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Asymptotic Expansion of Solutions of an Elliptic Equation Related to the Nonlinear Schrödinger Equation*

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We study the radially symmetric blow-up solutions of the nonlinear Schrödinger equation. We give a method for developing such a solution in a series which represents it asymptotically.

KEY WORDS: Nonlinear Schrödinger equation; blow-up solution; asymptotic series representation.

1. INTRODUCTION

In this paper we will study the solutions of the nonlinear Schrödinger equation (NLS)

$$\begin{aligned} & i\,\frac{\partial\phi}{\partial t}+\Delta\phi+|\phi|^{p-1}\,\phi=0 \qquad (t>0,\, x\in\mathbb{R}^n) \\ & \phi(0,x)=\phi_0(x) \end{aligned} \tag{NLS}$$

which blow up in finite time. We assume that the dimension n is at least two and that $1+\frac{4}{n}\leqslant p\leqslant p^*$, where $p^*=\infty$ if $n=2$ and $p^*=\frac{n+2}{n-2}$ if $n\geqslant 3$. The quantity $1+\frac{4}{n}$ is the so-called critical exponent for the NLS. We will study

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the asymptotic behavior of the blow-up solutions near the blow-up time, and in so doing we will refine the information on the asymptotic behavior of the solutions given in [2–4]. We will use the well-known method of dynamical scaling [5–8], then apply the classical method of asymptotic analysis [1, 9] together with the arguments of [2] to obtain our results. The method of dynamical scaling allows one to define and study an asymptotic profile for blow-up solutions; that profile is a solution of an appropriate limiting equation.

Let us review first the dynamical scaling method. Assume that the solution ϕ is a radially symmetric function; then ϕ depends only on t and on $s=|x|$. Here and below, $|\cdot|$ denotes the euclidean norm in $\mathbb{R}^n$. Suppose that ϕ blows up at a finite time t^*. We rescale as follows:

$$\xi=\frac{s}{L(t)},\qquad \tau=\int_0^t\frac{du}{L^2(u)},\qquad \phi(t,s)=\frac{1}{L(t)^{\frac{1}{\sigma}}}w(\tau,\xi),$$

where $\sigma=\frac{p-1}{2}$ and $L(t)$ will be specified in a moment. Then $w(\tau,\xi)$ satisfies the following equation:

$$\begin{aligned}&i\frac{\partial w}{\partial\tau}+\frac{\partial^2 w}{\partial\xi^2}+\frac{n-1}{\xi}\frac{\partial w}{\partial\xi}+|w|^{p-1}w-ia(\tau)\left[\frac{w}{\sigma}-\xi\frac{\partial w}{\partial\xi}\right]=0\\&w(0,\xi)=L(0)^{\frac{1}{\sigma}}\phi_0(L(0),\xi),\end{aligned}\tag{1.1}$$

where $a(\tau)=L\frac{dL}{dt}=\frac{d}{d\tau}\log(L)$.

The idea now is to choose L (and hence a) in such a way that the solution of (1.1) is well-behaved near the blow-up time t^*. A standard choice of L is:

$$L(t)\sim c(t^*-t)^{\frac{1}{2}}\qquad(t\nearrow t^*).$$

We then obtain $a(\tau)\to-a_\infty$ as $\tau\to\infty$, where a_∞ is a nonnegative real number. We are led to study the limit equation obtained by setting $a(\tau)=-a_\infty$:

$$i\frac{\partial w}{\partial\tau}+\frac{\partial^2 w}{\partial\xi^2}+\frac{n-1}{\xi}\frac{\partial w}{\partial\xi}+|w|^{p-1}w-ia_\infty\left[\frac{w}{\sigma}-\xi\frac{\partial w}{\partial\xi}\right]=0.$$

As suggested in [5–7], we now set

$$w(\tau,\xi)=e^{ic_\infty\tau}Q(\xi).$$

Then the "profile" $Q(\xi)$ satisfies the equation:

$$\begin{aligned} &\frac{\partial^2 Q}{\partial \xi^2}+\frac{n-1}{\xi}\frac{\partial Q}{\partial \xi}-c_\infty Q+|Q|^{p-1}\, Q-ia_\infty\left[\frac{Q}{\sigma}+\xi\,\frac{\partial Q}{\partial \xi}\right]=0 \\ &\frac{dQ}{d\xi}(0)=0, \qquad 0\neq Q(0)\in\mathbb{R}. \end{aligned} \tag{1.2}$$

We will consider the case $a_\infty>0$, whose analysis is rendered more complicated by the oscillation of the corresponding solutions of (1.2).

As suggested in [2], we put

$$Q(\xi)=q(\xi)\, e^{-\frac{i}{4}a_\infty \xi^2}.$$

We obtain the following equation for $q(\xi)$:

$$\begin{aligned} &\frac{d^2 q}{d\xi^2}+\frac{n-1}{\xi}\frac{dq}{d\xi}-c_\infty q+|q|^{p-1}\, q+\frac{1}{4}\,a_\infty^2\xi^2 q+ia_\infty\left[\frac{1}{\sigma}-\frac{n}{2}\right]q=0 \\ &\frac{dq}{d\xi}=0, \qquad 0\neq q(0)\in\mathbb{R}. \end{aligned} \tag{1.3}$$

Next introduce the quantities

$$u(r)=c_\infty^{-\frac{1}{p-1}}q(\xi), \qquad r=\sqrt{c_\infty}\;\xi, \qquad \lambda=\frac{a_\infty^2}{4c_\infty^2}, \qquad B=\left(\frac{n}{2}-\frac{1}{\sigma}\right)\frac{a_\infty}{c_\infty}.$$

Observe that r is a time-scaled version of the norm $|x|$ of $x\in\mathbb{R}^n$. One sees that $u(r)$ satisfies the following equation:

$$\begin{aligned} &u''(r)=\frac{n-1}{r}\,u'(r)+(\lambda r^2-1)\,u(r)+|u(r)|^{p-1}\,u(r)-iBu(r)=0 \\ &u'(0)=0, \qquad 0\neq u(0)\in\mathbb{R}. \end{aligned} \tag{1.4}$$

In the critical case when $p=1+\frac{4}{n}$, we obtain $B=0$, and then (1.4) takes the simpler form

$$\begin{aligned} &u''(r)+\frac{n-1}{r}\,u'(r)+(\lambda r^2-1)\,u(r)+|u(r)|^{p-1}\,u(r)=0 \\ &u'(0)=0, \qquad 0\neq u(0)\in\mathbb{R}. \end{aligned} \tag{1.5}$$

The rest of the paper is organized as follows. In Section 2, we study the asymptotic behavior of solutions of Eq. (1.5). We will use the classical methods of [1, 9] together with the result of [2]. Then, in Section 3, we will analyze the asymptotic behavior of the solutions of (1.4). We use again the method of [9]; however the discussion is rendered more complicated by the presence of the term $-iBu(r)$.

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2. THE CRITICAL CASE

We study the behavior as $r \to \infty$ of the solution of the following problem:

$$
\begin{aligned}
&u''(r)+\frac{n-1'}{u}(r)+(\lambda r^2-1)\,u(r)+|u(r)|^{p-1}\,u(r)=0 \\
&u'(0)=0, \qquad 0 \neq u(0) \in \mathbb{R}.
\end{aligned}
\tag{1.5}
$$

The local existence and uniqueness of solutions of (1.5) can be proved using the contraction mapping theorem. It can also be proved that solutions exist on $0<r<\infty$. Our main interest is in the case $p=1+\frac{4}{n}$, but our analysis will be valid for all $p \geqslant 1+\frac{4}{n}$.

Let us first review the discussion of (1.5) which is given in [2]. Define

$$
v(r)=r^{\frac{n-1}{2}}\,u(r), \qquad k=\frac{(n-1)(n-3)}{4}.
$$

Then

$$
v''+\left[\lambda r^2-1-\frac{k}{r^2}-r^{-\frac{(n-1)(p-1)}{2}}\,|v|^{p-1}\right]v=0. \tag{2.1}
$$

Next set

$$
t=r^2, \qquad x(t)=v(r), \qquad y(t)=\frac{dx}{dt},
$$

so that (2.1) takes the form

$$
\begin{aligned}
x'&=y \\
y'&=-\frac{\lambda}{4}\,x-\frac{y}{2t}+\frac{1}{4}\left(\frac{1}{t}+\frac{k}{t^2}\right)x-\frac{1}{4}\,t^{-\sigma}\,|x|^{p-1}\,x,
\end{aligned}
\tag{2.2}
$$

where the prime $'$ in (2.2) denotes the differentiation with respect to t, and $\sigma=1+\frac{1}{4}(n-1)(p-1)$.

The first step in the asymptotic analysis of the solutions of (2.2) is carried out in [2]. Introduce the Lyapunov-type function

$$H(t)=\frac{y^2}{2}+\frac{\lambda}{8}x^2-\frac{1}{8}\left(\frac{1}{t}+\frac{k}{t^2}\right)x^2+\frac{|x|^{p+1}}{4(p+1)\,t^\sigma}.$$

Then

$$\frac{dH}{dt}=-\frac{y^2}{2t}+\frac{1}{8}\left(\frac{1}{t^2}+\frac{2k}{t^3}\right)x^2-\frac{\sigma}{4(p+1)}\frac{|x|^{p+1}}{t^{\sigma+1}}.$$

We can find $t_0>0$ with the property that, if $t\geqslant t_0$, then $H(t)>0$ and

$$\frac{H'(t)}{H(t)}\leqslant\frac{4}{\lambda t^2}\qquad \text{if } t \text{ is large.}$$

It follows that $H(t)$ is bounded for $t\geqslant t_0$, so $x(t)$ and $y(t)$ exist for all $t\geqslant t_0$ and are bounded as $t\to\infty$.

Next introduce the polar variable ρ, θ defined by

$$x=\frac{2}{\sqrt{\lambda}}\rho\cos\theta,\qquad y=\rho\sin\theta.$$

Arguing as in [2], one shows that

$$\rho(t)=\rho^* t^{-\frac{1}{4}}+O(t^{-\frac{1}{4}-\beta})\tag{2.3}$$

as $t\to\infty$, where $\beta=\min\{1,\frac{1}{4}(n-1)(p-1)\}$ and ρ^* is a positive constant. Although it is not noted explicitly in [2], one can improve (2.3) by substituting it in [2, Eq. (3.8), p. 780]; one obtains

$$\rho(t)=\rho^* t^{-\frac{1}{4}}+O(t^{-\frac{1}{2}-\beta})\qquad (t\to\infty).\tag{2.4}$$

This agrees with the remainder estimate of [3]. Our goal in the following discussion is that of obtaining a complete asymptotic expansion of a given solution of (2.2). With respect to this complete expansion, the relation (2.4) will correspond to the zeroth-order information.

We begin by studying the linear equation obtained by omitting the nonlinear term $\frac{1}{4}t^{-\sigma}|x|^{p-1}x$ in Eq. (2.2). Writing this linear equation in second-order form, we have

$$x''+\frac{1}{2t}x'+\left(\frac{\lambda}{4}-\frac{1}{4t}-\frac{k}{4t^2}\right)x=0.\tag{2.5}$$

Putting $z = x \cdot t^{-\frac{1}{4}}$, one obtains

$$z'' + \left[\frac{\lambda}{4} - \frac{1}{4t} + \frac{1}{t^2}\left(\frac{3}{16} - \frac{k}{4}\right)\right] z = 0. \tag{2.6}$$

If we write

$$q(t) = \frac{\lambda}{4} - \frac{1}{4t} + \frac{1}{t^2}\left(\frac{3}{16} - \frac{k}{4}\right) = q_0 + \frac{q_1}{t} + \frac{q_2}{t^2},$$

then (2.6) has the form $z'' + q(t)\, z = 0$, and we are in the position to study the normal solutions [1, pp. 61 ff] of (2.6).

We briefly recall how the normal solutions are constructed; for details see [1]. First write

$$z = e^{\omega t} t^{-\rho} g$$

and plug this quantity into (2.6). Then

$$\omega^2 + q_0 = 0 = \omega^2 + \frac{\lambda}{4}$$

$$-2\omega\rho + q_1 = 0 = -2\omega\rho - \frac{1}{4}.$$

One obtains the solutions $\pm\omega = \pm\frac{i\sqrt{\lambda}}{2}$, $\pm\rho = \frac{\pm i}{4\sqrt{\lambda}}$.

Following the calculations of [1], one deduces that there exist two linearly independent solutions $x_\pm$ of (2.2)

$$x_\pm \sim t^{-\frac{1}{4}} e^{\pm\omega t} t^{\mp\rho} \sum_{k=0}^{\infty} \frac{c_k^\pm}{t^k}. \tag{2.7}$$

In particular the 0th order term $x_\pm$ is of the form

$$\text{const} \times t^{-\frac{1}{4}} \exp\left[\pm i\left(\frac{\sqrt{\lambda}}{2} t + \frac{1}{4\sqrt{\lambda}} \ln t\right)\right].$$

We return now to the nonlinear system (2.2), and give a general procedure for analyzing the corrections to this term in the expansion (2.7) which arise from the presence of the nonlinearity $-\frac{1}{4} t^{-\sigma} |x|^{p-1} x$. The nonlinearity is not analytic in x, so standard techniques cannot be directly applied. So we develop a method adapted to the problem at hand.

It is easiest to present the procedure in an abstract framework. Consider the Cauchy problem

$$\begin{aligned} \underline{x}' &= A(t)\, \underline{x} + f(t, \underline{x}) \qquad (t \geqslant t_0,\ x \in \mathbb{R}^n) \\ \underline{x}(t_0) &= \hat{\underline{x}} \end{aligned} \tag{2.8}$$

where $f(t, \underline{x})$ is smooth enough in its arguments to guarantee local existence and uniqueness of solutions of (2.8). We suppose that $A(\cdot)$ is analytic in an open sector S in the complex t-plane, with vertex at ∞ and containing the semi-axis $\{t \in \mathbb{R} \mid t \geqslant t_0\}$. We further suppose that $A(\cdot)$ admits an asymptotic expansion valid in the sector S:

$$A(t) \sim A_0 + \frac{A_1}{t} + \frac{A_2}{t^2} + \cdots.$$

Let $\Phi(t)$ be the matrix solution of $\underline{x}' = A(t)\, \underline{x}$ which satisfies $\Phi(t_0) = I$; we suppose that $\Phi(t)$ satisfies

$$\Phi(t) = t^{-\alpha} \Psi(t), \tag{2.9}$$

where $\alpha \geqslant 0$ and $\Psi(t)$ together with $\Psi(t)^{-1}$ are bounded as $t \to \infty$.

Turning to the nonlinear function f, we assume that

$$f(t, \underline{x}) = t^{-\sigma} g(t, \underline{x}) \qquad (t \geqslant t_0) \tag{2.10}$$

where $\sigma > 1$ and g satisfies

$$|g(t, \underline{x})| = O(|\underline{x}|^p) \qquad (|x| \to 0). \tag{2.11}$$

Here $p > 1$, and the estimate is assumed uniform for $t \geqslant t_0$. We further assume that $g(t, 0) = 0$, and that g is Lipschitz in the following sense with respect to the variable x:

$$|g(t, \underline{x} + y) - g(t, \underline{x})| \leqslant M(\underline{x})\, |\underline{y}| \quad \text{where } M(\underline{x}) = O(|\underline{x}|^{p-1}) \quad (|x| \to 0). \tag{2.12}$$

Now suppose that $\underline{x}(t)$ is a solution of (2.8) which satisfies

$$|\underline{x}(t)| = O(t^{-\alpha}) \qquad (t \to \infty). \tag{2.13}$$

By the variation of constants formula:

$$\underline{x}(t) = \Phi(t)\, \hat{\underline{x}} + \int_{t_0}^{t} \Phi(t)\, \Phi(s)^{-1} f(s, \underline{x}(s))\, ds$$

where $\hat{\underline{x}}(t) = \underline{x}(t_0)$. It is convenient to rewrite this expression as

$$\underline{x}(t) = \underline{x}_0(t) + \int_{\infty}^{t} \Phi(t)\, \Phi(s)^{-1}\, f(s, \underline{x}(s))\, ds \tag{2.14}$$

where

$$\begin{aligned} \underline{x}_0(t) &= \Phi(t)\, \hat{\underline{x}} + \underline{x}_c(t) \\ \underline{x}_c(t) &= \int_{t_0}^{\infty} \Phi(t)\, \Phi(s)^{-1}\, f(s, \underline{x}(s))\, ds. \end{aligned} \tag{2.15}$$

We see that the "correction" term $x_c(t)$ is a solution of the linear equation $x' = A(t)\, \underline{x}$. It is well defined because of (2.10), (2.11), and (2.13), and in fact $|\underline{x}_c(t)| = O(t^{-\alpha})$ as $t \to \infty$. In principle it can be computed with arbitrary accuracy by approximating the integral $\int_{t_0}^{\infty} \Phi(t)\, \Phi(s)^{-1}\, f(s, \underline{x}(s))\, ds$ and solving the linear equation $\underline{x}' = A(t)\, \underline{x}$.

Define now

$$\underline{y}_1(t) = \underline{x}(t) - \underline{x}_0(t) = \int_{\infty}^{t} \Phi(t)\, \Phi(s)^{-1}\, f(s, \underline{x}(s))\, ds,$$

so that $\underline{x}(t) = \underline{x}_0(t) + \underline{y}_1(t)$. Using the estimates (2.10), (2.11), and (2.13), we see that

$$|\underline{y}_1(t)| = O(t^{-\sigma - \alpha p + 1}).$$

Writing

$$f_1(t, \underline{y}) = f(t, \underline{x}_0(t) + \underline{y}) - f(t, \underline{x}_0(t)) \qquad (t \geqslant t_0)$$

we see that

$$\underline{y}_1(t) = \int_{\infty}^{t} \Phi(t)\, \Phi(s)^{-1}\, f(s, \underline{x}_0(s))\, ds + \int_{\infty}^{t} \Phi(t)\, \Phi(s)^{-1}\, f_1(s, \underline{y}_1(s))\, ds.$$

Using the estimate (2.12), we have

$$|f_1(t, \underline{y}_1(t))| = O(t^{-\sigma} \cdot t^{-\alpha(p-1)} \cdot t^{-\sigma - \alpha p + 1}) = O(t^{-2\sigma - 2\alpha p + \alpha + 1}).$$

Next write

$$\underline{x}_1(t) = \int_\infty^t \Phi(t)\, \Phi(s)^{-1}\, f(s, \underline{x}_0(s))\, ds$$

$$\underline{y}_2(t) = \int_\infty^t \Phi(t)\, \Phi(s)^{-1}\, f_1(s, \underline{y}_1(s))\, ds.$$

Then $\underline{y}_1 = \underline{x}_1 + \underline{y}_2$ and $\underline{x} = \underline{x}_0 + \underline{x}_1 + \underline{y}_2$. The following estimates hold:

$$|\underline{x}_1(t)| = O(t^{-\sigma - \alpha p + 1})$$
$$|\underline{y}_2(t)| = O(t^{-2\sigma - 2\alpha p + \alpha + 2}).$$

It is now clear how to continue the development of $\underline{x}(t)$. For each $k = 2, 3, \ldots$ we write

$$\begin{aligned} f_k(t, \underline{y}) &= f_{k-1}(t, \underline{x}_{k-1}(t) + \underline{y}) - f_{k-1}(t, \underline{x}_{k-1}(t)) \\ &= f(t, \underline{x}_0 + \cdots + \underline{x}_{k-1}(t) + \underline{y}) - f(t, \underline{x}_0 + \cdots + \underline{x}_{k-1}(t)), \end{aligned}$$

$$\underline{x}_k(t) = \int_\infty^t \Phi(t)\, \Phi(s)^{-1}\, f_{k-1}(s, \underline{x}_{k-1}(s))\, ds,$$

$$\underline{y}_{k+1}(t) = \int_\infty^t \Phi(t)\, \Phi(s)^{-1}\, f_k(s, \underline{y}_k(s))\, ds.$$

Then $\underline{y}_k = \underline{x}_k + \underline{y}_{k+1}$ and

$$\underline{x} = \underline{x}_0 + \cdots + \underline{x}_k + \underline{y}_{k+1}. \tag{2.16}$$

For each $k \geqslant 1$ we have:

$$\begin{aligned} |\underline{x}_k(t)| &= O(t^{-k\sigma - k\alpha p + k + (k-1)\alpha}) \\ |\underline{y}_k(t)| &= O(t^{-k\sigma - k\alpha p + k + (k-1)\alpha}). \end{aligned} \tag{2.17}$$

Since $\sigma > 1$ and $p > 1$, we see that the development (2.16) gives rise to an asymptotic expansion in (perhaps fractional) powers of t of the solution $\underline{x}(t)$ of (2.8). Of course $\underline{x}(t)$ must satisfy the *a priori* bound (2.13).

Let us note that, if $\underline{x}_k(t)$ vanishes identically for some $k \geqslant 1$, then $\underline{x}_l(t) \equiv 0$ for all $l \geqslant k$. In this case we have

$$\underline{x}(t) = \sum_{l=0}^{k-1} \underline{x}_l(t) + \underline{r}(t)$$

where $\underline{r}(t)$ is small to all orders of t, as $t \to \infty$. This amplifies the discussion in [3] of relation (3.25) of that paper.

Let us return to the blow-up solution of the nonlinear Schrödinger equation which motivated our discussion. Set $\sigma = 1 + \frac{n-1}{n}$, $p = 1 + \frac{4}{n}$ and $\alpha = \frac{1}{4}$. Combining (2.7) and (2.15), and letting $x(t)$ denote the first component of the vector $\underline{x}(t) = \binom{x(t)}{y(t)}$, we see that

$$x(t) = x_0(t) + x_1(t) + \cdots + x_n(t) + \cdots$$

where

$$x_0(t) = at^{-\frac{1}{4}} \cos\left(\frac{\sqrt{\lambda}}{2}\, t + \frac{t}{4\sqrt{\lambda}} + b\right)$$

for constants a and b, and

$$|x_n(t)| = O(t^{-n}) \qquad (n = 1, 2, \ldots).$$

Observe in particular that the linear contribution $x_0(t)$ is determined by the constants a and b. All the remaining terms in the expansion are thus determined when a and b are known.

3. THE SUPER-CRITICAL CASE

Now we study Eq. (1.4) when $B > 0$. We first review some preliminary calculations which are given in [2]. Write $u = u_1 + iu_2$ where u_1 and u_2 are real quantities. Then

$$\begin{gathered} u_1'' + \frac{n-1}{r} u_1' + (\lambda r^2 - 1)\, u_1 + (u_1^2 + u_2^2)^{\frac{p-1}{2}} u_1 + Bu_2 = 0 \\ u_2'' + \frac{n-1}{r} u_2' + (\lambda r^2 - 1)\, u_2 + (u_1^2 + u_2^2)^{\frac{p-1}{2}} u_2 - Bu_1 = 0 \\ u_1'(0) = u_2'(0) = 0, \qquad u_2(0) = 0, \qquad u_1(0) = u_0 \neq 0. \end{gathered} \tag{3.1}$$

Introduce the quantities

$$t = r^2, \qquad \sigma = 1 + \frac{1}{4}(n-1)(p-1), \qquad k = \frac{(n-1)(n-3)}{4}.$$

Then writing $x_j(t) = r^{\frac{n-1}{2}} u_j(r)$, $y_j(t) = \frac{dx_j}{dt}$ $(j = 1, 2)$, one obtains from (3.1):

$$
\begin{aligned}
x_1' &= y_1 \\
y_1' &= -\frac{\lambda}{4} x_1 + \frac{1}{4}\left(\frac{1}{t} + \frac{k}{t^2}\right) x_1 - \frac{y_1}{2t} - \frac{B}{4t} x_2 - \frac{1}{4} t^{-\sigma} (x_1^2 + x_2^2)^{\frac{p-1}{2}} x_1 \\
x_2' &= y_2 \\
y_2' &= -\frac{\lambda}{4} x_2 + \frac{1}{4}\left(\frac{1}{t} + \frac{k}{t^2}\right) x_2 - \frac{y_2}{2t} + \frac{B}{4t} x_1 - \frac{1}{4} t^{-\sigma} (x_1^2 + x_2^2)^{\frac{p-1}{2}} x_2.
\end{aligned}
\tag{3.2}
$$

Here the prime $'$ indicates differentiation with respect to t. Writing the linear part of (3.2) in vector form with

$$
z = \begin{pmatrix} x_1 \\ y_1 \\ x_2 \\ y_2 \end{pmatrix},
$$

we obtain

$$
z' = \left(C_0 + \frac{C_1}{t} + \frac{C_2}{t^2}\right) z
$$

where

$$
C_0 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -\frac{\lambda}{4} & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -\frac{\lambda}{4} & 0 \end{pmatrix}, \qquad
C_1 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ \frac{1}{4} & -\frac{1}{2} & -\frac{B}{4} & 0 \\ 0 & 0 & 0 & 0 \\ \frac{B}{4} & 0 & \frac{1}{4} & -\frac{1}{2} \end{pmatrix},
$$

$$
C_2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ \frac{k}{4} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{k}{4} & 0 \end{pmatrix}.
$$

We diagonalize A_0 via the transformation $z = Qw$, where

$$
Q = \begin{pmatrix} 1 & 0 & 1 & 0 \\ -i\frac{\sqrt{\lambda}}{2} & 0 & i\frac{\sqrt{\lambda}}{2} & 0 \\ 0 & 1 & 0 & 1 \\ 0 & -i\frac{\sqrt{\lambda}}{2} & 0 & i\frac{\sqrt{\lambda}}{2} \end{pmatrix};
$$

then

$$Q^{-1}C_0Q = \begin{pmatrix} -i\frac{\sqrt{\lambda}}{2} & 0 & 0 & 0 \\ 0 & -i\frac{\sqrt{\lambda}}{2} & 0 & 0 \\ 0 & 0 & i\frac{\sqrt{\lambda}}{2} & 0 \\ 0 & 0 & 0 & i\frac{\sqrt{\lambda}}{2} \end{pmatrix},$$

$$Q^{-1}C_1Q = \begin{pmatrix} -\frac{1}{4}+\frac{i}{4\sqrt{\lambda}} & -\frac{iB}{4\sqrt{\lambda}} & \frac{1}{4}+\frac{i}{4\sqrt{\lambda}} & -\frac{iB}{4\sqrt{\lambda}} \\ \frac{iB}{4\sqrt{\lambda}} & -\frac{1}{4}+\frac{i}{4\sqrt{\lambda}} & \frac{iB}{4\sqrt{\lambda}} & \frac{1}{4}+\frac{i}{4\sqrt{\lambda}} \\ \frac{1}{4}-\frac{i}{4\sqrt{\lambda}} & \frac{iB}{4\sqrt{\lambda}} & -\frac{1}{4}-\frac{i}{4\sqrt{\lambda}} & \frac{iB}{4\sqrt{\lambda}} \\ -\frac{iB}{4\sqrt{\lambda}} & \frac{1}{4}-\frac{i}{4\sqrt{\lambda}} & -\frac{iB}{4\sqrt{\lambda}} & -\frac{1}{4}-\frac{i}{4\sqrt{\lambda}} \end{pmatrix}.$$

Writing $\hat{A}_i = Q^{-1}C_iQ$ $(i = 0, 1, 2)$ we obtain

$$w' = \left(\hat{A}_0 + \frac{\hat{A}_1}{t} + \frac{\hat{A}_2}{t^2}\right) w. \tag{3.3}$$

We now apply the method expounded in [W, pp. 54–55] to formally block-diagonalize Eq. (3.3). One looks for a change of variables of the form

$$w = P(t)\,\xi, \tag{3.4}$$

where $P(\cdot)$ is analytic in a sector S in the complex t-plane which has vertex at $t = \infty$ and which contains some real segment $\{t \in \mathbb{R} \mid t \geqslant t_0\}$. It is required that P admit a formal series expansion in the sector S:

$$P(t) \sim I + \sum_{k=1}^{\infty} \frac{P_k}{t^k}, \tag{3.5}$$

where I is the 4×4 identity matrix and each P_k has the form

$$P_k = \begin{pmatrix} 0 & P_k^{12} \\ P_k^{21} & 0 \end{pmatrix} \tag{3.6}$$

with 2×2 blocks P_k^{12}, P_k^{21} $(k = 1, 2,...)$. It turns out that one can find a sector S satisfying the condition above, together with a 4×4 matrix function $P(\cdot)$ which is holomorphic in S and which admits an asymptotic expansion in S, satisfying (3.5) and (3.6), such that, in the ξ-variable, (3.3) has the form

$$\xi' = A(t)\,\xi \tag{3.7}$$

where $A(t)$ is holomorphic in S, and admits an asymptotic expansion in S of the form

$$A(t) \sim A_0 + \frac{A_1}{t} + \sum_{k=2}^{\infty} \frac{A_k}{t^k}. \tag{3.8}$$

Furthermore $P(\cdot)$ can be chosen so that A is block-diagonal with 2×2 blocks (it follows that each A_k is block-diagonal as well), and so that

$$A_0 = C_0 = \begin{pmatrix} \frac{-i\sqrt{\lambda}}{2} & 0 & 0 & 0 \\ 0 & \frac{-i\sqrt{\lambda}}{2} & 0 & 0 \\ 0 & 0 & \frac{i\sqrt{\lambda}}{2} & 0 \\ 0 & 0 & 0 & \frac{i\sqrt{\lambda}}{2} \end{pmatrix}$$

and

$$A_1 = \begin{pmatrix} A_1^{11} & 0 \\ 0 & A_1^{22} \end{pmatrix} = \begin{pmatrix} C_1^{11} & 0 \\ 0 & C_1^{22} \end{pmatrix}.$$

Thus

$$A_1^{11} = \begin{pmatrix} -\frac{1}{4} + \frac{i}{4\sqrt{\lambda}} & -\frac{iB}{4\sqrt{\lambda}} \\ \frac{iB}{4\sqrt{\lambda}} & -\frac{1}{4} + \frac{i}{4\sqrt{\lambda}} \end{pmatrix}$$

$$A_1^{22} = \begin{pmatrix} -\frac{1}{4} - \frac{i}{4\sqrt{\lambda}} & \frac{iB}{4\sqrt{\lambda}} \\ -\frac{iB}{4\sqrt{\lambda}} & -\frac{1}{4} - \frac{i}{4\sqrt{\lambda}} \end{pmatrix} = \overline{A_1^{11}}.$$

Now set

$$\beta = \frac{B}{4\sqrt{\lambda}}.$$

By choosing the constant c_∞ in the introduction in an appropriate way, we can and will arrange that

$$\beta = \frac{n}{4} - \frac{1}{p-1}.$$

We see that, when p lies in the range $1 + \frac{4}{n} \leqslant p < \frac{n+2}{n-2}$, β lies in the interval $[0, \frac{1}{2})$. The eigenvalues of the matrices A_1^{11} and A_1^{22} are respectively

$-\frac{1}{4}\pm\beta+\frac{i}{4\sqrt{\lambda}}$ and $-\frac{1}{4}\pm\beta-\frac{i}{4\sqrt{\lambda}}$. Observe that if $\beta\in(0,\frac{1}{2})$, the eigenvalues do not differ by an integer. Observe further that, if $\beta=0$, we are in the critical case studied in Section 2. Assume from now on that $0<\beta<\frac{1}{2}$.

We apply the results of [9] to the system (3.7) to determine a matrix solution $\Phi(t)$, which is holomorphic in an open subsector S' of S containing $\{t\in\mathbb{R}\mid t\geqslant t_0\}$, and which takes the form

$$\Phi(t)=\hat{\Phi}(t)\,t^{A_1}e^{A_0t} \tag{3.9}$$

where $\hat{\Phi}$ admits an asymptotic expansion in S' of the form

$$\hat{\Phi}(t)\sim\sum_{k=0}^{\infty}\frac{\Phi_k}{t^k}.$$

See especially [9, Theorem 5.5, p. 25] and the discussion of [9, pp. 100–101].

Let us assume from now on that $t_0>0$. Multiplying $\Phi(t)$ on the right by an appropriate constant matrix K, we can assume that $\Phi(t_0)=I$. It is clear that $\Phi(t)=t^{-\frac{1}{4}+\beta}\Psi_1(t)$ and $\Phi(t)^{-1}=t^{\frac{1}{4}+\beta}\Psi_2(t)$, where $\Psi_1(t)$ and $\Psi_2(t)$ are matrix functions which are bounded for $t\geqslant t_0$.

We now carry out an analysis similar to that of Section 2; the main difference will consist in the corrections due to the presence of the quantity β. Let us write

$$\hat{f}(t,z)=\begin{pmatrix}0\\ -\frac{1}{4}t^{-\sigma}(x_1^2+x_2^2)^{\frac{p-1}{2}}x_1\\ 0\\ -\frac{1}{4}t^{-\sigma}(x_1^2+x_2^2)^{\frac{p-1}{2}}x_2\end{pmatrix}$$

and

$$f(t,\xi)=\hat{f}(t,P(t)\,Q\xi).$$

Then (3.2) takes the form

$$\xi'=A(t)\,\xi+f(t,\xi). \tag{3.10}$$

Observe that

$$|f(t,\xi)|=O(t^{-\sigma}\,|\xi|^p)\qquad\text{as}\quad t\to\infty, \tag{3.11}$$

uniformly in $\xi \in \mathbb{R}^4$. Observe further that

$$|f(t, \xi+\eta)-f(t, \xi)| \leqslant t^{-\sigma} M(\xi)\, \eta \tag{3.12}$$

where $M(\xi)=O(|\xi|^{p-1})$ as $\xi \to \infty$.

Let us now apply the scheme developed in Section 2. Let

$$w(t)=\begin{pmatrix} x_1(t) \\ x_2(t) \\ y_1(t) \\ y_2(t) \end{pmatrix}$$

be the solution of Eq. (3.2) which corresponds to the initial conditions indicated in (3.1).

According to [2; relation following Eq. (2.43)], we have for $j=1, 2$:

$$x_j(t)=\frac{2}{\sqrt{\lambda}}\, \rho^* t^{-\frac{1}{4}+\beta} e^{-\beta\epsilon(t)} \cos\theta_j(t)+O(t^{-\frac{1}{4}+\beta})$$

$$y_t(t)=\rho^* t^{-\frac{1}{4}+\beta} e^{-\beta\epsilon(t)} \cos\theta_j(t)+O(t^{-\frac{1}{4}-\beta}).$$

Here $\epsilon(t)$ is a positive function which tends to zero as $t \to \infty$ and furthermore

$$\theta_1(t)=-\frac{\sqrt{\lambda}}{2}\, t+\frac{1}{4\sqrt{\lambda}} \log t+\frac{c-\theta_0}{2}+\beta\mu(t)+O(t^{-2\beta})$$

$$\theta_2(t)=-\frac{\sqrt{\lambda}}{2}\, t+\frac{1}{4\sqrt{\lambda}} \log t+\frac{c+\theta_0}{2}-\beta\mu(t)+O(t^{-2\beta}),$$

where $\mu(t) \to 0$ as $t \to \infty$: indeed $\lim_{t \to \infty} \frac{\epsilon(t)}{\mu^2(t)}=2\beta$. The constant θ_0 depends on u_0.

We now refine these asymptotic relations in a way that seems to shed some light on the behavior, for large t, of the functions $\epsilon(t)$ and $\mu(t)$. Let $\bar{\xi}$ be the initial condition which corresponds to $w(t_0)$; explicitly $w(t_0)=P(t_0)\, Q\bar{\xi}$. Moreover let $\xi(t)$ be defined by $w(t)=P(t)\, Q\xi(t)$, so that $|\xi(t)|=O(t^{-\frac{1}{4}+\beta})$ as $t \to \infty$. We set

$$\xi_0(t)=\Phi(t)\, \bar{\xi}+\xi_c(t)$$

where

$$\xi_c(t) = \int_{t_0}^{\infty} \Phi(t)\,\Phi(s)^{-1}\, f(s, \xi(s))\, ds.$$

Then

$$|\xi_0(t)| = O(t^{-\frac{1}{4}+\beta}) \qquad \text{as} \quad t \to \infty.$$

Next we set

$$\eta_1(t) = \xi(t) - \xi_0(t) = \int_{\infty}^{t} \Phi(t)\,\Phi(s)^{-1}\, f(s, \xi(s))\, ds.$$

One then has from (3.11)

$$|\eta_1(t)| = O(t^{\gamma}) \qquad \text{as} \quad t \to \infty,$$

where

$$\gamma = \beta + \beta(1+p) - \sigma - \frac{p}{4} + 1.$$

Now one also has

$$\left(-\frac{1}{4} + \beta\right) - \gamma = \frac{p+1}{p-1} - \frac{n}{2} := \tau$$

where the quantity $\tau = \frac{p+1}{p-1} - \frac{n}{2}$ is positive if p takes values in the interval $[1 + \frac{4}{n}, \frac{n+2}{n-2})$. This means that $\gamma < -\frac{1}{4} + \beta$ when p takes values in this interval.

Next set

$$\xi_1(t) = \int_{\infty}^{t} \Phi(t)\,\Phi(s)^{-1}\, f(s, \xi_0(s))\, ds$$

$$f_1(t, \eta) = f(t, \xi_0(t) + \eta) - f(t, \xi_0(t))$$

$$\eta_2(t) = \int_{\infty}^{t} \Phi(t)\,\Phi(s)^{-1}\, f_1(s, \eta_1(s))\, ds,$$

and observe that

$$\eta_1(t) = \xi_1(t) + \eta_2(t)$$

$$\xi(t) = \xi_0(t) + \xi_1(t) + \eta_2(t).$$

Using (3.12), we obtain

$$|\eta_2(t)| = O(t^{\gamma-\tau}) \qquad \text{as} \quad t \to \infty.$$

It is now clear how to construct further approximations to the solution $\xi(t)$. For each $k \geqslant 1$ we have

$$\xi(t) = \xi_0 + \xi_1 + \cdots + \xi_k + \eta_{k+1}(t)$$

where $|\xi_k(t)| = O(t^{-\frac{1}{4}+\beta-k\tau})$, $|\eta_{k+1}(t)| = O(t^{\gamma-k\tau})$ for $t \to \infty$. Returning to the variables x_j, y_j, via the transformation $w = P(t)\,Q\xi$, and keeping in mind that $P(t) \to I$ as $t \to \infty$, we obtain developments for these quantities as well.

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