

Structure theorems for positive radial solutions of the generalized scalar curvature equation.

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Abstract

We give some structure results for positive radial solutions of the following equation:

$$\Delta_p u + K(r)u|u|^{\sigma-2} = 0$$

where $K(r)$ is a function bounded above and below by positive constants. Here $r = |x|$, $x \in \mathbb{R}^n$, $\frac{2n}{n+2} \leq p \leq 2$, $n > p > 1$, and $\sigma = \frac{np}{n-p}$. In particular we manage to prove the existence of ground states and singular ground states when $K(r)$ is monotone as $r \rightarrow 0$ and as $r \rightarrow \infty$. The results are new even when $p = 2$, that is when we consider the usual Laplacian.

The proofs make use of a new Emden-Fowler transform which allow us to consider a 2-dimensional dynamical system thus giving a geometrical point of view on the problem. A key role in the analysis is played by an energy function which is a dynamical interpretation of the Pohozaev function used in [20] and [21].

Keywords

p-laplace equations, radial solution, regular/singular ground state, Fowler inversion, invariant manifold.

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Abbreviated form of the title: Non-autonomous p -Laplace equations

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1 Introduction

The aim of this paper is to discuss the existence and the asymptotic behavior of radial solutions of the following equation

$$\Delta_p u + K(|x|)u|u|^{\sigma-2} = 0 \quad (1.1)$$

plaplace

where $\Delta_p u = \operatorname{div}(|Du|^{p-2}Du)$, $p > 1$, denotes the p -Laplace operator, $x \in \mathbb{R}^n$, $\frac{2n}{n+2} \leq p \leq 2$, $\sigma = \frac{np}{n-p}$ is the Sobolev critical exponent.

We are particularly interested in Ground States, Singular Ground States and crossing solutions. By Ground state (G.S.) we mean a positive solution $u(x)$ defined in the whole space $\mathbb{R}^n$ such that $\lim_{|x| \rightarrow \infty} u(x) = 0$, and by Singular Ground State (S.G.S.) we mean a G.S. which is not defined at the origin and satisfies $\lim_{|x| \rightarrow 0} u(x) = +\infty$. By crossing solution we mean a solution $u(x)$ such that $u(x) > 0$ if $|x| < R$ and $u(x) = 0$ if $|x| = R$, therefore such a solution can also be regarded as a Dirichlet solution in a ball of radius R .

We will only deal with radial solutions, so we shall consider the following O.D.E.

$$(u'|u'|^{p-2})' + \frac{n-1}{r}u'|u'|^{p-2} + K(r)u|u|^{\sigma-2} = 0 \quad (1.2)$$

eq.na

where $r = |x|$ and we commit the following abuse of notation: we write $u(r)$ for $u(x)$ where $|x| = r$; here and later $'$ denotes derivation with respect to r .

We introduce now some notation that will be in force throughout all the paper. We will use the term “regular solution” to refer to a solution $v(x)$ of Eq. (1.2) satisfying $u(0) = u_0 > 0$ and $u'(0) = 0$.

We will use the term “singular solution” to refer to a solution $v(x)$ of Eq. (1.2) such that $\lim_{|x| \rightarrow 0} v(x) = +\infty$.

Furthermore when we write that $u(r) \sim r^{-\alpha}$ as $r \rightarrow c$ we mean that the limits $\liminf_{r \rightarrow c} u(r)r^\alpha$ and $\limsup_{r \rightarrow c} u(r)r^\alpha$ are both finite and positive.

Equations with Δ and Δ_p have been extensively studied by many authors in recent years and nowadays the autonomous Eq. (1.2) is almost completely understood; see [12] for a survey on the topic. Important papers on the case $p \neq 2$ are [20], and [21] where the authors, in particular give a structure result for regular positive solutions when $K(r)$ is monotone or admits exactly one critical point and it is a maximum. In [9] we have been able to complete this result in the monotone case, in the sense that we give a better estimate on the asymptotic behavior, we prove the existence of positive singular solutions and we classify all such solutions. It can be proved, see Proposition 2.5, that

if $K(r)$ is monotone for r small and for r large, then positive solutions of (1.2) can just have two asymptotic behaviours, both as $r \rightarrow 0$ and as $r \rightarrow \infty$. To be more precise a positive solution $u(r)$ can just be regular that is $u(0) = A > 0$, or singular that is $u(r) \sim r^{-\frac{n-p}{p}}$ as $r \rightarrow 0$; furthermore a positive solution $u(r)$ can just have fast decay, that is $u(r) \sim r^{-\frac{n-p}{p-1}}$, or slow decay that is $u(r) \sim r^{-\frac{n-p}{p}}$, as $r \rightarrow \infty$.

Putting together the results explained in [21] and in [9] and restricting our attention to the case when $\frac{2n}{n+2} \leq p \leq 2$, we can state the following classification result:

1.1 Proposition. *Consider Eq. (1.2) and assume that $K(r) \in C^1$ is monotone decreasing. Then we can classify all the positive solutions as follows*

- *All the regular solutions are monotone decreasing G.S. with slow decay.*
- *There exists at least one monotone decreasing S.G.S. with slow decay.*
- *There exist uncountably many Dirichlet solutions $u(r)$ in exterior domains; that is, there exists $R > 0$ such that $u(R) = 0$ and $u(r) \sim r^{-\frac{n-p}{p-1}}$, as $r \rightarrow \infty$.*

Assume that $K(r)$ is monotone increasing, then

- *All the regular solutions are crossing solutions.*
- *There exists one monotone decreasing S.G.S. with slow decay.*
- *There exist uncountably many monotone decreasing S.G.S. with fast decay.*

In both the cases there are no solutions $u(r)$ positive as $r \rightarrow 0$, except the ones described.

The hypotheses of the Proposition may be weakened: if we let p take values in $(1, \infty)$, we still have the result concerning regular solutions, see [20]. We think in fact that the hypothesis $\frac{2n}{n+2} \leq p \leq 2$ is just technical, and that, if we remove it, also the result concerning singular solutions should hold.

When $K(r)$ is not monotone the structure of positive solutions becomes richer and less understood. Bianchi in [3], proved in particular that when

$p = 2$ and there is $r = R$ such that $K(r)$ is monotone decreasing for $r \leq R$ and monotone increasing for $r \geq R$, then all the solution of (1.1) are radial. Then Bianchi and Egnell in [4] and Bianchi in [2] give several different type of conditions on $K(r)$ which are sufficient for the existence of ground states with fast decay. In particular they proved the existence of such solutions when $(\int_0^\infty K'(s)s^n ds)(\int_0^\infty K'(s)s^{-n} ds) < 0$ and the limit $\lim_{r \rightarrow \infty} K(r)$ and $\lim_{r \rightarrow \infty} K'(r)$ exist, are positive and bounded and $K(r)$ is flat enough as $r \rightarrow 0$ and as $r \rightarrow \infty$. Bianchi in [2] also gives conditions sufficient for the non-existence of radial ground state and the existence of non-radial ground states with fast decay. Yanagida and Yotsutani in [23] consider eq. (1.2) assuming $p = 2$ and $\sigma > 2$. When σ is the Sobolev critical exponent (that is the case considered in this paper) they proved the existence of open sets of crossing solutions and ground states with slow decay disconnected by a non-empty set of ground states with fast decay, in particular when $K(r)$ is increasing for r small and decreasing for r large.

The case $p \neq 2$ is less understood. In [21] the authors consider the function $J(r) = \int_0^r K'(s)s^n$ and they prove a structure result assuming that there is $R > 0$ such that $J(r) \geq 0$ for $r \in (0, R)$ and $K'(r) \leq 0$ for $r > R$. They state that positive regular solutions could have either the structure of the monotone increasing case either the structure of the monotone decreasing case, or a richer situation, that is the coexistence of all the three different families of regular solutions. Kabeya, Yanagida and Yotsutani in [18], proved a result concerning nodal solutions. If we restrict to eq. (1.2) with σ critical they proved the existence of ground state with fast decay, both positive for any $r > 0$ or with a prescribed finite number of 0, assuming that $K(r)$ is increasing for r small and decreasing for r large and satisfy some further mild assumptions. In particular they assume that $\lim_{r \rightarrow 0} K(r) = 0 = \lim_{r \rightarrow \infty} K(r)$.

In this paper we want to complete the analysis, started in [9] with the monotone case, of the results obtained in [20] and [21] and in particular we want to classify singular solutions and to show when we can find Ground States with fast decay.

We consider the case when $K(r)$ is bounded above and below by positive constants, that is the complementary situation with respect to [18]. We need to assume that $\frac{2n}{n+2} \leq p \leq 2$, while the methods used in [18] and in [21] does not need such requirements. However we think that these requirements are technical and can only affect the asymptotic behaviour. In this setting, we are able to state natural conditions which are sufficient to have the richer structure for positive solutions (that is structure A, see below). In this case,

under suitable hypotheses, we will find one of the following structures for positive solutions:

- A**
- There exist uncountably many monotone decreasing G.S. with slow decay.
 - There exist uncountably many crossing solutions.
 - There exists a non empty set of monotone decreasing G.S. with fast decay disconnecting the first two sets.
 - There exist uncountably many solutions $u(r)$ of the Dirichlet problem in exterior domains; that is, there exists $R > 0$ such that $u(R) = 0$ and $u(r) \sim r^{-\frac{n-p}{p-1}}$, as $r \rightarrow \infty$.
 - There exist uncountably many monotone decreasing S.G.S. with fast decay.
 - There exist uncountably many monotone decreasing S.G.S. with slow decay.
- B**
- There exist uncountably many crossing solutions.
 - There exist uncountably many solutions $u(r)$ of the Dirichlet problem in exterior domains; that is, there exists $R > 0$ such that $u(R) = 0$ and $u(r) \sim r^{-\frac{n-p}{p-1}}$, as $r \rightarrow \infty$.
 - There exists a non empty set of monotone decreasing G.S. with fast decay.
 - There exist uncountably many monotone decreasing S.G.S. with fast decay.

Here we enumerate the main hypotheses that will be used in this paper and the main results.

Hypotheses

- α^- There exists $\rho > 0$ such that $K(r)$ is monotone decreasing, for any $0 \leq r \leq \rho$.
- α^+ There exists $\rho > 0$ such that $K(r)$ is monotone increasing, for any $0 \leq r \leq \rho$.
- Ω^- There exists $R > 0$ such that $K(r)$ is monotone decreasing, for any $r \geq R$.

Ω^+ There exists $R > 0$ such that $K(r)$ is monotone increasing, for any $r \geq R$.

finito **1.2 Theorem.** *Consider Eq. (1.2) and assume $K(r) \in C^1$ is strictly positive and bounded. Assume that hypotheses α^+ and Ω^- are satisfied, then positive solutions have a structure of type A.*

altro **1.3 Theorem.** *Consider Eq. (1.2) and assume $K(r) \in C^1$ is strictly positive and bounded. Assume that hypotheses α^- and Ω^+ are satisfied, then positive solutions have a structure of type B.*

Thus we manage to complete the analysis of positive solutions of (1.2) by giving a classification of singular positive solutions. Moreover we manage to give a condition which is sufficient to prove the existence of all the families of regular solutions and to extend the results to a wider class of functions. Furthermore we can refine the estimates on the asymptotic behavior of the solutions. We remark that the recent paper [19] treat the scalar curvature equation ($p = 2$) with a function $K(\cdot)$ which has properties significantly different from those of the functions $K(\cdot)$ considered here.

We make use of the techniques developed by Johnson, Pan, Yi and Battelli in [16], [5], [5], [6] for problem with the Laplacian and in [9] for the problem with the p -Laplacian. We also take some of the basic ideas of [11], developed for the problem where $K(r)$ is a perturbation of a constant, and we manage to reapply them here. Thus we can extend some of the results obtained for functions $K(r)$ which are regular or singular perturbations of a constant, to “generic” strictly positive $K(r)$ exhibiting the same oscillatory behavior.

We exploit the transform of Fowler type introduced in [10], which transforms Eq. (1.2) to a dynamical system and enables us to give a geometrical interpretation to the problem. The refining of the estimate on the asymptotic behavior follows by an application to the problem of invariant manifold theory, extended to non-autonomous systems. A crucial role will be played by a function H which enables us to give a dynamical interpretation of the Pohozaev identity.

We finish this introduction by giving some terminology. Recall that given a system of the form

$$\dot{x} = f(x, t)$$

and a solution $x(t)$, the α -limit set of $x(t)$ is the set

$$A = \left\{ P : \exists t_n \rightarrow -\infty \text{ such that } \lim_{n \rightarrow \infty} x(t_n) = P \right\},$$

while the ω -limit set is the set

$$W = \{P : \exists t_n \rightarrow +\infty \text{ such that } \lim_{n \rightarrow \infty} x(t_n) = P\}.$$

One can show that, if $x(t)$ is bounded on $\mathbb{R}$, then these sets are compact. Moreover, if the system is autonomous, these sets are invariant for the flow generated by the system.

2 Preliminaries

We begin by introducing a transform which generalizes to the p -Laplacian the well known Fowler transform which works for the classic Laplacian.

$$\begin{aligned} x_1 = u(r)r^\alpha & \quad x_2 = u'(r)|u'(r)|^{p-2}r^\beta & r = e^t & \quad \phi(t) = K(e^t) \\ \text{where } \alpha = \frac{n-p}{p} & \text{ and } \beta = \frac{n(p-1)}{p} \end{aligned} \quad (2.1) \quad \boxed{\text{transf}}$$

This change of variables allows us to transform the singular O.D.E (1.2) into the following dynamical system:

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} \alpha & 0 \\ 0 & -\alpha \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} x_2|x_2|^{\frac{2-p}{p-1}} \\ -\phi(t)x_1|x_1|^{\sigma-2} \end{pmatrix} \quad (2.2) \quad \boxed{\text{si.na}}$$

Here and later “ $\cdot$ ” denotes derivation with respect to t . Note that the preceding equation is C^1 if and only if $\frac{2n}{2+n} \leq p \leq 2$, thus we will restrict our analysis only to this case. Moreover we have a close relationship between trajectories $x(t)$ of our system and solutions $u(r)$ of our problem.

corrispondenze

2.1 Remark. The solutions $u(r)$ of Eq. (1.2) correspond to the trajectories $x(t)$ of system (2.2) having the origin as α -limit point. Moreover if $u(r) > 0$ then $x_1(t) > 0$ and $u'(r) > 0$ implies $x_2(t) > 0$.

Since we are mainly interested in positive solutions $u(r)$ we will focus our attention on the halfplane where $x_1 \geq 0$, which will be denoted by $\mathbb{R}_+^2$.

It will be useful to embed system (2.2) in the following one parameter family of systems:

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} \alpha & 0 \\ 0 & -\alpha \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} x_2|x_2|^{\frac{2-p}{p-1}} \\ -\phi(t+\tau)x_1|x_1|^{\sigma-2} \end{pmatrix} \quad (2.3) \quad \boxed{\text{si.natau}}$$

Let $x(\tilde{\tau}, x_0; t)$ be the trajectory of system (2.3) departing at $t = 0$ from $x = x_0$, for $\tau = \tilde{\tau}$. Assume that ϕ is uniformly continuous, then using the theory developed in [14] and in [15], we can state the existence of C^1 leaves

$$W_{loc}^u(\tau) = \{x_0 \mid \lim_{t \rightarrow -\infty} x(\tau, x_0; t)\} \quad \text{and} \quad W_{loc}^s(\tau) = \{x_0 \mid \lim_{t \rightarrow \infty} x(\tau, x_0; t)\},$$

defined in a neighborhood of the origin. These leaves are C^1 graphs respectively on the unstable and stable manifold of the autonomous system linearized in the origin (that in this case are respectively the x_1 and x_2 axis). The uniform continuity of ϕ ensures that we can choose $W_{loc}^u(\tau)$ and $W_{loc}^s(\tau)$ in such a way that their diameter is greater than a positive small constant which is independent from τ .

Then, following the techniques explained in [16] and in [5], we can construct a global stable and unstable manifold as follows:

$$W^u(\tau) = \bigcup_{t \in \mathbb{R}} \{x_0 \mid x(\tau - t, x_0; t) \in W_{loc}^u(\tau)\}$$

$$W^s(\tau) = \bigcup_{t \in \mathbb{R}} \{x_0 \mid x(\tau - t, x_0; t) \in W_{loc}^s(\tau)\}.$$

It can be proved that these manifolds are C^1 , furthermore they vary C^1 smoothly in τ , see [16], Theorem 2.1 at page 1051, and [15]. Define $\Phi_\tau(t, x_0)$ to be the diffeomorphism which associates to a point x_0 its image through the flow of system (2.3) at time t , that is $x(\tau, x_0; t)$. Note that $\Phi_\tau(t, W^{u,s}(\tau)) = W^{u,s}(\tau + t)$.

Now we give the definitions of the following three sets of the dynamical system:

$$U^+ := \{(x_1, x_2, x_3) \mid x_1 \geq 0 \quad x_2 \leq 0 \quad \text{and} \quad \dot{x}_1 > 0\}$$

$$U^- := \{(x_1, x_2, x_3) \mid x_1 \geq 0 \quad x_2 \leq 0 \quad \text{and} \quad \dot{x}_1 < 0\}$$

$$c := \{(x_1, x_2, x_3) \mid x_1 \geq 0 \quad x_2 \leq 0 \quad \text{and} \quad \dot{x}_1 = 0\}.$$

We will also need the following extended autonomous system obtained from system (2.4) by adding the extra variable $x_3 = \tau + t$:

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{pmatrix} = \begin{pmatrix} \alpha & 0 & 0 \\ 0 & -\alpha & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} + \begin{pmatrix} x_2 |x_2|^{\frac{2-p}{p-1}} \\ -\phi(x_3)x_1 |x_1|^{\sigma-2} \\ 1 \end{pmatrix} \quad (2.4) \quad \boxed{\text{si.naa}}$$

We will make use also of the following system, where we set $x_3 = e^{\xi t}$.

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{pmatrix} = \begin{pmatrix} \alpha & 0 & 0 \\ 0 & -\alpha & 0 \\ 0 & 0 & \xi \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} + \begin{pmatrix} x_2|x_2|^{\frac{2-p}{p-1}} \\ -\phi(\xi; x_3)x_1|x_1|^{\sigma-2} \\ 0 \end{pmatrix} \quad (2.5) \quad \boxed{\text{si.naaxi}}$$

where $\phi(\xi; x_3) = \phi(\frac{\log(x_3)}{\xi})$.

If we set $\xi > 0$, this system is useful to investigate the asymptotic behaviour of the solutions as $t \rightarrow -\infty$, while if we set $\xi < 0$ it can be used to analyze the asymptotic behaviour as $t \rightarrow \infty$. In fact when $\xi > 0$ the α -limit set of any bounded trajectory of system (2.5) is contained in the plane $x_3 = 0$, while when $\xi < 0$ this plane contains the ω -limit set of any bounded trajectory. Note that when $\xi < 0$ the origin admits a two dimensional stable manifold W^s . Intersecting W^s with the plane $x_3 = e^{\xi\tau}$ we obtain the manifolds $W^s(\tau)$ already described. Analogously when $\xi > 0$ the origin admits a two dimensional unstable manifold W^u . Intersecting W^u with the plane $x_3 = e^{\xi\tau}$ we obtain the manifolds $W^u(\tau)$ already described.

Furthermore if we pass from (2.5) to (2.4) through the change of variables $x_3 \rightarrow \frac{\log x_3}{\xi}$, we have that W^u and W^s are transformed into two dimensional manifolds which will be denoted with the same name. Note that, intersecting these manifolds with the plane $x_3 = \tau$, we obtain respectively $W^u(\tau) \times \{\tau\}$ and $W^s(\tau) \times \{\tau\}$.

We introduce now a function closely related to the Pohozaev identity, and already used in a more general setting in [9], which will play a key role in the following analysis. It is in fact the transposition in this dynamical context of the function $P(r)$ of [21]. Let us consider a trajectory $x(t)$ of system (2.3), we define

$$H(x(t); t) := \alpha x_1 x_2 + \frac{p-1}{p} |x_2|^{\frac{p}{p-1}} + \phi(\tau + t) \frac{|x_1|^\sigma}{\sigma}.$$

Observe that by differentiating we get

$$\frac{d}{dt} H(x(t), t) = \frac{d}{dt} \phi(\tau + t) \frac{|x_1|^\sigma}{\sigma}.$$

We now begin our analysis of Eq. (2.3). To do this we need to recall some results about the autonomous system (2.2) where $\phi \equiv \text{const} > 0$, see [9]. First of all observe that such a system admits exactly three critical points which are the origin, $P = (P_1, P_2)$, where $P_2 < 0 < P_1$ and $-P$. Note that P

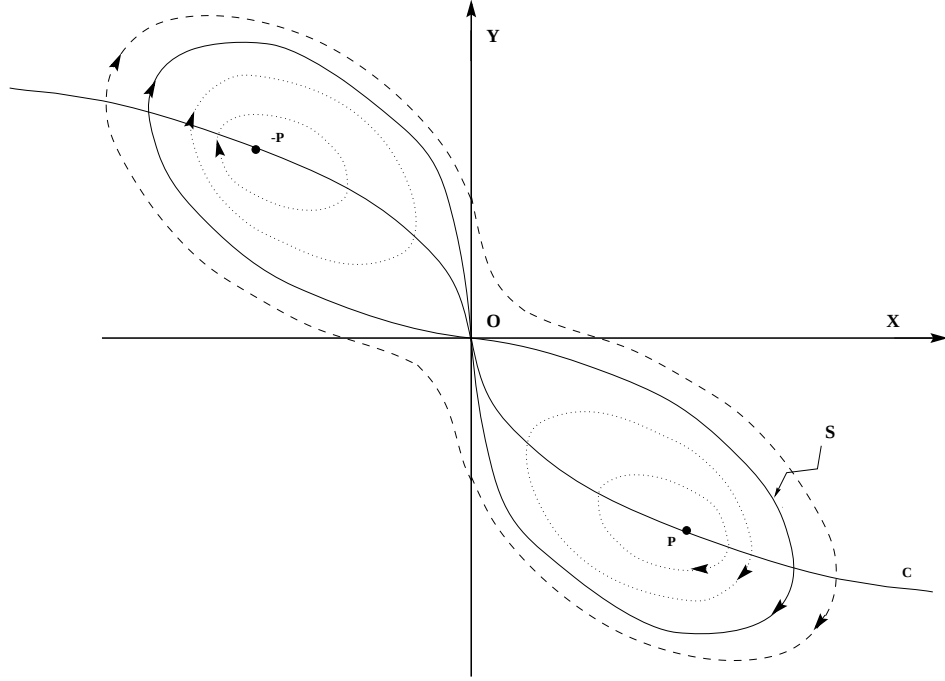


Figure 1: Sketch of the level sets of the function $H(x_1, x_2, T)$, for T fixed. The solid line is the level set C_0 , the dotted and the dashed lines represent some level sets C_b where respectively $b < 0$ and $b > 0$. livelli

depends on ϕ . We need now the following Lemma concerning the shape of the level sets of the function $H(\cdot, \cdot; t)$, for t fixed, see Figure 1. Let us define

$$C_b := \{(x_1, x_2) \mid H(x_1, x_2; T) = b\}.$$

For any fixed T such that $\phi(T)$ is positive and finite, the level sets C_b of the function $H(\cdot, \cdot; T)$ are as sketched in Figure (1). This claim follows easily from Lemma 2.9 in [9] and Lemma 2.6 in [10], therefore it will be skipped. Recalling that H is a first integral for the autonomous system we easily deduce the following.

autonomo

2.2 Proposition. *Consider Eq. (1.2) when $K \equiv \text{const} > 0$. All the regular solutions $u(r)$ of Eq. (1.2) are monotone decreasing G.S. with fast decay, that is $u(r) \sim r^{-\frac{n-p}{p-1}}$. They correspond to a unique homoclinic trajectory of (2.2), belonging to the closed 4th quadrant.*

There exist uncountable many S.G.S. $v(r)$ with slow decay that is $v(r) \sim r^{-\alpha}$, both as $r \rightarrow 0$ and as $r \rightarrow \infty$. They correspond to the periodic trajectories contained in the open 4th quadrant, therefore we can find $a, b > 0$ such that $ar^{-\alpha} \leq v(r) \leq br^{-\alpha}$, for any $r > 0$.

There exists a S.G.S. with slow decay that can be explicitly computed, that is $\bar{v}(r) = P_1 r^{-\alpha}$; it corresponds to the critical point P .

In the autonomous case we can also write the exact expression of the Ground States, see [12] for example. Therefore we can deduce the exact expression of the homoclinic trajectories. We denote by $U^K(t) = (U_1^K(t), U_2^K(t))$ the homoclinic trajectory of the system (2.2) where $K(r) \equiv K$ and crossing the isocline c at $t = 0$.

$$U_1^K(\tau, t) = \left[\frac{1}{D(e^{-t} + e^{\frac{1}{p-1}t})} \right]^{\frac{n-p}{p}} K^{-\frac{n-p}{p^2}}$$

where $D = (n-p)^{\frac{p-1}{p}} n^{\frac{1}{p}}$ is a constant.

Note that the autonomous system is invariant for translations in t , therefore all the homoclinic trajectories have the same graph.

Now we need a technical Lemma which shows that unbounded trajectories $x(t)$ of (2.2) cannot correspond to positive solutions $u(r)$ of (1.2).

illimitate

2.3 Lemma. Consider equation (1.2) and the corresponding system (2.2), where $K(r)$ is strictly positive and bounded. Then, if $x(t)$ is unbounded, it rotates clockwise crossing infinitely many times the x_1 and x_2 axes.

Proof. By assumption there exist $M > m > 0$ such that $m < K(r) < M$ for any r . Consider a trajectory $x(t)$ which becomes unbounded as $t \rightarrow c$ where c can also be ∞ . The proof in the case of a trajectory that becomes unbounded going backwards in t is analogous.

Fix t_0 and the corresponding point $P^1 = x(t_0)$ in $\mathbb{R}^2$. Assume that $P \in U^+$. Consider at first system (2.2), where $\phi(t) \equiv M$. Recall that the solution of (2.2) which are not homoclinic to $(0,0)$ and which do not coincide with equilibria are periodic, hence the corresponding trajectories define closed curves in $\mathbb{R}^2$. We choose a periodic solution $x^{1,M}(t)$ of (2.2) which crosses the coordinate axes and such that P^1 lies in the exterior of the disc $D^{1,M}$ enclosed by $x^{1,M}(t)$ for $t \in \mathbb{R}$. Such a choice is always possible, since we can choose $|P^1|$ as large as we wish. In a similar way, consider system (2.2), where $\phi(t) \equiv m$. We choose a periodic solution $x^{1,m}(t)$ of (2.2)

which crosses the coordinate axes and such that P lies in the open disc $D^{1,m}$ enclosed by $x^{1,m}(t)$, for $t \in \mathbb{R}$. We can choose $D^{1,M}$ and $D^{1,m}$ in such a way that $D^{1,m} \supset D^{1,M}$. Let $\partial D^{1,M}$ and $\partial D^{1,m}$ denote the boundary respectively of $D^{1,M}$ and $D^{1,m}$. Let us denote by $R^+ := D^{1,m} - D^{1,M}$ and by ∂R^+ its boundary.

We return to the non-autonomous system (2.2). We claim that the flow on $\partial R^+ \cap U^+$ is always going towards the interior of R^+ and that $P^1 \in R^+ \cap U^+$. In fact choose $Q \in \partial D^{1,m} \cap U^+$; denote by $x^{1,m}(Q, t)$ and $x(Q, t)$ the trajectories passing through Q at $t = 0$ respectively of system (2.2) where $\phi \equiv m$ and of the non autonomous system (2.2). Then we have $\dot{x}_1(Q, 0) = \dot{x}_1^{1,m}(Q, 0)$ and $\dot{x}_2(Q, 0) \leq \dot{x}_2^{1,m}(Q, 0)$. The proof for the case $Q \in \partial D^{1,M} \cap U^+$ is completely analogous, so the claim is proved

Consider now the unbounded trajectory $x(t)$. Note that it lies in $R^+ \cap U^+$ for $t \geq t_0$, until it crosses the isocline c in a point P^2 . Thus there exists $t_1 > t_0$ such that $x(t_1) = P^2 \in c$ and $x(t)$ enters U^- for $t > t_1$.

Once again we consider the autonomous system where $\phi(t) \equiv m$. We choose a periodic solution $x^{2,m}(t)$ of (2.2) which crosses the coordinate axes and such that P^2 lies in the exterior of the disc $D^{2,m}$ enclosed by $x^{2,m}(t)$ for $t \in \mathbb{R}$. In a similar way, we choose a periodic solution $x^{2,M}(t)$ of (2.2) where $\phi(t) \equiv M$ which crosses the coordinate axes, and such that P^2 lies in the open disc $D^{2,M}$ enclosed by $x^{2,M}(t)$, where $t \in \mathbb{R}$. We choose $D^{1,M}$ and $D^{2,M}$ in such a way that $D^{2,M} \supset D^{2,m}$ and define $R_2^- := D^{2,M} - D^{2,m}$. Now we return to the non-autonomous system (2.2). Reasoning as above we find that $x(t) \in R_2^-$, for all $t > t_1$ such that $x(t) \in U^-$. Recalling that $x(t)$ is unbounded we conclude that there exists $t_2 < t_1$ such that $x(t_2) \in c$. Therefore $x(t)$ rotates clockwise crossing the x_2 and x_1 negative semi-axes, then it enters U^+ for $t > t_2$.

Iterating the reasoning we obtain that $x(t)$ must cross the coordinate axes infinitely many times. $\square$

With a similar argument we get also the following result

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2.4 Lemma. *Consider a trajectory $x(t)$ of system (2.3) and assume that*

$\liminf_{t \rightarrow \infty} H(x(t), t) > 0$. Then, if we follow $x(t)$ forward in t we find that it must cross the positive x_2 semi-axis for some $t = t_2$. Analogously consider a trajectory $\bar{x}(t)$ and assume that $\liminf_{t \rightarrow -\infty} H(\bar{x}(t), t) > 0$. Then, if we follow $\bar{x}(t)$ backwards in t we find that it must cross the positive x_2 semi-axis for some $t = t_1$.

Proof. Assume that $\liminf_{t \rightarrow \infty} H(x(t), t) > 0$; then there are $a > 0$ small and $T > 0$ large so that $H(x_1(t), x_2(t), t) > 2a$ for any $t \geq T$. Let us denote by $\bar{D}^a(t) := \{(x_1, x_2) \mid H(x_1, x_2) \leq a\}$. Observe that $\bar{D}^a(t)$ is compact and connected. Assume first that $x(T) = Q \in U^+$. Consider the trajectory $x_m(Q, t)$ of system (2.2) where $\phi \equiv m = \inf_{t \in \mathbb{R}} \phi(t)$, departing from Q at $t = 0$. We denote by A the bounded subset enclosed by the trajectory $x_m(Q, t)$ and by $A^+ = A \cap U^+$. Then $x(t)$ is forced to stay in $A^+ \cap \bar{D}^a(t)$ for $t > T$, until it reaches the isocline $\dot{x} = 0$ at $t = T_1 \leq \infty$. Since $|\dot{x}(t)| + |\dot{y}(t)|$ is strictly positive for $t \in [T, T_1]$ it follows that $T_1 < \infty$.

Therefore we can assume that $Q \in U^-$. Let us denote by $x_M(Q, t)$ the trajectory of system (2.2) where $\phi \equiv M = \sup_{t \in \mathbb{R}} \phi(t)$, departing from Q at $t = 0$, and denote by B the bounded set enclosed by $x_M(Q, t)$. We choose $\|Q\|$ large enough so that $\bar{D}^a(t) \subset B$ for any $t \in \mathbb{R}$, and denote by $B^-(t) = (B \cap U^-) \setminus \bar{D}^a(t)$. It follows that $x(t)$ is forced to stay in $B^-(t)$ for any $t > T$, until it crosses the x_2 axes. Reasoning as above it follows that $x(t)$ crosses the x_2 axes at a certain $t = T_2$ finite $T_2 > T$. Hence the first part of the Lemma is proved; the second can be proved with a similar argument so we will skip it. $\square$

It is easy to show now that we can have only two kinds of asymptotic behaviour, both as $r \rightarrow 0$ and as $r \rightarrow \infty$ also when $K(r)$ is monotone.

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2.5 Proposition. *Consider Eq. (1.2) and assume that $K(r)$ is strictly positive and bounded. Consider a solution $u(r)$ which is well defined and positive for r small. Assume that $K(r)$ is monotone for $r \rightarrow 0$ and define $K(0) = A > 0$. Then we can only have two behaviour as $r \rightarrow 0$.*

$$0 < u(0) < \infty \quad (\text{regular solution}) \quad \text{or} \quad u(r) \sim r^{-\alpha} \quad (\text{singular solution})$$

Furthermore for each singular solution $u(r)$ there exists a S.G.S. $v(r)$ of the frozen Eq. (1.2) where $K(r) \equiv A$ such that

$$\lim_{r \rightarrow 0} (u(r) - v(r))r^\alpha = 0.$$

Analogously consider a solution $u(r)$ of Eq. (1.2) which is well defined and positive for r large. Assume that $K(r)$ is monotone for r large and define $\lim_{r \rightarrow \infty} K(r) = B > 0$. Then we can only have two behaviour as $r \rightarrow \infty$.

$$\liminf_{r \rightarrow \infty} u(r) \sim r^{-\frac{n-p}{p-1}} \quad (\text{fast decay}) \quad \text{or} \quad u(r) \sim r^{-\alpha} \quad (\text{slow decay})$$

Furthermore for each slowly decaying solution $u(r)$ there exists a S.G.S. $v(r)$ of the frozen Eq. (1.2) where $K(r) \equiv B$ such that

$$\lim_{r \rightarrow \infty} (u(r) - v(r))r^\alpha = 0.$$

Proof. For this proof we follow the ideas developed in [9]. Assume that $K(r)$ is monotone for r large and consider a solution $u(r)$ well defined and positive for r large. Consider the corresponding trajectory $x(t)$ of system (2.5) where $\xi < 0$. Observe that the ω -limit set of the trajectory $x(t)$ must belong to the plane $x_3 = 0$. Furthermore the dynamics in this plane is the one of the autonomous system (2.4) where $\phi(t) \equiv B$. Note that the limit $\lim_{t \rightarrow \infty} H(x_1(t), x_2(t), t) = l$ exists. In fact $H(x_1(t), x_2(t), t)$ is monotone for t large, and both $x(t)$ and $\phi(t)$ are bounded. Furthermore note that this limit individuates exactly one trajectory in the plane $x_3 = 0$, that corresponds to the level set $H = l$. Since we have assumed that $u(r) > 0$ for any r we have $l \leq 0$.

Assume at first $l < 0$, then the ω -limit set of the trajectory $x(t)$ is the periodic trajectory of the plane $x_3 = 0$ corresponding to the level set $H = l$ of the function H . This periodic trajectory corresponds to a S.G.S. with slow decay $v(r)$ of the autonomous system. Therefore $u(r)$ has slow decay and $\lim_{r \rightarrow \infty} (u(r) - v(r))r^\alpha = 0$.

Assume now $l = 0$. Then it is easy to observe that $x(t)$ must have the origin as ω -limit set. Thus $u(r) = o(r^{-\alpha})$, as $r \rightarrow \infty$. Observing that system (2.3) admits an exponential dichotomy and that $x(t)$ departs from a point in $W^u(\tau)$ we get $u(r) = o(r^{-2\alpha+\epsilon})$. This asymptotic estimate can be improved through some integral manipulations based on the ideas suggested in [21], Theorem 5.2, and developed in details in [9]. Therefore we get $u(r) \sim r^{-\frac{n-p}{p-1}}$.

The proof for solutions $u(r)$ defined for r small is obtained considering system (2.5) where $\xi > 0$. $\square$

From this proof we can also deduce the following

2.6 Remark. A solution $u(r)$ is a regular solution if and only if the corresponding trajectory of system (2.3) has the origin as α -limit point. A solution $u(r)$ can have fast decay if and only if the corresponding trajectory of system (2.3) has the origin as ω -limit point.

Now we want to get some information about the shape of the first branches of the manifolds $W^u(\tau)$ and $W^s(\tau)$. We recall that we can find $M > 0$ large and $m > 0$ small enough to have that $m < K(r) < M$ for any r . Let

us call respectively $U^m(t) = (U_1^m(t), U_2^m(t))$ and $U^M = (U_1^M(t), U_2^M(t))$ the homoclinic trajectories of system (2.3) where $\phi \equiv m$ and $\phi \equiv M$. We call $\dot{U}^{m,M}(t) = (\dot{U}_1^m(t), \dot{U}_2^m(t))$ their derivative with respect to t . Note that $\dot{U}^m(t)$ and $\dot{U}^M(t)$ are tangent respectively to $U^m(t)$ and $U^M(t)$.

Let us consider now the non-autonomous system (2.3): the curves U^m and U^M are not anymore trajectories. We call E the set delimited by the origin and these two curves; moreover we define $E^+ = E \cap U^+$ and $E^- = E \cap U^-$. We claim that the intersection between $W^u(\tau)$ (resp. $W^s(\tau)$) and the isocline $\dot{x} = 0$ is nonempty. Follow $W^u(\tau)$ (resp. $W^s(\tau)$) from the origin towards $\mathbb{R}_+^2$; we denote by $P^u(\tau)$ (resp. by $P^s(\tau)$) the first intersection with the isocline $\dot{x} = 0$. We denote by $\tilde{W}^u(\tau)$ (resp. $\tilde{W}^s(\tau)$) the component of $W^u(\tau)$ (resp. $W^s(\tau)$) connecting the origin to $P^u(\tau)$ (resp. by $P^s(\tau)$). In [11] it is shown that if K is either a regular or a singular perturbation of a constant (that is respectively $K(r) = 1 + \epsilon k(r)$ and $K(r) = k(r^\epsilon)$, with $\epsilon > 0$ small and $k \in \mathcal{C}^2$ and bounded), the intersections between $\tilde{W}_\tau^{u,s}$ and the isocline $\dot{x} = 0$ is transversal: this might not be the case in this setting. We will denote by $x^s(\tau; t)$ and $x^u(\tau; t)$ the trajectory of system (2.3) departing at $t = 0$ resp. from $P^s(\tau)$ or from $P^u(\tau)$. Now we can state the following result.

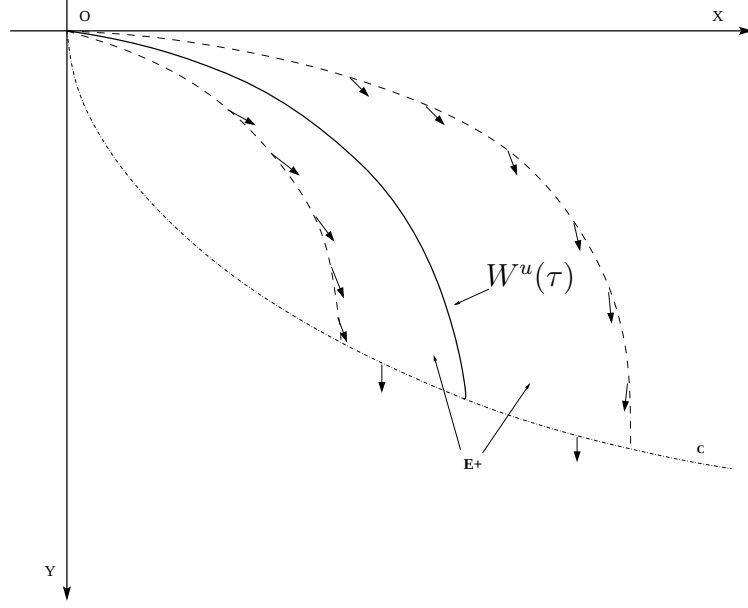
forma

2.7 Lemma. *The manifolds $W^u(\tau)$ and $W^s(\tau)$ intersect the isocline $\dot{x} = 0$, for any $\tau \in \mathbb{R}$. Moreover $\tilde{W}^u(\tau)$ (respectively $\tilde{W}^s(\tau)$) belong to E^+ (resp. E^-) and the trajectory $x^u(\tau; t)$ (resp. $x^s(\tau; t)$) is contained in E^+ for any $t < 0$ (resp. E^- for any $t > 0$), for any $\tau \in \mathbb{R}$.*

Proof. Consider the non-autonomous system (2.3); reasoning as in Lemma 2.3 we can prove that the flow on ∂E^+ points towards the interior of E^+ , while on ∂E^- the flow points towards the exterior of E^- , see Figure 2. Therefore any solution belonging to $W^u(\tau)$ departs from the origin, gets into E^+ and cross the isocline c for some finite t , or it stays in E^+ for any t and touches the isocline c as $t \rightarrow \infty$.

Now we claim that any solution $x(t)$ belonging to $W^u(\tau)$ reaches the isocline c for finite t . In fact consider a solution $\bar{x}(t)$ of system (2.3) belonging to $W^u(\tau)$. Observe that there exists the limit $\lim_{t \rightarrow \infty} H(\bar{x}(t), t) = l$. Consider now system (2.5) where $\xi < 0$ and call $P = (P_1, P_2, 0)$ the only critical point of this system in $\mathbb{R}_+^2 \times \mathbb{R}$. We call $\bar{X}(t)$ the trajectory $(\bar{x}(t), x_3(t))$ of system (2.5) corresponding to the trajectory $\bar{x}(t)$ of system (2.3). Assume for contradiction that $\bar{X}(t) \in U^+$ for any t .

Assume that $\lim_{r \rightarrow \infty} K(r) = B > 0$, and that $K(r) < B$; we define the

Figure 2: Sketch of the first branch of $W^u(\tau)$.

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following auxiliary function:

$$H_B(x_1, x_2) := \alpha x_1 x_2 + \frac{p-1}{p} |x_2|^{\frac{p}{p-1}} + \frac{B}{\sigma} |x_1|^\sigma.$$

Note that the minimum of this function is reached at the critical points P and $-P$. Furthermore, if $x(t)$ is a solution of system (2.3) differentiating with respect to t we get the following:

$$\frac{d}{dt} H_B(x_1(t), x_2(t)) := (B - \phi(t)) x_1 |x_1|^{\sigma-2} \frac{dx_1}{dt}.$$

Recalling that $\frac{d\bar{x}_1}{dt} > 0$ for any t we have that $H_B(\bar{x}_1(t), \bar{x}_2(t), t)$ is monotone increasing for t large. Thus the limit $\lim_{t \rightarrow +\infty} H_B(x_1(t), x_2(t))$ exists and it is strictly larger than $H_B(P_1, P_2)$. It follows that $x(t)$ has a periodic trajectory as ω -limit set, see Proposition 2.5, so it crosses the isocline $\dot{x} = 0$, a contradiction. So the Lemma is proved.

Now we remove the assumption on $K(r)$. Fix $\bar{\tau}$, we want to prove that $W^u(\bar{\tau})$ crosses the isocline c . We can construct a smooth function $\psi(t)$ such that $\psi(t + \bar{\tau}) = \phi(t + \bar{\tau})$ for any $t < 0$, and $\psi(t + \bar{\tau})$ is monotone increasing

for any $t > 1$. We denote by $W_\psi^u(\tau)$ the unstable manifold of the system with potential $\psi(\tau + t)$, and simply by $W^u(\tau)$ the unstable manifold of the system with potential $\phi(\tau + t)$.

Note that the unstable manifold $W_\psi^u(\tau)$ coincides with $W^u(\tau)$ for any $\tau \leq \bar{\tau}$. Furthermore we have just shown that $W_\psi^u(\tau)$ crosses the isocline c for any τ , since $\psi(\tau + t)$ is monotone increasing for t large. Therefore $W^u(\bar{\tau})$ crosses the isocline c .

Analogously, following backwards in t any solution belonging to $W^s(\tau)$, we notice that it departs from the origin, gets into E^- and crosses the isocline c for some t finite.

Recalling that the global manifolds $W^{u,s}(\tau)$ are constructed from the local manifolds $W_{loc}^{u,s}(\tau)$ using the flow of (2.3), the proof of the Lemma easily follows. $\square$

Observe now that $\lim_{t \rightarrow -\infty} H(x^u(\tau; t)) = 0$ and $\lim_{t \rightarrow \infty} H(x^s(\tau; t)) = 0$, thus

$$\begin{aligned} H(P^u(\tau), 0) &= \int_{-\infty}^0 \dot{\phi}(t + \tau) \frac{|x_1^u|^\sigma}{\sigma} dt \\ H(P^s(\tau), 0) &= - \int_0^\infty \dot{\phi}(t + \tau) \frac{|x_1^s|^\sigma}{\sigma} dt, \end{aligned} \tag{2.6} \quad \boxed{\text{integrale}}$$

where we have used the notation $x^u(\tau; t) = (x_1^u(\tau; t), x_2^u(\tau; t))$ and $x^s(\tau; t) = (x_1^s(\tau; t), x_2^s(\tau; t))$.

3 Oscillatory Potentials

Now we turn to consider potentials $K(r)$ which are monotone for r large and for r small. As usual we always assume that $K(r) \in C^1$ is bounded above and below by positive constants and that $\frac{2n}{n+2} \leq p \leq 2$. First of all we need the following Lemma:

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3.1 Lemma. *Consider (1.2) and assume that $K(r)$ is strictly positive and bounded.*

- *Assume that hypothesis α^+ is satisfied, then there is T_α^+ such that $H(P^u(\tau), 0) > 0$ and $H(P^s(\tau), 0) < 0$, for any $\tau < T_\alpha^+$. Assume that hypothesis α^- is satisfied, then there is T_α^- such that we have $H(P^u(\tau), 0) > 0$ and $H(P^s(\tau), 0) < 0$, for any $\tau < T_\alpha^-$.*

- Assume that hypothesis Ω^+ is satisfied, then there is T_ω^+ such that $H(P^u(\tau), 0) > 0$ and $H(P^s(\tau), 0) < 0$, for any $\tau > T_\omega^+$.
Assume that hypothesis Ω^- is satisfied, then there is T_ω^- such that we have $H(P^u(\tau), 0) > 0$ and $H(P^s(\tau), 0) < 0$, for any $\tau > T_\omega^-$.

Proof. We begin from the first claim. Consider any point $P^u(\tau)$ such that $\tau < T_0 = \log \rho$; observe that $\dot{\phi}(\tau) > 0$ for any $\tau < T_0$. Therefore recalling (2.6) we have

$$H(P^u(\tau), 0) = \int_{-\infty}^0 \dot{\phi}(\tau + t) \frac{|x_1^u(\tau, t)|^\sigma}{\sigma} dt > 0$$

for any $\tau < T_0$.

Moreover observe that

$$\begin{aligned} H(P^s(\tau), 0) &= - \int_0^{+\infty} \dot{\phi}(\tau + t) \frac{|x_1^s(\tau, t)|^\sigma}{\sigma} dt = \\ &= - \int_{T_0 - \tau}^{+\infty} \dot{\phi}(\tau + t) \frac{|x_1^s(\tau, t)|^\sigma}{\sigma} dt - \int_0^{T_0 - \tau} \dot{\phi}(\tau + t) \frac{|x_1^s(\tau, t)|^\sigma}{\sigma} dt \end{aligned} \quad (3.1) \quad \boxed{\text{calcolo1}}$$

We want to show that $|I(\tau)| = |\int_{T_0 - \tau}^{+\infty} \dot{\phi}(\tau + t) \frac{|x_1^s(\tau, t)|^\sigma}{\sigma} dt|$ becomes small as $\tau \rightarrow \infty$.

Let us denote by $Q^u(\tau) = x^s(\tau, T_0 - \tau)$. We want to show that $|Q^s(\tau)| \rightarrow 0$ as $\tau \rightarrow -\infty$; hence in particular $x_1^s(\tau, T_0 - \tau) \rightarrow 0$ as $\tau \rightarrow -\infty$. We recall that $x^s(\tau, t) \in E^-$ for any $t > 0$, thus $\dot{x}^s(\tau, t) < 0$ for $t > 0$. Suppose for contradiction that there is $l > 0$ and a sequence $\tau_n \rightarrow -\infty$ such that $x_1^s(\tau_n, T_0 - \tau_n) \geq l$. Then $x_1^s(\tau_n, t) > l$ and $x_1^s(\tau_n, t) \in E^-$ for any $t \in (0, T_0 - \tau_n)$; it follows that for any $\epsilon > 0$ there is $\delta > 0$ such that $\dot{x}_1^s(\tau_n, t) < -\epsilon$ for $t \in (\delta, T_0 - \tau_n)$. Hence

$$x_1^s(\tau_n, T_0 - \tau_n) - x_1^s(\tau_n, \delta) = - \int_{T_0 - \tau_n}^{\delta} \dot{x}_1^s(\tau_n, t) dt > \epsilon(\delta - \tau_n + T_0).$$

But, as $\tau_n \rightarrow -\infty$ the left hand side member is finite while the right hand side tends to $+\infty$, a contradiction; it follows that $x_1^s(\tau, t) \rightarrow 0$ as $\tau \rightarrow -\infty$.

Now, recalling that $x_1^s(\tau, t) \rightarrow 0$ exponentially fast as $t \rightarrow +\infty$, we deduce that, for any $\varepsilon > 0$, there exists $\tau(\varepsilon)$ such that $|I(\tau)| < \varepsilon$. Therefore, using (3.1) we deduce that we can find $T_\alpha^+ < T_0$ such that

$$H(P^s(\tau), 0) < \varepsilon - \int_0^{T_0 - \tau} \dot{\phi}(\tau + t) \frac{|x_1^s(\tau, t)|^\sigma}{\sigma} dt < 0, \quad (3.2) \quad \boxed{\text{calcolon}}$$

Analogously assume that hypothesis Ω^- is satisfied, then there exists $T_1 = \log(R)$ such that $\dot{\phi}(t) < 0$ for any $t > T_1$. Therefore

$$H(P^s(\tau), 0) = - \int_0^{+\infty} \dot{\phi}(\tau + t) \frac{|x_1^s(\tau, t)|^\sigma}{\sigma} dt > 0$$

for any $\tau > T_1$.

Moreover, reasoning as above we find

$$\begin{aligned} H(P^u(\tau), 0) &= \int_{-\infty}^0 \dot{\phi}(\tau + t) \frac{|x_1^u(\tau, t)|^\sigma}{\sigma} dt = \\ &= \int_{-\infty}^{T_1-\tau} \dot{\phi}(\tau + t) \frac{|x_1^u(\tau, t)|^\sigma}{\sigma} dt + \int_{T_1-\tau}^0 \dot{\phi}(\tau + t) \frac{|x_1^u(\tau, t)|^\sigma}{\sigma} dt < \\ &< \varepsilon + \int_{T_1-\tau}^0 \dot{\phi}(\tau + t) \frac{|x_1^u(\tau, t)|^\sigma}{\sigma} dt < 0 \end{aligned}$$

for $\tau > T_\omega^-$ large enough. The other claims can be proved reasoning in the same way. $\square$

We need to introduce the following surface for system (2.4)

$$S := \{(x_1, x_2, x_3) \mid H(x_1, x_2, x_3) = 0 \text{ and } (x_1, x_2) \in \mathbb{R}_+^2\}.$$

Note that, for any τ such that $\phi(\tau) > 0$ we have that $S(\tau) = S \cap \{(x_1, x_2, x_3) \mid x_3 = \tau\}$ is a closed bounded curve. This is a straightforward consequence of Lemma 2.4. Furthermore if $\phi(t)$ is bounded above and below by positive constants, then $S(\tau)$ is uniformly bounded and its diameter has a uniform lower bound. Now we are ready to state the first Theorem of this paper. This result will be completed afterwards, however it is interesting in itself. In fact the part concerning the singular solutions is new even for the Laplacian.

parte **3.2 Theorem.** *Consider Eq. (1.2) and assume that $K(r) \in C^1$ is strictly positive and bounded.*

- *Assume that hypothesis α^+ is satisfied. Then there are uncountably many S.G.S. with fast decay.*
- *Assume that hypothesis α^- is satisfied. Then there are uncountably many Dirichlet solutions $u(r)$ in exterior domains; that is, there exists $R > 0$ such that $u(R) = 0$ and $u(r) \sim r^{-\frac{n-p}{p-1}}$, as $r \rightarrow \infty$.*

- Assume that hypothesis Ω^+ is satisfied. Then there are uncountable many crossing solutions.
- Assume that hypothesis Ω^- is satisfied. Then there are uncountable many ground states with slow decay.

Proof. We begin from the first claim, thus assume that hypothesis α^+ is satisfied. Then, recalling Lemma 3.1 there is T_α^+ such that $H(P^s(\tau), 0) < 0$ and $\dot{\phi}(\tau + t) > 0$, for any $\tau < T_\alpha^+$ and for any $t < 0$. Therefore we have

$$H(x^s(\tau, t), t) = \int_0^t \dot{\phi}(\tau + t) \frac{|x^s(\tau, t)|^\sigma}{\sigma} + H(P^s(\tau), 0) < H(P^s(\tau), 0) < 0,$$

for any $\tau < T_\alpha^+$ and for any $t < 0$. Thus the trajectory $(x^s(\tau, t), \tau + t)$ of system (2.4) is forced to stay inside the surface S , so it is bounded as $t \rightarrow -\infty$. Therefore the corresponding solution $v(r)$ of (1.2) is a S.G.S. with slow decay.

Now assume that hypothesis α^- is satisfied. Reasoning as above we can show that there is T_α^- such that $\lim_{t \rightarrow -\infty} H(x^s(\tau, t), t) > 0$ for any $\tau < T_\alpha^-$. Therefore the corresponding solution $v(r)$ of (1.2) is a solution of the Dirichlet problem in the exterior of a ball, see Lemma (2.4). Reasoning in the same way we can see that, if hypothesis Ω^- is satisfied, there is T_ω^- such that $\lim_{t \rightarrow \infty} H(x^u(\tau, t), t) < 0$ for any $\tau > T_\omega^-$, while if hypothesis Ω^+ is satisfied, there is T_ω such that $\lim_{t \rightarrow \infty} H(x^u(\tau, t), t) > 0$ for any $\tau > T_\omega$. In the former case the corresponding solution $u(r)$ of (1.2) is a ground state with slow decay, while in the latter case $u(r)$ is a crossing solution, see again Lemma 3.1. $\square$

3.3 Remark. The proof of this and of the other Theorems of this paper work even if $K(r)$ is only locally Lipschitz. In this case we should replace the derivative $\dot{\phi}$ in the integral expression containing it with the weak derivative. It is also possible to rewrite the expression, used for example in (3.1), without using the term $\dot{\phi}$, simply by integrating by parts, as it is done in [21]. However this computation is beyond the purpose of this analysis so it is left to the interested reader.

We define now the following function which measures the distance between the stable and the unstable manifold along the isocline c .

$$G(\tau) = H(P^u(\tau), 0) - H(P^s(\tau), 0).$$

Note that whenever $G(\tau) > 0$ we have that $P^u(\tau)$ is on the right of $P^s(\tau)$, while when $G(\tau) < 0$ it is on the left. Here and later we think of the x_1 axis as horizontal, and of the x_2 axis as vertical. Moreover observe that $G(\tau) = 0$ implies $P^s(\tau) = P^u(\tau)$, therefore in this case we have that $x^u(\tau) \equiv x^s(\tau)$ is a homoclinic trajectory. Thus the condition $G(\tau) = 0$ is sufficient for the existence of a G.S. with fast decay for Eq. (1.2).

Note also that the function $G(\tau)$ is continuous. In fact the functions $P^u(\tau)$ and $P^s(\tau)$ are C^1 , due to invariant manifold theory, and H is continuous as well. We recall incidentally that in the perturbative case the functions $P^{u,s}(\tau)$ have, in fact, the same regularity as ϕ , see [7] and [11].

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3.4 Proposition. *Assume that hypothesis α^+ is satisfied, then $G(\tau) > 0$ for any $\tau < T_\alpha^+$, while if α^- is satisfied $G(\tau) < 0$ for any $\tau < T_\alpha^-$.*

Assume that hypothesis Ω^+ is satisfied, then $G(\tau) > 0$ for any $\tau > T_\omega^+$, while if Ω^- is satisfied we have $G(\tau) < 0$ for any $\tau > T_\omega^-$.

Exploiting this analysis we can deuce the following result.

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3.5 Corollary. *Consider Eq. (1.2) and assume $K(r) \in C^1$ is strictly positive and bounded.*

- *Assume that Hypothesis α^- is satisfied. Then there is $R^- > 0$, such that for each $0 < \rho < R^-$ there exists a solution $u(r)$ of the Dirichlet problem in the exterior of the ball of radius ρ . Therefore we have $u(\rho) = 0$, $u(r) > 0$ for any $r > \rho$ and $u(r) \sim r^{-\frac{n-p}{p-1}}$ as $r \rightarrow \infty$.*
- *Assume that Hypothesis Ω^+ is satisfied. Then there is $R^+ > 0$, such that for each $\rho > R^+$ there exists a solution $u(r)$ of the Dirichlet problem in any ball of radius ρ .*

Proof. We begin from the first claim, thus assume that hypothesis Ω^+ is satisfied. First of all note that $G(\tau) > 0$ for $\tau \rightarrow \infty$, according to Proposition 3.4. In particular we know that there exists T_ω^+ small enough so that for any $\tau > T_\omega^+$ the trajectories $x^u(\tau, t)$ are crossing solutions. Therefore, for any $\tau > T_\omega^+$, there exists $T_+(\tau) > 0$ such that $x_1^u(\tau, T(\tau)) = 0$, which implies that for the corresponding solution $u(r)$ of Eq. (1.2) we have $u(\exp(\tau + T_+(\tau))) = 0$. We just need to prove that $T_+(\tau)$ is bounded as $\tau \rightarrow -\infty$, then the first part of the Corollary easily follows.

Observe that $\dot{x}_1^u(\tau, t) < 0$ for any $t > 0$ and that there is a positive constant $C > 0$ independent of τ such that $\dot{x}_2^u(\tau, 0) < -C$. Then we can find

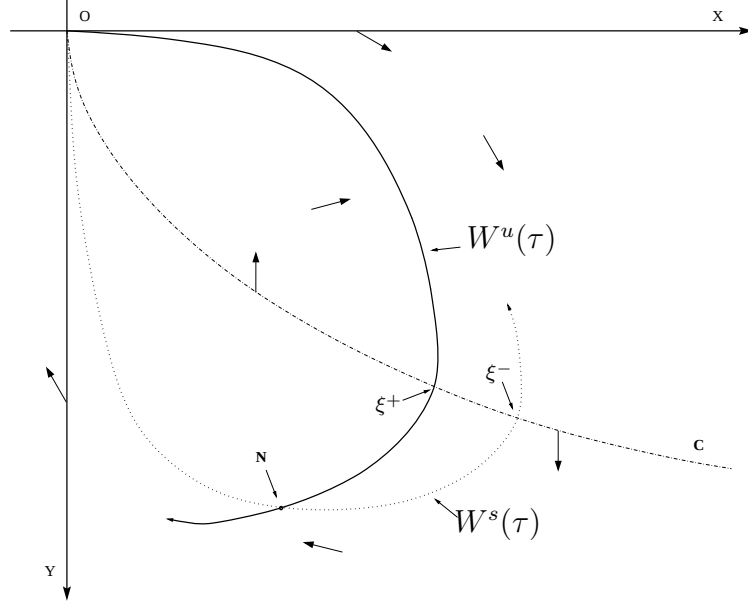


Figure 3: Sketch of the curve $Z(\tau)$ when $K'(r)$ admits exactly one critical point which is a maximum.

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$\epsilon > 0$ and $t_0 > 0$ independent of τ such that $\dot{x}_1^u(\tau, t) < -\epsilon$ for $t > t_0$ and for any τ . Let us denote by $Q^r = (Q_1^r, Q_2^r)$ and by $Q^l = (Q_1^l, Q_2^l)$ the two points of intersection between the isocline c and ∂E , and let $Q_1^r > Q_1^l$. Then $x_1^u(\tau, t_0) < P_1^u(\tau) < Q_1^r$ for any τ therefore there is $T_+(\tau) < t_0 + Q_1^r/\epsilon < \infty$ such that $x_1^u(\tau, T_+(\tau)) = 0$.

The second claim can be proved following backwards in t the trajectories $x^s(\tau, t)$ for $\tau < T_\alpha^-$ where $\tilde{\tau}$ is such that $G(\tau) < 0$ for $\tau < T_\alpha^-$ and reasoning in the same way. $\square$

Now we are ready to give the main Theorem of the paper which completes the result given in [21].

Proof of Theorem 1.2. Using Corollary 3.5 and Theorem 3.2, we already know the results concerning G.S. with fast and slow decay, crossing solutions Dirichlet solutions in exterior domains and S.G.S. with fast decay. The only thing left to prove is the existence of S.G.S with slow decay when α^+ and ω^- are satisfied. Then, in this case, we have classified all the positive solutions $u(r)$ of (1.2). In fact we have covered all the possible asymptotic behavior as $r \rightarrow 0$ and as $r \rightarrow \infty$.

Now we want to prove the existence of infinitely many S.G.S. with slow decay. The proof relies on a geometrical analysis of the phase portrait, thus we need to deepen our knowledge of the mutual positions of the manifolds W^u and W^s .

Reasoning as done in the proof of Theorem 4.1 in [11], we can construct a bounded subset of the 4th quadrant to which some trajectories of system (2.3) must belong. Since the construction is the same as the one given in [11] we will just sketch it.

Fix τ , we want to construct a closed bounded curve belonging to the 4th quadrant, made up of branches of the manifolds $W^u(\tau)$ and $W^s(\tau)$. We follow $W^u(\tau)$ starting from the origin until we reach the first crossing between $W^u(\tau)$ and $W^s(\tau)$, denoted by $P_1(\tau)$. Then we follow $W^s(\tau)$ towards the origin until we reach its further crossing with $W^u(\tau)$, denoted by $P_2(\tau)$. Then we follow $W^u(\tau)$ until the next crossing and so on. Eventually we will end with a last crossing denoted by $P_\infty(\tau)$. Then we follow $W^s(\tau)$ until we reach the origin. Let us call $Z(\tau)$ the union of the origin and the curve just constructed, and Z the surface of system (2.4) obtained letting τ takes values in the whole of $\mathbb{R}$.

Note that, for any τ , $Z(\tau)$ belongs to the 4th quadrant. In fact both $W^u(\tau)$ and $W^s(\tau)$ can cross the axes, but the branches have been chosen in order to have that $Z(\tau)$ belongs to the bounded set delimited by the curve $U^m(t)$ for $t \in \mathbb{R}$, see figg. 3 and 4. A detailed proof of this fact is given in [11]; however it depends on two facts. First $\tilde{W}^u(\tau)$ and $\tilde{W}^s(\tau)$ are contained in E for any τ . Second, choose an intersection $P_k(\tau)$ and denote by $\Phi_\tau(t, x)$ the flow of system (2.3) at time t evaluated in x . We can find T_k such that $\Phi_\tau(T_k, P_k(\tau)) = P_k(\bar{\tau}_k) \in C$, where $\bar{\tau}_k = T_k + \tau$. Then, if k is odd we have that when $\tau < \bar{\tau}_k$, $P^s(\tau)$ is on the left of $P^u(\tau)$, while when $\tau > \bar{\tau}_k$, $P^u(\tau)$ is on the left of $P^s(\tau)$. If k is even we have the opposite situation.

Let us call $D(\tau)$ the bounded subset delimited by $Z(\tau)$. Note that $D(\tau) - (W^u(\tau) \cup W^s(\tau))$ contains uncountably many points. Let us consider a trajectory $X(t)$ of the extended system (2.4), departing from one of these points.

Observe that it is forced to stay inside Z for any t and that it cannot converge to the origin, nor as $t \rightarrow -\infty$ neither as $t \rightarrow \infty$. Thus the corresponding solution $v(r)$ of Eq. (1.2) is a S.G.S. with slow decay.

We wish to remark that, exploiting the curve $Z(\tau)$ and this kind of analysis, we could give a different proof of Theorems 3.2 and 1.2, as it is done in [11]. $\square$

We observe now that putting together the results of Theorem 3.2 and Corollary 3.5 we can also prove easily Theorem 1.3.

Let us recall the definition of the function $J^+(r) = \int_0^r K'(s)s^n ds$ and define the following analogous function $J^-(r) = \int_r^\infty K'(s)s^{-\frac{np}{p-1}} ds$. Let us call $u_a(r)$ the regular solution of (1.2) satisfying $u(0) = a > 0$ and $u'(0) = 0$. Combining the ideas of Theorem 1.2 with the results of Theorem 1 and Proposition 4.2 in [21] we find the following.

3.6 Theorem. *Consider Eq. (1.2) and assume that $K(r) \in C^1$ is strictly positive and bounded; moreover assume that the hypotheses α^+ and Ω^- are satisfied. Furthermore assume that there exists $R > 0$ such that one of the following hypotheses is satisfied*

- $J^+(r) \geq 0$ for any $r < R$ and $K'(r) \leq 0$ for any $r > R$.
- $J^-(r) \leq 0$ for any $r > R$ and $K'(r) \geq 0$ for any $r > R$.

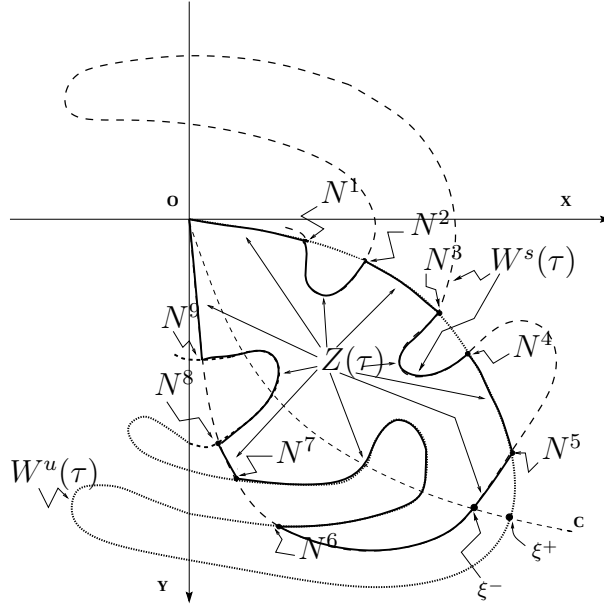


Figure 4: Sketch of the curve $Z(\tau)$ when there are 9 intersections between stable and unstable manifolds.

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Then positive solutions have a structure of type A. Furthermore we have that there exists $A > 0$ such that $u_A(r)$ is a G.S. with fast decay, each $u_A(r)$ is

a crossing solutions for any $a > A$, while $u_a(r)$ is a G.S. with slow decay if $0 < a < A$. Therefore we have the uniqueness of the G.S. with fast decay.

Note that α^+ implies that $J^+(r)$ is positive for r small and Ω^- implies that $J^-(r)$ is negative for r large; moreover α^+ is very close to the hypothesis on J^+ and Ω^- is very close to the hypothesis on $J^-(r)$ in many practical examples. If the second hypothesis concerning $J^-(r)$ is satisfied to prove the uniqueness of the G.S. with fast decay we have to consider the solutions $v(r)$ with fast decay of (1.2) and repeat all the reasonings developed in [21]. This way we obtain the same structure result as if the first hypothesis is satisfied, therefore the uniqueness of the G.S. with fast decay is proved.

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