FOWLER TRANSFORMATION AND RADIAL SOLUTIONS FOR QUASILINEAR ELLIPTIC EQUATIONS.
PART 2: NONLINEARITIES OF MIXED TYPE.

Abstract. We discuss the existence and the asymptotic behavior of positive radial solutions for the following equation:

$$\Delta_p u(\mathbf{x}) + f(u, |\mathbf{x}|) = 0,$$

where $\Delta_p u = div(|Du|^{p-2}Du)$, $\mathbf{x} \in \mathbb{R}^n$, $n > p > 1$, and we assume that $f \geq 0$ is subcritical for u large and $|\mathbf{x}|$ small and supercritical for u small and $|\mathbf{x}|$ large, with respect to the Sobolev critical exponent.

We give sufficient conditions for the existence of ground states with fast decay. As a corollary we also prove the existence of ground states with slow decay and of singular ground states with fast and slow decay.

For the proofs we use a Fowler transformation that enables us to use dynamical arguments. This approach allows to unify the study of different types of nonlinearities and to complete the results already appeared in literature with the analysis of singular solutions.

1. Introduction

The purpose of this paper is to investigate positive radial solutions of the following quasi-linear elliptic equation

$$\Delta_p u(\mathbf{x}) + f(u, |\mathbf{x}|) = 0, \tag{1.1}$$

where $\Delta_p u = div(|Du|^{p-2}Du)$ is the so called p-Laplacian, $\mathbf{x} \in \mathbb{R}^n$, $n > p > 1$, and $f(u, |\mathbf{x}|)$ is a continuous function which is positive and locally Lipschitz in the u variable for $u > 0$ and it is null for $u = 0$. We also assume f to be super-half linear, see hypotheses **F0** and **G0** below. The prototypical nonlinearities we are interested in are

$$f(u, |\mathbf{x}|) = k(|\mathbf{x}|)u|u|^{q-2}, \tag{1.2}$$

where $q > p$ and k is positive and continuous, and

$$f(u, |\mathbf{x}|) = k_2(|\mathbf{x}|)\frac{u|u|^{q_2-2}}{1 + k_1(|\mathbf{x}|)u|u|^{q_1-2}}, \tag{1.3}$$

where $q_2 > q_1 > 0$, $q_2 - q_1 > p - 1$ and the functions $k_1(|\mathbf{x}|)$ and $k_2(|\mathbf{x}|)$ are nonnegative and continuous. Another function f that matches the hypothesis is

$$f(u, |\mathbf{x}|) = k(|\mathbf{x}|) \times \begin{cases} u|u|^{q_1-2}, & \text{if } |u| \geq 1; \\ u|u|^{q_2-2}, & \text{if } |u| \leq 1 \end{cases} \tag{1.4}$$

We consider just radial solutions and we commit the following abuse of notation: we write $u(r)$ for $u(\mathbf{x})$ where $|\mathbf{x}| = r$. Since we only deal with radial solutions we will in fact consider the following singular O.D.E.

$$(u'|u'|^{p-2})' + \frac{n-1}{r}u'|u'|^{p-2} + f(u, r) = 0. \tag{1.5}$$

Here $'$ denotes the derivative with respect to r. We call "regular" the positive solutions $u(d, r)$ of (1.5) satisfying the following initial condition

$$u(0) = d > 0 \qquad\qquad u'(0) = 0. \tag{1.6}$$

We call "singular" the positive solutions $u(r)$ which are singular in the origin, that is $\lim_{r\to 0} u(r) = +\infty$.

In particular we focus our attention on the problem of existence of ground states (G.S.), of singular ground states (S.G.S.) and of crossing solutions. By G.S. we mean a positive regular solution $u(r)$ defined for any $r \geq 0$ such that $\lim_{r\to\infty} u(r) = 0$. A S.G.S. of equation (1.1) is a singular positive solution $v(r)$ such that $\lim_{|r|\to 0} v(r) = +\infty$ and $\lim_{|r|\to+\infty} v(r) = 0$. Crossing solutions are radial solutions $u(r)$ such that $u(r) > 0$ for any $0 \leq r < R$ and $u(R) = 0$ for a certain $R > 0$, so they can be considered as solutions of the Dirichlet problem in the ball of radius R.

It is possible to prove in a very general context, see Lemma 2.2 below, that the limit $\lim_{r\to+\infty} u(r) r^{\frac{n-p}{p-1}} = L$ exists and it is positive for all the solutions $u(r)$ which are positive for r large. If L is finite we say that $u(r)$ has fast decay, while if $L = +\infty$ we say that it has slow decay.

Let us denote by $F(u,r) = \int_0^u f(s,r)ds$ and by $\mathfrak{F}(u,r) = f(u,r)/|u|^{p-1}$. We will usually consider functions f satisfying the following:

F0: There are $M > 0$ and $R > 0$ such that $\mathfrak{F}(u,r)$ is increasing in u whenever $(u,r) \in \big([M,+\infty) \times (0,1/R]\big) \cup \big((0,1/M] \times [R,+\infty)\big)$.

In fact we will consider a slightly more general assumption introduced in [16], which will be stated precisely in the next section. When f is of type (1.2) it matches our hypothesis whenever $q > p$. It is well known that in this case the structure of positive solutions changes drastically when the parameter q is larger or smaller than some critical values: $p^* := \frac{np}{n-p}$ the Sobolev critical exponent and $\sigma = p\frac{n-1}{n-p}$ see e.g. [13, 25].

In the last 20 years this family of equations has received a lot of interest, both for the intrinsic mathematical interest and for the applications it has in different areas such as astrophysics, differential geometry for $p = 2$, and elasticity theory and the study of non Newtonian fluids for $p \neq 2$, see [3, 5, 16, 18, 22]. We wish to point out also that radially symmetric solutions are particularly important for these problems. In fact when the domain has radial symmetry it is known that, for a large family of spatial independent nonlinearities f, G.S. and solutions of the Dirichlet problem in balls have to be radially symmetric. This fact has been proved using moving plane techniques in [8, 9, 32, 34]. The same result has been proved when $p = 2$, f is of type (1.2) and it is subcritical for u small and supercritical for u large, see [3], and it is a general characteristic of solutions of (1.1), even if there are some interesting counterexamples, see [3]. Moreover the α-limit and the ω-limit set of certain parabolic equations associated to (1.1) is made up by the union of radially symmetric ground states of (1.1), see [26].

There are several results for existence of G.S. and solutions of the Dirichlet problem, even in non-symmetric domains, obtained via variational techniques. However it is difficult to apply these methods when the non-linearity f is either critical or supercritical, e. g. f is of type (1.2) and $q \geq p^*$, due to the lack of compactness of the problem. In order to recover some compactness, especially in the supercritical case, the analysis is often restricted to radial solutions. Another key ingredient for the analysis of these equation is the Pohozaev identity, see [23, 24, 25, 27, 29] et al. which is a clever way to restate Green formula for the solutions of these problems; see also [28] for a more detailed discussion of this tool also in different context.

When f is of type (1.2) or (1.3) roughly speaking positive solutions exhibit two typical structures separated by a third one that lies in the border between them, see [16].

Sub: All the regular solutions $u(d,r)$ are crossing solutions and they have negative slope at their first zero $R(d)$, there is a unique S.G.S. with slow decay and uncountably many S.G.S. with fast decay. No G.S. can exist.

Crit: All the regular solutions are G.S. with fast decay, there are uncountably many S.G.S. with slow decay. No other positive solutions can exist.

Sup: All the regular solutions are G.S. with slow decay, there is a unique S.G.S. and has slow decay. There are uncountably many solutions $u(r)$ of the Dirichlet problem in the exterior of balls, i. e. there is $R > 0$ such that $u(R) = 0$, $u(r) > 0$ for $r > R$ and $u(r)$ has fast decay.

When f is of type (1.2) and it is spatial independent it is well known that we have structure **Sup** for $q > p^*$, **Crit** for $q = p^*$ and **Sub** for $\sigma < q < p^*$. When $p < q \le \sigma$ regular solutions have structure **Sub** but singular solutions do not exist. Sufficient conditions to have one of these structures for positive solutions for f of type (1.2) are given in [13, 14, 16, 23]. Similar classification results were observed in the case $p = 2$ and f spatial independent for equations of type (1.3) in [7]. The general case of a function f satisfying **F0** and $p > 1$ is considered in [16].

When the nonlinearity f exhibits both subcritical and supercritical behavior the situation becomes richer and more interesting. When $p = 2$ and f is spatial independent and it is subcritical for u large and supercritical for u small (as in (1.3) where $q_2 - q_1 < p^* < q_2$ or in (1.4) where $p < q_1 < p^* < q_2$ and k, k_1, k_2 are positive constants), Erbe and Tang in [10] classified regular solutions. More recently Chern and Yotsutani in [7] managed to classify singular solutions as well, obtaining the following structure for positive solutions:

Mix: There are uncountably many crossing solutions, uncountably many G.S. with slow decay, and at least one G.S. with slow decay. There are uncountably many S.G.S. with fast decay, uncountably many solution of the Dirichlet problem in exterior domains, and uncountably many S.G.S. with slow decay.

In fact in [7] the authors just proved the existence and conjectured the uniqueness of the S.G.S. with slow decay, but as a Corollary of Theorem 3.10 we obtain uncountably many such solutions. In [33] the authors obtain the same structure for regular solutions in the case $p = 2$, assuming that f is of type (1.2) and k is "sufficiently increasing" for r small and "sufficiently decreasing" for r large (e.g. when $q = p^*$ and $k(r)$ behaves like a positive power for r small and $k(r)$ like a negative power for r large). Recently these results have been extended to the $p \neq 2$ case in [20]. In all these papers the results depend strongly on the Pohozaev identity, and use essentially ODE techniques.

This paper is thought as the sequel of [16] and the main purpose is to apply the dynamical method developed there to obtain structure **Mix** for positive solutions. The advantages of this approach lie in the possibility to discuss together nonlinearities of type (1.2) and (1.3) inserting them in a wider family and to extend the previous results to the case $p \neq 2$ and by giving also the classification of singular solutions. Moreover we are also able to refine the asymptotic estimates for singular and slow decay solutions. So we generalize the results given in [7], since we extend the results to the spatial dependent case and to the setting $p \neq 2$, and the ones in [20] since we consider more generic nonlinearities and we discuss singular solutions.

In the proofs we follow the way paved by Johnson, Pan and Yi in [21, 22], and later followed by Johnson, Battelli in [2] and Bamon Flores Dal Pino [1, 12]. So we introduce a dynamical system through a change of coordinates that generalizes the well known Fowler transform and we pass to a dynamical system. Then we use this new point of view on the problem; so we prove the existence of unstable and stable sets $W^u(\tau)$ and $W^s(\tau)$ which are made up of initial conditions which correspond respectively to regular and fast decay solutions. Then we establish their mutual position using the transposition of the Pohozaev function for this dynamical context and we conclude with elementary analysis of the phase portrait.

We remark that when f is of type (1.2), $q = p^*$ and $k(r)$ is uniformly positive and bounded the situation is more delicate. In fact Bianchi in [4] found some example in which no radial G.S. may exist and in [5] Bianchi and Egnell have provided several very generic conditions that are sufficient for the existence of them. See also [2] and [18] for some multiplicity results when k is a perturbation of a constant. In this setting it is possible to prove the existence of G.S. with fast decay even if $k(r)$ is decreasing for r small and increasing for r large, that is the opposite situation with respect to the one we discuss in this paper. We stress that in that case anyway we do not have G.S. with slow decay neither S.G.S. with fast decay, so positive solutions do not have structure **Mix**. That situation has some analogy to the case where f is subcritical for u small and supercritical for u large, that is the opposite setting with respect to the one considered in this paper. We stress that there are very few papers concerning that context which seems to be the most delicate. Among them we wish to quote [12] and [1] in which the authors use a dynamical approach similar to the one introduced in [22] and followed also in this paper.

This paper is divided as follows. In section 2 we review some known facts: in section 2 we introduce the generalized Fowler transformation and we establish the existence of unstable and stable sets $W^u(\tau)$ and $W^s(\tau)$ for the non-autonomous problem via Wazewski's principle. In section 3 we use the Pohozaev function to establish the mutual position of $W^u(\tau)$ and $W^s(\tau)$ and we prove the main result, Theorem 3.10: this Theorem is very general but difficult to be understood. So in section 4 we apply it to specific cases to obtain easier to read Corollaries. In the Appendix we explain some technical construction needed to construct the stable and the unstable sets.

We stress that all the results of this paper can be extended to more generic equations involving spatial dependent generalizations of the p-Laplace operator, see Remark 3.12 and [19], where this fact was first noted. In fact, after [19], this generalization has been worked out in details also for similar equations governed by different nonlinearities, and thus exhibiting different structures for positive solutions: we quote [6, 17, 30] for the case where f is negative for u small and positive for u large, and [11, 31] for the case where f is negative.

2. Preliminary result: Fowler transformation and construction of stable and unstable sets

In this section we recall some known facts proved in [16]. First of all we introduce a dynamical system through the following change of coordinates depending on the parameter $l > p$:

$$\begin{aligned} &\alpha_l = \tfrac{p}{l-p}, \qquad \beta_l = \tfrac{(p-1)l}{l-p}, \qquad \gamma_l = \beta_l - (n-1), \\ &x_l = u(r)r^{\alpha_l} \qquad y_l = u'(r)|u'(r)|^{p-2}r^{\beta_l} \qquad r = e^t \\ &g_l(x_l,t) = f(xe^{-\alpha_l t}, e^t)e^{\alpha_l(l-1)t} \end{aligned} \tag{2.1}$$

Using (2.1) we pass from (1.5) to the following system:

$$\begin{pmatrix} \dot{x}_l \\ \dot{y}_l \end{pmatrix} = \begin{pmatrix} \alpha_l & 0 \\ 0 & \gamma_l \end{pmatrix} \begin{pmatrix} x_l \\ y_l \end{pmatrix} + \begin{pmatrix} y_l|y_l|^{\frac{2-p}{p-1}} \\ -g(x_l,t) \end{pmatrix} \tag{2.2}$$

Obviously positive solution $u(r)$ of (1.5) correspond to trajectories $\mathbf{x}_l(t) = (x_l(t), y_l(t))$ such that $x_l(t) > 0$ and $u'(r) < 0$ implies $y_l(t) < 0$ and viceversa. We denote by $G_l(x_l,t) := \int_0^{x_l} g_l(\xi,t)d\xi = F(x_l e^{-\alpha_l t}, e^t)e^{\alpha_l l t}$ and by $\mathfrak{G}_l(x_l,t) := g_l(x_l,t)/|x_l|^{p-1}$ $= \mathfrak{F}(x_l e^{-\alpha_l t}, e^t)e^{pt}$.

This change of coordinates generalizes the well known classical Fowler transformation that works in the $p = 2$ case. It was first introduced in [13] for functions f of the form (1.2). In this case we have $g_l(x,t) = h_l(t)x|x|^{q-2}$ and $G_l(x,t) = h(t)\frac{|x|^q}{q}$, where $h_l(t) = k(e^t)e^{\delta_l t}$ and $\delta_l = \alpha_l(l-q)$. Moreover, if we set $l = q$, we simply find $g_l(x,t) = k(e^t)x_l|x_l|^{q-2}$ and if k does not depend on t we find a 2 dimensional autonomous system which is completely understood, see [13]. In the whole paper we will assume (without explicitly mentioning it anymore) the following hypothesis which generalizes slightly **F0**, in order to guarantee that the basic features of (2.2) with $g(x,t) = k(e^t)x|x|^{q-2}$ are maintained.

> **G0:** There is $N > 0$ such that for any $|t| > N$ the function $\mathfrak{G}_l(x,t) = g_l(x,t)/x^{p-1}$ is such that $\mathfrak{G}_l(0,t) = 0$, $\lim_{x\to\infty}\mathfrak{G}_l(x,t) = \infty$ and it is increasing in x for any $|t| > N$.

Observe that **G0** holds for (1.2), (1.3), (1.4). Let us denote by $\mathcal{G}_l(x,t) = G_l(x,t)/x^p$: note that if $\mathfrak{G}_l(x,t)$ satisfy **G0** then $\mathcal{G}_l(x,t)$ has the same property, see Remark 2.2 in [16].

A key role in this analysis will be played by the Pohozaev function $P(u,u',r)$, introduced by Pucci and Serrin in [27], which is one of the main tool for the investigation of this family of equations. In fact we use its transposition for this dynamical setting $H_l(x_l,y_l,t)$, which acts as a sort of Lijapunov function:

$$P(u,u',r) := r^n\left[\frac{n-p}{p}\frac{uu'|u'|^{p-2}}{r} + \frac{p-1}{p}|u'|^p + F(u,r)\right]$$

$$H_l(x_l,y_l,t) := \frac{n-p}{p}x_l y_l + \frac{p-1}{p}|y_l|^{\frac{p}{p-1}} + G_l(x_l,t)$$

If $\mathbf{x}_{p^*}(t) = (x_{p^*}(t), y_{p^*}(t))$ and $\mathbf{x}_l(t) = (x_l(t), y_l(t))$ are the trajectories of (2.2) corresponding to $u(r)$, we have

$$P(u(r),u'(r),r) = H_{p^*}(x_{p^*}(t),y_{p^*}(t),t) = e^{-(\alpha_l+\gamma_l)t}H_l(x_l(t),y_l(t),t)\,. \tag{2.3}$$

The key observation is that when $G_{p^*}(x,t)$ is differentiable with respect to t, for any trajectory $\mathbf{x}_{p^*}(t)$ we have the following:

$$\frac{d}{dt}H_{p^*}(x_{p^*}(t),y_{p^*}(t),t) = \frac{\partial}{\partial t}G_{p^*}(x_{p^*}(t),t) \tag{2.4}$$

This is in fact another way to restate the Pohozaev identity, which is one of the main tool to investigate this equation, see e. g. [23, 24].

We recall some basic facts concerning positive solutions of (1.5), see [16].

2.1. *Remark.* For any $d > 0$ there is a $\rho(d) > 0$ such that (1.5), (1.6) admits a unique solution $u(d,r)$ which is positive and decreasing for $r \in [0,\rho(d))$. Moreover for any $R < \rho(d_1)$ and any $\epsilon > 0$ there is $\delta > 0$ such that $\max_{r\in[0,R]}|u(d_1,r) - u(d_2,r)| + |u'(d_1,r) - u'(d_2,r)| < \epsilon$ whenever $|d_1 - d_2| < \delta$.

2.2. **Lemma.**
- *If a solution $u(r)$ of (1.5) is positive for $0 < r \le R$, then $u'(r) < 0$ for $0 < r < R$.*
- *Assume that for any $x > 0$ $\limsup_{t\to+\infty} g_\sigma(x,t) < \infty$. If a solution $v(r)$ of (1.5) is positive and decreasing for any $r > R$, then $v(r)r^{\frac{n-p}{p-1}}$ is strictly increasing for any $r > R$. Moreover if $w(r)$ is such that $P(w(r),w'(r),r) < 0$ for $r \in (R_1,R_2)$, then $w(r)r^{\frac{n-p}{p-1}}$ is increasing in that interval.*

We say that a positive solution $v(r)$ has fast decay when the limit $v(r)r^{\frac{n-p}{p-1}}$ is finite and that it has slow decay when this limit is infinity. From the previous Lemma we know that the limit always exist.

We review quickly the results concerning the autonomous case, that is $g_l(x,t) \equiv \bar{g}_l(x)$ for a certain $l > p$. This will provide us some sub and super solutions for the original non-autonomous problem. First of all when $l > \sigma$, system (2.2) with $g_l(x,t) \equiv \bar{g}_l(x)$ admits three critical points: the origin, $\mathbf{P} = (P_x, P_y)$, where $P_x =$

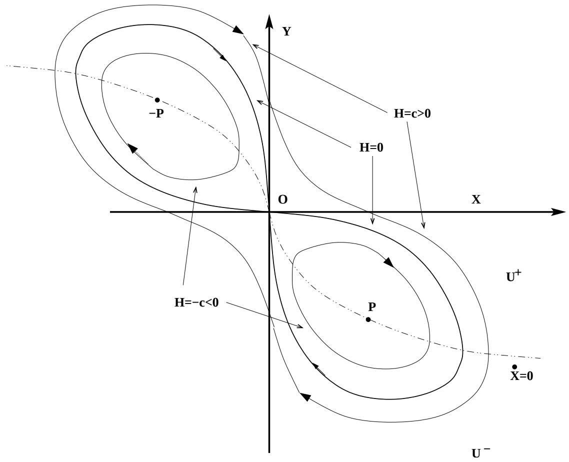

FIGURE 1. The level sets of the function $H(x,y,t)$ for t fixed

$\mathfrak{G}^{-1}(|\gamma_l(\alpha_l)^{p-1}|) > 0$ and $P_y = -(\alpha_l P_x)^{p-1} < 0$, and $-\mathbf{P}$. Furthermore $H_l(\mathbf{P}) = -C < 0$ and $\mathbf{P}$ is a repulser for $\sigma < l < p*$, a center for $l = p^*$, and an attractor for $l > p^*$. If $l \neq p^*$ there are no periodic trajectories, and if $p < l \leq p^*$, there are no critical points but the origin, see [16]. We introduce now some notation that will be in force through the whole paper:

$$\mathbb{R}^2_+ := \{(x,y) \mid x \geq 0\} \qquad \mathbb{R}^2_\pm := \{(x,y) \mid y < 0 < x\}$$

$$U_l^+ := \{(x,y) \in \mathbb{R}^2_+ \mid \alpha_l x + y|y|^{\frac{2-p}{p-1}} > 0\} \quad U_l^- := \{(x,y) \in \mathbb{R}^2_+ \mid \alpha_l x + y|y|^{\frac{2-p}{p-1}} < 0\}$$

$$U_l^0 := \{(x,y) \in \mathbb{R}^2_+ \mid \alpha_l x + y|y|^{\frac{2-p}{p-1}} = 0\}$$

Observe that $U^0_{l_2} \subset U^+_{l_1}$ and $U^0_{l_1} \subset U^-_{l_2}$ if $l_2 > l_1$. We denote by capital letters the trajectories of the autonomous system to distinguish them from the ones of the non-autonomous system. We denote by $\mathbf{X}_{\bar{l}}(t,\tau;\mathbf{Q},\bar{g}_{\bar{l}}) = (X_{\bar{l}}(t,\tau;\mathbf{Q},\bar{g}_{\bar{l}}), Y_{\bar{l}}(t,\tau;\mathbf{Q},\bar{g}_{\bar{l}}))$ the trajectory of the autonomous system (2.2) where $l = \bar{l}$ and $g_{\bar{l}}(x,t) \equiv \bar{g}_{\bar{l}}(x)$, departing from $\mathbf{Q}$ at $t = \tau$. We denote by $\mathbf{x}_{\bar{l}}(t,\tau;\mathbf{Q})$ the trajectory of the non-autonomous system (2.2) departing from $\mathbf{Q}$ at $t = \tau$.

We stress that when $p > 2$ local uniqueness on the coordinate axes is not guaranteed anymore. However it is possible to prove that the origin admits an unstable manifold, denoted by $M^u(\bar{g})$, whenever $l > p$ and a stable set which is compact and connected (it is the union of locally Lipschitz trajectories) denoted by $M^s(\bar{g})$, see [15] and [16]. Let $u(r)$ be a solution of (1.5) and $\mathbf{X}_l(t,\tau,\mathbf{Q})$ the corresponding trajectory of (2.2). In [16] it is proved that if $\mathbf{Q} \in M^u(\bar{g})$ then $u(r)$ is a regular solution and viceversa, while if $\mathbf{Q} \in M^s(\bar{g})$ then $u(r)$ has fast decay and viceversa.

Using the Pohozaev function H_{p^*} it is possible to establish their mutual positions, see [16], and to draw a picture of the phase portrait. Using this picture it is possible to classify completely positive solutions, both regular and singular, see [16].

We introduce now some hypotheses that guarantee the existence of stable and unstable sets for the non-autonomous system (2.2). Choose $\tau \in \mathbb{R}$, we introduce the following functions

$$\begin{array}{ll} a_l^\tau(x_l) = \inf_{t \leq \tau} 1/2 g_l(x_l,t) & b_l^\tau(x_l) = \sup_{t \leq \tau} 2 g_l(x_l,t) \\ A_l^\tau(x_l) = \inf_{t \geq \tau} 1/2 g_l(x_l,t) & B_l^\tau(x_l) = \sup_{t \geq \tau} 2 g_l(x_l,t) \end{array} \tag{2.5}$$

These functions are monotone increasing in x for any τ and satisfy $\mathbf{G0}$, when they are not identically null or infinity. We will make use of some of these assumptions:

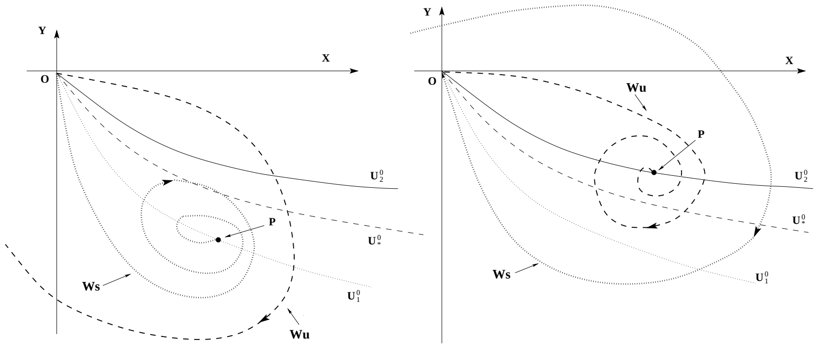

FIGURE 2. A sketch of M_l^u and of M_l^s when $l > p^*$ (on the left) and $\sigma < l < p^*$ (on the right). We have denoted by $U_1^0 = U_{l_1}^0$, by $U_2^0 = U_{l_2}^0$, by $U_*^0 = U_{p^*}^0$, where $l_1 < p^* < l_2$.

- **G1:** There is $l_1 > p$ such that for any $x > 0$ the function $g_{l_1}(x,t)$ converges to a t-independent locally Lipschitz function $g_{l_1}^{-\infty}(x) \not\equiv 0$ as $t \to -\infty$, uniformly on compact intervals.
- **G1′:** There is $l_1 > p$ such that for any $\tau \in \mathbb{R}$ the functions $a_{l_1}^\tau$ and $b_{l_1}^\tau$ are locally Lipschitz. Moreover for any $x > 0$ and any $\tau < 0$, we have $0 < a_{l_1}^\tau(x) < b_{l_1}^\tau(x) < \infty$.
- **G1″:** There are $j_1 \geq i_1 > p$, such that for any $\tau \in \mathbb{R}$ the functions $a_{j_1}^\tau$ and $b_{i_1}^\tau$ are locally Lipschitz. Moreover for any $x > 0$ we have $a_{j_1}^\tau(x) > 0$, and $b_{i_1}^\tau(x) < \infty$.
- **G2:** There is $l_2 > \sigma$ such that for any $x > 0$ the function $g_{l_2}(x,t)$ converges to a t-independent locally Lipschitz function $g_{l_2}^{+\infty}(x) \not\equiv 0$ as $t \to +\infty$, uniformly on compact intervals.
- **G2′:** There is $l_2 > \sigma$ such that such that for any $\tau \in \mathbb{R}$ the functions $A_{l_2}^\tau$ and $B_{l_2}^\tau$ are locally Lipschitz. Moreover for any $x > 0$ and any $\tau > 0$, we have $0 < A_{l_2}^\tau(x) < B_{l_2}^\tau(x) < \infty$.
- **G2″:** There are $j_2 \geq i_2 > \sigma$, such that for any $\tau \in \mathbb{R}$ the functions $A_{i_2}^\tau$ and $B_{j_2}^\tau$ are locally Lipschitz. Moreover for any $x > 0$ we have $A_{i_2}^\tau(x) > 0$, and $B_{j_2}^\tau(x) < \infty$.

Obviously **G1** implies **G1′** which implies **G1″**, and **G2** implies **G2′** which implies **G2″**.

In order to construct stable and unstable sets in the non-autonomous case and to localize them we introduce the following barrier sets. Assume first **G1′** and **G2′**; we recall that $M_{l_1}^u(a_{l_1}^\tau)$ is the unstable manifold of the autonomous systems (2.2) where $g_{l_1}(x_{l_1}, t) \equiv a_{l_1}^\tau(x_{l_1})$. Follow $M_{l_1}^u(a_{l_1}^\tau)$ from the origin towards $\mathbb{R}_+^2$: it intersects the isocline $U_{l_1}^0$ in a point denoted by $\tilde{\mathbf{Q}}_{l_1}^u(a_{l_1}^\tau)$. We denote by $\tilde{M}_{l_1}^u(a_{l_1}^\tau)$ the branch of the unstable manifold of $M_{l_1}^u(a_{l_1}^\tau)$ of the autonomous systems (2.2) where $g_{l_1}(x_{l_1}, t) \equiv a_{l_1}^\tau(x_{l_1})$, between the origin and $\tilde{\mathbf{Q}}_{l_1}^u(a_{l_1}^\tau)$; we give the analogous definition for $\tilde{\mathbf{Q}}_{l_1}^u(b_{l_1}^\tau)$ and $\tilde{M}_{l_1}^u(b_{l_1}^\tau)$. Moreover $\tilde{M}_{l_1}^u(a_{l_1}^\tau)$ and $\tilde{M}_{l_1}^u(b_{l_1}^\tau)$ do not intersect and the former is on the right of the latter (here and later we think of the x axis as horizontal and the y axis as vertical), see Lemma 3.1 in [16]. We denote by $\tilde{c}_{l_1}^u(\tau)$ the branch of $U_{l_1}^0$ between $\tilde{\mathbf{Q}}_{l_1}^u(a_{l_1}^\tau)$ and $\tilde{\mathbf{Q}}_{l_1}^u(b_{l_1}^\tau)$. Finally we denote by $\tilde{E}_{l_1}^u(\tau)$ the bounded set enclosed by $\tilde{M}_{l_1}^u(a_{l_1}^\tau)$, $\tilde{M}_{l_1}^u(b_{l_1}^\tau)$ and $\tilde{c}_{l_1}^u(\tau)$.

Analogously we follow $M_{l_2}^s(A_{l_2}^\tau)$ from the origin towards $\mathbb{R}_+^2$ and we denote by $\tilde{\xi}_{l_2}^s(A_{l_2}^\tau)$ its first intersection with $U_{l_2}^0$; in fact $\tilde{\xi}_{l_2}^s(A_{l_2}^\tau)$ is a compact connected set.

We denote by $\tilde{M}^s_{l_2}\big(A^\tau_{l_2}\big)$ the branch of $M^s_{l_2}\big(A^\tau_{l_2}\big)$ between the origin and $\tilde{\xi}^s_{l_2}\big(A^\tau_{l_2}\big)$, and we give the analogous definitions for $\tilde{\xi}^s_{l_2}\big(B^\tau_{l_2}\big)$ and $\tilde{M}^s_{l_2}\big(B^\tau_{l_2}\big)$. Again $\tilde{M}^s_{l_2}\big(A^\tau_{l_2}\big)$ is on the right of $\tilde{M}^s_{l_2}\big(B^\tau_{l_2}\big)$, and they do not intersect. It follows that $\tilde{\xi}^s\big(A^\tau_{l_2}\big)$ is on the right of $\tilde{\xi}^s\big(B^\tau_{l_2}\big)$ as well: we denote by $\tilde{\mathbf{Q}}^s_{l_2}\big(A^\tau_{l_2}\big)$ the left endpoint of $\tilde{\xi}^s_{l_2}\big(A^\tau_{l_2}\big)$ and by $\tilde{\mathbf{Q}}^s_{l_2}\big(B^\tau_{l_2}\big)$ the right endpoint of $\tilde{\xi}^s_{l_2}\big(B^\tau_{l_2}\big)$, again see [16] Lemma 3.1. We denote by $\tilde{c}^s_{l_2}(\tau)$ the branch of $U^0_{l_2}$ between $\tilde{\mathbf{Q}}^s_{l_2}\big(A^\tau_{l_2}\big)$ and $\tilde{\mathbf{Q}}^s_{l_2}\big(B^\tau_{l_2}\big)$. Then we denote by $\tilde{E}^s_{l_2}(\tau)$ the bounded set enclosed by $\tilde{M}^s_{l_2}\big(A^\tau_{l_2}\big)$, $\tilde{M}^s_{l_2}\big(B^\tau_{l_2}\big)$ and $\tilde{c}^s_{l_2}(\tau)$.

Let us denote by

$$W^u_l(\tau) := \{\mathbf{Q} \in \mathbb{R}^2 \,|\, \mathbf{x}_l(t,\tau;\mathbf{Q}) \in U^+_l \text{ for any } t<\tau\}$$
$$W^s_l(\tau) := \{\mathbf{Q} \in \mathbb{R}^2 \,|\, \mathbf{x}_l(t,\tau;\mathbf{Q}) \in U^-_l \text{ for any } t>\tau\}$$

It is easy to prove that if $\mathbf{Q^u} \in W^u_{l_1}(\tau)$ respectively $\mathbf{Q^s} \in W^s_{l_2}(\tau)$ then $\lim_{t\to-\infty}\mathbf{x}_l(t,\tau;\mathbf{Q^u}) = (0,0)$ resp. $\lim_{t\to+\infty}\mathbf{x}_l(t,\tau;\mathbf{Q^s}) = (0,0)$, see [16]. A priori these sets may be empty. In [16] it is proved that the flow of the non-autonomous system (2.2) with $l=l_1$ on $\tilde{M}^u_{l_1}\big(a^\tau_{l_1}\big)$ and $\tilde{M}^u_{l_1}\big(b^\tau_{l_1}\big)$ points towards the interior of $\tilde{E}^u_{l_1}(\tau)$, and the flow of (2.2) with $l=l_2$ on $\tilde{M}^s_{l_2}\big(A^\tau_{l_2}\big)$ and $\tilde{M}^s_{l_2}\big(B^\tau_{l_2}\big)$ points towards the exterior of $\tilde{E}^s_{l_2}(\tau)$. Exploiting this fact we can prove the following, see [16] Lemma 3.4.

2.3. **Lemma.** *Assume* $\mathbf{G1'}$*, then for any* $\tau \in \mathbb{R}$ *there is a compact connected subset* $\tilde{W}^u_{l_1}(\tau) \subset W^u_{l_1}(\tau)$ *which contains the origin and intersects* $U^0_{l_1}$ *in a compact connected set* $\tilde{\xi}^u_{l_1}(\tau)$*. Analogously assume* $\mathbf{G2'}$*, then for any* $\tau \in \mathbb{R}$ *there is a compact connected subset* $\tilde{W}^s_{l_2}(\tau) \subset W^s_{l_2}(\tau)$ *which contains the origin and intersects* $U^0_{l_2}$ *in a compact connected set* $\tilde{\xi}^s_{l_2}(\tau)$*.*

Using the flow of (2.2) we can define global stable and unstable sets:

$$\mathfrak{W}^u_{l_1}(\tau) := \cup_{T\in\mathbb{R}}\{\mathbf{P} \,|\, \exists \mathbf{Q} \in \tilde{W}^u_{l_1}(T) \; s.t. \; \mathbf{P} = \mathbf{x}_{l_1}(\tau,T;\mathbf{Q})\}\,,$$
$$\mathfrak{W}^s_{l_2}(\tau) := \cup_{T\in\mathbb{R}}\{\mathbf{P} \,|\, \exists \mathbf{Q} \in \tilde{W}^s_{l_2}(T) \; s.t. \; \mathbf{P} = \mathbf{x}_{l_2}(\tau,T;\mathbf{Q})\}$$

Obviously if $\mathbf{P_1} \in \mathfrak{W}^u_{l_1}(\tau)$ and $\mathbf{P_2} \in \mathfrak{W}^s_{l_2}(\tau)$, then $\lim_{t\to-\infty}x_{l_1}(t,\tau;\mathbf{P_1}) = (0,0)$ and $\lim_{t\to+\infty}x_{l_2}(t,\tau;\mathbf{P_2}) = (0,0)$. In fact it can be shown that trajectories $\mathbf{x}_{l_1}(t,\tau,\mathbf{Q})$, where $\mathbf{Q} \in \tilde{W}^u_{l_1}(\tau)$, correspond to regular solutions of (1.5) while if $\mathbf{Q} \in \tilde{W}^s_{l_2}(\tau)$ they correspond to fast decay solutions of (1.5), as in the autonomous case. So this is also a way to prove the existence of fast decay solutions for our problem in the general case $p \neq 2$: when $p=2$ the existence of such solutions follows directly from the Kelvin inversion, see e. g. [33], but this tool is not anymore available when $p \neq 2$. In fact we can push these correspondences a bit further, but we need to introduce further barrier sets, see [16] and figure 3.

We start by assuming $\mathbf{G1'}$ and $\mathbf{G2'}$. We set $\bar{\mathbf{B}}^u_{l_1}(\tau) = (\bar{B}^u_x(\tau), \bar{B}^u_y(\tau)) := \tilde{\mathbf{Q}}^u(b^\tau_{l_1})$ and we denote by $\bar{\mathbf{A}}^u_{l_1}(\tau)$ the point of intersection between $\tilde{M}^u_l(a^\tau_{l_1})$ and the line $x = \bar{B}^u_x(\tau)$. Then we denote by $\partial\bar{E}^{u,a}_{l_1}(\tau)$ the branch of $\tilde{M}^u_{l_1}(a^\tau_{l_1})$ between the origin and $\bar{\mathbf{A}}^u_{l_1}(\tau)$. We denote by $\bar{c}^u_{l_1}(\tau)$ the line between $\bar{\mathbf{B}}^u_{l_1}(\tau)$ and $\bar{\mathbf{A}}^u_{l_1}(\tau)$. Finally we denote by $\bar{E}^u_{l_1}(\tau)$ the subset enclosed by $\partial\bar{E}^{u,a}_{l_1}(\tau)$, $\partial\bar{E}^{u,b}_{l_1}(\tau)$ and $\bar{c}^u_{l_1}(\tau)$.
Analogously we set $\bar{\mathbf{B}}^s_{l_2}(\tau) = (\bar{B}^s_x(\tau), \bar{B}^s_y(\tau)) := \tilde{\mathbf{Q}}^\mathbf{s}(B^\tau_{l_2})$ and we denote by $\bar{\mathbf{A}}^s_{l_2}(\tau)$ the point of intersection between $\tilde{M}^s_{l_2}(A^\tau_{l_2})$ and the line $x = \bar{B}^s_x(\tau)$. Then we denote by $\partial\bar{E}^{s,a}_{l_2}(\tau)$ the branch of $\tilde{M}^s_{l_2}(A^\tau_{l_2})$ between the origin and $\bar{\mathbf{A}}^s_{l_2}(\tau)$. We denote by $\bar{c}^s_{l_2}(\tau)$ the line between $\bar{\mathbf{B}}^s_{l_2}(\tau)$ and $\bar{\mathbf{A}}^s_{l_2}(\tau)$. Finally we denote by $\bar{E}^s_{l_2}(\tau)$ the subset enclosed by $\partial\bar{E}^{s,a}_{l_2}(\tau)$, $\partial\bar{E}^{s,b}_{l_2}(\tau)$ and $\bar{c}^s_{l_2}(\tau)$. Once again the flow of (2.2) with $l=l_1$ on $\partial\bar{E}^{u,a}_{l_1}(\tau)$, $\partial\bar{E}^{u,b}_{l_1}(\tau)$ points towards the interior of $\bar{E}^u_{l_1}(\tau)$ for any $t \le \tau$, while the flow of (2.2) with $l=l_2$ on $\partial\bar{E}^{s,a}_{l_2}(\tau)$, $\partial\bar{E}^{s,b}_{l_2}(\tau)$ points towards the exterior of

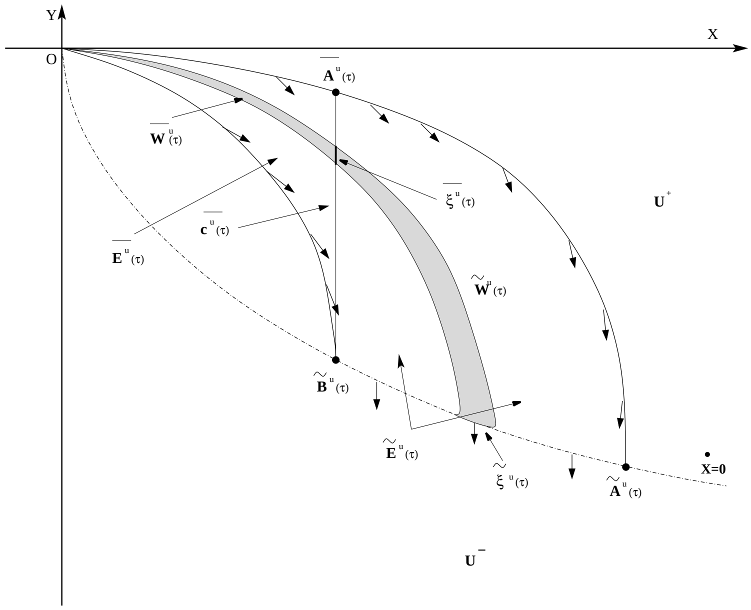

FIGURE 3. Construction of the unstable sets $\bar{W}^u_l(\tau)$ and $\tilde{W}^u_l(\tau)$.

$\bar{E}^s_{l_2}(\tau)$, see [16]. So using a kind of sub-super solution method we can construct the sets $\bar{W}^u_{l_1}(\tau)$ and $\bar{W}^s_{l_2}(\tau)$. These sets are closed and connected and join the origin with $\bar{c}^u_{l_1}(\tau)$ and $\bar{c}^s_{l_2}(\tau)$ respectively and have the following property:

$$\bar{W}^u_{l_1}(\tau) \subset \{\mathbf{Q} \in \mathbb{R}^2_+ \mid \mathbf{x}_{l_1}(t,\tau;\mathbf{Q}) \in \bar{E}^u_{l_1}(\tau) \text{ for any } t \le \tau\}$$
$$\bar{W}^s_{l_2}(\tau) \subset \{\mathbf{Q} \in \mathbb{R}^2_+ \mid \mathbf{x}_{l_2}(t,\tau;\mathbf{Q}) \in \bar{E}^s_{l_2}(\tau) \text{ for any } t \ge \tau\}$$

Let us denote by $\bar{\xi}^u_{l_1}(\tau) = \bar{c}^u_{l_1}(\tau) \cap \bar{W}^u_{l_1}(\tau)$ and $\bar{\xi}^s_{l_2}(\tau) = \bar{c}^s_{l_2}(\tau) \cap \bar{W}^s_{l_2}(\tau)$; observe that we can (and will) choose $\bar{W}^u_{l_1}(\tau) \subset \tilde{W}^u_{l_1}(\tau)$ and $\bar{W}^s_{l_2}(\tau) \subset \tilde{W}^s_{l_2}(\tau)$ for any $\tau \in \mathbb{R}$. Moreover from Lemma 3.5. in [16] we get the following.

2.4. **Lemma.** *Assume* $\mathbf{G1'}$ *and* $\mathbf{G2'}$*, and let* $\bar{\mathbf{Q}}^{\mathbf{u}} \in \bar{W}^u_{l_1}(\tau)$ *and* $\bar{\mathbf{Q}}^{\mathbf{s}} \in \bar{W}^s_{l_2}(\tau)$*. Then the solution of (1.5) corresponding to the trajectory* $\boldsymbol{x}_{l_1}(t,\tau;\bar{\mathbf{Q}}^{\mathbf{u}})$ *is a regular solution* $u(d,r)$*, where* $d = d(\tau;\bar{\mathbf{Q}}^{\mathbf{u}})$*. Moreover if* $\bar{\mathbf{Q}}^{\mathbf{u}} \in \bar{\xi}^u_{l_1}(\tau)$ *we have* $d(\tau;\bar{\mathbf{Q}}^{\mathbf{u}}) \to +\infty$ *as* $\tau \to -\infty$ *and viceversa,* $\tau \to +\infty$ *if* $d(\tau;\bar{\mathbf{Q}}^{\mathbf{u}}) \to 0$*, and if* $l_1 \le l_2$ *the viceversa holds as well.*

Analogously the solution of (1.5) corresponding to the trajectory $\boldsymbol{x}_{l_2}(t,\tau;\bar{\mathbf{Q}}^{\mathbf{s}})$ *is a fast decay solution* $v(L,r)$*, where* $L = L(\tau;\bar{\mathbf{Q}}^{\mathbf{s}}) = \lim_{r\to+\infty} v(L,r) r^{(n-p)/(p-1)}$*. Moreover if* $\bar{\mathbf{Q}}^{\mathbf{s}} \in \bar{\xi}^s_{l_2}(\tau)$ *we have* $L(\tau;\bar{\mathbf{Q}}^{\mathbf{s}}) \to +\infty$ *as* $\tau \to +\infty$ *and viceversa,* $\tau \to -\infty$ *as* $L(\tau;\bar{\mathbf{Q}}^{\mathbf{s}}) \to 0$ *as and if* $l_1 \le l_2$ *the viceversa holds as well.*

If **G1** holds we can prove something more, see [16].

2.5. *Remark.* Assume **G1** with $l_1 < p^*$. Then there is $D > 0$ such that $u(d,r)$ is a crossing solution for $d > D$ and its first zero $R(d)$ is such that $R(d) \to 0$ as $d \to +\infty$.

Let us introduce now the changes of coordinates from (2.2) with $l = l_u$ to (2.2) with $l = l_s$:

$$\aleph^t_{l_s,l_u}(x,y) = \big(x \exp[(\alpha_{l_s} - \alpha_{l_u})t], y \exp[(\beta_{l_s} - \beta_{l_u})t]\big)\,. \tag{2.6}$$

Let $u(r)$ be a solution of (1.5) and $\mathbf{x}_{l_u}(t)$ and $\mathbf{x}_{l_s}(t)$ be the trajectories of (2.2) with $l = l_u$ and $l = l_s$ respectively corresponding to $u(r)$; then $\aleph^t_{l_s,l_u}\big(\mathbf{x}_{l_u}(t)\big) = \mathbf{x}_{l_s}(t)$. If

$\mathbf{X}_{l_u}(t)$ is a trajectory of the autonomous system with $l_u > p$ and $\lim_{t\to-\infty}\mathbf{X}_{l_u}(t) = (0,0)$, then $\mathbf{X}_{l_s}(t) := \aleph^t_{l_s,l_u}\big(\mathbf{X}_{l_u}(t)\big)$ is such that $\lim_{t\to-\infty}\mathbf{X}_{l_s}(t) = (0,0)$, whenever $l_s > p$, see [16]. Analogously if $l_u > \sigma$ and $\lim_{t\to+\infty}\mathbf{X}_{l_u}(t) = (0,0)$, then $\lim_{t\to+\infty}\mathbf{X}_{l_s}(t) = (0,0)$ for any $l_s > \sigma$.

Let us introduce now the following modified polar coordinates:

$$x_l|x_l|^{p-2} = \rho_l\cos(\theta_l) \qquad\qquad y = \rho_l\sin(\theta_l) \tag{2.7}$$

Note that $\tan(\theta_l(t)) = u'(e^t)e^t/u(e^t)$, so in fact $\theta_l(t) = \theta(t)$ is independent of l and the sets defined by $\theta = const$ are invariant for $\aleph^t_{l_s,l_u}$. In particular U^0_l, which is defined by the relation $\theta = 1/(\alpha_l)\arctan(|\alpha_l|^{p-1})$, is invariant for $\aleph^t_{l_s,l_u}$. These diffeomorphisms allows us to introduce the unstable sets and the stable sets also for $l \neq l_1$ and $l \neq l_2$, and if we just assume $\mathbf{G1''}$ or $\mathbf{G2''}$; the construction of these sets with these weaker assumption is rather technical so it is postponed to the appendix.

More precisely we can construct the compact connected sets $\tilde{W}^u_{l^u}(\tau)$ and $\tilde{\xi}^u_{l^u}(\tau)$ for any $\tau \in \mathbb{R}$ whenever $l^u > p$ if $j_1 \le p^*$ and just for $l^u \ge j_1$ otherwise; analogously we can construct $\tilde{W}^s_{l^s}(\tau)$ and $\tilde{\xi}^s_{l^s}(\tau)$ for any $\tau \in \mathbb{R}$ whenever $l^s > \sigma$ if $i_2 \ge p^*$ and just for $l^s \le i_2$ otherwise. Moreover we can reprove Lemma 2.4 also with these weaker assumptions. Again the role of l_1 is played by j_1 and the role of l_2 is played by i_2.

Assume $\mathbf{G1''}$ and $\mathbf{G2''}$, then we can define global stable and unstable sets as follows: $\mathfrak{W}^u_{l^u}(\tau) = \aleph^\tau_{l^u,j_1}\big(\mathfrak{W}^u_{j_1}(\tau)\big)$, and $\mathfrak{W}^s_{l^s}(\tau) = \aleph^\tau_{l^s,i_2}\big(\mathfrak{W}^s_{i_2}(\tau)\big)$. Then if $\mathbf{Q} \in \mathfrak{W}^u_{l^u}(\tau)$, $\lim_{t\to-\infty}\mathbf{x}_{l^u}(t,\tau,\mathbf{Q}) = (0,0)$, while if $\mathbf{Q} \in \mathfrak{W}^s_{l^s}(\tau)$, $\lim_{t\to+\infty}\mathbf{x}_{l^s}(t,\tau,\mathbf{Q}) = (0,0)$. Then we follow $\mathfrak{W}^u_{l^u}(\tau)$ and $\mathfrak{W}^s_{l^s}(\tau)$ from the origin towards $\mathbb{R}^2_\pm$ and we denote by $\tilde{\xi}^u_{l^u}(\tau)$ the first intersection of $\mathfrak{W}^u_{l^u}(\tau)$ with $U^0_{l^u}$ and by $\tilde{\xi}^s_{l^s}(\tau)$ the first intersection of $\mathfrak{W}^s_{l^s}(\tau)$ with $U^0_{l^s}$. Then we denote by $\tilde{W}^u_{l^u}(\tau)$ the branch of $\mathfrak{W}^u_{l^u}(\tau)$ between the origin and $\tilde{\xi}^u_{l^u}(\tau)$, and by $\tilde{W}^s_{l^s}(\tau)$ the branch of $\mathfrak{W}^s_{l^s}(\tau)$ between the origin and $\tilde{\xi}^s_{l^s}(\tau)$.

We look for intersections between $\mathfrak{W}^u_l(\tau)$ and $\mathfrak{W}^s_l(\tau)$. For this purpose it is convenient to understand better the case $l = p^*$, and we need to assume $\mathbf{G1''}$ and $\mathbf{G2''}$ with $j_1 \le p^* \le i_2$. Observe first that as $\tau \to -\infty$ $|\bar{\mathbf{B}}^u_l(\tau)|$ becomes unbounded if $l > j_1$ and tends to 0 if $l < j_1$. Fix $x > 0$, then $b^\tau_{p^*}(x)$ and $B^\tau_{p^*}(x)$ are both positive and bounded for any τ. So we consider the autonomous system (2.2) with $l = p^*$, and the points $\tilde{\mathbf{Q}}^u_{p^*}(b^\tau_{p^*})$ and $\tilde{\mathbf{Q}}^s_{p^*}(B^\tau_{p^*})$. We denote by $\mathbf{B}^+(\tau) = (B^+_x(\tau), B^+_y(\tau)) := \tilde{\mathbf{Q}}^u_{p^*}(b^\tau_{p^*})$ and by $\mathbf{B}^-(\tau) = (B^-_x(\tau), B^-_y(\tau)) := \tilde{\mathbf{Q}}^s_{p^*}(B^\tau_{p^*})$. We set $c^+_{p^*}(\tau) := \{(x,y) \in U^+_{p^*} \mid x = B^+_x(\tau)\}$ and $c^-_{p^*}(\tau) := \{(x,y) \in U^-_{p^*} \mid x = B^-_x(\tau)\}$.

We have already seen that $\mathbf{x}_{p^*}(t,\tau,\mathbf{Q})$ corresponds to a regular solution of (1.5) whenever $\mathbf{Q} \in \mathfrak{W}^u_{p^*}(\tau)$ and to a fast decay solution whenever $\mathbf{Q} \in \mathfrak{W}^s_{p^*}(\tau)$. Moreover we can control $\mathfrak{W}^u_{p^*}(\tau)$ and $\mathfrak{W}^s_{p^*}(\tau)$ until they cross $U^0_{p^*}$. In the next section we will discuss in detail the case where the function $G_{p^*}(x,t)$ is increasing as $t \to -\infty$ and decreasing as $t \to +\infty$, roughly speaking. In such a case it is possible to obtain some further results concerning $\mathfrak{W}^u_{p^*}(\tau)$ and $\mathfrak{W}^s_{p^*}(\tau)$. In particular we can follow $\mathfrak{W}^u_{p^*}(\tau)$ until it crosses $c^-_{p^*}(\tau)$ and $\mathfrak{W}^s_{p^*}(\tau)$ until it crosses $c^+_{p^*}(\tau)$.

2.6. **Lemma.** *Assume $\mathbf{G1''}$ and that there is $T^+ < 0$ such that $G_{p^*}(x,t)$ is increasing in t for $t < T^+$ and any x. Then there is $\breve{M}^u > 0$ such that for any $\tau < -\breve{M}^u$, $\mathfrak{W}^u_{p^*}(\tau)$ contains a closed connected subset $\breve{W}^u_{p^*}(\tau)$ which contains the origin and intersects $c^-_{p^*}(\tau)$ in a set denoted by $\breve{\xi}^u_{p^*}(\tau)$. Moreover if $\mathbf{Q} \in \breve{W}^u_{p^*}(\tau)$, then $\mathbf{x}_{p^*}(t,\tau,\mathbf{Q}) \in \mathbb{R}^2_\pm$ for any $t \le \tau$.*

Analogously assume $\mathbf{G2''}$ and that there is $T^- > 0$ such that $G_{p^*}(x,t)$ is decreasing in t for any x and $t > T^-$. Then there is $\breve{M}^s > 0$ such that for any $\tau > \breve{M}^s$, $\mathfrak{W}^s_{p^*}(\tau)$ contains a closed connected subset $\breve{W}^s_{p^*}(\tau)$ which contains the *origin and intersects* $c^+_{p^*}(\tau)$ *in a set denoted by* $\breve{\xi}^s_{p^*}(\tau)$*. Moreover if* $\mathbf{Q} \in \breve{W}^s_{p^*}(\tau)$*, then* $\mathbf{x}_{p^*}(t,\tau,\mathbf{Q}) \in \mathbb{R}^2_\pm$ *for any* $t \geq \tau$*.*

The proof of this Lemma is postponed to the appendix. Since $\tilde{W}^u_l(\tau)$ and $\tilde{\xi}^u_l(\tau)$, $\tilde{W}^s_l(\tau)$ and $\tilde{\xi}^s_l(\tau)$ have been constructed through the continuous flow of (2.2) restricted to U^+_l and U^-_l respectively, they vary continuously in τ. More precisely we have the following: denote by $B(\mathbf{Q},\epsilon)$ the open ball of center $\mathbf{Q}$ and radius ϵ, and if A is a set denote by $B(A,\epsilon) = \cup_{\mathbf{Q}\in A} B(\mathbf{Q},\epsilon)$.

2.7. *Remark.* Assume that $\bar{\xi}^u_l(\tau)$ and $\bar{\xi}^u_l(\tau_0)$ exist. Then for any $\epsilon > 0$ there is $\delta > 0$ such that $B(\bar{\xi}^u(\tau_0),\epsilon) \cap \bar{\xi}^u(\tau) \neq \emptyset$ whenever $|\tau - \tau_0| < \delta$, for any $\tau_0 \in \mathbb{R}$.

From the previous Remark we easily get the following property.

2.8. **Lemma.** *Let* $\tau_1 \leq \tau_2$ *and* $\mathbf{Q_1} \in \bar{\xi}^u_{l_1}(\tau_1)$*,* $\mathbf{Q_2} \in \bar{\xi}^u_{l_1}(\tau_2)$*. If* $H : \mathbb{R}^2 \to \mathbb{R}$ *is a continuous function and* $H(\mathbf{Q_1}) = \alpha$*,* $H(\mathbf{Q_2}) = \beta$ *with* $\alpha < \beta$*, then for any* $\gamma \in (\alpha,\beta)$ *there is* $\tau \in [\tau_1,\tau_2]$ *and* $\mathbf{Q} \in \bar{\xi}^u_{l_1}(\tau)$ *such that* $H(\mathbf{Q}) = \gamma$*.*

The same continuity property is fulfilled by $\bar{W}^u(\tau)$, $\tilde{W}^u(\tau)$, $\breve{W}^u(\tau)$, $\tilde{\xi}^u(\tau)$, $\breve{\xi}^u(\tau)$, $\bar{W}^s(\tau)$, $\tilde{W}^s(\tau)$, $\breve{W}^s(\tau)$, $\bar{\xi}^s(\tau)$, $\tilde{\xi}^s(\tau)$, $\breve{\xi}^s(\tau)$.

3. The main result.

We begin this section with some basic facts, proved in [16] section 3.2, concerning the value of the function H_{p^*} along regular and fast decay solutions. First of all recall that for any finite value of t the level set $\{\mathbf{x} \in \mathbb{R}^2 \,|\, H_l(\mathbf{x},t) = 0\} \subset \mathbb{R}^2_\pm$ is bounded, see figure 1. Therefore if $H_l(\mathbf{x}_l(t),t) < 0$ for any $t \in (T_1,T_2)$ it follows that $\mathbf{x}_l(t) \in \mathbb{R}^2_\pm$ for any $t \in (T_1,T_2)$ and if T_1 and T_2 are finite $\mathbf{x}_l(t)$ can be continued till $t = T_1$ and $t = T_2$ and belongs to $\mathbb{R}^2_\pm$. This fact will be used several times in this paper.

3.1. *Remark.*
- Assume that for any $x > 0$ $\limsup_{t\to+\infty} g_\sigma(x,t) < \infty$. Then, if a solution $u(r)$ of (1.5) has fast decay, for the corresponding trajectory $\mathbf{x}_{p^*}(t)$ of (2.2) we have $\lim_{t\to+\infty} H_{p^*}(\mathbf{x}_{p^*}(t),t) = 0$.
- Assume $F(u,r)r^n \to 0$ as $r \to 0$ for any $u > 0$. Then, if $u(d,r)$ is a regular solution of (1.5), for the corresponding trajectory $\mathbf{x}_{p^*}(t)$ of (2.2) we have $\lim_{t\to-\infty} H_{p^*}(\mathbf{x}_{p^*}(t),t) = 0$.

3.2. **Lemma.** *Consider a trajectory* $\mathbf{x}_{p^*}(t)$ *of (2.2) and the corresponding solution* $u(r)$ *of (1.5) and denote by* $h^+ = \liminf_{t\to+\infty} H_{p^*}(\mathbf{x}_{p^*}(t),t)$*. Assume that there is* $l > p$ *such that* $\liminf_{t\to+\infty} G_l(x,t) > 0$ *for any* $x > 0$*. Follow* $\mathbf{x}_{p^*}(t)$ *forward in* t*, then if* $h^+ > 0$*,* $\mathbf{x}_{p^*}(t)$ *has to cross the negative* y_{p^*} *semi-axes. Moreover if* $u(r)$ *is positive and has slow decay then* $h^+ < 0$*.*

Analogously denote by $h^- = \liminf_{t\to-\infty} H_{p^*}(\mathbf{x}_{p^*}(t),t)$*, and assume that there is* $l > p$ *such that* $\liminf_{t\to-\infty} G_l(x,t) > 0$ *for any* $x > 0$*. Follow* $\mathbf{x}_{p^*}(t)$ *backwards in* t*, then if* $h^- > 0$ $\mathbf{x}_{p^*}(t)$ *has to cross the positive* y_{p^*} *semi-axes. Moreover if* $u(r)$ *is a singular solution then* $h^- < 0$*.*

Note that the assumptions of Remark 3.1 and Lemma 3.2 are satisfied if **G1″** and **G2″** hold. The proof of this Lemma is given explicitly in [16] Lemma 3.10, when h^+ and h^- are positive. But in fact from a careful analysis the proof of the part concerning singular and slow decay solutions follows.

Summing up, fast decay and regular solutions are such that H_{p^*} goes to 0 respectively as $t \to +\infty$ and as $t \to -\infty$. If H_{p^*} is uniformly positive along a trajectory, the corresponding regular solution cannot be positive. In [16] it is proved that if H_{p^*} evaluated along a trajectory $\mathbf{x}_{p^*}(t)$ is definitely negative, either as $t \to -\infty$ or as $t \to +\infty$, then $\mathbf{x}_{p^*}(t)$ corresponds respectively to a regular or to a slow decay solutions of (1.5). More precisely we have the following asymptotic estimates and existence results.

3.3. **Proposition.** *Consider a trajectory $\mathbf{x}_{p^*}(t)$ of (2.2) which is well defined for $t < -N$, and assume $\limsup_{t\to-\infty} H_{p^*}(\mathbf{x}_{p^*}(t),t) < 0$. Then the corresponding solution $u(r)$ of (1.5) is a singular solution, hence $\lim_{r\to 0}u(r) = +\infty$. Moreover, if $\mathbf{G1}''$ is satisfied, then*

$$\liminf_{r\to 0} u(r)r^{\frac{n-p}{p}} > 0 \quad \textit{and} \quad 0 < \limsup_{r\to 0} u(r)r^{\frac{p}{j_1-p}} < \limsup_{r\to 0} u(r)r^{\frac{p}{i_1-p}} < \infty\,.$$

Analogously consider a trajectory $\mathbf{x}_{p^*}(t)$ of (2.2) which is well defined for $t > N$ and assume that $\limsup_{t\to+\infty} H_{p^*}(\mathbf{x}_{p^*}(t),t) < 0$. Then the corresponding solution $u(r)$ of (1.5) has slow decay. Moreover, if $\mathbf{G2}''$ is satisfied

$$\liminf_{r\to \infty} u(r)r^{\frac{n-p}{p}} > 0 \quad \textit{and} \quad 0 < \limsup_{r\to \infty} u(r)r^{\frac{p}{i_2-p}} < \limsup_{r\to \infty} u(r)r^{\frac{p}{j_2-p}} < \infty\,.$$

The proof of Proposition 3.3 follows from the proof of Proposition 3.12 in [16], with some trivial changes. We repeat here also Proposition 3.11 of [16].

3.4. **Proposition.** *Assume $\mathbf{G1}$ with $l_1 > \sigma$, and that either $l_1 \neq p^*$ or $l_1 = p^*$ and $G_{p^*}(x,t)$ is monotone in t for any x and any $t \leq -M$, for a certain $M > 0$. Then there is a trajectory $\bar{\mathbf{x}}_{l_1}(t)$ of the non-autonomous system (2.2) such that $\lim_{t\to-\infty}\bar{\mathbf{x}}_{l_1}(t) = \mathbf{P}^{-\infty}$, where $\mathbf{P}^{-\infty}$ is the unique critical point in $\mathbb{R}^2_+\setminus\{(0,0)\}$ of the autonomous system where $g_{l_1}(x,t) \equiv g_{l_1}^{-\infty}(x)$.*

Analogously assume that $\mathbf{G2}$ is satisfied with $l_2 > \sigma$ and that either $l_2 \neq p^*$ or $l_2 = p^*$ and $G_{p^*}(x,t)$ is monotone in t for any x and any $t \geq M$, for a certain $M > 0$. Then there is a trajectory $\bar{\mathbf{x}}_{l_2}(t)$ of the non-autonomous system (2.2) such that $\lim_{t\to+\infty}\bar{\mathbf{x}}_{l_2}(t) = \mathbf{P}^{+\infty}$, where $\mathbf{P}^{+\infty}$ is the unique critical point in $\mathbb{R}^2_+\setminus\{(0,0)\}$ of the autonomous system where $g_{l_1}(x,t) \equiv g_{l_1}^{-\infty}(x)$.

In this section we will assume the following.

- $\mathbf{F_a^+}$**:** There are M_a^+ and $N^A < -N$ such that $G_{p^*}(x,t)$ is increasing in t for any $(x_{p^*},t) \in \mathfrak{U}^a := \{(x,t)\,|\, x > M_a^+ e^{\alpha_{p^*}t}, \quad t \leq N^A\}$.
- $\mathbf{F_z^-}$**:** There are M_z^- and $N^Z > N$ such that $G_{p^*}(x,t)$ is decreasing in t for any $(x,t) \in \mathfrak{S}^z := \{(x_{p^*},t)\,|\, x > M_z^- e^{-\frac{n-p}{p(p-1)}t}, \quad t \geq N^Z\}$.

To understand better the previous hypotheses observe that, when $\mathbf{G1}''$ and $\mathbf{G2}''$ hold, if $G_{p^*}(x,t)$ is increasing in t then $j_1, j_2 \in (p, p^*]$ while if it is decreasing then $i_1, i_2 \in [p^*, +\infty)$, see [16]. In fact with $\mathbf{F_a^+}$ we require that $G_{p^*}(x,t)$ is increasing when evaluated along trajectories corresponding to regular solutions $u(d,r)$ with d large and r small, while with $\mathbf{F_z^-}$ we require that $G_{p^*}(x,t)$ is decreasing when evaluated along trajectories corresponding to fast decay solutions $v(L,r)$ with L small for r large. In fact we ask the system to be subcritical for u large and r small and supercritical for u small and r large.

In the whole paper we will require also the following without explicitly mentioning:

- **H0:** The function $g_{p^*}(x,t)$ converges uniformly on compact subsets to a function $g_{p^*}^{-\infty}(x)$ as $t\to-\infty$ and to $g_{p^*}^{+\infty}(x)$ as $t\to+\infty$.

When $\mathbf{F_a^+}$ and $\mathbf{F_z^-}$ are satisfied the convergence is ensured, and if f also belongs to one of the families (1.2), (1.3), (1.4) or it is a sum of functions of these type **H0** holds. Let us denote by $\mathbf{X}_{p^*}^{-\infty}(t,\mathbf{Q})$ and $\mathbf{X}_{p^*}^{+\infty}(t,\mathbf{Q})$ the trajectories of the autonomous system (2.2) where respectively $g_{p^*}(x,t) \equiv g_{p^*}^{-\infty}(x)$ and $g_{p^*}(x,t) \equiv g_{p^*}^{+\infty}(x)$, departing from $\mathbf{Q}$ at $t = 0$.

When $\mathbf{G1''}$ and $\mathbf{G2''}$ are satisfied and $j_1 < p^* < i_2$ we have $g_{p^*}^{-\infty}(x) \equiv 0 \equiv g_{p^*}^{+\infty}(x)$ for any $x \geq 0$. In this case we set

$$\tilde{M}^u(g_{p^*}^{-\infty}) = \{(x,0) \,|\, x > 0\}, \quad \tilde{M}^s(g_{p^*}^{+\infty}) = \{(x, -[x(n-p)/(p-1)]^{p-1}) \,|\, x > 0\},$$

and if $\mathbf{Q^u} = (d,0) \in \tilde{M}^u(g_{p^*}^{-\infty})$ and $\mathbf{Q^s} = (L, -[L(n-p)/(p-1)]^{p-1}) \in \tilde{M}^s(g_{p^*}^{+\infty})$, then $\mathbf{X}_{p^*}^{-\infty}(t,\mathbf{Q^u}) = (de^{\alpha_{p^*}t}, 0)$ and $\mathbf{X}_{p^*}^{+\infty}(t,\mathbf{Q^s}) = (Le^{-\frac{n-p}{p(p-1)}t}, \quad -e^{-\frac{n-p}{p}t}\big(L\frac{n-p}{p-1}\big)^{p-1})$. Now we can introduce the following hypotheses:

H1: Assume $\mathbf{G1''}$ with $j_1 < p^*$ and that there are $d_1 > 0$ and $N^z \geq N^Z$ such that

$$G_{p^*}(de^{\alpha_{p^*}t}, t) - d\alpha_{p^*}\int_{-\infty}^{t} g_{p^*}(de^{\alpha_{p^*}s}, s)e^{\alpha_{p^*}s}ds < 0,$$

for any (d,t) such that $0 < d < d_1$ and $t > N^z$.

H2: Assume $\mathbf{G2''}$ with $p^* < i_2$ and that there are $L_2 > 0$ and $N^a \leq N^A$ such that

$$G_{p^*}(Le^{-\frac{n-p}{p(p-1)}t}, t) - L\frac{n-p}{p(p-1)}\int_{t}^{+\infty} g_{p^*}(Le^{-\frac{n-p}{p(p-1)}s}, s)e^{-\frac{n-p}{p(p-1)}s}ds < 0,$$

for any (L,t) such that $0 < L < L_2$ and $t < N^a$.

If either $g_{p^*}^{-\infty} \not\equiv 0$ or $g_{p^*}^{+\infty} \not\equiv 0$, the trajectories $\mathbf{X}_{p^*}^{-\infty}(t,\mathbf{Q^u})$ and $\mathbf{X}_{p^*}^{+\infty}(t,\mathbf{Q^s})$ have a different expression that in general is unknown.

However when f is of type (1.2) we know the exact formula for the homoclinic trajectories of the autonomous system: let $\tilde{X}(t,\tau,k)$ be the trajectory of (2.2) with $g = kx|x|^{q-2}$ departing from $\tilde{\mathbf{Q}}^{\mathbf{u}} = \tilde{\mathbf{Q}}^{\mathbf{s}}$ at $t = \tau$, we have

$$\tilde{X}(t,\tau,k) = \left[\frac{1}{D_1(e^{-(t+\tau)} + D_2 e^{\frac{1}{p-1}(t+\tau)})}\right]^{\frac{n-p}{p}} k^{-\frac{n-p}{p^2}},$$

where $D_1 = (n-p)^{\frac{p-1}{p}} n^{\frac{1}{p}}$ and $D_2 = [(p-1)/d]^{(p-1)/p}$. So if $\mathbf{Q^u} \in \tilde{M}^u(g_{p^*}^{-\infty})$ (respectively $\mathbf{Q^s} \in \tilde{M}^s(g_{p^*}^{+\infty})$), then $\mathbf{X}_{p^*}^{-\infty}(t,\mathbf{Q^u})$ has the form $\tilde{X}_{p^*}(t,\tau,k^0)$ where $k^0 = \lim_{r\to 0} k(r)r^\delta$ (resp. $\mathbf{X}_{p^*}^{+\infty}(t,\mathbf{Q^s})$ has the form $\tilde{X}_{p^*}(t,\tau,k^\infty)$ where $k^\infty = \lim_{r\to\infty} k(r)r^\delta$), for a certain τ and $\delta = \frac{n-p}{p}(p^* - q)$. So we denote these trajectories as $\tilde{\mathbf{X}}_{p^*}^{-\infty}(t,\tau)$ (resp. $\tilde{\mathbf{X}}_{p^*}^{+\infty}(t,\tau)$) and we introduce the following modified hypotheses:

[$\mathbf{H1'}$] Assume f is of type (1.2), that $\mathbf{G1}$ holds with $l_1 = p^*$, and that there is $N^z \geq N^Z$ such that

$$G_{p^*}(X_{p^*}^{-\infty}(\tau,\tau),\tau) - \int_{-\infty}^{\tau} g_{p^*}(X_{p^*}^{-\infty}(t,\tau),t)dt < 0,$$

for any $\tau > N^z$.

[$\mathbf{H2'}$] Assume f is of type (1.2), that $\mathbf{G2}$ holds with $l_2 = p^*$, and that there is $N^a \leq N^A$ such that

$$G_{p^*}(X_{p^*}^{+\infty}(\tau,\tau)e^{-\frac{n-p}{p-1}\tau},\tau) - \int_{\tau}^{+\infty} g_{p^*}(X_{p^*}^{+\infty}(t,\tau)e^{-\frac{n-p}{p-1}t},t)\frac{d}{dt}\big[e^{-\frac{n-p}{p-1}t}X_{p^*}^{+\infty}(t,\tau)\big]dt < 0,$$

for any $\tau < N^a$.

Observe that from $\mathbf{F_z^-}$, $\mathbf{F_a^+}$ we know that the limits $\lim_{\tau\to+\infty} b_{p^*}^\tau(x) = b_{p^*}^{+\infty}(x)$ and $\lim_{\tau\to-\infty} B_{p^*}^\tau(x) = B_{p^*}^{-\infty}(x)$ are finite, so the sets $\bar{E}_{p^*}^u(\tau)$ are uniformly bounded even if $\tilde{E}_{p^*}^u(\tau)$ becomes unbounded as $\tau \to +\infty$ see figure 3; analogously the sets $\bar{E}_{p^*}^s(\tau)$ are uniformly bounded even if $\tilde{E}_{p^*}^s(\tau)$ becomes unbounded as $\tau \to -\infty$. We recall that $\tilde{B}_x^u(\tau)$ is the x component of $\tilde{\mathbf{B}}_{p^*}^u(\tau)$ and $\tilde{B}_x^s(\tau)$ is the x component of $\tilde{\mathbf{B}}_{p^*}^s(\tau)$. We denote by $\rho^u := \inf_{\tau\in\mathbb{R}} \bar{B}_x^u(\tau) = \bar{B}_x^u(+\infty)$, by $R^u := \sup_{\tau\in\mathbb{R}} \bar{B}_x^u(\tau)$

$\rho^s := \inf_{\tau\in\mathbb{R}} \bar{B}^s_x(\tau) = \bar{B}^s_x(-\infty)$ and by $R^s := \sup_{\tau\in\mathbb{R}} \bar{B}^s_x(\tau)$: observe that ρ^u, R^u, ρ^s and R^s are positive and finite.

We need to embed (2.2) in the following one parameter family of non-autonomous systems where we have added the translation parameter τ.

$$\begin{pmatrix} \dot{x}_l \\ \dot{y}_l \end{pmatrix} = \begin{pmatrix} \alpha_l & 0 \\ 0 & \gamma_l \end{pmatrix} \begin{pmatrix} x_l \\ y_l \end{pmatrix} + \begin{pmatrix} y_l |y_l|^{\frac{2-p}{p-1}} \\ -g_l(x_l, t+\tau) \end{pmatrix} \tag{3.1}$$

We denote by $\mathbf{x}^\tau_l(t,\mathbf{Q})$ the trajectory of (3.1) departing from $\mathbf{Q}$ at $t=0$ so that $\mathbf{x}^\tau_l(t,\mathbf{Q}) \equiv \mathbf{x}(t+\tau,\tau;\mathbf{Q})$. So we can rewrite $\bar{W}^s_l(\tau)$ and $\bar{W}^u_l(\tau)$ as follows

$$\begin{aligned} \bar{W}^u_l(\tau) &= \{\mathbf{Q} \mid \mathbf{x}^\tau_l(t,\mathbf{Q}) \in \bar{E}^u_l(\tau) \text{ for any } t \le 0\} \\ \bar{W}^s_l(\tau) &= \{\mathbf{Q} \mid \mathbf{x}^\tau_l(t,\mathbf{Q}) \in \bar{E}^s_l(\tau) \text{ for any } t \ge 0\} \end{aligned}$$

and we have an analogous result for the sets $\tilde{W}^s_l(\tau)$ and $\tilde{W}^u_l(\tau)$.
Let us consider the set of trajectories

$$\begin{aligned} \mathfrak{U} &= \cup_{\tau\ge 0}\{\mathbf{x}^\tau_{p^*}(t,\mathbf{Q}) \mid \mathbf{Q} \in \bar{W}^u_{p^*}(\tau) \quad t \le 0\} \\ \mathfrak{S} &= \cup_{\tau\le 0}\{\mathbf{x}^\tau_{p^*}(t,\mathbf{Q}) \mid \mathbf{Q} \in \bar{W}^s_{p^*}(\tau) \quad t \ge 0\} \end{aligned}$$

Observe that any trajectory $\mathbf{x}^\tau_{p^*}(t,\mathbf{Q}) \in \mathfrak{U}$ is contained in the compact set $\bar{\mathcal{E}}^u_{p^*}(+\infty) = \{(x,y) \in U^+_{p^*} \cup U^0_{p^*} \mid 0 \le x \le R^u\}$, for any $t \le 0$; analogously $\mathbf{x}^\tau_{p^*}(t,\mathbf{Q}) \in \mathfrak{S}$ is contained in the compact set $\bar{\mathcal{E}}^s_{p^*}(-\infty) = \{(x,y) \in U^+_{p^*} \cup U^0_{p^*} \mid 0 \le x \le R^s\}$ for any $t \ge 0$. Therefore the sets of functions $\mathfrak{U}$ and $\mathfrak{S}$ are equibounded.

Observe also that if $\mathbf{x}^\tau_{p^*}(t,\mathbf{Q}) \in \mathfrak{U}$ (respectively $\mathfrak{S}$), it solves (3.1) and satisfies $\lim_{t\to-\infty}\mathbf{x}^\tau_{p^*}(t,\mathbf{Q}) = (0,0)$ (resp. $\lim_{t\to+\infty}\mathbf{x}^\tau_{p^*}(t,\mathbf{Q}) = (0,0)$). Moreover the function $t \to g_{p^*}(x,t+\tau)$ is bounded for $t<0$, uniformly in x and τ for any $x \in \bar{E}^u_{p^*}(\tau)$; analogously $t \to g_{p^*}(x,t+\tau)$ is bounded for any $t>0$, uniformly in x and τ, for any $x \in \bar{E}^s_{p^*}(\tau)$. So from (3.1) we easily find that the functions of $\mathfrak{U}$ and $\mathfrak{S}$ are also equi-Lipschitz.

Let us choose a sequence $(\tau_n, \mathbf{Q}^n)$ such that $\tau_n \to +\infty$ and $\mathbf{Q}^n \in \bar{W}^u_{p^*}(\tau_n)$ converges to a point $\mathbf{Q}^{+\infty}$, and consider the sequence of functions $(\mathbf{x}^{\tau_n}_{p^*}(t,\mathbf{Q}^n)) \subset \mathfrak{U}$. It follows that $\mathbf{x}^{\tau_n}_{p^*}(t,\mathbf{Q}^n)$ converges pointwise to the solution of $\mathbf{X}^{+\infty}_{p^*}(t,\mathbf{Q}^{+\infty})$, therefore $\mathbf{Q}^{+\infty} \in \tilde{M}^u_{p^*}(g^{+\infty}_{p^*})$. Analogously choose a sequence $(\tau_m, \mathbf{Q}^m)$ such that $\tau_m \to -\infty$ and $\mathbf{Q}^m \in \bar{W}^s_{p^*}(\tau_m)$ converges to a point $\mathbf{Q}^{-\infty}$. We find that $\mathbf{Q}^{-\infty} \in \tilde{M}^u_{p^*}(g^{-\infty}_{p^*})$ and $\mathbf{x}^{\tau_m}_{p^*}(t,\mathbf{Q}^m)$ converges pointwise to the solution of $\mathbf{X}^{-\infty}_{p^*}(t,\mathbf{Q}^{-\infty})$ of the autonomous system.

Exploiting the fact that the functions in $\mathfrak{U}$ and $\mathbf{X}^{+\infty}_{p^*}(t,\mathbf{Q}^{+\infty})$ tend to 0 exponentially as $t \to -\infty$ and using a Theorem of Ascoli-Arzelà type, it is in fact possible to show that $\mathbf{x}^{\tau_n}_{p^*}(t,\mathbf{Q}^n)$ converges uniformly to $\mathbf{X}^{+\infty}_{p^*}(t,\mathbf{Q}^{+\infty})$ for $t \le 0$. The analogous property holds for functions in $\mathfrak{S}$: we do not give the details since this property is not really needed in the proof. Now we can easily prove the following result.

3.5. **Lemma.** *Assume that either* **H1** *or* **H1′** *are satisfied. Then there is* $\bar{T}^z > N^z$ *such that* $H_{p^*}(\mathbf{Q},\tau) < 0$ *for any* $\tau > \bar{T}^z$ *and any* $\mathbf{Q} \in \bar{W}^u_{p^*}(\tau)$.

Analogously assume that either **H2** *or* **H2′** *are satisfied. Then there is* $\bar{T}^a < N^a$ *such that* $H_{p^*}(\mathbf{Q},\tau) < 0$ *for any* $\tau < \bar{T}^a$ *and any* $\mathbf{Q} \in \bar{W}^s_{p^*}(\tau)$.

Proof. Choose $\mathbf{Q} = (Q_x, Q_y) \in \bar{W}^u_{p^*}(\tau)$; from (2.4) and Remark 3.1 we have

$$\begin{aligned} H_{p^*}(\mathbf{Q},\tau) &= G_{p^*}(Q_x,\tau) - \int_{-\infty}^{\tau} \dot{x}_{p^*}(s,\tau;\mathbf{Q}) g_{p^*}(x_{p^*}(s,\tau;\mathbf{Q}),s)ds = \\ &= G_{p^*}(Q_x,\tau) - \int_{-\infty}^{0} \dot{x}^\tau_{p^*}(t,\mathbf{Q}) g_{p^*}(x^\tau_{p^*}(t,\mathbf{Q}),t+\tau)dt \end{aligned} \tag{3.2}$$

We have seen that, if we choose $\tau \to +\infty$ and $\mathbf{Q}(\tau) \to \mathbf{Q}(+\infty)$, the trajectory $\mathbf{x}^{\tau}_{p^*}(t, \mathbf{Q}(\tau))$ converges pointwise to $\mathbf{X}^{+\infty}_{p^*}(t, \mathbf{Q}^{+\infty})$ as $\tau \to +\infty$. Furthermore, using the fact that both $\mathbf{X}^{+\infty}_{p^*}(t, \mathbf{Q}^{+\infty})$ and $\mathbf{x}^{\tau}_{p^*}(t, \mathbf{Q})$ converge to 0 as $t \to -\infty$ with the same exponential rate (they both are $O(e^{\alpha_{p^*} t})$), for τ sufficiently large we find

$$\begin{aligned} &\left|\dot{x}^{\tau}_{p^*}(s, \mathbf{Q}) g_{p^*}(x^{\tau}_{p^*}(s, \mathbf{Q}), s+\tau) - \dot{X}^{+\infty}_{p^*}(s, \mathbf{Q}^{+\infty}) g_{p^*}(X^{+\infty}_{p^*}(s, \mathbf{Q}), s+\tau)\right| \leq \\ &\leq K\left|\dot{X}^{+\infty}_{p^*}(s, \mathbf{Q}^{+\infty}) g_{p^*}(X^{+\infty}_{p^*}(s, \mathbf{Q}), s+M)\right| \in L^1\left[(-\infty, 0)\right] \end{aligned}$$

where K and M are sufficiently large constants. So from Lebesgue Convergence Theorem we conclude that for any $\epsilon > 0$ there is $\bar{T}^z > N^z$ such that, for any $\tau > \bar{T}^z$ we have

$$\left|H_{p^*}(\mathbf{Q}, \tau) - G_{p^*}(Q_x^{+\infty}, \tau) + \int_{-\infty}^{0} \dot{X}^{+\infty}_{p^*}(s, \mathbf{Q}^{+\infty}) g_{p^*}(X^{+\infty}_{p^*}(s, \mathbf{Q}), s+\tau)\right| < \epsilon$$

So, if either **H1** or **H1′** hold, we can choose $\epsilon > 0$ small enough so that $H_{p^*}(\mathbf{Q}, \tau) < 0$ for any $\mathbf{Q} \in \bar{W}^u_{p^*}(\tau)$, whenever $\tau > \bar{T}^z$. The proof concerning the stable set $\bar{W}^s_{p^*}(\tau)$ is analogous, so we omit it. □

3.6. **Lemma.** *Assume* $\mathbf{G1''}$ *and* $\mathbf{F^+_a}$*; then there is* $\breve{T}^a < 0$ *such that for* $\tau < \breve{T}^a$ *we have* $H_{p^*}(\mathbf{Q}, \tau) > 0$ *for any* $\mathbf{Q} \in \breve{\xi}^u_{p^*}(\tau)$*. Analogously assume* $\mathbf{G2''}$ *and* $\mathbf{F^-_z}$*; then there is* $\breve{T}^z$ *such that for any* $\tau > \breve{T}^z$ *we have* $H_{p^*}(\mathbf{Q}, t) > 0$ *for any* $\mathbf{Q} \in \breve{\xi}^s_{p^*}(\tau)$*.*

Proof. Assume $\mathbf{G1''}$ and $\mathbf{F^+_a}$: it follows that $j_1 \leq p^*$. We construct the function $\bar{g}_{p^*}(x,t)$ defined as follows:

$$\bar{g}_{p^*}(x,t) := \begin{cases} g_{p^*}(x,t), & \text{if } (x,t) \in \mathfrak{U}^a; \\ g_{p^*}(x, \min\{N^a, 1/\alpha_{p^*} \ln(x/M^+_a)\}), & \text{if } (x,t) \notin \mathfrak{U}^a. \end{cases}$$

Consider system (2.2) where $l = p^*$ and we have replaced the original function $g_{p^*}(x,t)$ with the truncated function $\bar{g}_{p^*}(x,t)$. From $\mathbf{F^+_a}$ it follows that $\bar{G}_{p^*}(x,t) = \int_0^x \bar{g}_{p^*}(s,t)ds$ is increasing in t for any $(x,t) \in \mathbb{R}^2$. So we can apply Lemma 2.6 to conclude the existence of the manifold $\breve{W}^u_{p^*}(\tau)$ for any $\tau \in \mathbb{R}$, for the truncated system. Choose $\breve{\mathbf{Q}}^{\mathbf{u}} = (\breve{Q}^u_x, \breve{Q}^u_y) \in \breve{\xi}^u_{p^*}(\tau)$, where $\tau < N^a$, and consider the trajectory $\mathbf{x}_{p^*}(t, \tau, \breve{\mathbf{Q}}^{\mathbf{u}})$ and the corresponding regular solution $u(d,r)$ of the truncated problem. Set $\breve{T}^a = \min\{N^a, 1/\alpha_{p^*} \ln(\rho^u/M^+_a)\}$ so that $\breve{Q}^u_x e^{-\alpha_{p^*}\tau} > \rho^u e^{-\alpha_{p^*}\breve{T}^a} > M^+_a$ for any $\tau < \breve{T}^a$. From Lemma 2.2 we know that $u(d,r)$ is decreasing for $t \leq \tau$. So $u(d,r) > M^+_a$ for any $r \leq e^{\tau}$ and $(\mathbf{x}_{p^*}(t, \tau, \breve{\mathbf{Q}}^{\mathbf{u}}), t) \in \mathfrak{U}^a$ for any $t \leq \tau$. So $\mathbf{x}_{p^*}(t, \tau, \breve{\mathbf{Q}}^{\mathbf{u}})$ is a trajectory of the original problem as well. It follows that for any $\tau < T^a$ the set $\breve{W}^u_{p^*}(\tau) \subset \mathbb{R}^2_{\pm}$ exists for the original problem, and that it joins the origin with $\breve{\xi}^u_{p^*}(\tau)$. Note also that the sets $\breve{\xi}^u_{p^*}(\tau)$ for the original and the truncated problem coincide but the sets $\breve{W}^u_{p^*}(\tau)$ do not (even if they both have the properties described in Lemma 2.6). Since $\left(\mathbf{x}_{p^*}(t, \tau, \breve{\mathbf{Q}}^{\mathbf{u}}), t\right) \in \mathfrak{U}^a$ for any $t \leq \tau$, from $\mathbf{F^+_a}$ we find that

$$H_{p^*}(\breve{\mathbf{Q}}^{\mathbf{u}}, \tau) = \int_{-\infty}^{\tau} \frac{\partial}{\partial t} G_{p^*}(\mathbf{x}_{p^*}(t, \tau, \breve{\mathbf{Q}}^{\mathbf{u}}), t) dt > 0\,.$$

Analogously assume $\mathbf{G2''}$ and $\mathbf{F^-_z}$; it follows that $i_2 \geq p^*$. Consider the function $\bar{g}_{p^*}(x,t)$ defined as follows:

$$\bar{g}_{p^*}(x,t) := \begin{cases} g_{p^*}(x,t), & \text{if } (x,t) \in \mathfrak{S}^z; \\ g_{p^*}(x, \max\{N^z, \frac{p(p-1)}{n-p}|\ln(x/M^-_z)|\}), & \text{if } (x,t) \notin \mathfrak{S}^z. \end{cases}$$

and the modified problem where $g_{p^*}(x,t) = \bar{g}_{p^*}(x,t)$. Observe that $\bar{G}_{p^*}(x,t)$ is decreasing in t for any $(x,t) \in \mathbb{R}^2$. Reasoning as above we prove the existence of a stable set $\breve{W}^s_{p^*}(\tau)$ for any $\tau \in \mathbb{R}$. Set $\breve{T}^z := \max\{N^z, \frac{p(p-1)}{n-p} \ln(M^z_-/\rho^u)\}$; it follows that for any $\breve{\mathbf{Q}}^{\mathbf{s}} \in \breve{\xi}^s_{p^*}(\tau)$ and $\tau > \breve{T}^z$ we have $x_{p^*}(\tau,\tau,\breve{\mathbf{Q}}^{\mathbf{s}})\exp[\tau(n-p)/[p(p-1)] > M^z_-$. Moreover from Lemma 2.2 we have $x_{p^*}(t,\tau,\breve{\mathbf{Q}}^{\mathbf{s}})\exp\big(t(n-p)/[p(p-1)]\big) > M^z_-$ for any $t \ge \tau$ so $\mathbf{x}_{p^*}(t,\tau,\breve{\mathbf{Q}}^{\mathbf{s}}) \in \mathfrak{S}^z$ whenever $t \ge \tau$. Therefore these trajectories are actually solutions of the original problem and we can infer the existence of a stable set $\breve{W}^s_{p^*}(\tau) \subset \mathbb{R}^2_\pm$ which intersects $\breve{c}^s_{p^*}(\tau)$ in $\breve{\xi}^s_{p^*}(\tau)$ for any $\tau > \breve{T}^z$.

Again, since $G_{p^*}(x,t)$ is decreasing in $\mathfrak{S}^z$, from (2.4) we find

$$H_{p^*}(\breve{\mathbf{Q}}^{\mathbf{s}},\tau) = -\int_\tau^{+\infty} \frac{\partial}{\partial t} G_{p^*}(\mathbf{x}_{p^*}(t,\tau,\breve{\mathbf{Q}}^{\mathbf{s}}),t)dt > 0\,.$$

□

Observe that for $\tau > \breve{T}^z$ we have $\bar{W}^s_{p^*}(\tau) \subset \tilde{W}^s_{p^*}(\tau) \subset \breve{W}^s_{p^*}(\tau)$, while for $\tau < \breve{T}^a$ we have $\bar{W}^u_{p^*}(\tau) \subset \tilde{W}^u_{p^*}(\tau) \subset \breve{W}^u_{p^*}(\tau)$. Set $T^a = \min\{\bar{T}^a, \breve{T}^a\}$ and $T^z = \max\{\bar{T}^z, \breve{T}^z\}$.

We give now a Remark, that together with Proposition 3.3, allows to give better estimates on the asymptotic behavior of positive solutions.

3.7. *Remark.* Assume that **G1″**, **G2″**, $\mathbf{F}^-_{\mathbf{z}}$, $\mathbf{F}^+_{\mathbf{a}}$ are satisfied and consider a slow decay solution $u(r)$, a singular solution $v(r)$, and the corresponding trajectories $\mathbf{x}^u_{p^*}(t)$ and $\mathbf{x}^v_{p^*}(t)$.
Then $\lim_{t\to+\infty} H_{p^*}(\mathbf{x}^u_{p^*}(t),t) < 0$ and $\lim_{t\to-\infty} H_{p^*}(\mathbf{x}^v_{p^*}(t),t) < 0$.

Proof. From Lemma 3.2 we know that $\liminf_{t\to+\infty} H_{p^*}(\mathbf{x}^u_{p^*}(t),t) = h^+ < 0$. Therefore there is T large such that $H_{p^*}(\mathbf{x}^u_{p^*}(T),T) < 0$. From Lemma 2.2 $x^u_{p^*}(t)e^{\frac{n-p}{p(p-1)}t} \nearrow +\infty$ as $t \to +\infty$, so, possibly choosing a larger T, we can assume that $x^u_{p^*}(t)e^{\frac{n-p}{p(p-1)}t} > M^z_-$ for any $t > T$. Hence $\mathbf{x}^u_{p^*}(t) \in \mathfrak{S}^z$ for $t \ge T$ and $H_{p^*}(\mathbf{x}^u_{p^*}(t),t)$ is negative and decreasing for $t \ge T$; so it converges to a negative value. The proof concerning the singular solution $u(r)$ is completely analogous. □

Set $\mathcal{A}^- := \{d > 0 \,|\, u(d,r)$ is a G. S. with slow decay$\}$ and $\mathcal{A}^+ := \{d > 0 \,|\, u(d,r)$ is a crossing solution$\}$.

3.8. **Lemma.** *Assume* **G1″**, **G2″**, $\mathbf{F}^-_{\mathbf{z}}$, $\mathbf{F}^+_{\mathbf{a}}$*; then* $\mathcal{A}^+$ *and* $\mathcal{A}^-$ *are open.*

Proof. Let $\bar{d} \in \mathcal{A}^+$ and denote by $R(\bar{d}) = e^{T(\bar{d})}$ the first zero of $u(\bar{d},r)$. Let $\mathbf{x}_{j_1}(t,\tau,\bar{\mathbf{Q}}^u)$ be the trajectory corresponding to $u(\bar{d},r)$; we can choose τ such that $\bar{\mathbf{Q}}^u \in \bar{W}^u_{j_1}(\tau)\backslash\, \bar{\xi}^u_{j_1}(\tau)$. Choose $d > 0$ such that $|d-\bar{d}| < \delta$ and let $\mathbf{x}_{j_1}(t,\tau,\mathbf{Q}^u)$ be the trajectory corresponding to $u(d,r)$. If we show that $u(d,r)$ is a crossing solution, then it follows that $\mathcal{A}^+$ is open. From Remark 2.1 we know that for any $\epsilon > 0$ we can find $\delta > 0$ such that $\|\mathbf{Q}^u - \bar{\mathbf{Q}}^u\| < \epsilon$. So we can also assume that $\mathbf{Q}^u \in \bar{W}^u_{j_1}(\tau)$. Moreover we can assume that $\mathbf{x}_{j_1}(t,\tau,\mathbf{Q}^u)$ does not cross the y-negative semi-axis for $t \le T(\bar{d})$, otherwise we are done. Fix $\rho > 0$ small, and denote by $\bar{T}_\rho = \max\{t \le T(\bar{d}) \mid x_{j_1}(t,\tau,\mathbf{Q}^u) = \rho\}$. From the continuous dependence on initial data of (2.2) on $\mathbb{R}^2_\pm$ it follows that we can choose $\epsilon > 0$ small enough so that $\|\mathbf{x}_{j_1}(\bar{T}_\rho,\tau,\bar{\mathbf{Q}}^u) - \mathbf{x}_{j_1}(\bar{T}_\rho,\tau,\mathbf{Q}^u)\| < \rho/2$. Since we can choose $\rho > 0$ arbitrarily small and $\mathbf{x}_{j_1}(t,\tau,\bar{\mathbf{Q}}^u)$ crosses the y negative semi-axis transversally, it can be shown easily that $\mathbf{x}_{j_1}(t,\tau,\mathbf{Q}^u)$ has to cross the y negative semi-axis transversally as well, at a certain $t = T(d)$ close to $T(\bar{d})$. Hence $\mathcal{A}^+$ is open. □

Using Lemma 2.8, from elementary arguments we find that

3.9. *Remark.* Assume **G1″**, **G2″**, $\mathbf{F}^-_{\mathbf{z}}$, $\mathbf{F}^+_{\mathbf{a}}$, either **H1** or **H1′**, and either **H2** or **H2′**. Then there is $\tau^h \in \mathbb{R}$ such that $\tilde{\xi}^u_{p^*}(\tau^h) \cap \tilde{\xi}^s_{p^*}(\tau^h) \neq \emptyset$.

Now we are ready to prove the main result of the paper.

3.10. **Theorem.** *Assume* $\mathbf{G1}''$, $\mathbf{G2}''$, $\mathbf{F_z^-}$, $\mathbf{F_a^+}$*; moreover assume either* **H1** *or* $\mathbf{H1}'$*, and either* **H2** *or* $\mathbf{H2}'$*. Then there are* $d_* \leq d^*$ *such that* $u(d,r)$ *is a G.S. with fast decay if* $d \in \{d_*, d^*\}$*, a G.S. with slow decay if* $d \in (0, d_*)$*, and a crossing solution if* $d > d^*$*. Furthermore there are uncountably many S.G.S. with fast decay, uncountably many solutions of the Dirichlet problem in the exterior of balls, and S.G.S. with slow decay. So positive solutions have a structure of type* ***Mix****.*

Proof. Let us choose $\mathbf{Q} \in \tilde{\xi}^u_{p^*}(\tau^h) \cap \tilde{\xi}^s_{p^*}(\tau^h)$, see Remark 3.9. It follows that $\mathbf{x}_{p^*}(t, \tau^h, \mathbf{Q}) \in \tilde{E}^u_{p^*}(\tau^h)$ for any $t \leq \tau^h$ and $\mathbf{x}_{p^*}(t, \tau^h, \mathbf{Q}) \in \tilde{E}^s_{p^*}(\tau^h)$ for any $t \geq \tau^h$. In particular $\mathbf{x}_{p^*}(t, \tau, \mathbf{Q}) \in \mathbb{R}^2_{\pm}$ so the corresponding solution $u(d,r)$ is a monotone decreasing G.S. with fast decay.

We show now that any regular solution $u(d(\bar{\mathbf{Q}}^{\mathbf{u}}), r)$ corresponding to $\mathbf{x}_{p^*}(t, \tau, \bar{\mathbf{Q}}^{\mathbf{u}})$ where $\bar{\mathbf{Q}}^{\mathbf{u}} \in \bar{W}^u_{p^*}(\tau)$ and $\tau > T^z$, is a G.S. with slow decay. Then from Lemma 2.4 we deduce the existence of a $d^* > 0$ such that $u(d,r)$ is a G.S. with slow decay for any $0 < d < d^*$.

From Lemma 3.5 we know that $H_{p^*}(\bar{\mathbf{Q}}^{\mathbf{u}}, \tau) < 0$ for any $\bar{\mathbf{Q}}^{\mathbf{u}} \in \bar{\xi}^u_{p^*}(\tau)$ and $\tau > T^z$. Let us denote by $R(\bar{\mathbf{Q}}^{\mathbf{u}}) \leq +\infty$ the first zero of $u(d(\bar{\mathbf{Q}}^{\mathbf{u}}), r)$ and observe that $u(d(\bar{\mathbf{Q}}^{\mathbf{u}}), r) r^{\frac{n-p}{p-1}}$ is increasing for $0 < r < R$. Recall that $\bar{\mathbf{Q}}^{\mathbf{u}} = (\bar{Q}^u_x, \bar{Q}^u_y)$ is such that $\bar{Q}^u_x > \rho^u$. So we have $x_{p^*}(t, \tau, \bar{\mathbf{Q}}^{\mathbf{u}}) e^{\frac{n-p}{p(p-1)} t} > \rho^u e^{\frac{n-p}{p(p-1)} T^z} > M_z^-$ and $(\mathbf{x}_{p^*}(t, \tau, \bar{\mathbf{Q}}^{\mathbf{u}}), t) \in \mathfrak{S}^z$ for $t \in [\tau, \ln(R))$. It follows that $H_{p^*}(\mathbf{x}_{p^*}(t, \tau, \bar{\mathbf{Q}}^{\mathbf{u}}), t)$ is negative and decreasing for any $t \in [\tau, \ln(R))$, therefore $R = +\infty$ and $\mathbf{x}_{p^*}(t, \tau, \bar{\mathbf{Q}}^{\mathbf{u}}) \in \mathbb{R}^2_{\pm}$ for any $t \in \mathbb{R}$.
So $\lim_{t \to +\infty} H_{p^*}(\mathbf{x}_{p^*}(t, \tau, \bar{\mathbf{Q}}^{\mathbf{u}}), t) < 0$ and $u(d(\bar{\mathbf{Q}}^{\mathbf{u}}), r)$ is a G.S. with slow decay, see Proposition 3.3. Since we can repeat the argument for any $\tau > T^z$, from Lemma 2.4 we find that $u(d,r)$ is a G.S. with slow decay for d small enough.

Now we prove that there is d^* such that $u(d,r)$ is a crossing solution for any $d > d^*$. We choose $\tau < T^a$ and $\check{\mathbf{Q}}^{\mathbf{u}} \in \check{W}^u_{p^*}(\tau)$; from Lemma 3.5 and 3.6 it follows that $H_{p^*}(\check{\mathbf{Q}}^{\mathbf{u}}, \tau) > 0 > H_{p^*}(\bar{\mathbf{Q}}^{\mathbf{s}}, \tau)$ for any $\bar{\mathbf{Q}}^{\mathbf{s}} \in \bar{\xi}^s_{p^*}(\tau)$. Reasoning as above we can show that there is $\mathfrak{T} > \tau$ such that $\mathbf{x}_{p^*}(t, \tau, \check{\mathbf{Q}}^{\mathbf{u}}) \in \mathbb{R}^2_{\pm}$ for $t < \mathfrak{T}$ and crosses the y negative semi-axis at $t = \mathfrak{T}$. It follows that the corresponding solution $u(d,r)$ of (1.5) is regular and it is such that $u(d,r) > 0$ for $r \in [0, e^{\mathfrak{T}})$ and $u(d, e^{\mathfrak{T}}) = 0$, so it is a crossing solution. This proves the part of the Theorem concerning regular solutions. Note also that we have proved that $\mathcal{A}^+$ and $\mathcal{A}^-$ are non-empty and we know from Lemma 3.8 that they are open. So there is $d^* > 0$ which disconnect them, and this gives a different proof of the existence of G.S. with fast decay.
Using the same argument we can prove that the Dirichlet problem (1.5) in the exterior of the ball of radius R admits at least a solution for any R large enough.

Now we prove the existence of S.G.S. with fast decay. Choose $\tau < T^a$ and a point $\mathbf{Q} = (Q_x, Q_y) \in \bar{\xi}^s_{p^*}(\tau)$, so that $Q_x > \rho^s$. We denote by $T = \inf\{t \mid x_{p^*}(s, \tau, \mathbf{Q}) \in \mathbb{R}^2_{\pm}$ for any $s > t\}$; note that $T < \tau$. From Lemma 2.2 we know that $x_{p^*}(t, \tau, \mathbf{Q}) e^{-\alpha_{p^*} t} > \rho^s e^{-\alpha_{p^*} \tau} > M_a^+$ for any $t < \tau$, so $\mathbf{x}^\tau_{p^*}(t, \mathbf{Q}) \in \mathfrak{U}^a$ for $t \leq \tau$. From Lemma 3.5 we have that $H_{p^*}(\mathbf{x}_{p^*}(t, \tau, \mathbf{Q}), t)$ is negative for $t = \tau$ and increasing for $t \in (T, \tau)$. Hence $T = -\infty$ and $\mathbf{x}_{p^*}(t, \tau, \mathbf{Q}) \in \mathbb{R}^2_{\pm}$ for any $t \in \mathbb{R}$, and from Proposition 3.3 we find that $v(r)$ is a monotone decreasing S.G.S. with fast decay.

Now we prove the existence of uncountably many S.G.S. with slow decay, see figure 4. We stress that the multiplicity result is in fact new for nonlinearities of this type even in the classical case $p = 2$; in fact in [7] it was conjectured that the S.G.S. is unique. Choose $\tau < T^a$; we denote by $\mathcal{R}^u(\tau)$ the open subset enclosed by $\bar{W}^s_{p^*}(\tau)$, $\check{W}^u_{p^*}(\tau)$ and the vertical line between $\bar{\xi}^s_{p^*}(\tau)$ and $\check{\xi}^u_{p^*}(\tau)$. Analogously choose $\tau > T^z$; we denote by $\mathcal{R}^s(\tau)$ the open subset enclosed by $\bar{W}^u_{p^*}(\tau)$, $\check{W}^s_{p^*}(\tau)$ and the vertical line between $\bar{\xi}^u_{p^*}(\tau)$ and $\check{\xi}^s_{p^*}(\tau)$, see figure 4. Observe that $\mathcal{R}^u(\tau)$

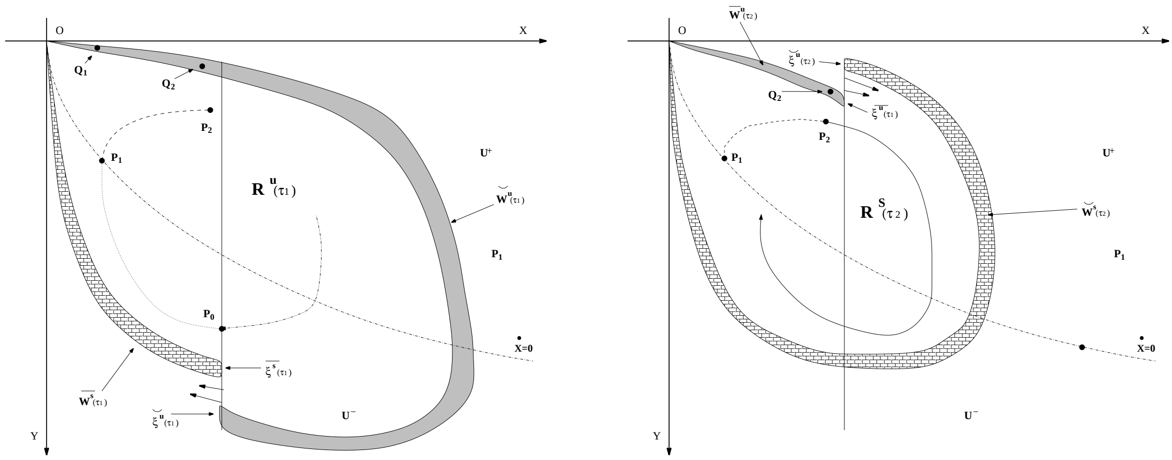

FIGURE 4. Construction of S.G.S. with slow decay.

is "negatively invariant" for any $\tau \leq T^a$; more precisely if $\mathbf{Q} \in \mathcal{R}^u(\tau)$ and $\tau \leq T^a$, then $\mathbf{x}_{p^*}(t,\tau,\mathbf{Q}) \in \mathcal{R}^u(t)$ for any $t \leq \tau$. Analogously $\mathcal{R}^s(\tau)$ is "positively invariant" for any $\tau \geq T^z$: if $\mathbf{Q} \in \mathcal{R}^s(\tau)$ and $\tau \geq T^z$, then $\mathbf{x}_{p^*}(t,\tau,\mathbf{Q}) \in \mathcal{R}^s(t)$ for any $t \geq \tau$.

We choose $T_2 > T^z$ and we look for a $\mathbf{P^1} \in \mathcal{R}^u(T_1)$ such that $\mathbf{x}_{p^*}(T_2,T_1,\mathbf{P^1}) \in \mathcal{R}^s(T_2)$ and $\mathbf{x}_{p^*}(t,T_1,\mathbf{P^1}) \in \mathbb{R}^2_{\pm}$ for any $t \in [T_1,T_2]$. Then it follows that $\mathbf{x}_{p^*}(t,T_1,\mathbf{P^1}) \in \mathbb{R}^2_{\pm}$ for any t. Then we will see that it is possible to choose $\mathbf{P^1}$ in such a way that there is $\delta < 0$ such that $H_{p^*}(\mathbf{x}_{p^*}(t,T_1,\mathbf{P^1}),t) < -\delta < 0$ for $|t|$ large enough. So it does not converge to the origin and we can conclude that the corresponding solution $u(r)$ of (1.5) is a S.G.S. with slow decay whose asymptotic behavior (both as $r \to 0$ and as $r \to +\infty$) is described by Proposition 3.3.

Let us choose $\mathbf{P^1} \in U^0_{p^*}$ and set $\delta = \|\mathbf{P^1}\|$. If $\delta > 0$ is not too large we have $\mathbf{P^1} \in \mathcal{R}^u(T_1)$ and $H_{p^*}(\mathbf{P^1},T_1) < 0$. If δ is small enough we can find $\mathbf{Q^1} \in \bar{W}^u_{p^*}(T_1)$ such that $\|\mathbf{Q^1}-\mathbf{P^1}\| < \delta$, $\|\mathbf{Q^1}\| < \delta$. Possibly choosing a smaller δ, we can assume that $\mathbf{x}_{p^*}(T_2,T_1,\mathbf{Q^1}) = \mathbf{Q^2} = (Q^2_x,Q^2_y) \in \bar{\xi}^u_{p^*}(T_2)$. Then from Lemma 3.5 there is $c > 0$ so that $H_{p^*}(\mathbf{Q^2},T_2) = -2c < 0$. Let us denote by $\mathbf{P^2} = \mathbf{x}_{p^*}(T_2,T_1,\mathbf{P^1})$; for any $\epsilon > 0$ we can find $\delta > 0$ such that $\|\mathbf{x}_{p^*}(t,T_1,\mathbf{P^1}) - \mathbf{x}_{p^*}(t,T_1,\mathbf{Q^1})\| < \epsilon$, for any $t \in [T_1,T_2]$, thanks to the continuous dependence on initial data of (2.2); in particular $\mathbf{x}_{p^*}(t,T_1,\mathbf{P^1}) \in \mathbb{R}^2_{\pm}$ for any $t \in [T_1,T_2]$. Moreover we can also assume that $|H_{p^*}(\mathbf{P^2},T_2) - H_{p^*}(\mathbf{Q^2},T_2)| < c$, so that $H_{p^*}(\mathbf{P^2},T_2) < -c$. Observe that if T_2 is large enough $x_{p^*}(t,T_1,\mathbf{P^1})e^{\frac{n-p}{p(p-1)}t} > (Q^2_x-\epsilon)e^{\frac{n-p}{p(p-1)}T_2} > \rho^u e^{\frac{n-p}{p(p-1)}T_2} > M^z_-$, for any $t > T_2$, thanks to Lemma 2.2. Hence $\mathbf{x}_{p^*}(t,T_1,\mathbf{P^1}) \in \mathfrak{S}^z$ for any $t > T_2$ and $H_{p^*}(\mathbf{x}_{p^*}(t,T_1,\mathbf{P^1}),t) \leq H_{p^*}(\mathbf{P^2},T_2) < 0$ for any $t > T_2$. So from Proposition 3.3 we get that the solution $v(r)$ corresponding to $\mathbf{x}_{p^*}(t,T_1,\mathbf{P^1})$ has slow decay.

Moreover there is $T_0 < T_1$ such that $\mathbf{x}_{p^*}(T_0,T_1,\mathbf{P^1}) = \mathbf{P^0} = (P^0_x,P^0_y) \in c^-_{p^*}(T_0) \setminus \bar{\xi}^s_{p^*}(T_0)$. Observe that for any $\mathbf{Q} = (P^0_x,Q_y) \in \bar{\xi}^s_{p^*}(T_0)$ by construction we have $P^0_y > Q_y$, so from an easy computation and Lemma 3.5 we get $H_{p^*}(\mathbf{P^0},T_0) < H_{p^*}(\mathbf{Q},T_0) < 0$. Thus from Lemma 2.2 we have $x_{p^*}(t,T_1,\mathbf{P^1}) > \rho^s \exp[\alpha_{p^*}(t-T_0)] > M^+_a \exp[\alpha_{p^*}t]$ for $t < T_0$; so $\big(\mathbf{x}_{p^*}(t,T_1,\mathbf{P^1}),t\big) \in \mathfrak{U}^a$ whenever $t < T_0$. It follows that $\lim_{t\to-\infty} H_{p^*}(\mathbf{x}_{p^*}(t,T_1,\mathbf{P^1}),t) < 0$, and $v(r)$ is a singular solution so it is a S.G.S. with slow decay. Since we can repeat the argument for uncountably many points $\mathbf{P^1}$ we find that there are uncountably many S.G.S. with slow decay. $\square$

If some further assumptions are satisfied it is possible to understand something more concerning the behavior of regular solutions.

3.11. **Corollary.** *Assume that we are in the hypothesis of Theorem 3.10. Moreover assume that* **G1** *is satisfied with $l_1 < p^*$ and denote by $R(d)$ the first zero of a crossing solution $u(d,r)$ where $d > d^*$. Then $R(d)$ depends continuously on d and tends to 0 as $d \to +\infty$ and to $+\infty$ as $d \to d^*$.*

Moreover, if we assume that there is T such that, for any $x > 0$, $G_{p^*}(x,t)$ is increasing in t for any $t < T$ and decreasing for any $t > T$, then we have $d_* = d^*$, so the G.S. with fast decay is unique.

Proof. The claim concerning the first zero of regular solutions follows from Lemma 2.5 and Theorem 3.10. The proof of the uniqueness of the G.S. with fast decay follows from [24]. More precisely in that paper the claim is proved just for f of type (1.2). But the argument relies on an intersection property of regular solutions (and of fast decay solutions) that depends on the fact that any G.S. with fast decay $u(d,r)$ is such that $P(u(d,r), u'(d,r), r) > 0$ for any $r > 0$. It is easily observed that this fact holds also with our hypotheses.

Then the proof of the uniqueness goes through following [24] almost with no changes. □

We also stress that if **G1** and **G2** hold then the asymptotic estimates for singular and slow decay solutions simplify to $u(r)r^{-\frac{p}{l_1-p}} \to P^u_x$ as $r \to 0$ and $u(r)r^{\frac{-p}{l_2-p}} \to P^s_x$ as $r \to \infty$, where P^u_x and P^s_x are positive computable constants, see Proposition 3.4.

3.12. *Remark.* We stress that following [19] (or [16]) it is possible to reduce the analysis of radial solutions of

$$\operatorname{div}(h(|\mathbf{x}|)\nabla u|\nabla u|^{p-2}) + \bar{f}(u, |\mathbf{x}|) = 0 \tag{3.3}$$

to the analysis of solutions of an equation of the form (1.5). Here again $\mathbf{x} \in \mathbb{R}^n$ and $h(|\mathbf{x}|) \geq 0$ for $|\mathbf{x}| \geq 0$. Set $a(r) = r^{n-1}g(r)$, the only requirement is that one of the Hypotheses below is satisfied

A1: $a^{-1/(p-1)} \in L^1[1,\infty]\setminus L^1[0,1]$
A2: $a^{-1/(p-1)} \in L^1[0,1]\setminus L^1[1,\infty)$

4. Applications

In this section we give some applications of Theorem 3.10 and Corollary 3.11 to functions of the form (1.2), (1.3) and (1.4). First of all we compute the functions g obtained in these cases. When f is of type (1.2) we have $g_l(x_l,t) = k(e^t)e^{\delta_l t}x_l|x_l|^{q-2}$, where $\delta_l = \alpha_l(l-q)$. If there is $l_1 > p$ such that $\lim_{t\to-\infty} k(e^t)e^{\delta_{l_1}t} = A > 0$, then **G1** holds, while if $k(e^t)e^{\delta_{l_1}t}$ varies between two positive values for $t < 0$ **G1′** holds, and **G1″** holds if there are $a > 0$ and $b > 0$ such that $k(e^t)e^{\delta_{j_1}t} > a$ and $k(e^t)e^{\delta_{i_1}t} < b$ for $t < 0$. Analogously if there is $l_2 > \sigma$ such that $\lim_{t\to+\infty} k(e^t)e^{\delta_{l_2}t} = A > 0$ then **G2** holds, and if $k(e^t)e^{\delta_{l_2}t}$ varies between two positive values for $t > 0$ then **G2′** holds, and **G2″** holds if there are $A > 0$ and $B > 0$ such that $k(e^t)e^{\delta_{i_2}t} > A$ and $k(e^t)e^{\delta_{j_2}t} < B$ for $t > 0$. In particular if $k(r)$ is uniformly positive and bounded we find that **G1′** and **G2′** hold with $l_1 = l_2 = q$.

When f is of type (1.3) and k_1 and k_2 are uniformly positive and bounded we can set $l_1 = q_2 - q_1$ to find

$$g_{l_1}(x_{l_1}, t) = \frac{k_2(e^t)x_{l_1}|x_{l_1}|^{q_2-2}}{e^{\frac{p}{q_2-q_1-p}t} + k_1(e^t)x_{l_1}|x_{l_1}|^{q_1-2}}$$

so that for any given $\tau \in \mathbb{R}$ we have that $a^{\tau}_{l_1}(x_{l_1})$ has the form $\frac{k_2 x_{l_1}^{q_2-1}}{2(\mathfrak{c}+k_1 x_{l_1}^{q_1-1})}$ and $b^{\tau}_{l_1}(x_{l_1}) = 2\frac{k_2}{k_1} x_{l_1}^{q_2-q_1}$, where $\mathfrak{c}$, k_1 and k_2 are positive constants. Analogously if we set $l_2 = q_2$ we obtain

$$g_{l_2}(x_{l_2}, t) = \frac{k_2(e^t) x_{l_2} |x_{l_2}|^{q_2-2}}{1 + k_1(e^t) e^{-\frac{p(q_1-1)}{q_2-p}t} x_{l_1} |x_{l_1}|^{q_1-2}},$$

hence $A^{\tau}_{l_2}(x_{l_2})$ has the form $\frac{k_2 x_{l_1}^{q_2-1}}{2(1+\mathfrak{c} x_{l_2}^{q_1-1})}$ and $B^{\tau}_{l_2}(x_{l_2}) = 2k_2 x_{l_2}^{q_2-1}$ where $\mathfrak{c}$, k_1 and k_2 are positive constants.

The function $g_{p^*}(x,t)$ has the following form:

$$g_{p^*}(x, t) = \frac{k_2(e^t) x |x|^{q_2-2}}{e^{\alpha_{p^*}(p^*-q_2)t} + k_1(e^t) e^{\alpha_{p^*}(p^*-q_2+q_1)t} x |x|^{q_1-2}}$$

When f is of type (1.4) and k is uniformly positive and bounded, **G1′** and **G2′** hold with $l_1 = q_1$ and $l_2 = q_2$ and we find respectively

$$(4.1)\qquad \begin{aligned} g_{q_1}(x_{q_1}, t) &= k(e^t) \times \begin{cases} x_{q_1}|x_{q_1}|^{q_1-2}, & \text{if } |x_{q_1}| \geq e^{\frac{pt}{q_1-p}}; \\ e^{-\frac{p(q_2-q_1)}{q_1-p}t} x_{q_1}|x_{q_1}|^{q_2-2}, & \text{if } |x_{q_1}| \leq e^{\frac{pt}{q_1-p}} \end{cases} \\ g_{q_2}(x_{q_2}, t) &= k(e^t) \times \begin{cases} e^{\frac{p(q_2-q_1)}{q_2-p}t} x_{q_2}|x_{q_2}|^{q_1-2}, & \text{if } |x_{q_2}| \geq e^{\frac{pt}{q_2-p}}; \\ x_{q_2}|x_{q_2}|^{q_2-2}, & \text{if } |x_{q_2}| \leq e^{\frac{pt}{q_2-p}} \end{cases} \end{aligned}$$

Let us denote by $\lambda = \alpha_{p^*}(q - p^*)$ and by $\lambda_i = \alpha_{p^*}(q_i - p^*)$ for $i = 1, 2$. After some easy computation we find the following.

4.1. *Remark.* Assume f is of type (1.2); if $k(r)r^{-\lambda}$ is increasing then $G_{p^*}(x,t)$ is increasing in t, while if it is decreasing $G_{p^*}(x,t)$ is decreasing. Moreover assume that there are $R > \rho > 0$ such that $k(r)r^{-\lambda}$ is increasing for $0 < r < \rho$ and decreasing for $r > R$, then $\mathbf{F}^+_{\mathbf{a}}$, $\mathbf{F}^{\mathbf{z}}_-$ hold. Assume further that respectively $l_1 < p^*$ or $l_1 = p^*$ and $(k(r)r^{-\lambda})' r^{-\frac{n}{p-1}} \notin L^1([0,1])$, then **H1** or respectively **H1′** hold; analogously if $l_2 > p^*$ or $l_2 = p^*$ and $(k(r)r^{-\lambda})' r^n \notin L^1([1,\infty))$, then **H2** or respectively **H2′** hold. So we can apply Theorem 3.10 and find structure **Mix** for positive solutions.

Finally assume $R = \rho$, then we can apply Corollary 3.11 and we find that the G.S. with fast decay is unique.

Proof. The only difficulty is to verify that **H1**, **H1′**, **H2** and **H2′** hold. But this follows from the fact that the integrals defined in these Hypotheses are always assumed to be diverging to $-\infty$. The proof can be obtained repeating the argument used in section 6 of [33], with some trivial changes. We stress also that in order to have structure **Mix** in the case $q = p^*$ and k uniformly positive and bounded in [33] it is required neither $(k(r)r^{-\lambda})' r^{-\frac{n}{p-1}} \notin L^1([0,1])$ nor $(k(r)r^{-\lambda})' r^n \notin L^1([1,\infty))$, but their proof in that case is uncorrect. $\square$

In particular we have the following results

4.2. **Corollary.** *Assume that f is of type (1.2) and $k(r) = A_0 r^{\lambda} + A_1 r^s + o(r^s)$ at $r = 0$ and $k(r) = B_0 r^{\lambda} + B_1 r^l + o(r^l)$ at $r = \infty$, where $-p < l < \lambda < s$ and $A_1, B_1 > 0$. Then hypotheses* **G1** *and* **G2** *hold with $l_1 = p^*$ if $A_0 > 0$ and $l_1 = p(s+q)/(s+p)$ if $A_0 = 0$, $l_2 = p^*$ if $B_0 > 0$ and $l_2 = p(l+q)/(l+p)$ if $B_0 = 0$. Moreover if either $A_0 = 0$ or $A_0 > 0$ and $s < \lambda + \frac{n}{p-1}$ and either $B_0 = 0$ or $B_0 > 0$ and $l > \lambda - n$, positive solution have a structure of type* ***Mix****. If $A_0 = B_0 = 0$ we also have that all the singular solutions $v(r)$ are such that $\lim_{r\to 0} v(r) r^{\frac{p+s}{q-p}} = C_1$ while all the slow decaying solutions $w(r)$ are such that $\lim_{r\to+\infty} w(r) r^{\frac{p+l}{q-p}} = C_2$,* *where C_1 and C_2 are computable constants (they are the x coordinates of the critical points $\mathbf{P}^{-\infty}$ and of $\mathbf{P}^{+\infty}$).*

Now we give an example in which $\mathbf{G1}'$ and $\mathbf{G2}'$ do not hold but $\mathbf{G1}''$ and $\mathbf{G2}''$ do.

4.3. **Corollary.** *Assume that f is of type (1.2) and $k(r) = Ar^{\lambda}/|\ln r|^s$. Then $\mathbf{G1}''$ and $\mathbf{G2}''$ hold with $i_1 = j_2 = p^* - \epsilon$ and $i_2 = j_1 = p^* + \epsilon$, for any $\epsilon > 0$. Moreover hypotheses $\mathbf{H1}$, $\mathbf{H2}$, $\mathbf{F^+_a}$, $\mathbf{F^z_-}$ are satisfied. So we can apply Theorem 3.10 and Corollary 3.11 and we find that positive solution have a structure of type **Mix** and that the G.S. with fast decay is unique.*

Similar results can be obtained for f of type (1.4): assume that $k(r) = A_0 r^{\lambda_1} + A_1 r^s + o(r^s)$ at $r = 0$ and $k(r) = B_0 r^{\lambda_2} + B_1 r^l + o(r^l)$ at $r = \infty$, where $\lambda_1 < s$, $-p < l < \lambda_2$ and $A_1, B_1 > 0$. Assume that either $A_0 = 0$ or $A_0 > 0$ and $s < \lambda_1 + \frac{n}{p-1}$ and either $B_0 = 0$ or $B_0 > 0$ and $l > \lambda_2 - n$, then we obtain the same results as in Corollary 4.2.

We give now an example in which $\mathbf{G1}'$ and $\mathbf{G2}'$ hold but $\mathbf{G1}$ and $\mathbf{G2}$ do not.

4.4. **Corollary.** *Assume f is of type (1.4) and $k(r) = 2 + \sin(a \ln(r))$ where $|\lambda_i| > |a|$ for $i = 1, 2$. If $q_1 < q_2 < p^*$, then $\mathbf{G1}'$ and $\mathbf{G2}'$ hold with $l_1 = q_1$ and $l_2 = q_2$, and we can apply Theorem 4.2 in* [16]*; in particular positive solutions have structure **Sub**. If $q_2 > q_1 > p^*$, we can apply Theorem 4.3 in* [16]*; in particular positive solutions have structure **Sup**. If $q_1 < p^* < q_2$, we can apply Theorem 3.10, so in particular positive solutions have structure **Mix**.*

4.5. **Corollary.** *Assume that f is of type (1.3), $p^* - 1 > q_2 - q_1 > p - 1$ and $q_2 > p^*$ and denote by $\lambda_a = \alpha_{p^*}(q_2 - q_1 - p^*)$ and by $\lambda_z = \alpha_{p^*}(q_2 - p^*)$. Assume that k_1 is a positive constant while $k_2(r)$ is such that and $k_2(r) = A_0 r^{\lambda_1} + A_1 r^s + o(r^s)$ at $r = 0$ and $k_2(r) = B_0 r^{\lambda_2} + B_1 r^l + o(r^l)$ at $r = \infty$, where $\lambda_1 < s$, $-p < l < \lambda_2$ and $A_1, B_1 > 0$. Then if either $A_0 = 0$ or $A_0 > 0$ and $s < \lambda_1 + \frac{n}{p-1}$ and either $B_0 = 0$ or $B_0 > 0$ and $l > \lambda_2 - n$, positive solution have a structure of type **Mix**. Again if $A_0 = B_0 = 0$ we also have that all the singular solutions $v(r)$ are such that $\lim_{r \to 0} v(r) r^{\frac{p+s}{q-p}} = C_1$ while all the slow decaying solutions $w(r)$ are such that $\lim_{r \to +\infty} w(r) r^{\frac{p+l}{q-p}} = C_2$, where $C_1 > 0$ and $C_2 > 0$ are computable constants.*

5. Appendix: Construction of stable and unstable set when $G1''$ and $G2''$ hold. Proof of Lemma 2.6.

Now we develop the construction of $W^u_{j_1}$ and $W^s_{i_2}$ in the case we just assume $\mathbf{G1}''$ and $\mathbf{G2}''$. First of all we need to introduce the three dimensional autonomous system obtained from (2.2) adding the extra variable $z = t$.

$$\begin{aligned} \dot{x}_{p^*} &= \alpha_{p^*} x_{p^*} + y_{p^*} |y_{p^*}|^{\frac{2-p}{p-1}} \\ \dot{y}_{p^*} &= \gamma_{p^*} y_{p^*} - g_{p^*}(x_{p^*}, z) \\ \dot{z} &= 1 \end{aligned} \tag{5.1}$$

We introduce the following further definitions:

$$M^u_{i_1}(a^\tau_{j_1}, z) := \aleph^z_{i_1, j_1}\Big(M^u_{j_1}(a^\tau_{j_1})\Big) \qquad \text{and} \qquad M^u_{j_1}(b^\tau_{i_1}, z) := \aleph^z_{j_1, i_1}\Big(M^u_{i_1}(b^\tau_{i_1})\Big).$$

We consider system (2.2) where $l = j_1$. We denote by $\tilde{\mathbf{A}}^u_{j_1}(\tau, z)$ the intersection of $\tilde{M}^u_{j_1}(a^\tau_{j_1}, z)$ with $U^0_{j_1}$ and by $\tilde{\mathbf{B}}^u_{j_1}(\tau, z)$ the intersection of $\tilde{M}^u_{i_1}(b^\tau_{j_1}, z)$ with $U^0_{j_1}$ (this intersection exists since $j_1 \geq i_1$). Fix $\tau \in \mathbb{R}$; we wish to stress that $\tilde{\mathbf{A}}^u_{j_1}(\tau, z)$ is in fact independent of z while $\tilde{\mathbf{B}}^u_{j_1}(\tau, z)$ does depend on z and $\lim_{z \to -\infty} \|\tilde{\mathbf{B}}^u_{j_1}(\tau, z)\| = 0$. However, for any fixed $z \leq \tau$, both $\|\tilde{\mathbf{A}}^u_{j_1}(\tau, z)\|$ and $\|\tilde{\mathbf{B}}^u_{j_1}(\tau, z)\|$ are positive and bounded.

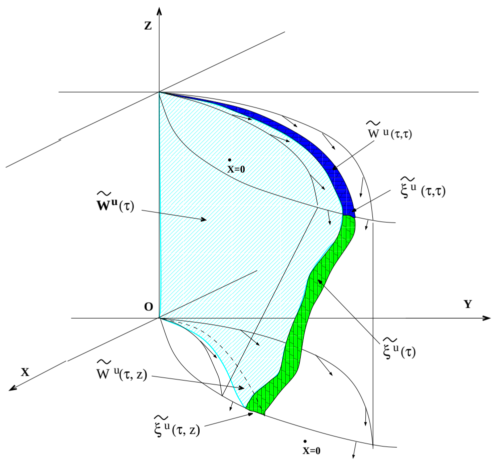

FIGURE 5. Construction of $\tilde{\mathbf{W}}^{\mathbf{u}}(\tau)$ and $\tilde{\xi}^{\mathbf{u}}(\tau)$ when $\mathbf{G1}''$ is assumed.

We denote by $\partial\tilde{E}^{u,a}_{j_1}(\tau,z) = \tilde{M}^u_{j_1}(a^\tau_{j_1},z)$ by $\partial\tilde{E}^{u,b}_{j_1}(\tau,z) = \tilde{M}^u_{j_1}(b^\tau_{i_1},z)$, and by $\partial\tilde{E}^u_{j_1}(\tau,z) = \partial\tilde{E}^{u,a}_{j_1}(\tau,z) \cup \partial\tilde{E}^{u,b}_{j_1}(\tau,z)$ and we claim that $\partial\tilde{E}^{u,a}_{j_1}(\tau,z)$ is on the right of $\partial\tilde{E}^{u,b}_{j_1}(\tau,z)$. We denote by $\tilde{E}^u_{j_1}(\tau,z)$ the bounded set enclosed by $\partial\tilde{E}^u_{j_1}(\tau,z)$ and $U^0_{j_1}$ for $z \le \tau$. Using elementary argument, see [16] we can show that the flow of (2.2) on $\partial\tilde{E}^u_{j_1}(\tau,z)$ points towards the interior of $\tilde{E}^u_{j_1}(\tau,z)$ for any $t = z \le \tau$. As a consequence we also find that $\tilde{M}^u_{i_1}(a^\tau_{i_1},\tau)$ is on the right of $\tilde{M}^u_{j_1}(b^\tau_{i_1},z)$ for any $\tau \in \mathbb{R}$ and any $z \le \tau$.

Assume $j_1 \le p^*$ and consider system (5.1) where $l = i_1$. We denote by $\tilde{\mathbf{A}}^u_{i_1}(\tau,z)$ the intersection of $M^u_{i_1}(a^\tau_{j_1},z)$ with $U^0_{i_1}$ and by $\tilde{\mathbf{B}}^u_{i_1}(\tau,z)$ the intersection of $\tilde{M}^u_{i_1}(b^\tau_{i_1},z)$ with $U^0_{i_1}$. Again $\tilde{\mathbf{B}}^u_{i_1}(\tau,z)$ is independent of z while $\tilde{\mathbf{A}}^u_{i_1}(\tau,z)$ does depend on z and $\lim_{z\to-\infty}\|\tilde{\mathbf{A}}^u_{i_1}(\tau,z)\| = +\infty$. However for any fixed $z \le \tau$, both $\|\tilde{\mathbf{A}}^u_{i_1}(\tau,z)\|$ and $\|\tilde{\mathbf{B}}^u_{i_1}(\tau,z)\|$ are positive and bounded. We denote by $\partial\tilde{E}^{u,a}_{i_1}(\tau,z) = \tilde{M}^u_{i_1}(a^\tau_{i_1},z)$, by $\partial\tilde{E}^{u,b}_{i_1}(\tau,z) = \tilde{M}^u_{i_1}(b^\tau_{j_1},z)$, and reasoning as above we find again that $\partial\tilde{E}^{u,a}_{i_1}(\tau,z)$ is on the right of $\partial\tilde{E}^{u,b}_{i_1}(\tau,z)$.

5.1. *Remark.* If $j_1 > p^*$, a priori $\tilde{M}^u_{i_1}(a^\tau_{j_1},z)$ may not intersect $U^0_{i_1}$, however we can repeat the argument replacing $U^0_{i_1}$ by $U^0_{j_1}$ finding a slightly smaller set $\tilde{E}^u_{j_1}(\tau,z) \subset U^+_{i_1}$. This allows anyway the construction of the sets $\bar{W}^u_{j_1}(\tau,z)$ and $\bar{W}^u_{j_1}(\tau)$ that will be defined below, and to prove Lemma 2.4 for a slightly smaller set $\bar{W}^u_{j_1}(\tau)$.

We denote by $\tilde{c}^u_l(\tau,z)$ the subset of U^0_l enclosed by $\tilde{\mathbf{A}}^u_l(\tau,z)$ and $\tilde{\mathbf{B}}^u_l(\tau,z)$, where $l = j_1$ or $l = i_1$, by $\partial\tilde{E}^u_l(\tau,z) = \partial\tilde{E}^{u,a}_l(\tau,z) \cup \partial\tilde{E}^{u,b}_l(\tau,z)$, and by $\tilde{E}^u_l(\tau,z)$ the two dimensional set enclosed by $\tilde{c}^u_l(\tau,z)$ and $\partial\tilde{E}^u_l(\tau,z)$. Reasoning as above we find that the flow of (2.2) on $\partial\tilde{E}^u_{i_1}(\tau,z)$ points towards the interior of $\tilde{E}^u_{i_1}(\tau,z)$ for any $t = z \le \tau$. Finally we turn to (5.1) and we define the following two dimensional sets $\tilde{\mathbf{C}}^{\mathbf{u}}_{\mathbf{l}}(\tau) = \{(\mathbf{Q},z) \mid \mathbf{Q} \in \tilde{c}^u_l(\tau,z)\ z \in \mathbb{R}\}$, $\partial\tilde{\mathbf{E}}^{\mathbf{u}}_{\mathbf{l}}(\tau) = \{(\mathbf{Q},z) \mid \mathbf{Q} \in \partial\tilde{E}^u_l(\tau,z),\ z \le \tau\}$, and we give the analogous definitions for $\tilde{\mathbf{E}}^{\mathbf{u}}_{\mathbf{l}}(\tau)$, $\partial\tilde{\mathbf{E}}^{\mathbf{u,a}}_{\mathbf{l}}(\tau)$, $\partial\tilde{\mathbf{E}}^{\mathbf{u,b}}_{\mathbf{l}}(\tau)$. Note that $\tilde{\mathbf{E}}^{\mathbf{u}}_{\mathbf{l}}(\tau)$ is the 3-dimensional set enclosed by $\partial\tilde{\mathbf{E}}^{\mathbf{u}}_{\mathbf{l}}(\tau)$ and $\tilde{\mathbf{C}}^{\mathbf{u}}_{\mathbf{l}}(\tau)$.

When $\mathbf{G1}''$ is satisfied, $\tilde{E}^u_{i_1}(\tau,z)$ and $\tilde{E}^u_{j_1}(\tau,z)$ have positive finite diameter for any finite value $z \le \tau$, and $\tilde{\mathbf{E}}^{\mathbf{u}}_{\mathbf{i_1}}(\tau)$ is as sketched in figure 5. Let $(\mathbf{Q},z) = (Q_x,Q_y,z)$ be a point in $\tilde{\mathbf{E}}^{\mathbf{u}}_{\mathbf{i_1}}(\tau)$; we denote by

$$\tilde{T}^u(\mathbf{Q}) = \inf\{T \le \tau \mid (\mathbf{x}_{i_1}(t,z,\mathbf{Q}),t) \in \tilde{\mathbf{E}}^{\mathbf{u}}_{\mathbf{i_1}}(\tau) \text{ for any } t \in (T,\tau]\}$$

and by $\Psi^{u,\tau}_{i_1}((\mathbf{Q},z)) = \lim_{t\to\tilde{T}^u(\mathbf{Q})} \mathbf{x}_{i_1}(t,z;\mathbf{Q})$. Reasoning as in section 3.1 of [16] we can prove that the function $\Psi^{u,\tau}_{j_1} : \tilde{\mathbf{E}}^{\mathbf{u}}_{\mathbf{j_1}}(\tau) \to \mathbb{R}^2_{\pm}$ is well defined and continuous; so we denote by

$$\mathbf{W}^{\mathbf{u}}_{\mathbf{j_1}}(\tau) = \{(\mathbf{Q},z) \in \tilde{\mathbf{E}}^{\mathbf{u}}_{\mathbf{j_1}}(\tau) \,|\, \Psi^{u,\tau}_{j_1}\big((\mathbf{Q},z)\big) = (0,0)\}.$$

Arguing as in section 3.1 of [16], through a topological lemma based on the idea of Wazewski's principle, we can prove that $\mathbf{W}^{\mathbf{u}}_{\mathbf{j_1}}(\tau)$ contains a compact connected set $\tilde{\mathbf{W}}^{\mathbf{u}}_{\mathbf{j_1}}(\tau)$ which intersects $(0,0,\bar{z})$ and $\tilde{c}^u_{j_1}(\tau,\bar{z})$. Let us denote by $\tilde{\mathbf{W}}^{\mathbf{u}}_{\mathbf{j_1}}(\tau) = \cup_{z\le\tau} \tilde{W}^u_{j_1}(\tau,z)\times\{z\}$; we claim that $\tilde{\mathbf{W}}^{\mathbf{u}}_{\mathbf{j_1}}(\tau)\backslash\{(0,0,z)\,|\, z\le\tau\}$ is connected.

In analogy to the $\mathbf{G1'}$ case we denote by $W^u_{j_1}(\tau)$ the set of the initial conditions of trajectories of (2.2) converging to the origin respectively as $t\to-\infty$, namely $W^u_{j_1}(\tau) = \{\mathbf{Q}\in\mathbb{R}^2_+ \,|\, (\mathbf{x}_{j_1}(t,\tau,\mathbf{Q}),t) \in \tilde{\mathbf{E}}^{\mathbf{u}}_{\mathbf{j_1}}(\tau) \text{ for } t\le\tau \ \}$ then we set

$$\tilde{W}^u_{j_1}(\tau) = \{(x_{j_1},y_{j_1}) \,|\, (x_{j_1},y_{j_1},\tau) \in \tilde{W}^u_{j_1}(\tau,\tau)\}\,.$$

With the same argument we can construct $W^s_{i_2}(\tau)$. It is easy to show that we can reprove Lemma 2.4 also with these weaker assumptions. The proof of this extension of Lemma 2.4 can be obtained from the original one with some trivial changes, see [16].

Now we prove Lemma 2.6, so we focus on the case $l = p^*$. In the original construction of $\bar{W}^u_{p^*}(\tau)$ we have to start from the barrier sets constructed through the autonomous system (2.2) with $l = j_1$, and then use $\aleph^\tau_{p^*,j_1}$ to obtain a barrier set for the system (2.2) with $l = p^*$. However for $l = p^*$ we can give a simpler construction of $\bar{W}^u_{p^*}(\tau)$ and $\bar{W}^s_{p^*}(\tau)$, which is more natural in this context. We set $\partial\bar{E}^{u,b}_{p^*}(\tau) = \tilde{M}^u_{p^*}(b^\tau_{p^*})$ and $\partial\bar{E}^{s,b}_{p^*}(\tau) = \tilde{M}^s_{p^*}(B^\tau_{p^*})$. We denote by $\partial\bar{E}^{u,a}_{p^*}(\tau) = \{(x,y)\in\tilde{M}^u_{p^*}(a^\tau_{p^*}) \,|\, 0\le x\le B^+_x(\tau)\}$ and by $\partial\bar{E}^{s,a}_{p^*}(\tau) = \{(x,y)\in\tilde{M}^s_{p^*}(A^\tau_{p^*}) \,|\, 0\le x\le B^-_x(\tau)\}$: note that in general $\tilde{M}^u_{p^*}(a^\tau_{p^*})$ may be the positive x semi-axis and $\tilde{M}^s_{p^*}(A^\tau_{p^*})$ the curve U^0_σ. Then we denote by $\bar{E}^u_{p^*}(\tau)$ the subset enclosed by $\partial\bar{E}^{u,a}_{p^*}(\tau)$, $\partial\bar{E}^{u,b}_{p^*}(\tau)$ and $c^+_{p^*}(\tau)$ and by $\bar{E}^s_{p^*}(\tau)$ the subset enclosed by $\partial\bar{E}^{s,a}_{p^*}(\tau)$, $\partial\bar{E}^{s,b}_{p^*}(\tau)$ and $c^-_{p^*}(\tau)$. Again the flow of (2.2) on $\partial\bar{E}^{u,a}_{p^*}(\tau)\cup\partial\bar{E}^{u,b}_{p^*}(\tau)$ points towards the interior of $\bar{E}^u_{p^*}(\tau)$ for $t\le\tau$ and on $\partial\bar{E}^{s,a}_{p^*}(\tau)\cup\partial\bar{E}^{s,b}_{p^*}(\tau)$ points towards the exterior of $\bar{E}^s_{p^*}(\tau)$, so we can construct subsets of $\mathfrak{W}^u_{p^*}(\tau)$ and $\mathfrak{W}^s_{p^*}(\tau)$, denoted with abuse of notation by $\bar{W}^u_{p^*}(\tau)$ and $\bar{W}^s_{p^*}(\tau)$, intersecting $c^+_{p^*}(\tau)$ and $c^-_{p^*}(\tau)$ respectively in $\xi^u_{p^*}(\tau)$ and $\xi^s_{p^*}(\tau)$, and satisfying Lemma 2.4. In fact it can be shown that these sets are subsets of $\bar{W}^u_{p^*}(\tau)$ and $\bar{W}^s_{p^*}(\tau)$ obtained through the original construction, so they have the properties described in Lemmas 2.3 and 2.4.

We have already seen that $\mathbf{x}_{p^*}(t,\tau,\mathbf{Q})$ corresponds to a regular solution of (1.5) whenever $\mathbf{Q}\in\mathfrak{W}^u_{p^*}(\tau)$ and to a fast decay solution whenever $\mathbf{Q}\in\mathfrak{W}^s_{p^*}(\tau)$. Moreover we can control $\mathfrak{W}^u_{p^*}(\tau)$ and $\mathfrak{W}^s_{p^*}(\tau)$ until they cross $U^0_{p^*}$. Lemma 2.6 claims that we can follow $\mathfrak{W}^u_{p^*}(\tau)$ until it crosses $c^-_{p^*}(\tau)$ and $\mathfrak{W}^s_{p^*}(\tau)$ until it crosses $c^+_{p^*}(\tau)$.

Proof of Lemma 2.6. We just prove the first claim of Lemma 2.6, since the second can be obtained reasoning in the same way. We follow again the idea of Lemmas 3.4 and 3.5 of [16], so we need to construct some barrier sets. It is easy to show that $j_1\le p^*\le i_2$. In particular, for any $x>0$, $b^\tau_{p^*}(x)$ and $B^\tau_{p^*}(x)$ are bounded for any $\tau\in\mathbb{R}$.

We fix $\tau\in\mathbb{R}$; for any $z\le\tau$ we denote by $\partial\breve{E}^{u,b}_{p^*}(z)$ the branch of the level set $H_{p^*}(x,y,z)=0$ joining the origin and $c^-_{p^*}(\tau)$, and contained in $\mathbb{R}^2_{\pm}$. We introduce the 2-dimensional surfaces $\partial\breve{\mathbf{E}}^{\mathbf{u,b}}_{p^*}(\tau) := \big(\bigcup_{z\le\tau}\partial\breve{E}^{u,b}_{p^*}(z)\times\{z\}\big)$, and $\mathbf{C}^-_{\mathbf{p^*}}(\tau) := \bigcup_{z\le\tau}\big(c^-_{p^*}(\tau)\times\{z\}\big)$.

The construction of the second barrier set is rather technical, so we start by assuming $j_1 = p^*$. We denote by $H_{p^*}(\mathbf{Q}, a^{\tau}_{p^*})$ the function

$$H_{p^*}(x,y,a^{\tau}_{p^*}) := \frac{n-p}{p}xy + \frac{p-1}{p}|y|^{\frac{p}{p-1}} + \int_0^x a^{\tau}_{p^*}(s)ds,$$

and we give the analogous definition for $H_{p^*}(x,y,b^{\tau}_{p^*})$. Observe that $H_{p^*}(\tilde{\mathbf{A}}^{\mathbf{u}}_{p^*}(\tau), a^{\tau}_{p^*}) = 0 < H_{p^*}(\tilde{\mathbf{A}}^{\mathbf{u}}_{p^*}(\tau), b^{\tau}_{p^*})$. Consider the trajectory $\mathbf{X}_{p^*}(t,0,\tilde{\mathbf{A}}^{\mathbf{u}}_{p^*}(\tau), b^{\tau}_{p^*})$ and observe that $H_{p^*}(\mathbf{X}_{p^*}(t,0,\tilde{\mathbf{A}}^{\mathbf{u}}_{p^*}(\tau), b^{\tau}_{p^*}), b^{\tau}_{p^*}) \equiv H_{p^*}(\tilde{\mathbf{A}}^{\mathbf{u}}_{p^*}(\tau), b^{\tau}_{p^*}) > 0$. It follows that there are $T^2(\tau) > T^1(\tau) > 0$ such that $\mathbf{X}_{p^*}(t,0,\tilde{\mathbf{A}}^{\mathbf{u}}_{p^*}(\tau), b^{\tau}_{p^*})$ intersects $c^-_{p^*}(\tau)$ at $t = T^1(\tau)$ and the y_{p^*} negative semi-axis at $t = T^2(\tau)$, see figure 2. Let us denote by

$$\partial\breve{E}^{u,a}_{p^*}(\tau) := \{\mathbf{X}_{p^*}(t,0,\tilde{\mathbf{A}}^{\mathbf{u}}_{p^*}(\tau), b^{\tau}_{p^*}) \mid 0 \le t \le T^1(\tau)\} \cup \tilde{M}^u_{p^*}(a^{\tau}_{p^*}).$$

Observe that by construction $\partial\breve{E}^{u,a}_{p^*}(\tau)$ and $\partial\breve{E}^{u,b}_{p^*}(z)$ do not intersect for any $z \le \tau$ and the former is on the right of the latter and the 2-dimensional surfaces $\partial\breve{\mathbf{E}}^{\mathbf{u},\mathbf{a}}_{p^*}(\tau) := \big(\bigcup_{z\le\tau} \partial\breve{E}^{u,a}_{p^*}(\tau) \times \{z\}\big)$ and we denote by $\breve{\mathbf{E}}^{\mathbf{u}}_{p^*}(\tau)$ the volume enclosed by $\partial\breve{\mathbf{E}}^{\mathbf{u},\mathbf{b}}_{p^*}(\tau)$, $\partial\breve{\mathbf{E}}^{\mathbf{u},\mathbf{a}}_{p^*}(\tau)$ and $\mathbf{C}^-_{\mathbf{p}^*}(\tau)$. Observe that $\breve{\mathbf{E}}^{\mathbf{u}}_{p^*}(\tau)$ is unbounded but its intersection with the planes $z = k$ is bounded for $k \le \tau$. Since $G_{p^*}(x,t)$ is increasing for $t \le T^+$ it follows that $H_{p^*}(\mathbf{x}_{p^*}(t), t)$ is increasing along the trajectories $\mathbf{x}_{p^*}(t)$ of (2.2) for any $t \le T^+$, see (2.4). Therefore the flow of (5.1) on $\partial\breve{\mathbf{E}}^{\mathbf{u},\mathbf{b}}_{p^*}(\tau)$ points towards the interior of $\breve{\mathbf{E}}^{\mathbf{u}}_{p^*}(\tau)$, for any $t \le \tau$.

We claim that the flow of (5.1) on $\partial\breve{\mathbf{E}}^{\mathbf{u},\mathbf{a}}_{p^*}(\tau)$ points towards the interior of $\breve{\mathbf{E}}^{\mathbf{u}}_{p^*}(\tau)$ as well. Observe in fact that $\partial\breve{E}^{u,a}_{p^*}(\tau)$ is independent of z and choose $\mathbf{Q} = (Q_x, Q_y) \in \tilde{M}^u_{p^*}(a^{\tau}_{p^*})$. Fix $z \le \tau$ and observe that $\tilde{M}^u_{p^*}(a^{\tau}_{p^*})$ is contained in the graph of $\mathbf{X}_{p^*}(t,z,\mathbf{Q},a^{\tau}_{p^*})$. Following Lemma 3.5 in [16] we see that $\dot{X}_{p^*}(z,z,\mathbf{Q},a^{\tau}_{p^*}) = \dot{x}_{p^*}(z,z,\mathbf{Q})$ but

$$\dot{y}_{p^*}(z,z,\mathbf{Q}) - \dot{Y}_{p^*}(z,z,\mathbf{Q},a^{\tau}_{p^*}) = a^{\tau}_{p^*}(Q_x) - g_{p^*}(Q_x,z) < 0.$$

So the flow of (5.1) on $(\mathbf{Q}, z)$ points towards the interior of $\breve{\mathbf{E}}^{\mathbf{u}}_{p^*}(\tau)$. Now choose $\mathbf{Q} = (Q_x, Q_y) \in \partial\breve{E}^{u,a}_{p^*}(\tau)\setminus \{\tilde{M}^u_{p^*}(a^{\tau}_{p^*})\}$. Reasoning as above we see that $\dot{x}_{p^*}(z,z,\mathbf{Q}) = \dot{X}_{p^*}(z,z,\mathbf{Q},b^{\tau}_{p^*})$, but

$$\dot{y}_{p^*}(z,z,\mathbf{Q}) - \dot{Y}_{p^*}(z,z,\mathbf{Q},b^{\tau}_{p^*}) = b^{\tau}_{p^*}(Q_x) - g_{p^*}(Q_x,z) > 0$$

for any $z \le \tau$. This way we have proved the claim for the whole $\partial\breve{\mathbf{E}}^{\mathbf{u},\mathbf{a}}_{p^*}(\tau)$.

The flow of (5.1) on $\mathbf{C}^-_{\mathbf{p}^*}(\tau)$ points towards the left, so we can use a topological idea based on Wazewski's principle, namely Lemma 3.3 of [16], to conclude the existence of a set $\breve{\mathbf{W}}^{\mathbf{u}}_{p^*}(\tau)$ of the form $\breve{\mathbf{W}}^{\mathbf{u}}_{p^*}(\tau) = \cup_{z\le\tau}\big(\breve{W}^u_{p^*}(z) \times \{z\}\big)$, with the following properties. If $\mathbf{Q} \in \breve{W}^u_{p^*}(\tau)$ then $\big(\mathbf{x}_{p^*}(z,\tau,\mathbf{Q}), z\big) \in \breve{\mathbf{W}}^{\mathbf{u}}_{p^*}(\tau) \subset \breve{\mathbf{E}}^{\mathbf{u}}_{p^*}(\tau)$ for any $z \le \tau$. Moreover $\breve{W}^u_{p^*}(z)$ is compact and connected for any z, contains the origin and intersects $c^-_{p^*}(\tau)$ in a compact set denoted by $\breve{\xi}^u_{p^*}(z)$. Furthermore the set $\breve{\xi}^u_{p^*}(\tau) := \cup_{z\le\tau}\big(\breve{\xi}^u_{p^*}(z) \times \{z\}\big)$ is connected.

Now assume $j_1 < p^*$, so that $a^{\tau}_{p^*}(x) \equiv 0$. In this case we follow the ideas developed above in this appendix to construct $\tilde{W}^u_{j_1}$ when $\mathbf{G}''$ is assumed. We consider the unstable manifold $\tilde{M}^u_{j_1}(a^{\tau}_{j_1})$ of the autonomous system (2.2) with $l = j_1$: it intersects $U^0_{p^*}$ in a point denoted by $\tilde{\mathbf{A}}(\tau)$. It follows that $\aleph^z_{p^*,j_1}(\tilde{M}^u_{j_1}(a^{\tau}_{j_1}))$ intersects $U^0_{p^*}$ in one point denoted by $\tilde{\mathbf{A}}(\tau, z)$, for any $z \le \tau$. We denote by $\tilde{M}^u_{p^*}(a^{\tau}_{j_1}, z)$ the branch of $\aleph^z_{p^*,j_1}(\tilde{M}^u_{j_1}(a^{\tau}_{j_1}))$ between the origin and $\tilde{\mathbf{A}}(\tau, z)$. Reasoning as above we find that the flow of the non-autonomous system (2.2) with $l = j_1$ on $\tilde{M}^u_{j_1}(a^{\tau}_{j_1})$ points towards the interior of the bounded set enclosed by $\tilde{M}^u_{j_1}(a^{\tau}_{j_1})$ and $U^0_{j_1}$. Since

$\aleph^z_{p^*,j_1}$ preserves orientation we have that the flow of (2.2) with $l=p^*$ on $\tilde{M}^u_{p^*}(a^\tau_{j_1},z)$ points towards the interior of the bounded set enclosed by $\tilde{M}^u_{p^*}(a^\tau_{j_1},z)$ and $U^0_{p^*}$ as well.

Then, arguing as above, we consider the trajectory $\mathbf{X}_{p^*}(t,0,\tilde{\mathbf{A}}(\tau,z),b^\tau_{p^*})$, and we find that there are $T^2(\tau,z)>T^1(\tau,z)>0$ such that $\mathbf{X}_{p^*}(t,0,\tilde{\mathbf{A}}(\tau,z),b^\tau_{p^*})$ intersects $c^-_{p^*}(\tau)$ at $t=T^1(\tau,z)$ and the y negative semi-axis at $t=T^2(\tau,z)$. Then we are ready to redefine the set

$$\partial\breve{E}^{u,a}_{p^*}(\tau,z):=\{\mathbf{X}_{p^*}(t,0,\tilde{\mathbf{A}}^{\mathbf{u}}_{p^*}(\tau,z),b^\tau_{p^*})\mid 0\le t\le T^1(\tau,z)\}\cup\tilde{M}^u_{p^*}(a^\tau_{l_1},z).$$

Then we introduce the 2-dimensional manifold $\partial\breve{\mathbf{E}}^{\mathbf{u,a}}_{\mathbf{p}^*}(\tau):=\cup_{z\le\tau}\partial\breve{E}^{u,a}_{p^*}(\tau,z)\times\{z\}$, and we exploit again the previously defined sets $\partial\breve{\mathbf{E}}^{\mathbf{u,b}}_{\mathbf{p}^*}(\tau)$, $\partial\mathbf{C}^-_{\mathbf{p}^*}(\tau)$ and $\breve{\mathbf{E}}^{\mathbf{u}}_{\mathbf{p}^*}(\tau)$. Once again the flow of (5.1) on $\partial\breve{\mathbf{E}}^{\mathbf{u,a}}_{\mathbf{p}^*}(\tau)$ points towards the interior of $\breve{\mathbf{E}}^{\mathbf{u}}_{\mathbf{p}^*}(\tau)$. The proof when $(\mathbf{Q},z)\in\partial\breve{\mathbf{E}}^{\mathbf{u,a}}_{\mathbf{p}^*}(\tau)\setminus\partial\tilde{M}^u_{p^*}(a^\tau_{l_1},z)$ can be obtained reasoning as above. Then the previous argument goes through and we get the thesis. $\square$

We can assume (when they all exist) $\bar{W}^u_{p^*}(\tau)\subset\tilde{W}^u_{p^*}(\tau)\subset\breve{W}^u_{p^*}(\tau)$, $\bar{W}^s_{p^*}(\tau)\subset\tilde{W}^s_{p^*}(\tau)\subset\breve{W}^s_{p^*}(\tau)$. With similar reasoning we obtain the following.

5.2. *Remark.* Assume **G1**$''$ and **G2**$''$ are satisfied and $j_1\le p^*\le i_2$. If $\mathbf{Q}\in U^+_{p^*}$ and $\mathbf{x}_{p^*}(t,\tau,\mathbf{Q})\in U^+_{p^*}$ for any $t\le\tau$ and $\lim_{t\to-\infty}\mathbf{x}_{p^*}(t,\tau,\mathbf{Q})=(0,0)$, then the corresponding solution $u(r)$ of (1.5) is a regular solution. Analogously if $\mathbf{Q}\in U^-_{p^*}$ and $\mathbf{x}_{p^*}(t,\tau,\mathbf{Q})\in U^-_{p^*}$ for any $t\ge\tau$ and $\lim_{t\to+\infty}\mathbf{x}_{p^*}(t,\tau,\mathbf{Q})=(0,0)$, then the corresponding solution $v(r)$ of (1.5) has fast decay.

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