observe that $v^j(r) < 1$ for any $r > \exp[\tau_j^0]$ and $v^j(\exp[\tau_j^0]) = 1$, so $\mathbf{x}_{\mathbf{q^s}}(t, \tau; \breve{\mathbf{Q}}_{\mathbf{q^s}}^{\mathbf{s}}(2j-1; L^u)) \equiv \mathbf{x}_\perp(t, \tau; \breve{\mathbf{Q}}_{\mathbf{q^s}}^{\mathbf{s}}(2j-1; L^u))$ for any $t \geq \tau \geq \tau_j^0$. It follows that $(\breve{\mathbf{Q}}_{\mathbf{q^s}}^{\mathbf{s}}(2j-1; L^u), \tau) \in \breve{\xi}^{\mathbf{s,j}}$ for any $\tau \geq \tau_j^0$.

For any $\mathbf{Q}_{\mathbf{q^s}}^{\mathbf{s}} = (X^s, Y^s) \in \breve{M}_{q^s}^s(2j-1; L^u)$, there is a unique $\bar{\tau}_j^s = \bar{\tau}_j^s(\mathbf{Q}_{\mathbf{q^s}}^{\mathbf{s}}) \in [T_j^0, \tau_j^0]$, such that $X^s = \exp[\alpha_{q^s}\bar{\tau}_j^s]$. Hence the trajectory $\mathbf{x}_\perp(t, \bar{\tau}_j^s; \mathbf{Q}_{\mathbf{q^s}}^{\mathbf{s}})$ of (4.3) coincides with $\mathbf{x}_{\mathbf{q^s}}(t, \bar{\tau}_j^s; \mathbf{Q}_{\mathbf{q^s}}^{\mathbf{s}})$ for $t \geq \bar{\tau}_j^s$.

We set $\breve{T}_j^s(\breve{\mathbf{Q}}_{\mathbf{q^s}}^{\mathbf{s}}(2j-1; L^u)) = \tau_j^0$ and for any $\mathbf{Q}_{\mathbf{q^s}}^{\mathbf{s}} \in \breve{M}_{q^s}^s(2j-1; L^u) \setminus \{\breve{\mathbf{Q}}_{\mathbf{q^s}}^{\mathbf{s}}(2j-1; L^u)\}$ we define

$$\breve{T}_j^s(\mathbf{Q}_{\mathbf{q^s}}^{\mathbf{s}}) := \inf\{T \mid \mathbf{x}_\perp(t, \bar{\tau}_j^s(\mathbf{Q}_{\mathbf{q^s}}^{\mathbf{s}}); \mathbf{Q}_{\mathbf{q^s}}^{\mathbf{s}}) \in A_{L^u}^- \text{ for any } T < t < \bar{\tau}_j^s(\mathbf{Q}_{\mathbf{q^s}}^{\mathbf{s}})\ \} \tag{4.8}$$

Set $\mathbf{Q}_{\mathbf{q^u}}^{\mathbf{s}} = \mathbf{Q}_{\mathbf{q^s}}^{\mathbf{s}} \exp[(\alpha_{q^u} - \alpha_{q^s})\bar{\tau}_j^s] \in W_{q^u}^s(\bar{\tau}_j^s)$ and observe that by construction $\mathbf{x}_\perp(t, \bar{\tau}_j^s; \mathbf{Q}_{\mathbf{q^s}}^{\mathbf{s}}) \equiv \mathbf{x}_{\mathbf{q^u}}(t, \bar{\tau}_j^s; \mathbf{Q}_{\mathbf{q^u}}^{\mathbf{s}}) \exp[(\alpha_{q^s} - \alpha_{q^u})t]$ for any $\breve{T}_j^s(\mathbf{Q}_{\mathbf{q^s}}^{\mathbf{s}}) \leq t \leq \bar{\tau}_j^s$. Moreover from Remark 2.4 we see that $\mathbf{x}_{\mathbf{q^u}}(t, \bar{\tau}_j^s; \mathbf{Q}_{\mathbf{q^u}}^{\mathbf{s}})$ crosses $A_{L^u}^0$ transversally at $t = \breve{T}_j^s(\mathbf{Q}_{\mathbf{q^s}}^{\mathbf{s}})$ and that $\breve{T}_j^s(\mathbf{Q}_{\mathbf{q^s}}^{\mathbf{s}})$ is bounded. Then reasoning as in Lemma 4.2 we see that the function $\breve{T}_j^s : \breve{M}_{q^s}^s(2j-1; L^u) \to \mathbb{R}$ is continuous. Let us define the function $\breve{\psi}_j^s : \breve{M}_{q^s}^s(2j-1; L^u) \to A_{L^u}^0$ as $\breve{\psi}_j^s(\mathbf{Q}) := \mathbf{x}_\perp(\breve{T}_j^s(\mathbf{Q}), \bar{\tau}_j^s; \mathbf{Q})$; from the continuity of (4.3) we find that $\breve{\psi}_j^s$ is continuous. Therefore the set $\breve{\mathbf{\Xi}}^{\mathbf{s,j}}$ defined as follows

$$\breve{\mathbf{\Xi}}^{\mathbf{s,j}} := \{(\breve{\psi}_j^s(\mathbf{Q}), \breve{T}_j^s(\mathbf{Q})) \mid \mathbf{Q} \in \breve{M}_{q^s}^s(2j-1; L^u)\} \cup \{(\breve{\mathbf{Q}}_{\mathbf{q^s}}^{\mathbf{s}}(2j-1; L^u), \tau) \mid \tau \geq \tau_j^0)\}$$

is connected and $\breve{\mathbf{\Xi}}^{\mathbf{s,j}} \subset \breve{\xi}^{\mathbf{s,j}}$. We stress that $\breve{\mathbf{Q}}^{\mathbf{s,j}}(\tau) := \breve{\mathbf{Q}}^{\mathbf{s}}_{\mathbf{q^s}}(2j-1; L^u)$ for $\tau < \tau^0_j$ is such that $H_*(\breve{\mathbf{Q}}^{\mathbf{s,j}}(\tau), \tau) < 0$ since $q^s < 2^*$. Now we are ready to state the following result, analogous to Lemma 4.4.

4.5. **Lemma.** *Fix $q^u > 2^*$, $k \in \mathbb{N}$, $k \geq 2$. Then we can choose $L^u > q^u$, $L^u - q^u$ small enough so that there are $\varepsilon_k(q^u) \in (0, \epsilon_k(L^u))$, $T^-_j < T^+_k$, $\breve{\mathbf{Q}}^{\mathbf{s,j}}(\tau)$ and $\tilde{\mathbf{Q}}^{\mathbf{u}}(\tau)$ such that $(\breve{\mathbf{Q}}^{\mathbf{s,j}}(\tau), \tau) \in \breve{\mathbf{\Xi}}^{\mathbf{s,j}}$ and $(\tilde{\mathbf{Q}}^{\mathbf{u}}(\tau), \tau) \in \tilde{\Xi}^u$ for any $T^-_j \leq \tau \leq T^+_k$, whenever $q^s \in (2^* - \varepsilon_k(q^u), 2^*)$ and $j = 2, \dots, k$. Moreover $H_*(\breve{\mathbf{Q}}^{\mathbf{s,j}}(T^-_j)) > 0 > H_*(\tilde{\mathbf{Q}}^{\mathbf{u}}(T^-_j))$ and $H_*(\breve{\mathbf{Q}}^{\mathbf{s,j}}(T^+_k)) < 0 < H_*(\tilde{\mathbf{Q}}^{\mathbf{u}}(T^+_k))$, for any $j = 2, \dots, k$.*

Proof. Fix $k \in \mathbf{N}$, $k \geq 2$, $q^u > 2^*$ and $q^s \in (2^* - \epsilon_k(q^u), 2^*)$: we choose $L^u > q^u$ so that $q^s \in (2^* - \epsilon_k(L^u), 2^*)$ too, and we fix $j \in 2, \dots, k$. We have just proved that $H_*(\breve{\mathbf{Q}}^{\mathbf{s,j}}(\tau)) < 0 < H_*(\tilde{\mathbf{Q}}^{\mathbf{u}}(\tau))$, for any $\tau \geq T^+_k := \max\{\tau^0_k, T^+\}$.

Denote by $\delta^j(q^s)$ the x-coordinate of $\breve{\mathbf{Q}}^{\mathbf{s}}_{\mathbf{q^s}}(2j-2; L^u)$ and observe that $0 < \delta^j(q^s) < \delta^{j+1}(q^s) < \delta^k(q^s)$ for any $q^s \in (2^* - \epsilon_k(L^u), 2^*)$, $j = 2, \dots, k-2$, and that $\delta^k(q^s) \to 0$ as $q^s \to 2^*$. Set

$$\mathbf{Q}^{\mathbf{s,j}}_{\mathbf{q^u}} := \breve{\mathbf{Q}}^{\mathbf{s}}_{\mathbf{q^s}}(2j-2; L^u) e^{(\alpha_{q^u} - \alpha_{q^s})T^0_j} = ([\delta^j(q^s)]^{\frac{q^s-2}{q^u-2}}, -\alpha_{L^u}[\delta^j(q^s)]^{\frac{q^s-2}{q^u-2}})$$

and consider the trajectory $\mathbf{x_{q^u}}(t, T^0_j; \mathbf{Q}^{\mathbf{s,j}}_{\mathbf{q^u}})$ of (2.4) where $q = q^u$ and the corresponding trajectory $u^j(r)$ of (1.4). We can find $\varepsilon_k(q^u) < \epsilon_k(L^u)$ so that

$$[\delta^k(q^u)]^{\frac{q^s-2}{q^u-2}} < [\delta^k(q^u)]^{\frac{2_*-2}{q^u-2}} < \Delta^s_{q^u}(L^u)$$

where the constant $\Delta^s_{q^u}(L^u)$ has been defined in Remark 2.13. It follows that $\mathbf{Q}^{\mathbf{s,j}}_{\mathbf{q^u}} \in E^s_{q^u}(L^u)$ for any $j = 2, \dots, k$; so from Remark 2.13 we find that there is $\breve{T}^s_j(\mathbf{Q}^{\mathbf{s,j}}_{\mathbf{q^u}}) < T^0_j$ such that $\mathbf{x_{q^u}}(t, T^0_j; \mathbf{Q}^{\mathbf{s,j}}_{\mathbf{q^u}}) \in A^-_{L^u}$ for any $\breve{T}^s_j(\mathbf{Q}^{\mathbf{s,j}}_{\mathbf{q^u}}) < t < T^0_j$, and $\mathbf{x_{q^u}}(\breve{T}^s_j(\mathbf{Q}^{\mathbf{s,j}}_{\mathbf{q^u}}), T^0_j; \mathbf{Q}^{\mathbf{s,j}}_{\mathbf{q^u}}) = \mathbf{C}^{\mathbf{j}}_{\mathbf{q^u}} \in A^0_{L^u}$ and $H_*(\mathbf{C}^{\mathbf{j}}_{\mathbf{q^u}}) > 0$. Set $\mathbf{C}^{\mathbf{j}}_{\mathbf{q^s}} = \mathbf{C}^{\mathbf{j}}_{\mathbf{q^u}} \exp[(\alpha_{q^s} - \alpha_{q^u})\breve{T}^s_j(\mathbf{Q}^{\mathbf{s,j}}_{\mathbf{q^u}})]$: from (2.8) we find that $H_*(\mathbf{C}^{\mathbf{j}}_{\mathbf{q^s}}) > 0$ too.

Moreover note that $\breve{T}^s_j(\mathbf{Q}^{\mathbf{s,j}}_{\mathbf{q^u}}) \to -\infty$ as $\delta^j(q^u) \to 0$, i.e. $q^u \to 2^*$; so we can assume $T^-_j := \breve{T}^s_j(\mathbf{Q}^{\mathbf{s,j}}_{\mathbf{q^u}}) < T^-$ for any $j = 2, \dots, k$ and set $\breve{\mathbf{Q}}^{\mathbf{s,j}}(T^-_j) := \mathbf{C}^{\mathbf{j}}_{\mathbf{q^s}}$ in order to get $(\breve{\mathbf{Q}}^{\mathbf{s,j}}(T^-_j), T^-_j) \in \breve{\mathbf{\Xi}}^{\mathbf{s,j}}$ and $H_*(\breve{\mathbf{Q}}^{\mathbf{s,j}}(T^-_j)) > 0 > H_*(\tilde{\mathbf{Q}}^{\mathbf{u}}(T^-_j))$. Finally we also get that for any $T^-_j \leq \tau \leq T^+_k$ there are $(\breve{\mathbf{Q}}^{\mathbf{s,j}}(\tau), \tau) \in \breve{\mathbf{\Xi}}^{\mathbf{s,j}}$ and $(\tilde{\mathbf{Q}}^{\mathbf{u}}(\tau), \tau) \in \tilde{\Xi}^u$, since $\breve{\mathbf{\Xi}}^{\mathbf{s}}$ and $\tilde{\Xi}^u$ are connected. □

Now we can easily prove Theorem 1.11.

Proof of Theorem 1.11. Fix $2_* < q^s < 2^*$, q^s, and choose $L^s < q^s$ and $\varepsilon_k(q^s)$ as in Lemma 4.4. Following the proof of Theorem 1.10 we denote by $\mathfrak{A}^-$ and $\mathfrak{A}^+$ the two open sets in which $\tilde{\Xi}^s$ divides $\mathbf{A}^{\mathbf{0}}_{\mathbf{L^s}}$. From Lemma 4.4 we find that there are $(\breve{\mathbf{Q}}^{\mathbf{u,j}}(T^-_k), T^-_k) \in \mathfrak{A}^- \cap \breve{\mathbf{\Xi}}^{\mathbf{u,j}}$, $(\breve{\mathbf{Q}}^{\mathbf{u,j}}(T^+_j), T^+_j) \in \mathfrak{A}^+ \cap \breve{\mathbf{\Xi}}^{\mathbf{u,j}}$, for any $j = 2, \dots, k$ and from Lemma 4.3 we find $(\breve{\mathbf{Q}}^{\mathbf{u}}(T^-), T^-) \in \mathfrak{A}^- \cap \breve{\Xi}^u$, $(\breve{\mathbf{Q}}^{\mathbf{u}}(T^+), T^+) \in \mathfrak{A}^+ \cap \breve{\Xi}^u$. Therefore there are $\tau^*_j \in (T^-_k, T^+_j)$ and $\breve{\mathbf{Q}}^{\mathbf{u,j}}(\tau^*_j)$ such that $(\breve{\mathbf{Q}}^{\mathbf{u,j}}(\tau^*_j), \tau^*_j) \in \breve{\Xi}^{u,j} \cap \tilde{\Xi}^s$ for any $j = 2, \dots, k$ and $(\breve{\mathbf{Q}}^{\mathbf{u}}(\tau^*_1), \tau^*_1) \in \breve{\Xi}^u \cap \tilde{\Xi}^s$. So the trajectory $\mathbf{x}_\top(t, \tau^*_j, \breve{\mathbf{Q}}^{\mathbf{u,j}}(\tau^*_j))$ of (4.1) are homoclinic and correspond to a G.S. with f.d. $u(d^j, r)$ of (1.4) for any $j = 1, \dots, k$. Note that the trajectories $\mathbf{x}_\top(t, \tau^*_j, \breve{\mathbf{Q}}^{\mathbf{u,j}}(\tau^*_j))$ are distinct since they rotates exactly j times around the critical point $\mathbf{P}(q^s)$. Thus we have k different G.S. with f.d.

The proof for the case $q^u > 2^*$ and $2^* - \varepsilon_k(q^s) < q^s < 2^*$ follows using Lemmas 4.3 and 4.5 and reasoning as above. □

Exploiting the ideas of Theorem 1.7 we can prove the analogous result in this setting. Once again we prove the result replacing $\epsilon_0(q^u)$ and $N_0(q^s)$ by the better constants $\tilde{\epsilon}_0(q^u)$ and $\tilde{N}_0(q^s)$ defined in (2.12).

Proof of Theorem 1.12. Let $q^u > 2^*$, $q^s \in (2_*, 2_* + \tilde{\epsilon}_0(q^u))$ so that $S(q^s) < U(q^u)$. Consider a regular solution $u(d,r)$ of (1.4) with f of type (1.3) and the corresponding trajectories $\mathbf{x}^{\mathbf{u}}_{\top}(t)$ and $\mathbf{x}^{\mathbf{u}}_{\perp}(t)$ of (4.1) and (4.3) respectively.

Assume first $d > 1$ and denote by T^0 the smallest positive value such that $u(\exp[T^0]) = 1$: we set $\mathbf{Q}^{\mathbf{u}}_{\mathbf{0}} = \mathbf{x}^{\mathbf{u}}_{\top}(T^0)$, $\mathbf{Q}^{\mathbf{s}}_{\mathbf{0}} = \mathbf{x}^{\mathbf{u}}_{\perp}(T^0)$. Observe that $\mathbf{x}^{\mathbf{u}}_{\top}(t) \equiv \mathbf{x}_{\mathbf{q^u}}(t, T^0; \mathbf{Q}^{\mathbf{u}}_{\mathbf{0}}) \in M^u_{q^u}$ for any $t \le T^0$, while $\mathbf{x}^{\mathbf{u}}_{\perp}(t) \equiv \mathbf{x}_{\mathbf{q^s}}(t, T^0; \mathbf{Q}^{\mathbf{s}}_{\mathbf{0}})$ for any $t \in (T^0, \ln(R))$, where R is the largest value (possibly $+\infty$) such that $u(d,r) > 0$ for $0 < r < R$.

Then $\mathbf{Q}^{\mathbf{u}}_{\mathbf{0}} \in M^u_{q^u} \subset (A^+_U \cup A^0_U)$, where $U = U(q^u)$, so $\mathbf{Q}^{\mathbf{s}}_{\mathbf{0}} = \mathbf{Q}^{\mathbf{u}}_{\mathbf{0}} \exp[(\alpha_{q^s} - \alpha_{q^u})T^0] \in (A^+_U \cup A^0_U)$ too. Since $M^s_{q^s} \subset (A^-_S \cup A^0_S) \subset A^-_U$ it follows that $\mathbf{Q}^{\mathbf{s}}_{\mathbf{0}} \notin M^s_{q^s}$; hence $u(d,r)$ cannot have fast decay. Moreover $\mathbf{P}(q^s) \in A^0_{q^s}$, so $\mathbf{P}(q^s) \neq \mathbf{Q}^{\mathbf{s}}_{\mathbf{0}}$ and $u(d,r)$ cannot have slow decay either; so it is a crossing solution. If $d \le 1$, then $\mathbf{x}^{\mathbf{u}}_{\perp}(t)$ is directly a solution of (2.4) where $q = q^s$, so $u(d,r)$ is a crossing solution, too.

Now consider a fast decay solution $v(r)$ of (1.4) with f of type (1.3) and the corresponding trajectories $\mathbf{x}^{\mathbf{s}}_{\top}(t)$ and $\mathbf{x}^{\mathbf{s}}_{\perp}(t)$ of (4.1) and (4.3) respectively. Again denote by τ^0 the largest positive values such that $v(\exp[\tau^0]) = 1$, and set $\mathbf{R}^{\mathbf{0}}_{\mathbf{s}} = \mathbf{x}^{\mathbf{s}}_{\perp}(\tau^0)$, $\mathbf{R}^{\mathbf{0}}_{\mathbf{u}} = \mathbf{x}^{\mathbf{s}}{}_{\top}(\tau^0)$. Reasoning as above we see that $\mathbf{R}^{\mathbf{0}}_{\mathbf{u}} \in (A^-_S \cup A^0_S)$, while $(M^u_{q^u} \cup \{\mathbf{P}(q^u)\}) \subset A^+_S$. Hence $\mathbf{R}^{\mathbf{0}}_{\mathbf{u}} \notin (M^u_{q^u} \cup \{\mathbf{P}(q^u)\})$ so $v(r)$ can be neither regular nor singular. It follows that there are $R_2 > R_1 > 0$ such that $v'(R_2) = 0 < v(R_2)$ and $v(R_1) = 0 < v'(R_1)$.

Reasoning as above we also find that there is $\rho > 0$ such that the unique singular solution $v(r)$ of (1.4) (corresponding to the critical point $\mathbf{P}(q^u)$) satisfies $v(\rho) = 0 > v'(\rho)$, and for the unique slow decaying solution $w(r)$ of (1.4) (corresponding to the critical point $\mathbf{P}(q^s)$) there are $\rho_2 > \rho_1 > 0$ such that $w'(\rho_2) = 0 < w(\rho_2)$ and $w(\rho_1) = 0 < w'(\rho_1)$.

The proof for $2_* < q^s < 2^*$, and $q^u > N_0(q^s)$ is completely analogous so will be omitted. □

From the previous proof we obtain in this context the analogous of Corollaries 3.4 and 3.5.

4.6. **Corollary.** *Consider f of type (1.3). Then (1.4) admits no positive solutions either regular or singular whenever $2_* < q^s \le \omega^s_* < 2^* < \omega^u_* < q^u$, where $\omega^s_* > \sigma_*$ and $\omega^u_* < \sigma^*$ are defined in Remark 3.5.*

Now we turn to the problem of existence of G.S. with s.d. and of S.G.S. with f.d.

Proof of Theorem 1.13. Fix $q^s \in (2_*, 2^*)$ and consider the critical point $\mathbf{P}(q^s) = (P_x(q^s), P_y(q^s))$ of (2.4), and denote by $v(r)$ the corresponding slow decay solution $v(r)$ of (1.4), i.e. $v(r) = P_x(q^s) r^{-\alpha_{q^s}}$. We denote by $T^0 = \ln(P_x(q^s))/\alpha_{q^s}$ the value such that $v(\exp(T^0)) = 1$, and by $\mathbf{R}(q^u, q^s) = \mathbf{P}(q^s)\exp[(\alpha_{q^u} - \alpha_{q^s})T^0] \in A^0_{q^s}$. Observe that the solution $w(r)$ of (1.4) corresponding to the trajectory $\mathbf{x}_{\top}(t, T^0; \mathbf{R}(q^u, q^s))$ of (4.1) coincides with $v(r)$ for $r \ge R^0 := \exp(T^0)$. Fix $k \in \mathbb{N}$ and consider the manifold $M^u_{q^u}$ for $q^u < 2^* + \epsilon_k(q^s)$: it intersects $A^0_{q^s}$ in at least $2k$ points. From Remark 2.8 and 2.9 we know that there is $\nu_k(q^s) < \epsilon_k(q^s)$ such that $c < H_*(\mathbf{Q}) < 0$ for any $\mathbf{Q} \in \tilde{M}^u_{q^u}(2k)$ whenever $q^u \in (2^*, 2^* + \nu_k(q^s))$. Consider the bounded set $\tilde{B}^u_{q^u}(2k)$ enclosed by $\tilde{M}^u_{q^u}(2k)$ and $A^0_{q^u}$, and observe that $\mathbf{R}(q^u, q^s)$ is in the interior of $\tilde{B}^u_{q^u}(2k)$.

From Theorem 1.12 and Remark 2.9 there is $\omega^u_* < \sigma^*$ such that $\tilde{B}^u_{q^u}(2k) \subset \tilde{B}^u_{q^u}(2) \subset A^+_{2^*}$, whenever $\omega^u_* < q^u < \sigma^*$, so that $\mathbf{R}(q^u, q^s) \in A^0_{q^s}$ lies in the exterior of $\tilde{B}^u_{q^u}(2k)$. So from Remark 2.9 we find that there is a value, denoted by $r^k(q^s)$ such that $\mathbf{R}(r^k(q^s), q^s)$ lies on the border of $\tilde{B}^u_{r^k(q^s)}(2k)$. Moreover, since

$\mathbf{R}(r^k(q^s), q^s) \in A^0_{q^s}$ and $q^s < 2^* < r^k(q^s)$, it follows that $\mathbf{R}(r^k(q^s), q^s) \notin A^0_{r^k(q^s)}$: thus $\mathbf{R}(r^k(q^s), q^s) \in \tilde{M}^u_{q^u}(2k)$. So consider the trajectory $\mathbf{x_{r^k(q^s)}}(t, T^0; \mathbf{R}(r^k(q^s), q^s))$ of (2.4) with $q = r^k(q^s)$ and the corresponding regular solution $u(r)$ of (1.4). It follows that the function $w(r)$ defined as $u(r)$ for $0 \le r \le \exp(T^0)$, and as $v(r)$ for $r \ge \exp(T^0)$, solves (1.4) with f of type (1.3) and $q^u = r^k(q^s)$ and it is a G.S. with s.d.

For the arbitrariness of k we find a whole decreasing sequence of values $r^k(q^s) \to 2^*$ as $k \to +\infty$ such that (1.4) with f of type (1.3) and $q^u = r^k(q^s)$ admits a G.S. with s.d. We think it is worthwhile to stress that the slow decaying solution is unique so the G.S. with s.d is the unique such solution if it exists.

Observe further that, if the functions $U_k(l)$ were monotone, the values $r^k(q^s)$ would be uniquely defined and decreasing in k. However we can always choose $r^k(q^s)$ to be decreasing in k, since $U_k(l) < U_{k+1}(l)$ for any l. Moreover by construction $r^k(q^s) \to 2^*$ as $q^s \to 2_*$, for any $k \in \mathbb{N}$.

The proof of the existence of S.G.S. with f.d. is completely analogous and will be omitted. □

Proof of Corollary 1.14. Here we wish to adapt to the context the ideas used by Flores in [11], so we want to use Lemma 3.8 and Remark 3.9. We start by assuming that (1.4) admits a G.S with s.d. $u(d_*, r)$ and $\hat{q}^s \in (\sigma_*, 2^*)$. We recall that $\tilde{\mathbf{Q}}^{\mathbf{s}}_{\mathbf{q^s}}(1) = (\tilde{X}^s(1), \tilde{Y}^s(1))$ and $\tau^s = \ln(\tilde{X}^s(1))/\alpha_{q^s}$. Consider system (4.3) and its stable manifold $W^s_{q^s}(\tau^s) \equiv M^s_{q^s}$: $W^s_{q^s}(\tau^s)$ is a C^1 spiral which joins the origin and $\mathbf{P}(\hat{q}^s)$ and it rotates clockwise infinitely many times around $\mathbf{P}(\hat{q}^s)$, so it can be parameterized by the function $S(s)$ defined in (3.4), where $\mathbf{P} = \mathbf{P}(\hat{q}^s)$ and $\theta(0) = +\infty$.

Observe that $d_* > 1$, otherwise $u(d_*, r)$ solves (1.4) with f of type (1.5) and $q = q^s$ too, so it is a crossing solution. Note that the trajectory of (4.3) corresponding to $u(d_*, r)$ is given by $\mathbf{x}_\perp(t, \tau^s, \mathbf{P}(\hat{q}^s))$. So there is a unique value $\tau_0 < \tau^s$ such that $u(d_*, \exp(\tau_0)) = 1$. Hence $\mathbf{x}_\perp(t, \tau^s, \mathbf{P}(\hat{q}^s)) \equiv \mathbf{P}(\hat{q}^s)$ for $t \ge \tau_0$ and $\mathbf{x}_\perp(t, \tau^s, \mathbf{P}(\hat{q}^s)) \in W^u_{\hat{q}^s}(t)$ for any $t \in \mathbb{R}$, since $u(d_*, r)$ is a regular solution.

Therefore $W^u_{\hat{q}^s}(\tau)$ contains a 1-dimensional path, say $\hat{W}^u_{\hat{q}^s}(\tau)$ that joins the origin and $\mathbf{P}(\hat{q}^s)$ with no self-intersections, for any $\tau \ge \tau_0$. We parameterize $\hat{W}^u_{\hat{q}^s}(\tau)$ as a locally Lipschitz family of curves $U(s, \tau) := \mathbf{P}(\hat{q}^s) + R(s, \tau)e^{iw(s,\tau)}$, such that $U(0, \tau) = \mathbf{P}(\hat{q}^s)$ and $U(0, \tau) = (0, 0)$ for $\tau \ge \tau_0$. We can assume w.l.o.g. that if $\mathbf{Q} = U(\bar{s}, \tau_0)$ then $\mathbf{x}_\perp(T, \tau; \mathbf{Q}) = U(\bar{s}, T)$ for any $\tau_0 \le T \le \tau^s$. Observe that $U(+\infty, \tau) = (0, 0) = S(+\infty)$ so $w(+\infty, \tau) = \theta(+\infty)$ are fixed and can be assumed to belong to $(-\pi, \pi)$, for any $\tau_0 \le \tau \le \tau^s$.

It is easy to check that $w(s, \tau_0) < w(+\infty, \tau_0) < \pi$ for any $s > 0$. We claim that, for any $s > 0$, $w(s, \tau_0)$ is uniformly bounded from below too.
We denote by $B^+ := \{(x, y) \in \mathbb{R}^2_\perp \,|\, x \ge \exp[\alpha_{\hat{q}^s}\tau_0]\}$ and by $B^- := \{(x, y) \in \mathbb{R}^2_\perp \,|\, x < \exp[\alpha_{\hat{q}^s}\tau_0]\}$, the sets corresponding to solutions $u(r)$ of (1.4) such that $u(\exp[\alpha_{\hat{q}^s}\tau_0]) \ge 1$ and $u(\exp[\alpha_{\hat{q}^s}\tau_0]) < 1$ respectively. We set $M^u(\tau_0) := \{\mathbf{Q}\exp[\alpha_{\hat{q}^s} - \alpha_{\hat{q}^u})\tau_0] \mid \mathbf{Q} \in M^u_{\hat{q}^u}\}$. Note that $\mathbf{P}(\hat{q}^s) \in M^u(\tau_0)$: we denote by $\hat{M}^u(\tau_0)$ the branch of $M^u(\tau_0)$ between the origin and $\mathbf{P}(\hat{q}^s)$. We stress that $\hat{W}^u_{\hat{q}^s}(\tau_0) \cap B^+ = \hat{M}^u(\tau_0) \cap B^+$, and by construction $\hat{W}^u_{\hat{q}^s}(\tau_0)$ is connected.

It is easy to check that $\hat{M}^u(\tau_0)$ rotates a finite number of times around $\mathbf{P}(\hat{q}^s)$ because it is not a spiral. Since $\mathbf{P}(\hat{q}^s)$ is on the line which separates B^- and B^+, the number of complete rotations of $\hat{W}^u_{\hat{q}^s}(\tau_0)$ around $\mathbf{P}(\hat{q}^s)$ equals the number of complete rotations of $\hat{M}^u(\tau_0)$ around $\mathbf{P}(\hat{q}^s)$, so they are both finite and the claim is proved.

Now we want to show that $w(s,\tau^s)$ is uniformly bounded for $s>0$ small enough, too. Given a value $0<s<1$ we consider the trajectory $\mathbf{x}_\perp(t,\tau^0;U(s,\tau^0))$ and the corresponding regular solution $u(d(s),r)$. We define the function $\mathfrak{T}_0:(0,1)\to\mathbb{R}$ that associates to s the first value $t=\mathfrak{T}_0(s)$ such that $u(d(s),e^t)=1$. I.e. $\mathfrak{T}_0(s):=\inf\{t\in\mathbb{R}\,|\,x_\perp(t,\tau_0;U(s,\tau_0))\exp[-\alpha_{\hat q^s}t]=1\}$. From the continuity of the flow of (4.3) it follows that $\mathfrak{T}_0$ is continuous and $\mathfrak{T}_0(0)=\tau_0$.

Let $\nu>0$ denote the imaginary part of the eigenvalues of the linearization of (2.4) where $q=\hat q^s$, around the critical point $\mathbf{P}(\hat q^s)$. Using the continuity of $\mathfrak{T}_0$ and approximating the flow of (2.4) with its linearization in $\mathbf{P}_{\hat{\mathbf{q}}}$, we find that there is $\delta>0$ such that

$$|w(s,\tau^s)-w(s,\tau_0)|<|w(s,\tau^s)-w(s,\mathfrak{T}_0(s))|+1<\frac{(\tau^s-\tau_0)\nu}{2\pi}+2\,, \tag{4.9}$$

for any $s\in(0,\delta)$. It follows that $\limsup_{s\to0^+}w(s,\tau^s)<\pi$ and $\liminf_{s\to0^+}w(s,\tau^s)$ is bounded and the claim is proved.

So, choosing $S(s)$ as parametrization of $W^s(\tau^s)$ and $U(s)\equiv U(s,\tau^s)$, we can use Lemma 3.8 and Remark 3.9 to conclude the existence of infinitely many points $\mathbf{Q}^{\mathbf{k}}_*\in\big(\hat W^u_{\hat q^s}(\tau^s)\cap W^s_{\hat q^s}(\tau^s)\big)$, $\mathbf{Q}^{\mathbf{k}}_*\to\mathbf{P}(\hat q^s)$ as $k\to\infty$. Then the trajectories $\mathbf{x}_\perp(t,\tau^s,\mathbf{Q}^{\mathbf{k}}_*)$ correspond to G.S. with f.d. for (1.4) with f of type (1.3) where $q^u=\hat q^u$ and $q^s=\hat q^s$. Furthermore, again from Lemma 3.8 and Remark 3.9 we see that finitely many of these intersections persist under small perturbations of the parameters q^u and q^s. Hence finitely many G.S. with f.d. persist, and the part of the Theorem concerning the case in which a G.S. with s.d. exists is proved.

Now suppose that we have a S.G.S. with s.d. Then we can simply repeat the proof but reversing the role of $S(s)$ and $U(s)$. I.e. $S(s)$ is the locally Lipschitz parametrization of $W^u_{\hat q^u}(\tau^u)$ which rotates infinitely many times around $\mathbf{P}(\hat q^u)$, where τ^u is the value fixed just before the proof of Theorem 1.10.

We denote by $v(r)$ the unique S.G.S. with f.d., and by T^0 the values such that $v(\exp(T^0))=1$. So we have $\mathbf{x}_\top(t,\tau^u;\mathbf{P}(\hat q^u))\equiv\mathbf{P}(\hat q^u)$ for $t\le T^0$ and $\mathbf{x}_\top(t,\tau^u;\mathbf{P}(\hat q^u))\in W^s_{\hat q^u}(t)$ for any $t\in\mathbb{R}$, since $v(r)$ has fast decay. Therefore $W^s_{\hat q^u}(\tau)$ contains a 1-dimensional path, say $\hat W^s_{\hat q^u}(\tau)$ that joins the origin and $\mathbf{P}(\hat q^u)$ with no self-intersections, for any $\tau\ge T^0$. So we can parameterize $\hat W^s_{\hat q^u}(\tau)$ as the locally Lipschitz family of curves $U(s,\tau):=\mathbf{P}(\hat q^u)+R(s,\tau)e^{iw(s,\tau)}$. Reasoning as above we see that that $\limsup_{s\to0^+}w(s,\tau^u)$ and $\liminf_{s\to0^+}w(s,\tau^u)$ are bounded, $w(+\infty,\tau^u)=\theta(+\infty)$ are bounded, while $\theta(0)=+\infty$.

So we can use again Lemma 3.8 and Remark 3.9, to conclude the existence of infinitely many points $\mathbf{Q}^{\mathbf{k}}_*\in\big(\hat W^s_{\hat q^u}(\tau^u)\cap W^u_{\hat q^u}(\tau^u)\big)$, $\mathbf{Q}^{\mathbf{k}}_*\to\mathbf{P}(\hat q^u)$ as $k\to\infty$. Then the trajectories $\mathbf{x}_\top(t,\tau^u,\mathbf{Q}^{\mathbf{k}}_*)$ correspond to G.S. with f.d. for (1.4) with f of type (1.3) where $q^u=\hat q^u$ and $q^s=\hat q^s$. Using again Lemma 3.8 and Remark 3.9 and reasoning as above we also get the persistence of finitely many G.S. with f.d. under small perturbations. □

Reasoning as at the end of section 3 we can reprove Proposition 3.12 for (1.4) and f of type (1.3). In fact the only difference is that the value ρ_1 is positive but not necessarily larger than 1. Also in this context the Proposition follows from the previous construction and Remark 2.5. Observe in fact that if $u(d,r)$ is a regular solution of (1.4) and $d\le1$, then it solves also (1.4) with f of type (1.5) and $q^s<2^*$. Hence it is a crossing solution and its first zero $R(d)$ is such that $R(d)\to+\infty$ as $d\to0$, see Remark 2.5. If $d\ge1$, then the corresponding trajectory $\mathbf{x}_\top(t)$ of (4.1) is such that $\mathbf{x}_\top(t)\in M^u_{q^u}$ until $u(d,\rho(d))=1$. Let us denote by $\tau_0(d)=\ln(\rho(d))$; obviously $\rho(1)=0$, but $R(1)-\rho(1)=R(1)>0$. Hence $R(d)$ is uniformly positive for d close to 1.

Denote by $R(+\infty)\le+\infty$ the first zero of the unique singular solution of (1.4)

with f of type (1.3). From the continuity of the flow of (4.1) and Remark 2.5 it follows easily that $R(d)$ is continuous and $R(d) \to R(+\infty)$ as $d \to +\infty$. Assume for contradiction that $\inf_{d>0} R(d) = 0$, then there is $D \in (1, +\infty)$ such that $R(D) = 0$, a contradiction. Thus $\inf_{d>0} R(d) = \rho_0 > 0$. So we have the following.

4.7. **Proposition.** *Proposition 3.12 holds for (1.4) with f of type (1.3), too.*

Analyzing the proof of Corollary 1.14 we see that there is a sequence $\mathbf{Q}^{\mathbf{j}}_* \in W^u(\tau^s) \cap W^s(\tau^s)$ such that $\mathbf{Q}^{\mathbf{j}}_* \to \mathbf{P}(\hat{q}^s)$. Exploiting this fact and using Remark 2.5 we obtain the analogous of Proposition 3.13.

4.8. **Proposition.** *Consider (1.4) with f of type (1.3) and assume $q^s \in (\sigma_*, 2^*)$ while $q^u = r^j(q^s) > 2^*$, so that there is a G.S. with s.d. $u(\bar{d}, r)$, see Theorem 1.13 and Corollary 1.14. Then there is a sequence $d_j \to \bar{d}$ such that $u(d_j, r)$ is a G.S. with f.d.*

Analogously assume that $q^u \in (2^*, \sigma^*)$ and $\delta^s = r^j(q^u)$, so that there is a S.G.S. with f.d., see Theorem 1.8 and Corollary 1.14. Then there is a sequence $d_j \to +\infty$ such that $u(d_j, r)$ is a G.S. with f.d.

Acknowledgment
The author wish to thank prof. R. Johnson for having introduced him to the subject.

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