Advanced Nonlinear Studies **4** (2004), xxx–xxx

Ground States and Singular Ground States for Quasilinear Partial Differential Equations with Critical Exponent in the Perturbative Case

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27 November 2003
Communicated by Kenneth Palmer

Abstract

We study the structure of the family of radially symmetric ground states and singular ground states for certain elliptic partial differential equations with p-Laplacian. We use methods of Dynamical systems such as Melnikov functions, invariant manifolds, and exponential dichotomy.

1991 Mathematics Subject Classification. 35J60, 34C35, 34C37
Key words. p-Laplacean, critical exponent, ground state, Fowler transform, singular perturbations

*Research supported by the Doctoral Program in Mathematics at the Università di Firenze and the G.N.A.M.P.A.
†Research supported by the M.I.U.R. and the G.N.A.M.P.A.

1 Introduction

In this paper we study the properties of positive radial solutions of the following equation:

$$\Delta_p u(x) + K(|x|)u(x)|u(x)|^{\sigma-2} = 0 \tag{1.1}$$

where $x \in \mathbb{R}^n$, $n > p > 1$ and $\sigma = \frac{np}{n-p}$, where σ is the so called Sobolev critical exponent, and $K(|x|)$ is a function which is assumed to be $\mathbb{C}^2$. Since we only deal with radial solutions we will in fact consider the following equation:

$$(u'(r)|u'(r)|^{p-2})' + \frac{n-1}{r}u'(r)|u'(r)|^{p-2} + K(r)u(r)|u(r)|^{\sigma-2} = 0 \tag{1.2}$$

where $r = |x|$.

We will call "regular" the solutions $u(r)$ of (1.2) satisfying the following initial condition

$$u(0) = u_0 > 0 \qquad\qquad u'(0) = 0$$

and "singular" the positive solutions $v(r)$ of (1.2) such that

$$\lim_{r \to 0} v(r) = +\infty.$$

We are mainly interested in the existence of ground states (G.S.), singular ground states (S.G.S.), and crossing solutions. A G. S. is a solution $u(r)$ of (1.2) defined and positive for $r \geq 0$ and such that $\lim_{r\to\infty} u(r) = 0$. A S.G.S. $v(r)$ is a solution defined and positive for $r > 0$ which satisfies

$$\lim_{r \to 0} v(r) = +\infty \qquad \text{and} \qquad \lim_{r \to \infty} v(r) = 0.$$

A crossing solution is a regular solution $u(r)$ of (1.2) such that there exists $R > 0$ with the property that $u(r) > 0$ for any $0 \leq r < R$ and $u(R) = 0$, so it can also be regarded as a solution of the Dirichlet problem in the ball of radius R. This equation has been studied by many authors, especially when $p = 2$. Let us observe that, even though only positive solutions are of interest, it will be convenient to consider solutions which a priori may have negative values; hence we write $u|u|^{\sigma-2}$ instead of $u^{\sigma-1}$ in (1.1).

We will use the following terminology: we write $v(r) \sim r^{-d}$ as $r \to c$ to mean that both $\limsup_{r\to c} v(r)r^d$ and $\liminf_{r\to c} v(r)r^d$ are positive and finite. Throughout all the paper c is a generic positive constant whose value may change from line to line. We recall that, if certain generic hypotheses are satisfied, positive solutions $u(r)$ can only have two asymptotic behaviors as $r \to 0$: regular that is $u(0) = u_0 > 0$, or singular that is $u(r) \sim r^{-\frac{n-p}{p}}$. Analogously only two asymptotic behaviors are possible for positive solutions as $r \to \infty$: fast decay that is $u(r) \sim r^{-\frac{n-p}{p-1}}$ and slow decay that is $u(r) \sim r^{-\frac{n-p}{p}}$. This result is already known, see [12], but we give a new proof in Proposition 2.2. If $K(r)$ oscillates indefinitely the estimates become less precise, see Proposition 2.5, but we will use the same terminology.

Kawano, Ni and Yotsutani in [12] have given a structure result for positive radial solutions, depending on the sign of the following function:

$$G(r) = \int_0^r K'(s)s^n ds.$$

The results have been refined in [6] where one finds a complete classification of all positive solutions, depending also on the sign of

$$J(r) = \int_r^\infty K'(s)s^n ds.$$

However an additional technical assumption on $\lim_{r\to 0} K'(r)$ and $\lim_{r\to\infty} K'(r)$ is needed. In particular it is proved that if $G(r) > 0$ for any $r > 0$, then each regular solution $u(r)$ is a crossing solution, while if $G(r) < 0$ then for any $r > 0$ each $u(r)$ is a G.S. with fast decay. Moreover, if $\frac{2n}{n+2} \le p \le 2$, we have that if $K(r)$ is monotone increasing, there exists at least one S.G.S. with slow decay and infinitely many S.G.S. with fast decay, while if $K(r)$ is monotone decreasing there exists at least one S.G.S. with slow decay. In both cases, there are no other solutions which are positive for r small. Moreover, Kawano, Yanagida and Yotsutani have proved in [13] that, if there exists $R > 0$ such that $G(r) > 0$ for any $0 < r < R$ and $K'(r) \le 0$, for $r > R$, then positive regular solutions must have one of the following structures:

- they are all G.S. with slow decay (this holds if $G(r) \le 0$ for any $r > 0$)
- they are all crossing solutions (this holds if $G(r) \ge 0$ for any $r > 0$)
- there exist infinitely many G.S. with slow decay, infinitely many crossing solutions and exactly one G.S. with fast decay separating the other two families of solutions.

One of the main purposes of this paper is to extend to the p-Laplacean the results proved for the Laplacean in [10], [1], [2], [3]. In particular we assume that $K(r)$ is a perturbation of a positive constant; namely, if $\epsilon > 0$ is a small constant, we assume

1 $K(r) = 1 + \epsilon k(r)$, where $k(r)$ is a bounded function.

2 $K(r) = k(r^\epsilon)$, where $k(r)$ is a bounded function, positive in some interval.

Note that if we are dealing with a function of the second type, we are allowing $K(r)$ to have a wide range of variation; in fact it can take negative values (but must vary slowly).

We will use techniques taken from dynamical systems theory, concerning in particular invariant manifolds and Melnikov functions. The first step in our analysis is to introduce the following Fowler inversion, taken from [6]:

$$x_1 = u(r)r^\alpha \qquad x_2 = u'(r)|u'(r)|^{p-2}r^\beta \qquad r = e^t \qquad \Phi(t) = K(e^t) \tag{1.3}$$
$$\text{where} \quad \alpha = \frac{n-p}{p} \quad \text{and} \quad \beta = \frac{n(p-1)}{p}$$

which transforms equation (1.2) in the following dynamical system:

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} \alpha & 0 \\ 0 & -\alpha \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} x_2|x_2|^{\frac{2-p}{p-1}} \\ -\Phi(t)x_1|x_1|^{\sigma-2} \end{pmatrix}. \tag{1.4}$$

Here and below we write " $\cdot$ " for $\frac{d}{dt}$ and " $'$ " for $\frac{d}{dr}$. Note that the right hand side of equation (1.2) is $\mathbb{C}^1$ if and only if $\frac{2n}{2+n} \leq p \leq 2$. In fact if this hypothesis is not satisfied the system is only Hoelder continuous on the coordinate axes, so that local uniqueness of the solutions is not a priori ensured; thus this hypothesis will be in force throughout the whole paper.

We will also maintain the following hypothesis without any further comment:

Hypothesis H
There exists $M > 0$ such that

$$\limsup_{r \to 0} |K'(r)r| \leq M \qquad \limsup_{r \to \infty} |K'(r)r| \leq M.$$

Note that if $K(r)$ is Lipschitz continuous, strictly positive and bounded, and monotone as $r \to \infty$ the preceding hypothesis is always satisfied. Hypothesis H gives a sufficient condition for the uniform continuity of $\Phi(t)$, which is the condition which is really needed. In any case note that, if we are assuming that $K(r)$ is continuous for $r = 0$, we can drop the assumption in Hypothesis H concerning the behaviour as $r \to 0$. In fact, in that case $K(r)$ is uniformly continuous and this implies the uniform continuity of $\Phi(t)$ for $t \to -\infty$.

We need to assume that $\Phi(t)$ is uniformly continuous, in order to apply invariant manifold theory for non-autonomous systems. In fact we will show that the unstable manifold, departing from the origin of (1.4), is made up of all and only the trajectories corresponding to regular solutions $u(r)$ of (1.2), while the stable manifold is made up of trajectories corresponding to solutions $u(r)$ with fast decay. Using Melnikov theory, we will prove that, for each non degenerate positive critical point of a singularly perturbed potential $\Phi(t)$, we have a G.S. with fast decay, corresponding to a crossing of stable and unstable manifolds, Theorem 3.2.

Moreover, assuming that $\Phi(t)$ is periodic, we will find that system (1.2) exhibits chaotic behavior. In particular we will prove the existence of a Cantor like set of solutions (a dense subset of which are periodic) corresponding to a set of S.G.S. with slow decay, Theorem 3.6 and Theorem 3.8. An analogous statement holds for regularly perturbed potentials, but the sufficient condition is a bit more complicated. The proof of this claim follows closely the analogous reasoning developed in [10] for the Laplacean and exploits the framework developed in [1] and [2]. In particular it is worthwhile to note that the theorems regarding Melnikov theory hold, even if we are dealing with a singularly perturbed potential $K(r)$ which changes sign.

We also give results which are of a new type. We need to introduce the following notation: we say that $f(r)$ is oscillatory as $r \to c$, if it has infinitely many local maxima and minima in a neighborhood of $r = c$. Assume that $K(r)$ is bounded and that it is a perturbation of a constant as already specified. We will consider the following hypotheses:

Hypotheses

$\mathbf{M}_1$ there exists $\rho > 0$ such that $K(\rho) > 0$ is a non degenerate maximum and $K(r)$ is strictly positive and monotone increasing for $0 \le r \le \rho$.

$\mathbf{M}_2$ there exists $R > 0$ such that $K(R) > 0$ is a non degenerate maximum and $K(r) > 0$ is strictly positive and monotone decreasing for $r \ge R$.

$\mathbf{O}_1$ $K(r)$ is oscillatory as $r \to 0$ and admits infinitely many positive non degenerate maxima.

$\mathbf{O}_2$ $K(r)$ is oscillatory as $r \to \infty$ and admits infinitely many positive non degenerate maxima.

We will give a structure result for positive solutions. It is convenient to distinguish the following situations for equation (1.2):

A
- There exist uncountably many monotone decreasing G.S. with slow decay.
- There exist uncountably many crossing solutions.
- There exist uncountably many solutions $u(r)$ of the Dirichlet problem in exterior domains; that is, there exists $R > 0$ such that $u(R) = 0$, $u(r) > 0$ for any $r > R$ and $u(r) = O(r^{-\frac{n-p}{p-1}})$, as $r \to \infty$, that is $u(r)$ has fast decay.
- There exists a non empty set of monotone decreasing G.S. with fast decay disconnecting the other two sets in a sense that will be made precise later (see the proof of Theorem 4.1).

B
- There exist uncountably many crossing solutions.
- There exist uncountably many monotone decreasing S.G.S. with fast decay.
- There exists a non empty set of monotone decreasing G.S. with fast decay disconnecting the last two sets.

C
- All the existence results at the points A and B are valid.
- There exist infinitely many monotone decreasing S.G.S. with slow decay; no other solutions $u(r)$, well defined and positive for all $r > 0$, can exist.

In Theorem 4.1 and in the Corollaries 4.2 and 4.3 we will prove that, if either hypothesis M_1 or O_1 is satisfied, then positive solutions have a structure of type B. If either hypothesis M_2 or O_2 is satisfied, then positive solutions have a structure of type A. Moreover if M_1 or O_1 holds and M_2 or O_2 holds then positive solutions have a structure of type C. Note that the results do not apply to functions $K(r) = k(r^\epsilon)$ where $K(r)$ is periodic, because in such a case Φ is not uniformly continuous. However they apply to functions Φ which are uniformly continuous and oscillate indefinitely, even if they have no recurrence properties. We also have a result concerning solutions of the Dirichlet problem in the interior and in the exterior of a ball, see Theorem 4.4.

We can prove an analogous statement for regularly perturbed potentials $K(r)$, i. e. for the so called Kazdan-Warner problem, if the sufficient condition for the existence of G.S. with fast decay is satisfied. Since the proofs are rather similar to the ones given for the singular perturbed problem we will just sketch them.

The paper is organized as follows. In section 2 we introduce the Fowler transform for the Laplacean and recall some known results concerning the autonomous case. In section 3 we extend to the p-Laplacean the results obtained in [10], [1], [2], [3] for the Laplacean in the singularly perturbed case. In section 4 we give a geometric construction which enables us to prove Theorems 4.1 and 4.4, and Corollaries 4.2 and 4.3 which contain results which are new even in the case $p = 2$. In section 5 we consider the regularly perturbed case obtaining results analogous to those explained in sections 3 and 4.

2 Preliminaries

We begin by recalling some known results about the asymptotic behavior of the solutions and about the autonomous case. Recall that given a system of the form

$$\dot{x} = f(x,t)$$

and a solution $x(t)$, the α-limit set of $x(t)$ is the set

$$A = \big\{ P \mid \exists t_n \to -\infty \quad \text{such that} \lim_{n\to\infty} x(t_n) = P \big\},$$

while the ω-limit set is the set

$$W = \big\{ P \mid \exists t_n \to +\infty \quad \text{such that} \lim_{n\to\infty} x(t_n) = P \big\}.$$

One can show that, if $x(t)$ is bounded on $\mathbb{R}$, then these sets are compact. Moreover, if the system is autonomous, these sets are invariant for the flow generated by the system. If the system is non-autonomous they are no longer invariant; however we will see that they are still useful for our present purposes.

Proposition 2.1 *Consider system (1.4) and assume that $\Phi(t)$ is bounded. Then a solution $u(r)$ of (1.2) is regular as $r \to 0$ if and only if the corresponding trajectory of system (1.4) has the origin as α-limit point.*
Moreover $u(r) \sim r^{-\frac{n-p}{p-1}}$ as $r \to \infty$ if and only if the corresponding trajectory of system (1.4) has the origin as ω-limit point.

Proof. In both the cases one implication is obvious, the other is a consequence of Lemma 2.10 and Observation 3.17 in [6]. □

In [6] the following statement is also proved.

Proposition 2.2 *Assume that $K(r)$ is strictly positive and bounded; moreover assume that $K(r)$ is monotone for $r \in (0,R)$, for some $R > 0$; then we can have one of the following asymptotic behaviors as $r \to 0$:*

$$0 < u(0) < \infty \text{ (regular)} \qquad \text{or} \qquad u(r) \sim r^{-\frac{n-p}{p}} \text{ (singular).}$$

Analogously assume that $K(r)$ is monotone in $r \in (R, \infty)$ for some $R > 0$. Then positive solutions can only have the following asymptotic behaviors as $r \to \infty$:

$$u(r) \sim r^{-\frac{n-p}{p-1}} \quad \textit{(fast decay)} \qquad \textit{or} \qquad u(r) \sim r^{-\frac{n-p}{p}} \quad \textit{(slow decay).}$$

In the former case, both as $r \to 0$ and as $r \to \infty$, the corresponding trajectories converge to the origin, respectively as $t \to -\infty$ and as $t \to \infty$, while in the latter the corresponding trajectories are bounded and bounded away from the x_1 and x_2 axes.

Now we recall some known results about the autonomous equation and give their interpretation in terms of the dynamical system (1.4).

Proposition 2.3 *Consider system (1.4) when $K \equiv const > 0$.*

- *All the regular solutions $u(r)$ of equation (1.2) are monotone decreasing G.S. with fast decay, and they correspond to a unique homoclinic trajectory of (1.4), which is contained in the closed 4^{th} quadrant.*
  Any trajectory of (1.4) which is not homoclinic to $(0,0)$ and which is not an equilibrium point, is defined by a periodic solution of (1.4) with positive period.
- *There exist infinitely many monotone decreasing S.G.S. with slow decay $v(r)$ of equation (1.2), corresponding to periodic trajectories, contained in the open 4^{th} quadrant.*
- *There exist infinitely many oscillating solutions $v(r)$ of equation (1.2), which have infinitely many positive maxima and negative minima both in a neighborhood of $r = 0$ and of $r = \infty$. They correspond to periodic trajectories of (1.4) which cross the x_1 and x_2 axes.*

Now we give some notations that will be used throughout the paper; the quantity $\dot{x}_1$ is defined by system (1.4):

$$A^+ := \{x \in \mathbb{R}^2 \quad | \quad \dot{x}_1 > 0\} \quad \text{and} \quad A^- := \{x \in \mathbb{R}^2 \quad | \quad \dot{x}_1 < 0\}$$
$$L := \{x \in \mathbb{R}^2 \quad | \quad \dot{x}_1 = 0\}$$

Note that we know the exact expression of the homoclinic trajectories $U(t)$, see for example [7].

$$U(t) = \left(\frac{K^{-\frac{\alpha}{p}}}{(e^{-t} + De^{\frac{t}{p-1}})^{-\alpha}}, \frac{-(2\alpha e^{-t})^{p-1}}{\left|K(e^{-t} + De^{\frac{t}{p-1}})\right|^{\beta}} \right) \tag{2.1}$$

where $D = (p-1)(n-p)n^{\frac{1}{p-1}}$ is a positive constant. Observe that the autonomous system is invariant for translations in t. Therefore $U(\tau, t) = U(t + \tau)$ is still a homoclinic trajectory. To have consistent notation, we set $U(0, t) = U(t)$; note that $U(0, 0) \in L$. Note that the homoclinic trajectories $U(\tau, \cdot)$ all have the same graph and that $U(\tau, t) = U(0, t + \tau)$.

Lemma 2.4 *Consider equation (1.2) and the corresponding system (1.4), where $K(r)$ is strictly positive and bounded. Then, if $x(t)$ is unbounded, it rotates clockwise crossing infinitely many times the x_1 and x_2 axes.*

Proof. By assumption there exists $N > 0$ such that $\frac{1}{N} < K(r) < N$ for any r. Consider a trajectory $x(t)$ which becomes unbounded as $t \to c$ where c can also be ∞. We will assume that $x(t)$ is well defined for $t < c$: the proof in the case of a trajectory that becomes unbounded going backwards in t is analogous.

Fix t_0 and the corresponding point $P = x(t_0)$ in $\mathbb{R}^2$. Assume that $P \in A^+$. Consider at first system (1.4), where $\Phi(t) \equiv N$. Recall that the solutions of (1.4) which are not homoclinic to $(0,0)$ and which do not coincide with equilibria are periodic, hence the corresponding trajectories define closed curves in $\mathbb{R}^2$. We choose a periodic trajectory $\hat{x}(t)$ of (1.4) which crosses the coordinate axes and such that P lies in the exterior of the disc $\hat{D}$ enclosed by $\hat{x}(\cdot)$. Such a choice is always possible, since we can choose $|P|$ as large as we wish, since we are assuming that $x(t)$ is unbounded. In a similar way, consider system (1.4), where $\Phi(t) \equiv \frac{1}{N}$. We choose a periodic solution $\breve{x}(t)$ of (1.4) which crosses the coordinate axes and such that P lies in the open disc $\breve{D}$ enclosed by $\breve{x}(\cdot)$. We can choose $\breve{D}$ and $\hat{D}$ in such a way that $P \in \breve{D} \supset \hat{D}$. Let $\partial\breve{D}$ and $\partial\hat{D}$ denote the boundary of $\breve{D}$ respectively $\hat{D}$. Let us denote with $R^+ := \breve{D} - \hat{D}$ and with ∂R^+ its boundary.

We return to the non-autonomous system (1.4). Note that the flow on $\partial R^+ \cap A^+$ is always going towards the interior of R^+ and that $P \in R^+ \cap A^+$. Consider now the unbounded trajectory $x(t)$. Note that it lies in $R^+ \cap A^+$ for $t \geq t_0$, until it crosses the isocline L in a point P_1. Thus there exists $t_1 > t_0$ such that $x(t_1) = P_1 \in c$ and $x(t)$ enters A^- for $t \geq t_1$.

Once again we consider the autonomous system where $\Phi(t) \equiv \frac{1}{N}$. We choose a periodic solution $\breve{x}_1(t)$ of (1.4) which crosses the coordinate axes and such that P_1 lies in the exterior of the open disc $\breve{D}_1$ enclosed by $\breve{x}(\cdot)$. In a similar way, we choose a periodic solution $\hat{x}(t)$ of (1.4) where $\Phi(t) \equiv N$ which crosses the coordinate axes, and such that P_1 lies in the open disc $\hat{D}_1$ enclosed by $\hat{x}(\cdot)$. We choose $\breve{D}_1$ and $\hat{D}_1$ in such a way that $\breve{D}_1 \subset \hat{D}_1$ and define $R_1^- := \breve{D}_1 - \hat{D}_1$. Now we return again to the non-autonomous system (1.4). Observe that $x(t) \in R_1^-$, for all $t > t_1$ such that $x(t) \in A^-$. Recalling that $x(t)$ is unbounded we conclude that there exists $t_2 > t_1$ such that $x(t_2) \in L$. Therefore $x(t)$ rotates clockwise crossing the x_2 and x_1 negative semi-axes, then it enters A^+ for $t > t_2$.

Iterating the reasoning we obtain that $x(t)$ must cross the coordinate axes infinitely many times. □

Now, putting together Proposition 2.1 and Lemma 2.4, we can give a result concerning the asymptotic behaviour of positive solutions, removing the monotonicity hypotheses. However the estimates are not as sharp as in Proposition 2.2.

Proposition 2.5 *Assume that $K(r)$ is strictly positive and bounded, then a positive solution can only have one of the following asymptotic behaviors as $r \to 0$*

$$0 < u(0) < \infty \qquad \textit{(regular behavior)} \qquad \textit{or}$$

$$c_1 \leq u(r) \leq c_2 r^{-\frac{n-p}{p}} \qquad \textit{(singular behavior).}$$

Analogously a positive solution can only have one of the following asymptotic behaviors as $r \to \infty$:

$$\begin{aligned} u(r) &\sim r^{-\frac{n-p}{p-1}} && \text{(fast decay)} \quad \text{or} \\ c_1 r^{-\frac{n-p}{p-1}} \le u(r) &\le c_2 r^{-\frac{n-p}{p}} && \text{(slow decay).} \end{aligned}$$

Here c_1, c_2 are positive constants.

Remark 2.6 We will use the terminology "singular solution" and "solution with slow decay" with different meaning according to the fact that $K(r)$ is monotone or oscillates indefinitely for r small and r large, see Propositions 2.2 and 2.5. The only exception is Theorem 3.6 in which we manage to prove the existence of S.G.S. with slow decay satisfying the estimates given in Proposition 2.2, even if $K(r)$ is oscillatory.

3 Singularly perturbed systems and Melnikov functions

Now we review some facts about invariant manifold theory for non-autonomous systems. We refer to [10] and [9] for a discussion of this topic in a general framework. We will combine elements of this theory with the use of Melnikov functions. Since we will follow rather closely the reasoning developed in [10] and [1], we will just suggest the main ideas. Moreover we will consider only the singular perturbation problem, since the abstract framework of the regular one is very similar; see [10] and [1] for details.

Consider a system of the form

$$\dot{x} = f(x, \tau + \epsilon t) \tag{3.1}$$

where $x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in \mathbb{R}^2$. Assume that $f(x, \tau + \epsilon t) = Ax + F(x, \tau + \epsilon t)$ where A is a 2×2 real matrix with eigenvalues $\lambda_- < 0 < \lambda_+$ and F is of class $\mathbb{C}^1$ on $\mathbb{R}^2 \times \mathbb{R}$ with $F(0, \tau) = F_x(0, \tau) = 0$. Furthermore assume that both the functions $F(x, \cdot)$ and $F_x(x, \cdot)$ are uniformly continuous in $Q \times \mathbb{R}$ for all compact subsets $Q \subset \mathbb{R}^2$. Moreover assume that the frozen equation

$$\dot{x} = f(x, \tau)$$

admits a homoclinic solution $U(\tau, t)$ for any τ.

It is now useful to introduce the extended system

$$\dot{x} = f(x, \tau), \qquad \dot{\tau} = \epsilon, \qquad \dot{\epsilon} = 0. \tag{3.2}$$

Denote by E a neighborhood of $\epsilon = 0$ in $\mathbb{R}$. We denote by $V^{\pm} \subset \mathbb{R}^2$ respectively the eigenspaces corresponding to the positive and the negative eigenvalues of A. Let

$V_0^\pm$ be neighborhoods of the origin in $V^\pm$, then if E and $V_0^\pm$ are sufficiently small, the following result holds, see [9] and [10].

There are $\mathbb{C}^1$ maps $\psi^\pm : \mathbb{R} \times E \times V_0^\mp \to V_0^\pm$ such that the submanifolds

$$N^\pm = \{(\tau, \epsilon, \rho + \psi^\pm(\tau, \epsilon, \rho)) : \tau \in \mathbb{R}, \epsilon \in E, \rho \in V_0^\mp\} \subset \mathbb{R} \times E \times \mathbb{R}^2$$

are locally invariant under the flow defined by (3.2). Note that N^+ and N^- are respectively the local center-stable and center-unstable manifolds of (3.2). Let $P = P_{\epsilon,\tau}$ be the plane in the four dimensional (τ, ϵ, x)-space obtained by fixing the values of τ and ϵ. We can now define the local stable and unstable leaves $W^{u,s}_{\epsilon,loc}(\tau) = P_{\epsilon,\tau} \bigcap N^\pm$.

These leaves are of class $\mathbb{C}^1$ and vary continuously with respect to τ and ϵ, in the $\mathbb{C}^1$ topology. Moreover, by transversality, they are one-dimensional and tangent in the origin to $N^\pm$. In the case we are considering this means that, for ϵ small and any τ, the manifolds $W^{u,s}_{\epsilon,loc}(\tau)$, in the origin, are tangent respectively to the negative x_2 semiaxis and to the positive x_1 semiaxis. Moreover it is important to remark that the contribution of the non-linear and non-autonomous terms deflects these manifolds towards the interior of the 4^{th} quadrant, whenever $K(\tau)$ is positive. We will usually commit an abuse of notation, calling stable and unstable manifolds these connected components $W^u_\epsilon(\tau)$ and $W^s_\epsilon(\tau)$. Since we are mainly interested in positive solutions $u(r)$ of (1.2), we will consider only trajectories $x(t)$ such that $x_1(t) > 0$. Thus we will restrict our attention to $\mathbb{R}^2_+$, that is the semiplane of $\mathbb{R}^2$ where $x_1 \geq 0$.

Now we use the semiflow defined by (3.2) to give a characterization of the global stable and unstable leaves $W^{u,s}_\epsilon(\tau) \subset \mathbb{R}^2$. Namely, if we call $z(t, z_0; \tau, \epsilon)$ the solution of (3.1) with $z(0, z_0; \tau, \epsilon) = z_0$, we have

$$W^u_\epsilon(\tau) = \bigcup \{z(t, z_0; \tau - \epsilon t, \epsilon) : z_0 \in W^u_{\epsilon,loc}(\tau - \epsilon t)\}$$

where we have left unsaid that t is in the domain of existence of z. An analogous characterization holds for $W^s_\epsilon(\tau)$, see [10], pp. 1063-1065, for more details. Note that if $\epsilon = 0$ the set $\{U(\tau, t) \quad | \quad t \in \mathbb{R}\}$ defined by the homoclinic orbit is a subset of both $W^u_0(\tau)$ and $W^s_0(\tau)$; to be more precise it is the connected component belonging to $\mathbb{R}^2_+$.

Now let $\tau_0 \in \mathbb{R}$ and let $L \subset \mathbb{R}^2$ be a line segment which contains $U(\tau_0, 0)$ in its interior and which is not parallel to $f(U(\tau_0, 0), \tau_0)$. Such a segment is called a transversal. Note that, perhaps reparametrizing the homoclinic orbits $U(\tau, t)$ we can assume that $U(\tau, 0) \in L$, for τ near τ_0. In our problem we will use as a transversal the curvilinear segment defined by the isocline $\dot{x}_1 = 0$.

Exploiting the fact that $W^{s,u}_\epsilon(\tau)$ is $\mathbb{C}^1$ close to $W^{s,u}_0(\tau)$ for ϵ small, we can determine "initial" branches $\tilde{W}^{s,u}_\epsilon(\tau)$ of the stable and unstable leaves which depart from the origin and cross transversally the isocline. We will call these first points of intersection respectively $\xi^-(\tau, \epsilon)$ and $\xi^+(\tau, \epsilon)$. By transversality we know that $\xi^\pm(\tau, \epsilon)$ are $\mathbb{C}^1$ functions of (τ, ϵ); we will prove in Lemma 3.4 that they actually have the same regularity as $K(r)$, so we can assume that they are $\mathbb{C}^2$.

We now define and study a Melnikov function for equation (3.1). This function allows us to measure the distance between $\xi^+(\tau,\epsilon)$ and $\xi^-(\tau,\epsilon)$. Thus it will give us a condition for the crossing of the stable and unstable leaves $\tilde{W}^{u,s}_\epsilon(\tau)$, which in fact means a sufficient condition for the existence of a homoclinic solution for the system. Such a trajectory corresponds to a G.S. with fast decay of equation (1.2), see Proposition 2.1.

Define now

$$M(\tau) = \frac{d}{d\epsilon}\left[\xi^-(\tau,\epsilon) - \xi^+(\tau,\epsilon)\right]\lfloor_{\epsilon=0} \wedge f(U(\tau,0),\tau)$$

where "$\wedge$" denotes the standard wedge product in $\mathbb{R}^2$. Then define

$$h(\tau,\epsilon) = \begin{cases} M(\tau) & \text{for} \quad \epsilon = 0 \\ \frac{\xi^-(\tau,\epsilon)-\xi^+(\tau,\epsilon)}{\epsilon} \wedge f(U(\tau,0),\tau) & \text{for} \quad \epsilon \neq 0. \end{cases}$$

We point out that the vector $\xi^-(\tau,\epsilon) - \xi^+(\tau,\epsilon)$ belongs to the transversal segment L, so we have

$$h(\tau,\epsilon) = 0 \qquad \Longleftrightarrow \qquad \xi^-(\tau,\epsilon) - \xi^+(\tau,\epsilon) = 0 \qquad \text{for} \quad \epsilon \neq 0.$$

Lemma 3.1 *Suppose $M(\tau_0) = 0$ and $M'(\tau_0) \neq 0$, then there exists a $\mathbb{C}^1$ function $\epsilon \to \tau(\epsilon)$ defined on a neighborhood of $\epsilon = 0$, such that $\tau(0) = \tau_0$, for which we have $\xi^-(\tau(\epsilon),\epsilon) = \xi^+(\tau(\epsilon),\epsilon)$. Therefore we have a homoclinic solution of the system (1.4).*

Proof. To prove the proposition it is enough to apply the implicit function theorem to $h(\tau,\epsilon)$ for (τ,ϵ) near $(\tau_0, 0)$. □

Now we want to compute explicitly the functions $M(\tau)$ and $M'(\tau)$ for our system, in order to give a simple sufficient condition for the existence of the homoclinic trajectory. The first step is to recall Theorem 3.2 of [10], page 1054, which gives the following formula for the Melnikov function:

$$M(\tau) = \int_{-\infty}^{+\infty} e^{-\int_0^t tr f_x(U(\tau,\sigma),\tau)d\sigma} t f_\tau(U(\tau,t),\tau) \wedge f(U(\tau,t),\tau)dt. \tag{3.3}$$

Now we consider our singularly perturbed problem $K(r) = k(r^\epsilon)$ and we introduce the dynamical system exploiting the Fowler transform:

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} \alpha & 0 \\ 0 & -\alpha \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} x_2|x_2|^{\frac{2-p}{p-1}} \\ -\phi(\tau+\epsilon t)x_1|x_1|^{\sigma-2} \end{pmatrix}. \tag{3.4}$$

Here $\phi(\epsilon t) = k(e^{\epsilon t}) = \Phi(t) = K(e^t)$ and τ is a translation parameter. Now, if we apply it to our system, recalling that $\phi(\tau) = K(e^\tau)$, we can rewrite (3.3) in the following way:

$$M(\tau) = \int_{-\infty}^{+\infty} t \begin{pmatrix} 0 \\ -\phi'(\tau)X_{\phi(\tau)}|X_{\phi(\tau)}|^{\sigma-2} \end{pmatrix} \wedge \begin{pmatrix} \dot{X}_{\phi(\tau)} \\ \dot{Y}_{\phi(\tau)} \end{pmatrix} dt$$

where we have denoted

$$U(\tau,t)=(X_{\phi(\tau)}(t),Y_{\phi(\tau)}(t)) \quad \text{and} \quad \frac{dU(\tau,t)}{dt}=(\dot{X}_{\phi(\tau)}(t),\dot{Y}_{\phi(\tau)}(t)),$$

so $M(\tau)$ is in fact a computable function. We observe now that

$$X_{\phi(\tau)}(t)=\phi(\tau)^{-\frac{\alpha}{p}}X_1(t)=\left(e^{-t}+De^{\frac{t}{p-1}}\right)^{-\alpha}\phi(\tau)^{-\frac{\alpha}{p}},$$

where X_1 is the homoclinic solution of the frozen equation (3.4) with $\phi(t)\equiv 1$. Therefore

$$M(\tau)=-\phi'(\tau)\phi(\tau)^{-\frac{n}{p}}\int_{-\infty}^{+\infty} tX_1|X_1|^{\sigma-2}\dot{X}_1 dt.$$

Integrating by parts we obtain

$$M(\tau)=-\phi'(\tau)\phi(\tau)^{-\frac{n}{p}}\int_{-\infty}^{+\infty}\frac{|X_1(t)|^{\sigma}}{\sigma}dt=-C\phi'(\tau)\phi(\tau)^{-\frac{n}{p}} \tag{3.5}$$

where $C>0$ is a positive constant deriving from the value of the integral, so it can be explicitly computed. Differentiating with respect to τ, we get

$$M'(\tau)=\left(-\phi''(\tau)\phi(\tau)^{-\frac{n}{p}}-\frac{n}{p}\phi'(\tau)^2\phi(\tau)^{-\frac{n+p}{p}}\right)C. \tag{3.6}$$

Thus, applying Lemma 3.1 to the system we are considering, we obtain the theorem that follows. As in [10], we use the uniform continuity of ϕ to prove that the leaves $W^{u,s}_{\epsilon}(\tau)$ have diameter uniformly bounded above zero; see [10], p. 1063. The Hypothesis H - which is in force throughout the paper - implies that ϕ is uniformly continuous.

Theorem 3.2 *Suppose that there exists $\bar{\tau}$ such that $\phi'(\bar{\tau})=0$ and $\phi''(\bar{\tau})\neq 0$; then there exists a $\mathbb{C}^1$ function $\epsilon\to\tau(\epsilon)$, defined for $|\epsilon|$ small, such that $\tau(0)=\bar{\tau}$ and the system (3.4) admits a homoclinic trajectory for $\tau=\tau(\epsilon)$, corresponding to a G.S. with fast decay of (1.2). Moreover there exists at least one such homoclinic solution for any positive critical point of the function ϕ.*

Remark 3.3 We do not need $\phi(t)$, and hence $K(r)$, to be always positive. We just need that it admit a positive critical point.

This is a rather amazing feature, already observed by Battelli and Johnson in [2]. It may be explained observing that trajectories belonging to $W^s_{\epsilon}(\tau)$ converge to the origin exponentially fast, and the same holds for trajectories belonging to $W^u_{\epsilon}(\tau)$. This fact can easily be observed using invariant manifold theory. Now, recalling that the influence of the potential is due to the term $x^{\sigma}(t)\phi(\tau+\epsilon t)$, we conclude that this contribution becomes exponentially small as $t\to\pm\infty$ for trajectories belonging to $W^{s,u}_{\epsilon}(\tau)$. So, the principal factor of this term is influenced just by the dynamics in compact intervals of t. Thus, assuming that $\phi(t)$ varies slowly, we can find some solutions which are influenced just from the intervals in which $\phi(t)$ is positive. For a more detailed analysis of the phenomenon see [3].

Now we want to prove that $W^u_\epsilon(\tau(\epsilon))$ and $W^s_\epsilon(\tau(\epsilon))$ intersect transversally. In fact, if this is the case, there exists a transversal homoclinic point; therefore, if we assume that $\phi(t)$ is periodic, we can deduce the existence of chaotic behavior, using the classical Smale horseshoe construction. Furthermore, we will prove that, from this fact, we can deduce the existence of a Cantor like set of S.G.S. with slow decay.

Here we follow rather closely the analogous analysis derived for the corresponding problem with the Laplacean in [1], [2], [3]. Now we need the following lemma:

Lemma 3.4 *Assume that the function $\phi(t)$ is $\mathbb{C}^r$ and strictly positive. Then the function $(\tau, \epsilon) \to \xi^{\pm}(\tau, \epsilon)$ is $\mathbb{C}^r$ as well.*

Proof. For this proof we have to exploit the analysis derived in [3], modifying it slightly for our purposes. In that paper one considers a non-autonomous two-dimensional dynamical system that is $\mathbb{C}^r$ apart from the x_2 axis in which it is just $\mathbb{C}^1$. Moreover, it is assumed that the frozen system always admits a homoclinic trajectory $U(t, \tau)$ belonging to a closed subsector of $\mathbb{R}^2_+$. Furthermore, some estimates on the behavior of the functions of the right hand side of the system and their derivatives are needed. Then using the tool of exponential dichotomy and a technique involving the introduction of suitable Banach spaces, the authors prove the higher regularity of $\xi^{\pm}(\tau, \epsilon)$.

The proof is developed in an abstract framework, so we will simply apply it to our problem. We just have to point out some facts: first of all we note that we are working only in the 4^{th} quadrant and that the system is only $\mathbb{C}^1$ in both the axes. Furthermore, to obtain $\xi^{\pm}(\tau, \epsilon)$, we are intersecting the stable and unstable leaves $\tilde{W}^{s,u}_\epsilon(\tau)$ with a curvilinear segment L, but we maintain the essential assumption of the transversality, so the proof still works.

The main difference is the following: in [3] the authors require that the homoclinic orbit of the frozen system is contained in a sector R, so that the trajectory cannot be tangent to the x_2 axis, where we have less regularity. Such an assumption does not hold in our case. However what is really needed in the proof is that the trajectories departing from points of the stable and unstable leaves $W^{s,u}_\epsilon(\tau)$ do not cross these "non regular" sets. But this holds also in our case, therefore we can still apply the theorem. Let us prove this. Note that the leaf $W^s_{\epsilon,loc}(\tau)$ is $\mathbb{C}^1$ close to $W^s_{0,loc}(\tau)$, when we are close to the origin. Assume for contradiction that there exists a trajectory $x(\tau, \epsilon; t) = (x_1(\tau, \epsilon; t), x_2(\tau, \epsilon; t))$ departing from a point in $W^s_{\epsilon,loc}(\tau)$, and such that $\lim_{t\to\infty} x(\tau, \epsilon; t) = (0, 0)$ and $x_1(\tau, \epsilon; 0) = 0$, $x_2(\tau, \epsilon; 0) < 0$. Recalling that on the axes the flow rotates clockwise, we see that $x(\tau, \epsilon; t)$ rotates clockwise through all the quadrants. Now we can argue that there exists $\bar{\tau} > \tau$ such that $W^s_\epsilon(\bar{\tau})$ crosses the negative x_1 semiaxis and then rotates towards the origin. But then it cannot be $\mathbb{C}^1$ close to $W^s_0(\bar{\tau})$. Thus we can drop the hypothesis on the angular distance of the homoclinic orbits from the axes, exploiting the fact that the flow always rotates clockwise on the coordinate axes. □

Now we want to prove that the crossing between the leaves $W^s_\epsilon(\tau(\epsilon))$ and $W^u_\epsilon(\tau(\epsilon))$ is transversal. Here we will follow the main ideas of [1], adapting them to our problem. For the convenience of the reader we will repeat briefly the main steps of the proofs carried out in that article. The basic idea is to introduce the vectors tangent to $W^s_\epsilon(\tau)$ and $W^u_\epsilon(\tau)$ near $\xi^\pm(\tau,\epsilon)$. Then we introduce a new version of the Melnikov function, in order to measure the angular distance between these two vectors. We will find out that this distance is different from 0 for $\epsilon \neq 0$, so we can conclude that the manifolds have a transversal crossing.

First of all we introduce some notations: let us call $y^+(t;\tau,\epsilon)$ the trajectory at time t of system (3.4), such that $y^+(0;\tau,\epsilon) = \xi^+(\tau,\epsilon)$, and analogously define $y^-(t;\tau,\epsilon)$ in order to have $y^-(0;\tau,\epsilon) = \xi^-(\tau,\epsilon)$. First of all observe that

$$y^\pm(a;\tau-\epsilon a,\epsilon) \in W^{u,s}_\epsilon(\tau)$$

and that

$$\frac{d}{da} y^\pm(a;\tau-\epsilon a,\epsilon)\lfloor_{a=0} = f(\xi^\pm(\tau,\epsilon),\tau) - \epsilon\frac{\partial y^\pm}{\partial \tau}(0,\tau,\epsilon)$$

does not vanish for ϵ small. Therefore $\frac{d}{da}y^\pm(a;\tau-\epsilon a,\epsilon)\lfloor_{a=0}$ are respectively tangent vectors to $W^{u,s}_\epsilon(\tau)$ in $\xi^\pm(\tau,\epsilon)$.

Now we introduce a Melnikov-like function, in order to prove that these vectors are not parallel. As in [1], we define

$$\tilde{M}(\epsilon,a) = [y^-(a;\tau(\epsilon)-\epsilon a,\epsilon) - y^+(a;\tau(\epsilon)-\epsilon a,\epsilon)] \wedge f(U(\tau_0,0),\tau_0).$$

Then differentiating, we get:

$$\frac{\partial \tilde{M}}{\partial a}(\epsilon,a) = \epsilon T(\epsilon) = \epsilon\left[\frac{\partial y^-}{\partial \tau}(0,\tau(\epsilon),\epsilon) - \frac{\partial y^+}{\partial \tau}(0,\tau(\epsilon),\epsilon)\right] \wedge f(U(\tau_0,0),\tau_0).$$

Observe that $\frac{\partial \tilde{M}}{\partial a}(0,0) = 0$; in fact, for $\epsilon = 0$ we have a smooth homoclinic orbit. If we manage to prove that for $\epsilon \neq 0$ we have $T'(\epsilon) \neq 0$ and hence $\frac{\partial \tilde{M}}{\partial a}(\epsilon,a) \neq 0$, we are done.

It is convenient to denote the derivatives with respect to x, ϵ and τ with subscripts. First of all, following [1], we have

$$T'(0) = [y^-_{\tau\epsilon}(0;\tau_0,0) - y^+_{\tau\epsilon}(0;\tau_0,0)] \wedge f(U(\tau_0,t),\tau_0).$$

Then recalling Lemma 2.4 on page 150 in [1], after some manipulations, we get

$$T'(0) = \int_{-\infty}^{\infty} e^{-\int_0^t tr f_x(U(\tau,\sigma),\tau)d\sigma}\{f_{xx}U_\tau[y_\epsilon - U_\tau\tau'(0)] +$$
$$+ f_{x\tau}[y_\epsilon - U_\tau\tau'(0)] + t[f_{x\tau}U_\tau + f_{\tau\tau}]\} \wedge f dt$$

and finally, applying once again Lemma 2.4 in [1], we conclude that $T'(0) = M'(\tau_0)$. See ([1], pages 150-152) for details. Hence we can conclude the following.

Proposition 3.5 *Consider system (3.4); assume that ϕ is strictly positive and bounded and that there exists τ for which $\phi(\tau)$ admits a non degenerate extremum. Then there exists a function $\epsilon \to \tau(\epsilon)$, of class $\mathbb{C}^1$, such that the unstable and stable leaves $W^u_\epsilon(\tau(\epsilon))$ and $W^s_\epsilon(\tau(\epsilon))$ cross transversally.*

Now applying the Smale horseshoe construction we can conclude that, if $\phi(t)$ is periodic of period T, the system exhibits chaotic behavior. To be more precise, let us denote by Ψ_t the flow defined by system (3.4). Then there exists a Cantor-like set $\wedge$ which is invariant for the flow, and there are integers $k \geq 2$ and $N \geq 1$, such that the discrete dynamical system, made up of $\wedge$ and the map $\Psi_{\frac{NT}{\epsilon}}$, is conjugated to the Bernoulli shift on the set of sequences of k symbols. In particular there exists a Cantor-like set of periodic orbits. We will see that these orbits correspond to S.G.S. with slow decay $v(r)$ satisfying $v(r) \sim r^{-\alpha}$ both as $r \to 0$ and as $r \to \infty$.

The proof can be obtained repeating step by step the proof developed in Theorem 5.4 in [10]. However we can also give a new proof, based on similar reasoning. Consider system (3.4). In section 4 we will construct a topological circle which is denoted by $H(\tau)$, made up joining branches of the stable and unstable manifolds $W^u_\epsilon(\tau)$ and $W^s_\epsilon(\tau)$. If ϕ is strictly positive $H(\tau)$ is contained in the 4^{th} quadrant, otherwise it is contained in $\mathbb{R}^2_+$. Consider now system (4.2); letting τ takes values in the whole of $\mathbb{R}$ we obtain a topological surface H. We can arrange $\wedge$ to be in the bounded subset enclosed by $H(\tau)$.

Note that the trajectory departing from the periodic points cannot have the origin as α or Ω limit set, thus cannot cross $W^u_\epsilon(\tau)$ and $W^s_\epsilon(\tau)$; therefore they are forced to stay in the set enclosed by H for any t, thus they correspond to positive solutions $v(r)$ of (1.2). Furthermore, since these trajectories $\hat{x}(t)$ are periodic, we have that $\hat{x}_1(t)$ is strictly positive and bounded, therefore the corresponding $v(r)$ is a S.G.S. with slow decay satisfying $v(r) \sim r^{-\alpha}$ both as $r \to 0$ and as $r \to \infty$.

Theorem 3.6 *Assume that $\phi(t)$ is a $\mathbb{C}^2$ periodic function which is strictly positive and admits non degenerate extrema, then equation (1.2) admits a Cantor-like set of monotone decreasing S.G.S. with slow decay $v(r)$, satisfying $v(r) = O(r^{-\alpha})$ both as $r \to 0$ and as $r \to \infty$. Thus we have proved the following theorem.*

Remark 3.7 Recall that the critical points of $\phi(t)$ correspond to critical point of $K(r)$, since $K(r) = \phi(\epsilon log(r))$. But if $\phi(t)$ is periodic $K(r)$ is not, moreover $K(r)$ will not even be well defined near the origin as $r \to 0$.

Note that when $p = 2$, Theorem 3.6 works even when $K(r)$ changes sign, but admits positive non degenerate maxima, see [1], [2] and [3]. We are not able to extend this result to the case $p \neq 2$, since in the sign changing case the trajectories $y^+(\tau; t)$ may cross the x_1 axis for $t < 0$; thus the construction developed in Lemma 3.4 fails. However when ϕ admits positive maxima it can be easily seen that there is a topological crossing between $W^u_\epsilon(\tau)$ and $W^s_\epsilon(\tau)$. Thus following the construction developed by Burns and Weiss in [4], in the periodic case we can still prove the existence of a horseshoe factor. Therefore there exist integers k and N, and a set $\acute{\wedge}$ such that the action of the map $\Psi(\frac{NT}{\epsilon})$ on $\acute{\wedge}$ is semiconjugated to the Bernoulli shift on the sequence of k symbols. In particular, there exists an infinite set $\wedge$ of distinct periodic points; thus we can repeat the proof of the previous theorem. The only difference is that $H(\tau)$ can intersect the x_1 axis, but it is in $\mathbb{R}^2_+$. Therefore the S.G.S. $v(r)$ are not a priori monotone.

Theorem 3.8 *Assume that $\phi(t)$ is a $\mathbb{C}^1$ periodic function which admits positive maxima, then equation (1.2) admits a Cantor-like set of S.G.S. with slow decay $v(r)$, satisfying $v(r) \sim r^{-\alpha}$ both as $r \to 0$ and as $r \to \infty$.*

4 Geometrical analysis of singularly perturbed systems

In the previous sections, we have essentially extended to the p-Laplacean the results known for the Laplacean and exposed in [10], [1], [2] and [3]. Now we will give some results of a new type, the proofs of which are based on the same geometrical approach. We will discuss the case of functions $K(r)$ which are singular perturbations of a constant, but these techniques apply also to the regular perturbation problem.

It is already well known (see [13] and [6]) that, if $K(r)$ (and hence $\phi(t)$) is monotone increasing, then positive solutions $v(r)$ of (1.2) can be classified as follows: all the regular solutions are crossing solutions, and there exists at least one S.G.S. with slow decay and infinitely many S.G.S. with fast decay; no other solutions $v(r)$ positive for $r > 0$ can exist. On the other hand, if $K(r)$ (and hence $\phi(t)$) is monotone decreasing, all the regular solutions are G.S. with slow decay and there exists at least one S.G.S. with slow decay.

Furthermore, when $K(r)$ is a constant then all the regular solutions are G.S. with fast decay and there exist infinitely many S.G.S. with slow decay. This last situation can be considered as lying at the border between the increasing and the decreasing cases. We want to prove now that if $\phi(t)$ is oscillatory we have the coexistence of all of these solution types. Furthermore in the previous section we have seen that, in any case, the number of G.S. with fast decay is greater or equal to the number of non degenerate positive critical points of $K(r)$. Now we can state the following result:

Theorem 4.1 *Consider equation (1.2) and assume that $K(r)$ is a $\mathbb{C}^2$ function which is strictly positive and bounded, and that it is a singular perturbation of a constant as already specified. Moreover assume either that $K(r)$ satisfies Hypothesis M_1 or O_1 as $r \to 0$ and either Hypothesis M_2 or O_2 as $r \to \infty$. Furthermore assume that at least one of the maxima is non degenerate; then the positive solutions have a structure of type C (see the classification given in the introduction).*

Proof. Consider system (1.4): we introduce now the extra variables τ and ϵ, to obtain the following autonomous dynamical system:

$$\begin{aligned}
\dot{x_1} &= \alpha x_1 + x_2 |x_2|^{\frac{2-p}{p-1}} \\
\dot{x_2} &= -\alpha x_2 - \phi(\tau) x_1 |x_1|^{\sigma-2} \\
\dot{\tau} &= \epsilon \\
\dot{\epsilon} &= 0.
\end{aligned} \tag{4.1}$$

Note that the origin is a critical point for the system which admits a center-stable and a center-unstable manifold. Moreover note that these manifolds have dimension

3 and they are both transversal to the planes $P(\bar{\tau}, \bar{\epsilon}) = \{\tau = \bar{\tau} \quad \epsilon = \bar{\epsilon}\}$, where $|\bar{\epsilon}| > 0$ is small and $\bar{\tau} \in \mathbb{R}$.

It is convenient to introduce the variable $x_3 = \frac{\tau}{\epsilon}$ for each fixed non-zero value of ϵ; then the equations (4.1) take the equivalent form

$$\begin{aligned} \dot{x_1} &= \alpha x_1 + x_2 |x_2|^{\frac{2-p}{p-1}} \\ \dot{x_2} &= -\alpha x_2 - \phi(x_3) x_1 |x_1|^{\sigma-2} \\ \dot{x_3} &= 1. \end{aligned} \tag{4.2}$$

Note that the x_3 axis is a center manifold and that there exist a two dimensional center-unstable manifold W^u_ϵ and a two dimensional center-stable manifold W^s_ϵ. Note that the leaves $W^u_\epsilon(\tau)$ can be obtained intersecting W^u_ϵ with the plane $x_3 = \tau$. We have already proved that there is a transversal crossing between $W^u_\epsilon(\tau(\epsilon))$ and $W^s_\epsilon(\tau(\epsilon))$. Therefore we have a transversal crossing between the center-unstable W^u_ϵ and the center-stable manifold W^s_ϵ, of system (4.2).

Consider the curves $(x_1(t), x_2(t))$ obtained by dropping the x_3-coordinate of the trajectories belonging to W^u_ϵ. We want to follow them forward in t and to prove that some of them will cross the negative x_2-semiaxis (so they correspond to crossing solutions of (1.2)), while some others are forced to stay in a bounded subset of the open 4^{th} quadrant (so they correspond to G.S. with slow decay of (1.2)). These trajectories $(x_1(t), x_2(t))$ make up two subsets which are disconnected by the homoclinic trajectories in the following sense. If we consider the set of the crossing solutions and the set of the G.S. with slow decay in W^u_ϵ, we find that they are open. The topological border of each of these two sets is the union of all the homoclinic trajectories.

Analogously, following backwards in t the curves $(x_1(t), x_2(t))$ obtained from W^s_ϵ, we find a set of curves which cross the positive x_1-semiaxis (so they correspond to solutions of the Dirichlet problem in exterior domains described in the conclusions of Theorem 4.1), and a set of trajectories which are forced to stay in a bounded subset of the open 4^{th} quadrant (so they correspond to S.G.S. with fast decay of (1.2)). Again these two sets are disconnected by the homoclinic trajectories in the sense that they are open in W^s_ϵ, and the set of all the homoclinic trajectories is their topological border.

We will analyze only W^u_ϵ, since W^s_ϵ can be studied in a completely analogous way. Observe that both $W^u_\epsilon(\tau)$ and $W^s_\epsilon(\tau)$ have a first crossing with the isocline $\dot{x}_1 = 0$. We will consider this isocline as the transversal for the Melnikov function. Observe now that, by assumption, we have a sequence of non degenerate extrema, which could be either finite or infinite in number. Assume at first that this number is finite and equals $n \in \mathbb{N}$. Using Theorem 3.2 we can say that there exists a sequence of values $\tau_1(\epsilon)$, $\tau_2(\epsilon)$, ... , $\tau_k(\epsilon)$ or simply τ_1, τ_2, ... , τ_k for which $h(\tau, \epsilon) = 0$, here $1 \leq k \leq n$; thus for these values we have a crossing between $W^s_\epsilon(\tau)$ and $W^u_\epsilon(\tau)$. Suppose that, for k odd, $\phi(\tau_k(0))$ is a minimum and that for k even, it is a maximum. Note that whenever $\tau \in (\tau_{2k}, \tau_{2k+1})$, $\xi^-(\tau, \epsilon)$ is on the left with respect to $\xi^+(\tau, \epsilon)$, while whenever $\tau \in (\tau_{2k-1}, \tau_{2k})$ $\xi^-(\tau, \epsilon)$ is on the right with respect to $\xi^+(\tau, \epsilon)$. Here we are thinking of the x_1 axis as horizontal and of the x_2 axis as

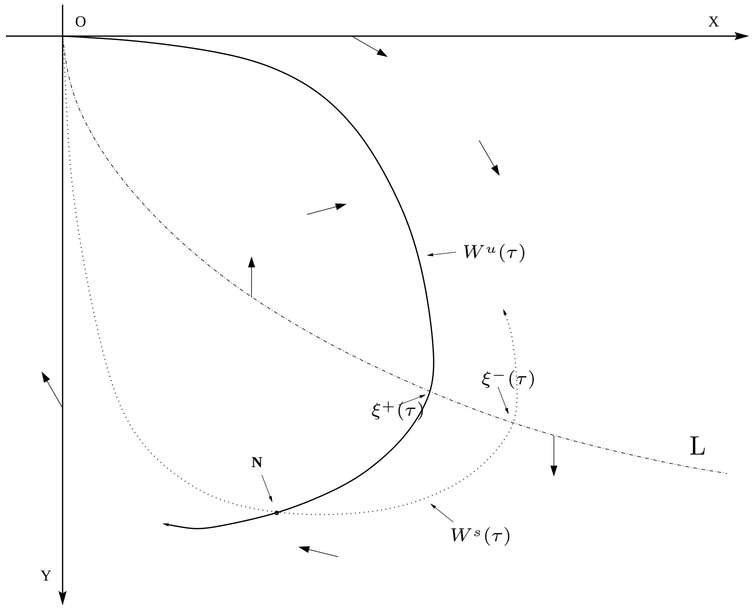

Figure 1: A sketch of the phase portrait in the plane $x_3 = \tau$, when $K(r)$ admits only one critical point which is a maximum. The curve $H(\tau)$ is contained in the 4^{th} quadrant.

vertical.

Let us define the following surfaces for system (4.2):

$$\begin{aligned} L &:= \{(x_1, x_2, x_3) \quad | \quad \dot{x}_1 = 0 \quad \text{and} \quad x_1 \geq 0\} \\ \tilde{W}^u_\epsilon &:= \{(x_1, x_2, x_3) \quad | \quad (x_1, x_2) \in \tilde{W}^u_\epsilon(\tau) \quad \text{and} \quad x_3 = \tau, \quad -\infty < \tau < \infty\} \\ \tilde{W}^s_\epsilon &:= \{(x_1, x_2, x_3) \quad | \quad (x_1, x_2) \in \tilde{W}^s_\epsilon(\tau) \quad \text{and} \quad x_3 = \tau, \quad -\infty < \tau < \infty\}. \end{aligned}$$

We recall that $y^\pm(t; \tau, \epsilon)$ is the trajectory of system (3.4) departing at $t = 0$ from $\xi^\pm(\tau, \epsilon)$.

We will use the following notation: let us call $Y^\pm_\tau(t) := (y^\pm(t; \tau, \epsilon), x^\tau_3(t))$ the trajectory of system (4.2) departing at $t = 0$, from $(\xi^\pm(\tau, \epsilon), \tau)$. We have already seen that the trajectories $y^+(t; \tau_k(\epsilon), \epsilon)$ are homoclinic.

Consider the trajectories $Y^+_\tau(t)$ of system (4.2), where $\tau \in (\tau_{2k-1}, \tau_{2k})$. Following them forward in time for $t > 0$ we note that they must lie in the exterior of the set delimited by the surfaces L and $\tilde{W}^s_\epsilon$. Note now that $\frac{d}{dt}\bar{x}_1(t) \leq 0$ and that $|\frac{d}{dt}\bar{x}_1(t)| + |\frac{d}{dt}\bar{x}_2(t)|$ is strictly positive for any $t > 0$. Thus $Y^+_\tau(t)$ ($\tau \in (\tau_{2k-1}, \tau_{2k})$) is forced to enter the subset where $x_1 \leq 0$ in finite time. Therefore the corresponding solutions $u(r)$ of (1.2) are crossing solutions. Analogously, following backwards in t the trajectories $Y^-_\tau(t)$ where $\tau \in (\tau_{2k}, \tau_{2k+1})$, we note that they must lie in the exterior of the set enclosed by L and S^u. Thus they correspond to solutions of the Dirichlet problem in exterior domains of (1.2), like the ones described in the conclusion of Theorem 4.1. Assume now that we have an infinite number of critical points of ϕ. We can still denote with an even subscript the maxima and with an odd subscript the minima. Furthermore, we can repeat the proof and arrive at exactly the same conclusions.

We denote with $N^k(\tau)$ the intersection with the trajectories $Y^+_{\tau_k}(t)$ of system (4.2) and the plane $x_3 = \tau$; as before k is an index of the critical points of ϕ. See Figures 1, 2 and 3. Let us fix τ; let us call $U^{k,k+1}_\tau(s)$, where $s \in [k, k+1]$, a parameterization of the segment of $W^u_\epsilon(\tau)$, joining $N^k(\tau)$ and $N^{k+1}(\tau)$, such that $U^{k,k+1}_\tau(k) = N^k(\tau)$, and $U^{k,k+1}_\tau(k+1) = N^{k+1}(\tau)$. Analogously, define $S^{k,k+1}_\tau(s)$ to be a parameterization of the segment of $W^u_\epsilon(\tau)$, joining $N^k(\tau)$ and $N^{k+1}(\tau)$. Assume at first that Hypotheses O_1 and O_2 are satisfied. Then we can define the following curve:

$$\tilde{H}(\tau, s) := \begin{cases} S^{k,k+1}_\tau(s) & s \in [k, k+1] \quad \text{if } k \text{ is odd,} \\ U^{k,k+1}_\tau(s) & s \in [k, k+1] \quad \text{if } k \text{ is even.} \end{cases}$$

Fixing τ, we get a curve $\tilde{H}(\tau, \cdot)$ such that $\lim_{s\to\pm\infty} \tilde{H}(\tau, s) = (0,0)$. Let us call $H(\tau)$ the topological manifold $H(\tau) = \{\tilde{H}(\tau, s) \quad | \quad s \in \mathbb{R}\} \bigcup (0,0)$, which is homeomorphic to a circle. Let us call $D(\tau)$ the disc delimited by $H(\tau)$. Assume now

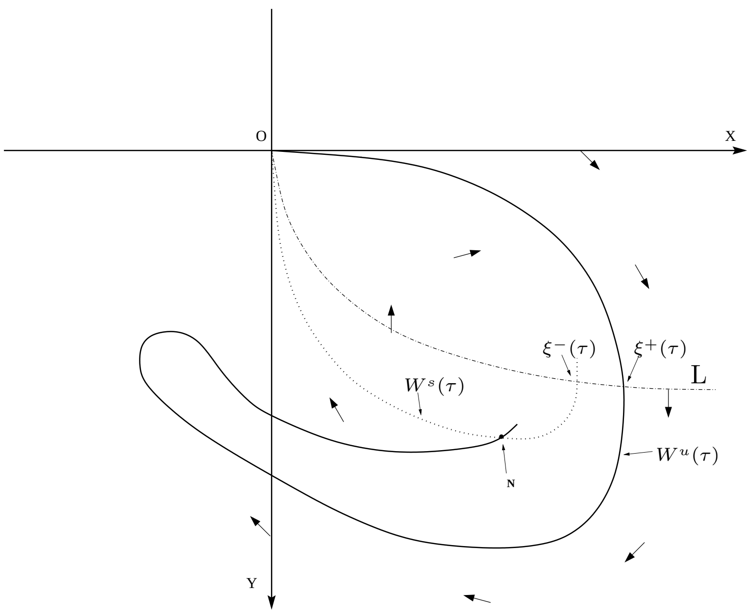

Figure 2: A sketch of the phase portrait in the plane $x_3 = \tau$, when $K(r)$ admits only one critical point which is a minimum. The curve $H(\tau)$ may cross the x_2 axis.

that Hypothesis M_1 and M_2 are satisfied, then there exist a minimum τ_1 and a maximum τ_m in the sequence τ_k. We define $U^{-\infty,1}_\tau$ to be the segment of $W^u(\tau)$

joining the origin to $N^1(\tau)$ and S_τ^m to be the segment of $W^s(\tau)$ joining $N^m(\tau)$ to the origin. We can define now a parameterization $U_\tau^{-\infty,1}(s)$ of $U_\tau^{-\infty,1}$, where $s \in (-\infty, 1]$, such that $\lim_{s\to-\infty} U_\tau^{0,1}(s) = (0,0)$ and $U_\tau^{-\infty,1}(1) = N^1(\tau)$. Analogously, we define a parameterization $S_\tau^{m,\infty}(s)$ of $S_\tau^{m,\infty}$, where $s \in [m, \infty)$, such that $S_\tau^{m,\infty}(m) = N^m(\tau)$ and $\lim_{s\to\infty} S_\tau^{m,\infty}(s) = (0,0)$. Then we can define the following curve:

$$\tilde{H}(\tau, s) := \begin{cases} U_\tau^{-\infty,1}(s) & s \in (-\infty, 1], \\ S_\tau^{k,k+1}(s) & s \in [k, k+1] \quad \text{if } 1 \le k \le m-1 \text{ is odd}, \\ U_\tau^{k,k+1}(s) & s \in [k, k+1] \quad \text{if } 1 \le k \le m-1 \text{ is even}, \\ S_\tau^{m,\infty}(s) & s \in [m, +\infty). \end{cases}$$

Once again we define the curve $H(\tau) = \{\tilde{H}(\tau, s) \quad | \quad s \in \mathbb{R}\} \bigcup (0,0)$, see Figure 3. We can define $H(\tau)$ in the same way if either Hypotheses M_1 and O_2 or Hypotheses M_2 and O_1 are satisfied.

Let us call $R(\tau)$ the curve made up by joining $\tilde{W}^u(\tau)$, the origin, $\tilde{W}^s(\tau)$ and the segment of the isocline L joining $\xi^-(\tau, \epsilon)$ to $\xi^+(\tau, \epsilon)$. Assume that either Hypothesis O_1 or M_1 is satisfied and moreover assume that either Hypothesis O_2 or M_2 is satisfied. Observe that $R(\tau)$ belongs to the 4^{th} quadrant for any τ, and that $H(\tau)$ is contained in the interior of $R(\tau)$. In fact, the criterion of selection used to construct $\tilde{H}(\tau, s)$, ensures that each segment of $\tilde{H}(\tau, s)$ lies on $R(\tau)$ or in the bounded subset enclosed in $R(\tau)$. Note that if such Hypotheses are not satisfied, then the surface $H(\tau)$ lies outside $R(\tau)$ and the construction fails, see Figures 1 and 2.

We now define the following set, observing that it contains infinitely many points

$$E(\tau) := D(\tau) - \{W_\epsilon^u(\tau) \cup W_\epsilon^u(\tau)\}.$$

Letting τ take values in $\mathbb{R}$, we have that $\bigcup\{H(\tau) \quad | \quad \tau \in \mathbb{R}\}$ defines a topological manifold of dimension 2, which is homeomorphic to a cylinder and which cannot be crossed by any trajectory $\breve{x}(t)$ departing from $Q \in E(\tau)$. Thus each $\breve{x}(t)$ corresponds to a monotone decreasing S.G.S. with slow decay.

Let us assume that either Hypothesis M_2 or O_2 is satisfied. We want to prove the existence of infinitely many G.S. with slow decay. Let us fix a value $\tau_j(\epsilon)$ such that j is even. Let us define $S_\tau^{j,\infty}$ to be the segment of $W^s(\tau)$ joining $N^j(\tau)$ to the origin. We can define now a parameterization $S_\tau^{j,\infty}(s)$ of $S_\tau^{j,\infty}$, where $s \in [j, +\infty)$, such that $S_\tau^{j,\infty}(j) = N^j(\tau)$ and $\lim_{s\to\infty} S_\tau^{j,\infty}(s) = (0,0)$. We need to define the following function of s for $\tau \ge \tau_j(\epsilon)$

$$\tilde{H}_j(\tau, s) := \begin{cases} \tilde{H}(\tau, s) & s \in (-\infty, j] \\ S_\tau^{j,\infty}(s) & s \in [j, +\infty). \end{cases}$$

Then we define $H_j(\tau)$ by adjoining the origin. Observe that the topological manifold $H_j(\tau)$ lies in $\mathbb{R}_+^2$ for any $\tau \ge \tau_j(\epsilon)$. In fact $S_\tau^{j,\infty}(s) \subset \tilde{W}^s(\tau)$ for any $s > j$. Furthermore note that, for any $s < j$ the segments of $\tilde{H}(\tau, s)$ are contained either in $R(\tau)$, or in the bounded subset enclosed in $R(\tau)$. We now define $D_j(\tau)$ to be the subset of system (1.4) enclosed in $H_j(\tau)$, and D_j the subset of system (4.2) enclosed in the topological surface H_j. We recall now that $\xi^+(\tau,\epsilon)$ lies on the left of $\xi^-(\tau,\epsilon)$, for $\tau \in (\tau_j,\tau_{j+1})$. Let us fix now $\delta > 0$ small enough: we can find infinitely many values $\tau \in (\tau_j,\tau_j+\delta)$ such that $\xi^+(\tau,\epsilon) \in W^u(\tau) - W^s(\tau)$. Let us consider the trajectory $Y^+_\tau(t)$ of system (4.2) departing from one of these points. Observe that it lies in the interior of the set bounded by the surface $\tilde{W}^s_\epsilon$ and L, for $t > 0$ small. If it lies in this set for any $t > 0$ we are done. Thus suppose that there exists a time $T(\tau)$ such that $Y^+_\tau(T(\tau)) \in L^+$, where L^+ is the subset of L where $x_1 > 0$ and $\dot{x}_2 > 0$. Note that $L^+ \subset D_j$, thus $Y^+_\tau(T(\tau)) \in D_j$. Recalling that $Y^+_\tau(t)$ cannot cross H_j for any $t > 0$, we have that $Y^\tau(t) \in D_j$ for any $t > 0$. Thus $Y^+_\tau(t)$ corresponds to a G.S. with slow decay. To prove the existence of S.G.S. with fast decay we have to follow backwards in t, the trajectories $Y^-_\tau(t)$ and to repeat the analysis just explained.

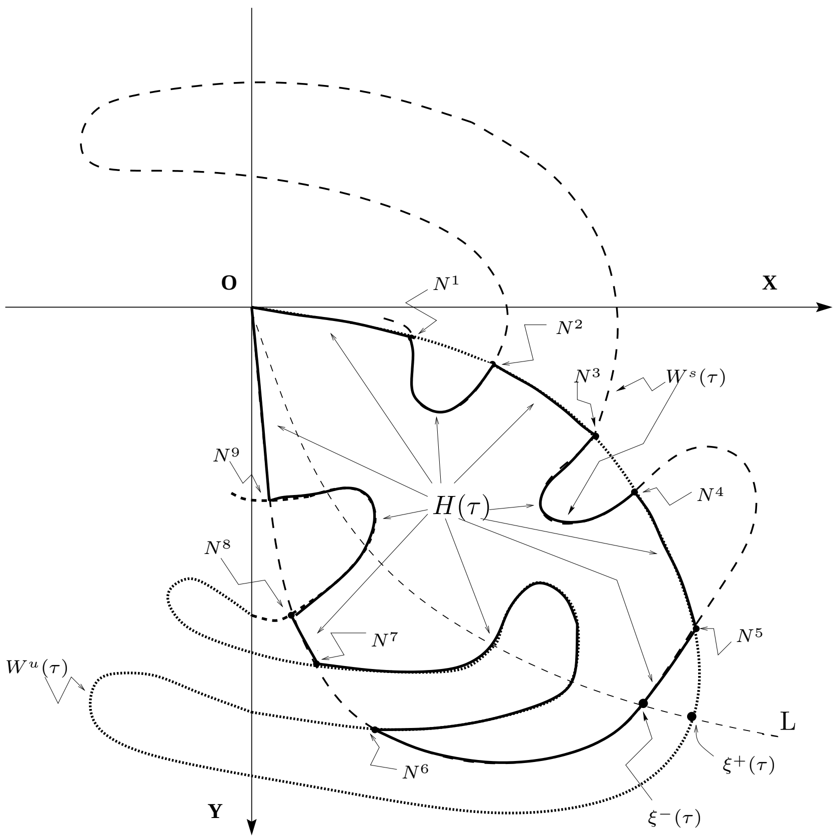

Figure 3: A sketch of the phase portrait in the plane $x_3 = \tau$, when $K(r)$ have 5 maxima and 4 minima. The curve $H(\tau, s)$, represented with a solid line, is obtained joining segments of the curve $W^u(\tau)$ (dotted line), and segments of the curve $W^s(\tau)$ (dashed line).

The non existence result is a consequence of Lemma 2.4. In fact we have described the behavior of the bounded trajectories and Lemma 2.4 describes the behavior of the unbounded ones. □

We can combine the techniques used in Theorem (4.1) to conclude the following:

Corollary 4.2 *Consider equation (1.2) and assume that $K(r)$ is strictly positive and bounded and that it is a singular perturbation of a constant in the sense already specified. Then we have at least as many G.S. with fast decay as the non degenerate critical points of $K(r)$. Moreover*

1. *Assume that either Hypothesis M_2 or Hypothesis O_2 is satisfied. Then the positive solutions of equation (1.2) have a structure of type A.*
2. *Assume that either Hypothesis M_1 or Hypothesis O_1 is satisfied. Then the positive solutions of equation (1.2) have a structure of type B.*
3. *Assume that both Hypotheses 1 and 2 are satisfied. Then the positive solutions of equation (1.2) have a structure of type C.*

Notice that we can weaken the hypotheses of the theorem as follows. In this corollary we will commit the following abuse of notation: we will say that we have a structure of type A, B, or C, even if **the monotonicity of the solutions is not anymore ensured**.

Corollary 4.3 *Consider equation (1.2) and assume that $K(r)$ is a singular perturbation of a constant, but that it may change its sign.*

1. *Assume that either Hypothesis M_2 or Hypothesis O_2 is satisfied. Then the positive solutions of equation (1.2) have a structure of type A (aside form the fact that the monotonicity of the solutions is not any more ensured).*
2. *Assume that either Hypothesis M_1 or Hypothesis O_1 is satisfied. Then the positive solutions of equation (1.2) have a structure of type B.*
3. *Assume that both Hypotheses 1 and 2 are satisfied. Then the positive solutions of equation (1.2) have a structure of type C, apart from the non-existence result. That is there exist infinitely many crossing solutions, G.S. with slow decay, S.G.S. with fast and slow decay, and solutions of the Dirichlet problem in exterior domains. Furthermore, the number of G.S. with fast decay is greater than or equal to the number of positive non degenerate critical points of ϕ. But the situation is even more rich, thus we cannot exclude the existence of other families of positive solutions.*

Proof. We want to adapt the proof of Theorem 4.1 to this case. We have to reprove the existence of $\xi^{\pm}(\tau,\epsilon)$ and that $\tilde{W}^u_\epsilon(\tau)$ and $\tilde{W}^s_\epsilon(\tau)$ belong to $\mathbb{R}^2_+$ for any τ. Another difficulty will be that the transversality of the crossing between $\tilde{W}^u_\epsilon(\tau)$ and $\tilde{W}^s_\epsilon(\tau)$ is not anymore ensured. However, what is really needed in the proof of Theorem 4.1 is that we can find points belonging to $\tilde{W}^u_\epsilon(\tau) - W^s_\epsilon(\tau)$ and in $\tilde{W}^s_\epsilon(\tau) - W^u_\epsilon(\tau)$. But this easily follows observing that the manifolds $W^u_\epsilon(\tau)$ and $W^s_\epsilon(\tau)$ do not coincide and from some elementary topological reasoning.

Consider system (1.4): first of all observe that, if $\phi(t)$ changes sign, the flow continues to rotate clockwise on the x_2 axis. Assume that Hypothesis 1 is satisfied; then the existence of a crossing between $W^{u,s}_\epsilon(\tau)$ and the isocline L, is ensured if $\phi(\tau) > 0$. But when $\phi(\tau) > 0$ we can also affirm that $\tilde{W}^u(\tau) \subset A^+$ and $\tilde{W}^s(\tau) \subset A^-$. Furthermore, applying Theorem 3.2, from the existence of positive non degenerate critical points, we can deduce the existence of homoclinic trajectories belonging to $\mathbb{R}^2_+$. Let us call $N^1(\tau)$, ..., $N^k(\tau)$ the intersections between such trajectories and the plane $x_3 = \tau$. Let us call τ_1, ..., τ_k, the values for which $N^j(\tau_j) \in L$, that is the values for which we have a crossing between $\tilde{W}^u_\epsilon(\tau_j)$ and $\tilde{W}^s_\epsilon(\tau_j)$. Let us call $U^{-\infty,j}(\tau)$ the submanifold of $W^u(\tau)$ connecting the origin and $N^j(\tau)$; analogously we define $S^{j,\infty}(\tau)$ to be the submanifold of $W^s(\tau)$ connecting the origin and $N^j(\tau)$. Note that these submanifolds are connected and in fact are $\mathbb{C}^1$ embedded curves in $\mathbb{R}^2$. Note that, for any $\tau < \tau_k$, $N^k(\tau) \in A^+$, thus we have that there exists $\xi^-(\tau,\epsilon) = \tilde{W}^s(\tau) \cap L$, for any $\tau < \tau_k$. Analogously for any $\tau > \tau_1$ there exists $\xi^+(\tau,\epsilon) = \tilde{W}^u(\tau) \cap L$; see Figure 4.

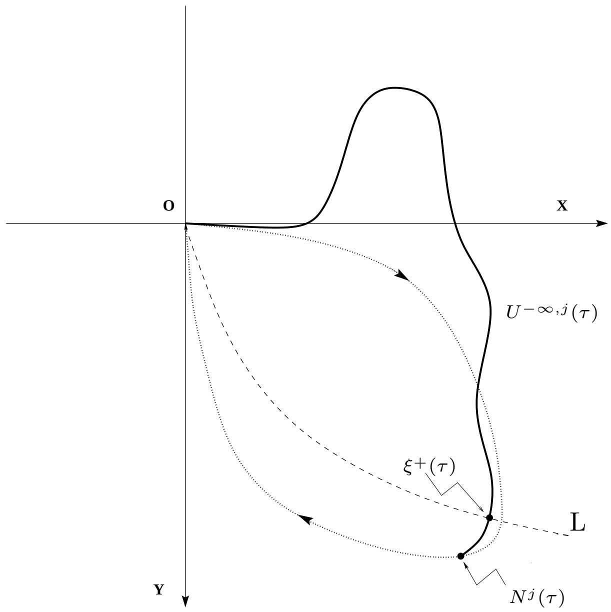

Figure 4: Existence of the points $\xi^+(\tau,\epsilon) = \tilde{W}^u(\tau) \cap L$. The solid line represents the curve $U^{-\infty,j}(\tau)$ while the dotted line represents the homoclinic trajectory departing from $N^j(\tau)$. The dashed line is the isocline L.

We want to prove now that $U^{-\infty,j}(\tau)$ is contained in A^+ for any $\tau < \tau_j$. Assume for contradiction that there exists $\bar{\tau} < \tau_j$ such that $U^{-\infty,j}(\bar{\tau})$ crosses the negative x_2 semiaxis in a point $P(\bar{\tau})$. We call $z(x_0, \tau; T)$ the solution of system (3.4) departing at $t=0$ from $x_0 \in \mathbb{R}^2$, evaluated at $t=T$. Let us fix $\tau = \bar{\tau}$ and consider system (3.4). Let us call $\theta(x,0)$ the angular coordinate of $x \in \mathbb{R}^2 - (0,0)$ and let $\theta(x_0,t)$ be the continuous function such that $\theta(x_0,t) = \theta(z(x_0,\bar{\tau};t),0)$. Let us call $\Psi(x,t)$ the evolution map of system (3.4). Observe that $\Psi(U^{-\infty,j}(\bar{\tau}), \tau_j - \bar{\tau}) = U^{-\infty,j}(\tau_j) \in \mathbb{R}^2_+ \cap A^+$. In particular, there is a point $P(\tau_j) = z(P(\bar{\tau}), \bar{\tau}; \tau_j - \bar{\tau}) \in U^{-\infty,j}(\tau_j)$.

We recall that $\theta(P(\bar{\tau}),0) = -\frac{\pi}{2}$; note that the flow of the solutions on the x_2 axis always rotates clockwise. Now noticing that $P(\tau_j)$ is in the 4^{th} quadrant we get that $\theta(P(\bar{\tau}), \tau_j - \bar{\tau}) < -2\pi$. Observe now that $\theta(N^j(\bar{\tau}), \tau_j - \bar{\tau}) > -\frac{\pi}{2}$ since the trajectory through $N^j(\bar{\tau})$ is homoclinic. We observe now that the function $\theta(\cdot, \tau_j - \bar{\tau})$, evaluated along the continuous path $U^{-\infty,j}(\bar{\tau})$ has a jump discontinuity between $P(\bar{\tau})$ and $N^j(\bar{\tau})$. Thus we have found a contradiction and we have proved that $U^{-\infty,j}(\tau) \subset A^+ \cap \mathbb{R}^2_+$ for any $\tau < \tau_j$.

Let us consider $y^+(\tau;t)$. Note that $y^+(\tau;t) \in \tilde{W}^u(\tau + \epsilon t)$. Thus, for any $\tau < \tau_j$ we have $y^+(\tau;t) \in A^+ \cap \mathbb{R}^2_+$, for any $t < 0$. Assume that there exists $\hat{\tau}$ such that $\phi(\hat{\tau})$ is a maximum and $\phi(\tau)$ is strictly positive, for any $\tau \geq \hat{\tau}$. Then we can apply the reasoning developed in the proof of Theorem 4.1, restricting our attention to the trajectories $y^+(\tau;t)$, where $\tau > \hat{\tau}$. This way we can prove that there exist infinitely many trajectories $y^+(\tau;t)$, departing from the origin and contained in $\mathbb{R}^2_+$ for any t, thus corresponding to G.S. with slow decay of equation (1.2). The proof of the existence of crossing solutions is analogous.

Now assume that $K(r)$ is oscillatory as $r \to \infty$. Then there exists a sequence τ_k of values of τ such that $\tilde{W}^u_\epsilon(\tau_k) \cap \tilde{W}^s_\epsilon(\tau_k) \neq \emptyset$. Therefore we also have a corresponding sequence of points $N^k(\tau)$ just like the ones described previously. Reasoning as above we can prove the existence of the points $\xi^+(\tau,\epsilon)$ and $\xi^+(\tau,\epsilon)$. Then we can apply the construction described in the proof of Theorem 4.1 and conclude the proof of Claim 1 of the Corollary. Claim 2 and 3 can be proved in the same way. $\square$

Theorem 4.4 *Let $K(r) = k(r^\epsilon) \in \mathbb{C}^2(\mathbb{R})$ be a singular perturbation of a constant as described above and consider equation (1.1).*

A. *Assume that there exists R such that $K(R)$ is a positive non degenerate minimum, and that $K(r)$ is monotone increasing for any $r > R$. Then there exists R^* such that the Dirichlet problem corresponding to (1.1) admits a positive radial solution in any ball of radius $r > R^*$.*

B. *Assume that there exists ρ such that $K(\rho)$ is a positive non degenerate maximum, and that $K(r)$ is positive and monotone decreasing for any $0 \leq r < \rho$.*

Then there exists R_* such that the Dirichlet problem corresponding to (1.1) admits a positive radial solution $u(r)$ in the exterior of any ball of radius $R < R^*$, that is there is a solution $u(r)$ of (1.1) such that $u(R) = 0$, $u(r) > 0$ for $r > R$ and $u(r) \sim r^{-\frac{n-p}{p-1}}$ as $r \to \infty$.

Proof. Consider system (3.4); we have already observed that there exists $\tau(\epsilon)$, such that $W^u_\epsilon(\tau(\epsilon))$ and $W^s_\epsilon(\tau(\epsilon))$ cross. Assume that Hypothesis A is satisfied, then, for any $\tau > \tau(\epsilon)$, we have that $\xi^+(\tau,\epsilon)$ lies on the right with respect to $\xi^-(\tau,\epsilon)$, thus the corresponding solution $u(r)$ can only be a G.S with fast decay or a crossing solution, see Theorem 4.1. First of all we want to prove that the first case can be excluded. Suppose for contradiction that the trajectory $y^+(\tau,\epsilon;t)$ departing from some $\xi^+(\bar{\tau},\epsilon)$, where $\bar{\tau} > \tau(\epsilon)$, represents a G.S with fast decay. Then $\xi^+(\bar{\tau},\epsilon) \in W^s(\bar{\tau})$. Recall now that $\xi^+(\bar{\tau},\epsilon)$ cannot belong to $\tilde{W}^s(\bar{\tau})$, since otherwise we would have another crossing in $\xi^-(\bar{\tau},\epsilon) = \xi^+(\bar{\tau},\epsilon)$, for $\bar{\tau} > \tau(\epsilon)$. Thus $\xi^+(\bar{\tau},\epsilon) \in W^s(\bar{\tau}) - \tilde{W}^s(\bar{\tau})$. Consider the trajectory $Y^+_{\bar{\tau}}(t) = (x^+_1(t), x^+_2(t), x^+_3(t))$ of system (4.2), departing at $t = 0$ from $(\xi^+(\bar{\tau},\epsilon),\bar{\tau})$. Note that $\dot{x}^{\bar{\tau}}_1 < 0$ for any $t > 0$, until the trajectory reaches the isocline L. But, since Y^+ cannot cross the manifold $\tilde{W}^s_\epsilon$, it cannot reach the isocline L. So there exists $T(\bar{\tau})$ such that $x^+_1(T(\bar{\tau})) = 0$. This contradicts the assumption that the trajectory $Y^+_{\bar{\tau}}(t)$ represents a G.S. Thus the corresponding $u(r)$ is a radial positive solution of the Dirichlet problem in the ball of radius $R^* = exp(\epsilon T(\bar{\tau}) + \bar{\tau})$.

We have proved for any $\tau \in (\tau(\epsilon), +\infty)$, the trajectories $y^+(\tau,\epsilon;t)$ represent crossing solutions. Thus, using a continuity argument and the fact that $T(\tau)$ is always positive, we can conclude that the Dirichlet problem (1.2) in any ball of radius $R > R^*$.

Claim B is obtained in the same way, following backwards in t the trajectories $y^-(\tau;t)$. It can be proved that there exists $\bar{\tau}$ such that, for any $\tau < \bar{\tau}$, there exist $T_1(\tau) < 0$ such that $y^-(\tau;T_1(\tau))$ crosses the positive x_1 semi-axis and $T_2(\tau) < T_1(\tau)$ such that $y^-(\tau;T_2(\tau))$ crosses the positive x_1 semi-axis. □

5 Regularly perturbed systems

Now we want to apply to the regular perturbation problem the techniques we have developed for the singular one. Therefore we consider a function $K(r) = 1 + \epsilon k(r)$, where $\epsilon > 0$ is small and k is a bounded function.

We follow the general framework developed in [10]. Let us consider a family of system of the form

$$\dot{x} = f(x) + \epsilon h(x, t + \tau),$$

where $\epsilon > 0$ is a small parameter, $x \in \mathbb{R}^2$ and $f, h \in \mathbb{C}^1(\mathbb{R}^2, \mathbb{R}^2)$. Assume that the autonomous system obtained setting $\epsilon = 0$ admits a family of homoclinic solution $U(\tau,t)$. Now consider the non-autonomous system; arguing as in section 2, for any τ we can find a stable leaf $W^s_\epsilon(\tau)$ and an unstable leaf $W^u_\epsilon(\tau)$. Following [10] we can construct a Melnikov function which measures the distance between $W^s_\epsilon(\tau)$ and $W^u_\epsilon(\tau)$ along a transversal L which can be computed as follows, see [10], page 1055:

$$\bar{M}(\tau) = \int_{-\infty}^{+\infty} f(U(\tau, t - \tau)) \wedge h(U(\tau, t - \tau), t)dt. \tag{5.1}$$

Now we go back to our problem, therefore we apply to (1.2) the change of variables (1.3), obtaining the following one parameter family of systems

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} \alpha & 0 \\ 0 & -\alpha \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} x_2|x_2|^{\frac{2-p}{p-1}} \\ -1 + \epsilon g(\tau + t)x_1|x_1|^{\sigma-2} \end{pmatrix}. \tag{5.2}$$

Note that we have set $\Phi(t) = 1 + \epsilon g(t)$, so that we will deal with $g(t) = k(e^t)$.

Thus applying (5.1) to our problem we find the following

$$\bar{M}(\tau) = -\int_{-\infty}^{+\infty} X_1 |X_1|^{\sigma-2}(t) \dot{X}_1(t) g(t+\tau) dt$$

where $U(\tau, t) = (X_1(t), Y_1(t))$ is the homoclinic solution of the frozen system (1.4) with $K(r) \equiv 1$. Then, integrating by parts, we get:

$$\bar{M}(\tau) = \int_{-\infty}^{+\infty} g'(t+\tau) \frac{|X_1|^\sigma}{\sigma} dt$$

and hence

$$\bar{M}'(\tau) = \int_{-\infty}^{+\infty} g''(t+\tau) \frac{|X_1|^\sigma}{\sigma} dt.$$

Remark 5.1 We recall that $X_1 = \left(e^{-t} + De^{\frac{t}{p-1}}\right)^{-\alpha}$ and $g(t) = k(e^t)$, therefore the function $\bar{M}(\tau)$ and its derivative $\bar{M}'(\tau)$ can be explicitly computed.

We will see that, when $\bar{M}(\tau) = 0$ and $\bar{M}'(\tau) \neq 0$, we can apply the construction already used for the singularly perturbed system. Now, reasoning as in Theorem 3.2, we can construct a function $h(\epsilon, \tau)$, analogous to the one used in section 3. Then, using the implicit function theorem, we deduce that the manifolds $W^u_\epsilon(\tau)$ and $W^s_\epsilon(\tau)$ have a crossing. Moreover, reasoning as in [10], page 1057, we can conclude that such a crossing is transversal.

We give a sketch of the proof of the transversality for the convenience of the reader. Consider equation (3.1). We can construct a segment $L(a)$, parallel to the transversal L, containing the point $U(\tau, a)$ in its interior. Thus we can deduce the existence of the points $\xi^+(\tau, \epsilon, a)$ and $\xi^-(\tau, \epsilon, a)$ which are the intersection points of $W^{u,s}_\epsilon(\tau)$ with $L(a)$, respectively.

Now we construct the following modified Melnikov function

$$\tilde{M}(\tau, a) = \frac{d}{d\epsilon} \left[\xi^-(\tau, \epsilon, a) - \xi^+(\tau, \epsilon, a)\right] \lfloor_{\epsilon=0} \wedge f(U(\tau, a)) = \bar{M}(\tau - a).$$

Now observe that

$$\frac{\partial}{\partial a} \tilde{M}(\tau, a)(\tau(\epsilon), a) \lfloor_{a=0} = -\bar{M}'(\tau(\epsilon)) \neq 0.$$

Thus we can deduce the transversality of the vectors

$$\frac{\partial}{\partial a} \xi^-(\tau(\epsilon), \epsilon, a) \quad \text{and} \quad \frac{\partial}{\partial a} \xi^+(\tau(\epsilon), \epsilon, a),$$

which are tangent respectively to $W^s_\epsilon(\tau(\epsilon))$ and $W^u_\epsilon(\tau(\epsilon))$. Thus, applying the Smale horseshoe construction we can state the following theorem:

Theorem 5.2 *Assume that $K(r) = 1 + \epsilon k(r)$ is a $\mathbb{C}^2$ function which is a regular perturbation of a constant. Then equation (1.2) admits a G.S. with fast decay for each non degenerate zero of $M(\tau)$. Assume in addition that $g \in \mathbb{C}^2(\mathbb{R}, \mathbb{R})$ is a periodic function. Then equation (1.2) admits a Cantor-like set of monotone decreasing S.G.S. with slow decay.*

Moreover, repeating the analysis drawn in section 4, we can get the following:

Theorem 5.3 *Consider equation (1.2), where $K(r) = 1 + \epsilon k(r)$, $|\epsilon|$ is small and $k(r)$ is a bounded function of class $\mathbb{C}^2$.*

1. *Assume either that there exists $\grave{T}$ for which $\bar{M}$ has a non degenerate zero and $\bar{M}(\tau) < 0$ for any $\tau > \grave{T}$, or that $\bar{M}(\tau)$ oscillates indefinitely as $\tau \to \infty$. Then the positive solutions of equation (1.2) have a structure of type A.*

2. *Assume either that there exists $\acute{T}$ for which $\bar{M}$ has a non degenerate zero and $\bar{M}(\tau) > 0$ for any $\tau < \acute{T}$, or that $\bar{M}(\tau)$ oscillates indefinitely as $\tau \to -\infty$. Then the positive solutions of equation (1.2) have a structure of type B.*

3. *Assume that both Hypotheses 1 and 2 are satisfied; then the positive solutions of equation (1.2) have a structure of type C.*

The proof is completely analogous to the one given for the singular perturbation problem in Theorem 4.1. The only difference comes from the fact that we are not able to reformulate the conditions on the Melnikov function $\bar{M}(\tau)$ in a simpler way. Now we reformulate in the regular setting the result given in Theorem (4.4).

Theorem 5.4 *Let $K(r) = 1+\epsilon k(r) \in \mathbb{C}^2(\mathbb{R})$ be a regular perturbation of a constant and consider equation (1.1).*

- *Assume that there exists $\bar{\tau}$ such that $\bar{M}(\tau) > 0$ for any $\tau > \bar{\tau}$. Then there exists R^* such that the Dirichlet problem corresponding to (1.1) admits a positive radial solution in any ball of radius $r > R^*$.*

- *Assume that there exists $\hat{\tau}$ such that $\bar{M}(\tau) > 0$ for any $\tau < \hat{\tau}$. Then there exists R_* such that the Dirichlet problem corresponding to (1.1) admits a positive radial solution $u(r)$ in the exterior of any ball of radius $R < R^*$, that is there is a solution $u(r)$ of (1.1) such that $u(R) = 0$, $u(r) > 0$ for $r > R$ and $u(r) \sim r^{-\frac{n-p}{p-1}}$ as $r \to \infty$.*

Proof. The proof is completely analogous to the one given for Theorem 4.4, at the end of section 4. □

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