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Periodic solutions of a periodically forced and undamped beam resting on weakly elastic bearings

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Abstract. We study the problem of existence of periodic solutions to a partial differential equation modelling the behavior of an undamped beam subject to an external periodic force. We assume that the ordinary differential equation associated to the first two modes of vibration of the beam has a symmetric homoclinic solution. By using methods borrowed by dynamical systems theory we prove that, if the period is non resonant with the (infinitely many) internal periods of the PDE, the equation has a weak periodic solution of the same period as the external force. In particular we obtain continua of periodic solutions for the undamped beam in absence of external forces. This result may be considered as an infinite dimensional analogue of a result obtained in [16] concerning accumulation of periodic solutions to homoclinic orbits in finite dimensional reversible systems.

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1. Introduction

In this paper we consider the problem of existence of weak, $\frac{2T}{\sqrt{\varepsilon}}-$periodic solutions of equation

$$\begin{aligned}
&u_{tt} + u_{xxxx} + \varepsilon\mu h(x, \sqrt{\varepsilon}t) = 0\,,\\
&u_{xx}(0,\cdot) = u_{xx}(1,\cdot) = 0\,,\\
&u_{xxx}(0,\cdot) = -\varepsilon f\big(\textstyle\int_0^1 u(x,\cdot)\varphi(x)dx\big),\\
&u_{xxx}(1,\cdot) = \varepsilon g\big(\textstyle\int_0^1 u(x,\cdot)\varphi(1-x)dx\big)\,.
\end{aligned} \tag{1}$$

We assume that $h(x,t)$ is a $2T-$periodic (in t) C^1-function on $[0,1]\times\mathbb{R}$, $f(x)$, $g(x)$ are sufficiently smooth functions such that $f(0) = g(0) = 0$ and $\varphi(x) = \varphi_a(x) \in L^2(\mathbb{R},\mathbb{R})$, is a non negative function whose support $\operatorname{supp}\varphi \subseteq [0,a]$, where a is a

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fixed positive number such that $0 < a < \frac{1}{3}$, and

$$\int_0^1 \varphi(x)dx = \int_{-\infty}^{\infty} \varphi(x)dx = 1. \tag{2}$$

By a weak $\frac{2T}{\sqrt{\varepsilon}}-$periodic solution of (1) we mean a function $u(x,t) \in C(\mathbb{R}, L^2([0,1]))$ that satisfies (1) in the distributional sense and it is $\frac{2T}{\sqrt{\varepsilon}}$-periodic in t. Clearly $u \in L^2_{loc}([0,1] \times \mathbb{R})$.

We may also consider the more general case where the condition on $u_{xxx}(1,\cdot)$ is replaced by

$$u_{xxx}(1,\cdot) = \varepsilon g\left(\int_0^1 u(x,\cdot)\widetilde{\varphi}(x)dx\right)$$

$\widetilde{\varphi}(x) \in L^1_{loc}(\mathbb{R},\mathbb{R})$, being another non negative function such that $\operatorname{supp}\varphi \subseteq [1-a,1]$ and (2) holds. Physically these conditions mean that the response at the end points of the beam depends on a small part of the beam near the end points. In (1) we assume that the response may be different ($f(x)$ may possibly be different than $g(x)$) but depend in a symmetric way on the beam. This assumption simplifies the analysis, however the result we obtain holds as well if we consider different functions $\varphi(x)$ and $\widetilde{\varphi}(x)$ as the reader may check making suitable changes at the appropriate places in this paper. To perform such a study we will use perturbation methods, that is we first look at equation (1) with $\varepsilon = 0$. This unperturbed equation has its own internal modes of vibration. As in [2, 4] we assume that the $4-$th dimensional equation in the direction of the first two modes (those associated to the zero eigenvalue) has a symmetric homoclinic solution $\Gamma(t)$. Then we look for a weak $\frac{2T}{\sqrt{\varepsilon}}-$periodic solution of (1) which is close to the homoclinic orbit when $|t| \leq \frac{T}{\sqrt{\varepsilon}}$. Our main result states that if $h(x,t) = h(x,-t)$, $\varepsilon > 0$ and μ are sufficiently small and the period $2T$ of $h(x,t)$ belongs to a certain non-zero measure subset of the interval $[2\widetilde{T}_0, 2\varepsilon^{-1/4}]$, with $\widetilde{T}_0$ sufficiently large, then equation (1) has a weak $\frac{2T}{\sqrt{\varepsilon}}-$periodic solutions.

Related results to those in this paper concerning existence of periodic solutions to partial differential equations are obtained for instance also in the papers [1, 4, 5, 6, 8, 9, 10, 13, 15, 17].

2. The integral equation

As a first step we take $u(x,t) \leftrightarrow u(x,\sqrt{\varepsilon}t)$ and get the equivalent problem

$$\begin{aligned}
&u_{tt} + \varepsilon^{-1}u_{xxxx} + \mu h(x,t) = 0\,,\\
&u_{xx}(0,\cdot) = u_{xx}(1,\cdot) = 0\,,\\
&u_{xxx}(0,\cdot) = -\varepsilon f(\textstyle\int_0^1 u(x,\cdot)\varphi(x)dx),\\
&u_{xxx}(1,\cdot) = \varepsilon g(\textstyle\int_0^1 u(x,\cdot)\varphi(1-x)dx).
\end{aligned} \tag{3}$$

Of course a weak $\frac{2T}{\sqrt{\varepsilon}}-$periodic solution of equation (1) corresponds to a weak $2T-$periodic solution of equation (3).

Since $\operatorname{supp}\varphi \subseteq [0,a]$ we have (here $*$ denotes convolution in the x variable)

$$\int_0^1 u(x,t)\varphi(x)dx = \int_{-\infty}^{\infty} u(x,t)\varphi(x)dx = [u(\cdot,t) * \hat{\varphi}](0)$$

where $\hat{\varphi}(x) = \varphi(-x)$ and

$$\int_0^1 u(x,t)\varphi(1-x)dx = \int_{-\infty}^{\infty} u(x,t)\varphi(1-x)dx = [u(\cdot,t) * \varphi](1)$$

Note that, here, $u(x,t)$ is any measurable extension of $u(x,t)$ to $\mathbb{R}\times\mathbb{R}$, the above integrals being independent on the choice of the extension since $\operatorname{supp}\varphi \subseteq [0,a] \subset [0,1]$.

By a weak $2T-$periodic solution of (3) (cf. [9, p. 135]), we mean any $u \in C(\mathbb{R}, L^2([0,1]))$ that is $2T$-periodic in t and satisfies the identity

$$\begin{aligned}&\int_{-T}^{T}\int_0^1 \Big\{u(x,t)\Big[v_{tt}(x,t) + \varepsilon^{-1}v_{xxxx}(x,t)\Big] + \mu h(x,t)v(x,t)\Big\}\,dx\,dt\\ &+\int_{-T}^{T}\Big\{f([u(\cdot,t) * \hat{\varphi}](0))v(0,t) + g([u(\cdot,t) * \varphi](1))v(1,t)\Big\}\,dt = 0\end{aligned} \tag{4}$$

for any $v(x,t) \in C_T^{\infty}([0,1]\times\mathbb{R})$ - the set of all $v(x,t) \in C^{\infty}([0,1]\times\mathbb{R},\mathbb{R})$ that are $2T$-periodic in t - and the following boundary value conditions hold

$$v_{xx}(0,\cdot) = v_{xx}(1,\cdot) = v_{xxx}(0,\cdot) = v_{xxx}(1,\cdot) = 0\,. \tag{5}$$

We intend to search for weak periodic solutions of equation (1), or equivalently (4), by perturbation methods, i.e. for small ε. Thus we let $\varepsilon \to 0$ in (1) and get the linear homogeneous equation with homogeneous boundary conditions

$$\begin{aligned}&u_{tt} + u_{xxxx}(x) = 0\\ &u_{xx}(0,\cdot) = u_{xx}(1,\cdot) = 0,\\ &u_{xxx}(0,\cdot) = u_{xxx}(1,\cdot) = 0\,.\end{aligned}$$

By separation of variables, i.e. setting $u(t,x) = U(x)T(t)$ (or else using Fourier series), we see that κ has to exists such that

$$\begin{aligned}&U^{(iv)}(x) = \kappa U(x),\\ &U''(0) = U''(1) = 0,\\ &U'''(0) = U'''(1) = 0\end{aligned} \tag{6}$$

and

$$T''(t) = -\kappa T(t).$$

Now, changing x with $\frac{\pi}{4}x$ we see that (6) is equivalent to

$$\begin{aligned} &U^{(iv)}(x) = \left(\tfrac{4}{\pi}\right)^4 \kappa U(x), \\ &U''(0) = U''(\pi/4) = 0, \\ &U'''(0) = U'''(\pi/4) = 0 \end{aligned}$$

and then, using a result in [9] and coming back to the old variable x we see that (6) may have nonzero solutions only for $\kappa = \mu^4 \geq 0$ with $\mu = \mu_k$, $k = -1, 0, 1, \cdots$ and $\mu_{-1} = \mu_0 = 0$, $\mu_k = \frac{\pi}{2}(2k+1) + r_k \geq 1$, for $k \geq 1$. Moreover, in Appendix 1 it is proved that, for any $k \in \mathbb{N}$, the following estimate holds:

$$|r_k| \leq c\frac{\pi}{4}e^{-k\pi} \tag{7}$$

where $c < 2.6$. The corresponding orthonormal system of eigenfunctions $\{w_i\}_{i=-1}^{\infty} \in L^2([0,1])$ is bounded, i.e. $\sup_{i \geq -1, x \in [0,1]} |w_i(x)| < \infty$. Moreover the eigenfunctions $w_{-1}(x)$ and $w_0(x)$ of the zero eigenvalue are:

$$w_{-1}(x) = 1, \quad w_0(x) = \sqrt{3}(2x-1).$$

We note that

$$[w_{-1} * \hat{\varphi}](0) = 1 = [w_{-1} * \varphi](1)$$

and similarly,

$$[w_0 * \hat{\varphi}](0) = -[w_0 * \varphi](1)$$

since $w_0(1-x) = -w_0(x)$. We set

$$k_\varphi = [w_0 * \varphi](1)\,.$$

Note that $k_\varphi > 0$ since $\operatorname{supp}\varphi \subseteq [0, \frac{1}{3}]$. Moreover:

$$\lim_{a \to 0} \int_{-\infty}^{\infty} w_0(x)\varphi(1-x)dx = w_0(1) = \sqrt{3}$$

since $w_0(x) \in C([0,1])$. We can also easily estimate the difference $k_\varphi - \sqrt{3}$. In fact we have, using also (2) and $w_0(1-x) = -w_0(x)$:

$$|k_\varphi - \sqrt{3}| = \left| \int_0^a [w_0(1-x) - \sqrt{3}]\varphi_a(x)\,dx \right| = 2\sqrt{3}\int_0^a x\varphi_a(x)\,dx \leq 2\sqrt{3}a. \tag{8}$$

We seek a solution $u(x,t)$ of (3) in the form

$$u(x,t) = y_1(t)w_{-1}(x) + y_2(t)w_0(x) + z(x,t)$$

where, for any $t \in [-T, T]$, $z(x,t)$ belongs to the infinite dimensional space spanned by $\{w_i\}_{i=1}^{\infty}$. To get the equations for $y_1(t)$, $y_2(t)$, and $z(x,t)$ we take $v(x,t) = \phi_1(t)w_{-1}(x) + \phi_2(t)w_0(x) + v_0(x,t)$ in (4) with $\phi_i \in C^\infty$, $v_0(x,t) \in C^\infty$, $2T$-periodic in t and $v_0(x,t)$ satisfying, besides (5), also:

$$\int_0^1 v_0(x,t)dx = \int_0^1 xv_0(x,t)dx = 0. \tag{9}$$

Note that conditions (9) correspond to the orthogonality of $v_0(x,t)$ to $w_{-1}(x)$ and $w_0(x)$, for any $t \in \mathbb{R}$. The same equations are also satisfied by $z(x,t)$. Plugging the above expression for $v(x,t)$ into (4) and using the orthonormality, we arrive at the system of equations

$$\begin{aligned}&\ddot{y}_1(t) + \mu \int_0^1 h(x,t)\,dx + f(y_1(t) - k_\varphi y_2(t) + [z(\cdot,t) * \hat{\varphi}](0)) \\ &+ g(y_1(t) + k_\varphi y_2(t) + [z(\cdot,t) * \varphi](1)) = 0\,,\end{aligned} \tag{10}$$

$$\begin{aligned}&\ddot{y}_2(t) + \sqrt{3}\mu \int_0^1 h(x,t)(2x-1)\,dx \\ &-\sqrt{3} f(y_1(t) - k_\phi y_2(t) + [z(\cdot,t) * \hat{\varphi}](0)) \\ &+\sqrt{3} g(y_1(t) + k_\phi y_2(t) + [z(\cdot,t) * \varphi](1)) = 0\,,\end{aligned} \tag{11}$$

$$\begin{aligned}&\int_{-T}^{T} \int_0^1 \Big\{ z(x,t)\Big[v_{tt}(x,t) + \varepsilon^{-1} v_{xxxx}(x,t)\Big] + \mu h(x,t) v(x,t) \Big\}\,dx\,dt \\ &+ \int_{-T}^{T} \Big\{ f([u(\cdot,t) * \hat{\varphi}](0)) v(0,t) + g([u(\cdot,t) * \varphi](1) v(1,t) \Big\}\,dt = 0\end{aligned} \tag{12}$$

where we wrote $v(x,t)$ instead $v_0(x,t)$. Thus, in equation (12), $v(x,t)$ is any function in $C_T^\infty([0,1] \times \mathbb{R})$ and the conditions (5), (9) hold.

We now assume that the following conditions hold:

H1) $f(0) = g(0) = 0$, $f'(0) < 0$, $g'(0) < 0$ and the system:

$$\begin{aligned}&\ddot{\xi}_1 + f(\xi_1 - k_\varphi \xi_2) + g(\xi_1 + k_\varphi \xi_2) = 0 \\ &\ddot{\xi}_2 - \sqrt{3}[f(\xi_1 - k_\varphi \xi_2) - g(\xi_1 + k_\varphi \xi_2)] = 0\end{aligned} \tag{13}$$

has a symmetric homoclinic solution $\Gamma(t) = (\Gamma_1(t), \Gamma_2(t)) \neq 0$, that is a non–trivial bounded solution such that $\Gamma(t) = \Gamma(-t)$ and $\lim_{t\to\pm\infty} \Gamma(t) = \lim_{t\to\pm\infty} \dot{\Gamma}(t) = 0$.

H2) The homoclinic solution $\Gamma(t)$ is non-degenerate, that is the linear system

$$\begin{aligned}\ddot{y}_1 =\ & -[f'(\Gamma_1(t) - k_\varphi \Gamma_2(t)) + g'(\Gamma_1(t) + k_\varphi \Gamma_2(t))] y_1 \\ & + k_\varphi [f'(\Gamma_1(t) - k_\varphi \Gamma_2(t)) - g'(\Gamma_1(t) + k_\varphi \Gamma_2(t))] y_2 \\ \ddot{y}_2 =\ & \sqrt{3}[f'(\Gamma_1(t) - k_\varphi \Gamma_2(t)) - g'(\Gamma_1(t) + k_\varphi \Gamma_2(t))] y_1 \\ & -\sqrt{3} k_\varphi [f'(\Gamma_1(t) - k_\varphi \Gamma_2(t)) + g'(\Gamma_1(t) + k_\varphi \Gamma_2(t))] y_2\end{aligned} \tag{14}$$

has the only bounded solution $(y_1(t), y_2(t), \dot{y}_1(t), \dot{y}_2(t)) = (\Gamma(t), \dot{\Gamma}(t))$ up to a multiplicative constant.

We also remark that assumption (H1) imply that $(y_1, y_2) = (0,0)$ is a hyperbolic equilibrium of system (13). In fact the Jacobian matrix at the point $(0,0)$ of the first order system associated to (13) is

$$A := \begin{pmatrix} 0 & 1 & 0 & 0 \\ -f'(0) - g'(0) & 0 & k_\varphi[f'(0) - g'(0)] & 0 \\ 0 & 0 & 0 & 1 \\ \sqrt{3}[f'(0) - g'(0)] & 0 & -\sqrt{3}k_\varphi[f'(0) + g'(0)] & 0 \end{pmatrix} \tag{15}$$

whose eigenvalues are the solutions of the equation

$$\mu^4 + (\sqrt{3}k_\varphi + 1)[f'(0) + g'(0)]\mu^2 + 4\sqrt{3}k_\varphi f'(0)g'(0) = 0.$$

Now, the discriminant Δ of the equation

$$\lambda^2 + (\sqrt{3}k_\varphi + 1)[f'(0) + g'(0)]\lambda + 4\sqrt{3}k_\varphi f'(0)g'(0) = 0 \tag{16}$$

satisfies

$$\frac{\Delta}{f'(0)^2} = (k_\varphi\sqrt{3} + 1)^2(s+1)^2 - 16k_\varphi\sqrt{3}s$$

where $s = \frac{g'(0)}{f'(0)} > 0$. The function on the right hand side has a minimum at the point

$$s = -\frac{3k_\varphi^2 - 6k_\varphi\sqrt{3} + 1}{(k_\varphi\sqrt{3} + 1)^2}$$

and its value at this point is

$$16\sqrt{3}k_\varphi \left(\frac{k_\varphi\sqrt{3} - 1}{k_\varphi\sqrt{3} + 1}\right)^2 > 0$$

if $k_\varphi\sqrt{3} \neq 1$. Since, as observed in (8), $|k_\varphi - \sqrt{3}| \leq 2\sqrt{3}a < \frac{2}{\sqrt{3}}$ and $f'(0) < 0$, $g'(0) < 0$, we see that for any $a \in (0, 1/3)$, equation (16) has two positive solutions. As a consequence the matrix A has two positive and two negative real eigenvalues.

We set

$$\begin{aligned} A_{11}(t) = A_{22}(t) &= f'(\Gamma_1(t) - k_\varphi\Gamma_2(t)) + g'(\Gamma_1(t) + k_\varphi\Gamma_2(t)) \\ A_{12}(t) = A_{21}(t) &= f'(\Gamma_1(t) - k_\varphi\Gamma_2(t)) - g'(\Gamma_1(t) + k_\varphi\Gamma_2(t)) \end{aligned}$$

and

$$A(t) = \begin{pmatrix} 0 & 1 & 0 & 0 \\ A_{11}(t) & 0 & -k_\varphi A_{12}(t) & 0 \\ 0 & 0 & 0 & 1 \\ -\sqrt{3}A_{21}(t) & 0 & \sqrt{3}k_\varphi A_{22}(t) & 0 \end{pmatrix}. \tag{17}$$

Since $A(t) \to A_0$ as $|t| \to \infty$, from [7] it follows that the first order linear system

$$\dot{y} = A(t)y \tag{18}$$

corresponding to (14) and obtained setting $y = (y_1, \dot{y}_1, y_2, \dot{y}_2)$ has an exponential dichotomy on both $\mathbb{R}_+$ and $\mathbb{R}_-$. This means that projections P_+ and P_- and constants $\hat{k} \geq 1$ and $\hat{\delta} > 0$ (called, respectively, the constant and the exponent of the dichotomy) exist such that the fundamental matrix $Y(t)$ of system (18) satisfies

$$\begin{array}{ll} \|Y(t)P_+Y^{-1}(s)\| \le \hat{k}e^{-\hat{\delta}(t-s)} & \text{if } 0 \le s \le t \\ \|Y(t)(\mathbb{I}-P_+)Y^{-1}(s)\| \le \hat{k}e^{\hat{\delta}(t-s)} & \text{if } 0 \le t \le s \\ \|Y(t)P_-Y^{-1}(s)\| \le \hat{k}e^{-\hat{\delta}(t-s)} & \text{if } s \le t \le 0 \\ \|Y(t)(\mathbb{I}-P_-)Y^{-1}(s)\| \le \hat{k}e^{\hat{\delta}(t-s)} & \text{if } t \le s \le 0 \end{array} \tag{19}$$

Moreover H2) implies that the space of bounded solutions on $\mathbb{R}$ is spanned by $(\Gamma_1(t), \dot{\Gamma}_1(t), \Gamma_2(t), \dot{\Gamma}_2(t))$. Throughout this paper $\hat{k}$ and $\hat{\delta}$ denote the constant and the exponent, respectively, of the dichotomy of (18) on $\mathbb{R}_+$ and $\mathbb{R}_-$.

Since we look for $2T-$periodic solutions of equations (10)–(12) such that the sup-norm in $[-T,T]$ of $y_1(t) - \Gamma_1(t)$, $y_2(t) - \Gamma_2(t)$ and the norm of $z(x,t) \in C([-T,T], L^2([0,1]))$ are small, we replace $y_j(t)$ with $y_j(t) + \Gamma_j(t)$, $j = 1,2$ in (10)–(12) and write $y(t)$ for $(y_1(t), \dot{y}_1(t), y_2(t), \dot{y}_2(t))$. We obtain:

$$\dot{y}(t) + A(t)y(t) = F(t, y_1(t), y_2(t), [z(\cdot,t) * \hat{\varphi}](0), [z(\cdot,t) * \varphi](1), \mu, \varepsilon), \tag{20}$$

$$\begin{aligned} &\int_{-T}^{T}\int_0^1 \Big\{ z(x,t)\Big[v_{tt}(x,t) + \varepsilon^{-1}v_{xxxx}(x,t)\Big] + \mu h(x,t)v(x,t)\Big\}\,dx\,dt \\ &+ \int_{-T}^{T} \Big\{ f([u(\cdot,t) * \hat{\varphi}](0))v(0,t) + g([u(\cdot,t) * \varphi](1))v(1,t)\Big\}\,dt = 0 \end{aligned} \tag{21}$$

where $u(x,t) = [y_1(t) + \Gamma_1(t)] + [y_2(t) + \Gamma_2(t)]w_0(x) + z(x,t)$, and

$$F(t, y_1, y_2, z_1, z_2, \mu, \varepsilon) = \begin{pmatrix} 0 \\ F_1(t, y_1, y_2, z_1, z_2, \mu, \varepsilon) \\ 0 \\ F_2(t, y_1, y_2, z_1, z_2, \mu, \varepsilon) \end{pmatrix} \tag{22}$$

with

$$\begin{aligned} F_1(t, y_1, y_2, z_1, z_2, \mu, \varepsilon) &= -f([y_1 + \Gamma_1(t)] - k_\varphi[y_2 + \Gamma_2(t)] + z_1) \\ &\quad -g([y_1 + \Gamma_1(t)] + k_\varphi[y_2 + \Gamma_2(t)] + z_2) \\ &\quad +f(\Gamma_1(t) - k_\varphi\Gamma_2(t)) + g(\Gamma_1(t) + k_\varphi\Gamma_2(t)) \\ &\quad -\mu\int_0^1 h(x,t)\,dx + A_{11}(t)y_1 - k_\varphi A_{12}(t)y_2 \\ \tfrac{1}{\sqrt{3}}F_2(t, y_1, y_2, z_1, z_2, \mu, \varepsilon) &= f([y_1 + \Gamma_1(t)] - k_\varphi[y_2 + \Gamma_2(t)] + z_1) \\ &\quad -g([y_1 + \Gamma_1(t)] + k_\varphi[y_2 + \Gamma_2(t)] + z_2) \\ &\quad -f(\Gamma_1(t) - k_\varphi\Gamma_2(t)) + g(\Gamma_1(t) + k_\varphi\Gamma_2(t)) \\ &\quad -\mu\textstyle\int_0^1 h(x,t)(2x-1)\,dx \\ &\quad -A_{21}(t)y_1 + k_\varphi A_{22}(t)y_2. \end{aligned}$$

Remark 1. Since equation (13) depends on the function $\varphi(x)$ one might wonder whether conditions (H1) and (H2) may be satisfied. Now, if we assume that $f(x) = g(x)$ (equal responses at the end points of the beam) we see that we may consider the case where $\Gamma_2(t) = 0$. In this case conditions H1) and H2) are replaced by

H3) the second order equation on $\mathbb{R}$: $\ddot{\xi} + f(\xi) = 0$ has a solution $\Gamma_0(t)$ homoclinic to the hyperbolic fixed point $\xi = 0$

H4) $\Gamma_0(t)$ is non degenerate that is the linear equation $\ddot{\xi} + \sqrt{3}k_\varphi f'(\Gamma_0(t))\xi = 0$ has no bounded solutions apart from the trivial one $\xi = 0$.

In fact, if H3) and H4) hold we can take $\Gamma_1(t) = \Gamma_0(\sqrt{2}t)$ and $\Gamma_2(t) = 0$. We note that the assumption $\Gamma(t) = \Gamma(-t)$ follows by requiring, without loss of generality, that $\dot{\Gamma}_0(0) = 0$. Moreover, since $\lim_{a\to 0} k_\varphi = \sqrt{3}$, we see that condition **H4**) is satisfied, provided $a > 0$ is sufficiently small and the equation $\ddot{\xi} + 3f'(\Gamma_0(t))\xi = 0$ has the only bounded solution $\xi = 0$.

Again in the case where $f(x) = g(x)$ but with the further condition $f(-x) = -f(x)$, we may also consider the case where $\Gamma_1(t) = 0$. In this case conditions H1) and H2) are replaced by

H5) the second order equation on $\mathbb{R}$: $\ddot{\xi} + f(\xi) = 0$ has a solution $\Gamma_0(t)$ homoclinic to the hyperbolic fixed point $\xi = 0$

H6) $\Gamma_0(t)$ is non degenerate that is the linear equation $\ddot{\xi} + \frac{1}{k_\varphi\sqrt{3}} f'(\Gamma_0(t))\xi = 0$ has no bounded solutions apart from the trivial one $\xi = 0$.

In fact, if H5) and H6) hold we can take $\Gamma_1(t) = 0$ and

$$\Gamma_2(t) = \frac{1}{k_\varphi}\Gamma_0(\sqrt[4]{12}\sqrt{k_\varphi}t).$$

Again the assumption $\Gamma(t) = \Gamma(-t)$ follows by requiring, without loss of generality, that $\dot{\Gamma}_0(0) = 0$. Moreover, as in the previous case we see that, if the second order equation $3\ddot{\xi} + f'(\Gamma_0(t))\xi = 0$ has the only bounded solution $\xi = 0$, the non degenerateness of $\Gamma_0(t)$ follows from roughness, provided $a > 0$ is sufficiently small.

We conclude this Section by noting some properties of the matrix $A(t)$ and the function $F(t, y_1, y_2, z_1, z_2, \mu, \varepsilon)$ that will be used in Section 4. Let

$$J = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}. \tag{23}$$

Then it is easy to see that

$$J^2 = J, \qquad JA(t) = -A(t)J, \qquad A(t) = A(-t) \tag{24}$$

the last equality following from $\Gamma(t) = \Gamma(-t)$, and moreover:

$$JF(t, y_1, y_2, z_1, z_2, \mu, \varepsilon) = -F(t, y_1, y_2, z_1, z_2, \mu, \varepsilon). \tag{25}$$

3. The linear equations

First we study the problem of existence of $2T-$periodic solutions of equation (21). We begin by considering the problem of existence of a $2T-$periodic solution of the following linear non-homogeneous equation in $\mathbb{R}$:

$$\ddot{z}(t) + \omega^2 z(t) = h(t)\,, \tag{26}$$

where $h(t) \in L^1([-T,T],\mathbb{R})$. We set

$$\|h\|_1 = \int_{-T}^{T} |h(t)|dt < \infty$$

and extend $h(t)$ to the whole $\mathbb{R}$ by $2T-$periodicity (i.e. $h(t+2T) = h(t)$ for any $t \in \mathbb{R}$).

Since the homogeneous equation has a periodic fundamental matrix of period $\frac{2\pi}{\omega}$ we can have a $2T-$periodic solution only if $\omega T \neq k\pi$, with $k \in \mathbb{Z}$. As a matter of fact, elementary computations show that, in this case, equation (26) has a (unique) $2T-$periodic solution which is given by

$$z(t) = \frac{1}{2\omega \sin \omega T} \int_{t-T}^{t+T} h(T+s) \cos \omega(t-s) ds \tag{27}$$

We assume that $T \in \mathbb{R}$ is such that $|\sin \omega T| \geq \sin \tilde{\delta}$ for some $\tilde{\delta} \in (0, \frac{\pi}{2})$. Of course this is equivalent to say that

$$|\omega T - k\pi| \geq \tilde{\delta}$$

for any $k \in \mathbb{Z}$.

Now, from (27) we obtain:

$$|z(t)| \leq \frac{1}{2\omega \sin(\tilde{\delta})} \|h\|_1 \tag{28}$$

Next, if $h(t)$ is differentiable in $[-T,T]$, and $\dot{h}(t) \in L^1([-T,T])$ we obtain, integrating by parts and using the periodicity of $h(t)$:

$$\begin{aligned} z(t) = \; & \frac{1}{2\omega \sin \omega T} \Big[\frac{2\sin(\omega T)}{\omega} h(t) + \int_{t-T}^{t+T} \frac{\sin \omega(t-s)}{\omega} \dot{h}(s+T) ds \\ & + \frac{\sin \omega(t-2kT)}{\omega} (h(T^+) - h(T^-)) \Big] \\ & = \frac{1}{\omega^2} h(t) + \frac{1}{2\omega^2 \sin \omega T} \Big[\int_{t-T}^{t+T} \dot{h}(s+T) \sin \omega(t-s) ds \\ & + \sin \omega(t-2kT)(h(T^+) - h(T^-)) \Big] \end{aligned} \tag{29}$$

for $(2k-1)T \leq t < (2k+1)T$, $k \in \mathbb{Z}$. Hence:

$$|z(t)| \leq \frac{1}{\omega^2} \left[\left(1 + \frac{1}{\sin \tilde{\delta}} \right) \|h\|_\infty + \frac{1}{2\sin(\tilde{\delta})} \|\dot{h}\|_1 \right] \tag{30}$$

where

$$\|h\|_\infty = \sup_{t\in[-T,T]} |h(t)|.$$

Now, for $j \in \mathbb{N}$, we consider the family of equations:

$$\ddot{z}_j(t) + \omega_j^2 z_j(t) = h_j(t)\,, \tag{31}$$

where

$$\omega_j = \frac{\mu_j^2}{\sqrt{\varepsilon}}$$

and $h(t) := \{h_j(t)\}_{j\in\mathbb{N}}$ is a family of $2T-$periodic functions on $\mathbb{R}$ such that

$$\|h\| := \sup_{j\in\mathbb{N}} \|h_j\|_1 < \infty.$$

Assume that, for some $\beta \in (1, \frac{3}{2})$, T satisfies the inequalities

$$|\omega_j T - k\pi| \geq \theta j^{-\beta} > 0 \tag{32}$$

for any $j \in \mathbb{N}$ and $k \in \mathbb{Z}$, where, according to (89) (see Appendix 2), we take θ satisfying

$$0 < \theta < \frac{\pi(\beta-1)}{2\beta}$$

and set $\tilde{\delta} = \theta j^{-\beta} \leq \theta < \pi/2$. It follows from Appendix 2 that the set of values of $T \in \mathbb{R}$ that satisfy (32) has positive measure (see also Remark 2 in Section 4). Hence from (32) we get $|\sin(\omega_j T)| \geq \sin(\theta j^{-\beta})$ for any $j \in \mathbb{N}$.

Let $z_j(t)$ be the corresponding $2T-$periodic solution of equation (31) and set

$$z(x,t) := \sum_{j=1}^{\infty} z_j(t) w_j(x). \tag{33}$$

Now, for any $t \in [-T,T]$, we evaluate the L^2-norm of the function $z(x,t)$. The usual integral norm on $L^2([0,1])$ is denoted by $\|\cdot\|_2$, i.e. we take $\|w\|_2 = \sqrt{\int_0^1 w(x)^2\,dx}$ for any $w \in L^2([0,1])$. Since $\{w_j(x)\}$ is an orthonormal system in $L^2([0,1])$ we have, according to (28) and using also $\sin\widetilde{\delta} \geq \widetilde{\delta}/2$ for any $\widetilde{\delta} \in [0,\pi/2]$ and $\mu_j > j\pi$ for any $j \in \mathbb{N}$,

$$\begin{aligned}\|z(x,t)\|_2^2 &= \sum_{j=1}^{\infty} |z_j(t)|^2 \leq \sum_{j=1}^{\infty} \|z_j(t)\|_\infty^2 \leq \frac{\varepsilon}{\theta^2} \sum_{j=1}^{\infty} \frac{j^{2\beta}}{\mu_j^4} \sup_{j\in\mathbb{N}} \|h_j\|_1^2 \\ &\leq \frac{\varepsilon}{\pi^4\theta^2} \sum_{j=1}^{\infty} \frac{1}{j^{2(2-\beta)}} \sup_{j\in\mathbb{N}} \|h_j\|_1^2 \leq \frac{2\varepsilon(2-\beta)}{\pi^4\theta^2(3-2\beta)} \sup_{j\in\mathbb{N}} \|h_j\|_1^2,\end{aligned}$$

since

$$\sum_{j=1}^{\infty} \frac{1}{j^{2(2-\beta)}} \leq 1 + \int_1^\infty s^{-2(2-\beta)}\,ds = \frac{2(2-\beta)}{3-2\beta}\,.$$

Recalling that $1 < \beta < \frac{3}{2}$, we find that $\|z(x,t)\|_2$ is bounded. Moreover, from the total convergence of the series

$$\sum_{j=1}^{\infty} \|z_j\|_\infty^2$$

we see that, for any $\widetilde{\sigma} > 0$ there is a $p \in \mathbb{N}$ such that $\displaystyle\sum_{j=p+1}^{\infty} \|z_j\|_\infty^2 < \widetilde{\sigma}/8$. Using the uniform continuity of the functions $z_j(t)$ in the compact interval $[-T,T]$, we can find $\widetilde{\rho} > 0$ so small that $\displaystyle\sum_{j=1}^{p} |z_j(t) - z_j(t_0)|^2 < \widetilde{\sigma}/2$, whenever $|t - t_0| < \widetilde{\rho}$. Consequently, we derive for $|t - t_0| < \widetilde{\rho}$

$$\begin{aligned}\|z(x,t) - z(x,t_0)\|_2^2 &= \sum_{j=1}^{\infty} |z_j(t) - z_j(t_0)|^2 \\ &\le \sum_{j=1}^{p} |z_j(t) - z_j(t_0)|^2 + 4 \sum_{j=p+1}^{\infty} \|z_j\|_\infty^2 < \widetilde{\sigma}\,.\end{aligned}$$

Hence the map $t \mapsto z(x,t)$ belongs to $C([-T,T],W)$ where

$$W = \left\{ w \in L^2([0,1]) \;\middle|\; \int_0^1 w(x)dx = \int_0^1 xw(x)dx = 0 \right\}. \tag{34}$$

Finally, it is clear that $z(x,-T) = z(x,T)$ a.e.

Now, let $H^1(x,t) \in L^1([0,1] \times [-T,T])$, $H^2(t), H^3(t) \in L^1([-T,T])$ and consider the equation

$$\begin{aligned}&\int_{-T}^{T}\int_0^1 \Big\{ z(x,t)\Big[v_{tt}(x,t) + \varepsilon^{-1} v_{xxxx}(x,t)\Big] + H^1(x,t)v(x,t)\Big\}\,dx\,dt \\ &+ \int_{-T}^{T} \Big\{ H^2(t)v(0,t) + H^3(t)v(1,t)\Big\}\,dt = 0\end{aligned} \tag{35}$$

where $v(x,t) \in C_T^\infty([0,1]\times\mathbb{R})$ satisfies the boundary conditions (5), (9). For $j \in \mathbb{N}$, we take $v(x,t) = \phi(t)w_j(x)$ where $\phi(t) \in C^\infty(\mathbb{R})$ is $2T$-periodic. Then (35) reads:

$$\begin{aligned}&\int_{-T}^{T}\int_0^1 \Big\{ z(x,t)\Big[\phi''(t)w_j(x) + \frac{\mu_j^4}{\varepsilon}\phi(t)w_j(x)\Big] + H^1(x,t)\phi(t)w_j(x)\Big\}\,dx\,dt \\ &+ \int_{-T}^{T} \Big\{ H^2(t)\phi(t)w_j(0) + H^3(t)\phi(t)w_j(1)\Big\}\,dt = 0\end{aligned} \tag{36}$$

and hence, writing $z(x,t)$ as in (33), we see that $z_j(t)$ has to satisfy equation (31) with

$$h_j(t) = -\Big(\int_0^1 H^1(x,t)w_j(x)\,dx + H^2(t)w_j(0) + H^3(t)w_j(1)\Big) \tag{37}$$

Here we have silently $2T$-periodically extended on $\mathbb{R}$ with respect to the $t-$variable, the functions H^1, H^2, H^3. Now:

$$\|h_j\|_1 \le M_1[\|H^1\|_1 + \|H^2\|_1 + \|H^3\|_1] \tag{38}$$

where

$$M_1 = \sup_{j\in\mathbb{N}} \|w_j\|_\infty \tag{39}$$

and

$$\|H^1\|_1 = \int_{-T}^{T}\int_0^1 |H^1(x,t)|dxdt.$$

As a consequence equation (35) has a unique solution $z(x,t) \in C([-T,T],W)$ that satisfies $z(x,-T) = z(x,T)$ a.e. and

$$\|z(\cdot,t)\|_2 \le C_{\beta,\theta}\sqrt{\varepsilon}M_1[\|H^1\|_1 + \|H^2\|_1 + \|H^3\|_1], \tag{40}$$

where $C_{\beta,\theta} = \frac{1}{\pi^2\theta}\sqrt{\frac{2(2-\beta)}{3-2\beta}}$.

For any $H^1(x,t) \in L^1([0,1]\times[-T,T])$, $H^2(t), H^3(t) \in L^1([-T,T])$ we denote with

$$L_\varepsilon(H^1,H^2,H^3) \in C([-T,T],W)$$

the unique function $z(x,t)$ that satisfies equation (35) and $z(x,-T) = z(x,T)$ a.e. It is obvious that $L_\varepsilon(H^1,H^2,H^3)$ is a linear function from $L^1([0,1]\times[-T,T]) \times L^1([-T,T]) \times L^1([-T,T])$ into $C([-T,T],W)$.

We have the following

Proposition 1. *For any given triple $(H^1(x,t),H^2(t),H^3(t)) \in L^1([0,1]\times[-T,T]) \times L^1([-T,T]) \times L^1([-T,T])$ with T as in* (32)*, equation* (35) *has a unique solution $z(x,t) \in C([-T,T],W)$ such that $z(x,-T) = z(x,T)$ a.e. Moreover $z(x,t)$ has the form*

$$z(x,t) = \sum_{j=1}^{\infty} z_j(t)w_j(x)$$

$z_j(t)$ being the unique $2T-$periodic solution of equation (31)*; furthermore $z(x,t)$ satisfies the estimate* (40)*. Finally, there exist positive constants $c_{1,\beta,\theta}$, $c_{2,\beta,\theta}$ such that if $(H^1(x,t),H^2(t),H^3(t)) \in L^\infty([0,1]\times[-T,T]) \times L^\infty([-T,T]) \times L^\infty([-T,T])$ the following hold:*

a) *if $H^1_t(x,t) \in L^1([0,1]\times[-T,T])$ and $\dot{H}^2(t), \dot{H}^3(t) \in L^1([-T,T])$, then*

$$\begin{aligned}\|z(\cdot,t)\|_2 \le\ & M_1\varepsilon\Big\{c_{1,\beta,\theta}[\|H^1\|_\infty + \|H^2\|_\infty + \|H^3\|_\infty] \\ & +c_{2,\beta,\theta}[\|H^1_t\|_1 + \|\dot{H}^2\|_1 + \|\dot{H}^3\|_1]\Big\}.\end{aligned} \tag{41}$$

b) *if* $H_t^1(x,t) \in L^2([0,1]\times[-T,T])$ *and* $\dot H^2(t), \dot H^3(t) \in L^2([-T,T])$*, then*

$$\begin{aligned}\|z(\cdot,t)\|_2 \leq\ & M_1\varepsilon\Big\{c_{1,\beta,\theta}[\|H^1\|_\infty + \|H^2\|_\infty + \|H^3\|_\infty] \\ & + c_{2,\beta,\theta}(\sqrt{T}+\varepsilon^{1/4})[\|H_t^1\|_2 + \|\dot H^2\|_2 + \|\dot H^3\|_2]\Big\}.\end{aligned} \tag{42}$$

Proof. Only (41) and (42) need to be proved. From the assumptions on $H^1(x,t)$, $H^2(t)$, $H^3(t)$ in a) it follows that $h_j(t)$ defined as in (37) satisfies $h_j(t) \in L^\infty([-T,T])$, $\dot h_j(t) \in L^1([-T,T])$. Moreover

$$\|h_j\|_\infty \leq M_1\left[\|H^1\|_\infty + \|H^2\|_\infty + \|H^3\|_\infty\right]$$

and

$$\|\dot h_j\|_1 \leq M_1\left[\|H_t^1\|_1 + \|\dot H^2\|_1 + \|\dot H^3\|_1\right].$$

Similarly if $H^1(x,t)$, $H^2(t)$, $H^3(t)$ satisfy the conditions in b), then $\dot h_j \in L^2([-T,T])$ and

$$\|\dot h_j\|_2 \leq M_1\left[\|H_t^1\|_2 + \|\dot H^2\|_2 + \|\dot H^3\|_2\right].$$

Thus, in case a), from (30) we get:

$$\begin{aligned}|z_j(t)| \leq\ & \tfrac{M_1\varepsilon}{\mu_j^4}\Big\{\left(1+\tfrac{2j^\beta}{\theta}\right)\left[\|H^1\|_\infty + \|H^2\|_\infty + \|H^3\|_\infty\right] \\ & + \frac{j^\beta}{\theta}\left[\|H_t^1\|_1 + \|\dot H^2\|_1 + \|\dot H^3\|_1\right]\Big\}\end{aligned}$$

and, since $\mu_j > 2j$, we find:

$$\begin{aligned}\sum_{j=1}^\infty |z_j(t)|^2 \leq\ & 2M_1^2\varepsilon^2\Big\{\sum_{j=1}^\infty \mu_j^{-8}\left(1+\frac{2j^\beta}{\theta}\right)^2\left[\|H^1\|_\infty + \|H^2\|_\infty + \|H^3\|_\infty\right]^2 \\ & + \sum_{j=1}^\infty \mu_j^{-8}\frac{j^{2\beta}}{\theta^2}\left[\|H_t^1\|_1 + \|\dot H^2\|_1 + \|\dot H^3\|_1\right]^2\Big\} \\ \leq\ & 4M_1^2\varepsilon^2\Big\{\left[\|H^1\|_\infty + \|H^2\|_\infty + \|H^3\|_\infty\right]^2\sum_{j=1}^\infty\left(\mu_j^{-8} + \frac{1}{\theta^2}\mu_j^{2(\beta-4)}\right) \\ & + \frac{1}{8\theta^2}\left[\|H_t^1\|_1 + \|\dot H^2\|_1 + \|\dot H^3\|_1\right]^2\sum_{j=1}^\infty \mu_j^{2(\beta-4)}\Big\} \\ \leq\ & M_1^2\varepsilon^2\Big\{c_{1,\beta,\theta}\left[\|H^1\|_\infty + \|H^2\|_\infty + \|H^3\|_\infty\right] \\ & + c_{2,\beta,\theta}\left[\|H_t^1\|_1 + \|\dot H^2\|_1 + \|\dot H^3\|_1\right]\Big\}^2\end{aligned}$$

with

$$c_{1,\beta,\theta} = 2\sqrt{\sum_{j=1}^\infty\left(\mu_j^{-8} + \frac{1}{\theta^2}\mu_j^{2(\beta-4)}\right)}, \qquad c_{2,\beta,\theta} = \frac{1}{\sqrt{2}\theta}\sqrt{\sum_{j=1}^\infty \mu_j^{2(\beta-4)}}. \tag{43}$$

The proof of the statement in b) is quite similar. We only have to note that

$$\begin{aligned}\left|\int_{t-T}^{t+T}\dot h(s+T)\sin\omega(t-s)ds\right| &\le \|\dot h\|_2\left(\int_{t-T}^{t+T}\sin^2\omega(t-s)ds\right)^{1/2}\\ &\le \left(T-\frac{\sin\omega T\cos\omega T}{\omega}\right)^{1/2}\|\dot h\|_2\ .\end{aligned}$$

Hence, using (29), we get

$$\begin{aligned}|z_j(t)| \le &\frac{1}{\omega_j^2}\Bigg[(1+|\sin\omega_j T|^{-1})\|h_j\|_\infty\\ &\qquad+\frac{1}{2|\sin\omega_j T|}\left(T-\frac{\sin\omega_j T\cos\omega_j T}{\omega_j}\right)^{1/2}\|\dot h_j\|_2\Bigg]\\ \le&\frac{1}{\omega_j^2}\left[(1+2j^\beta\theta^{-1})\|h_j\|_\infty+\frac{1}{2|\sin\omega_j T|}\left(\sqrt{T}+\sqrt{\frac{|\sin\omega_j T\cos\omega_j T|}{\omega_j}}\right)\|\dot h_j\|_2\right]\\ \le&\frac{\varepsilon}{\mu_j^4}\left[(1+2j^\beta\theta^{-1})\|h_j\|_\infty+\left(\frac{\sqrt{T}}{\theta}j^\beta+\frac{\varepsilon^{1/4}}{2\mu_j}\sqrt{\frac{|\cos\omega_j T|}{|\sin\omega_j T|}}\right)\|\dot h_j\|_2\right]\\ \le&\frac{\varepsilon}{\mu_j^4}\left[(1+2j^\beta\theta^{-1})\|h_j\|_\infty+\left(\frac{\sqrt{T}}{\theta}j^\beta+\frac{\varepsilon^{1/4}}{\sqrt{2}\mu_j}\sqrt{\frac{j^\beta}{\theta}}\right)\|\dot h_j\|_2\right]\end{aligned}$$

The conclusion now easily follows, so the proof is complete. □

Next, to study the problem of existence of $2T-$periodic solutions of equations (10)–(11) we consider the problem of existence of bounded solutions (on $[-T,T]$) of equation (20) that satisfy the boundary condition

$$y(T)-y(-T)=b(T):=-2\begin{pmatrix}0\\ \dot\Gamma_1(T)\\ 0\\ \dot\Gamma_2(T)\end{pmatrix} \tag{44}$$

To this end we need the following two Lemmas, the first being a slight variation of a result proved in [11] (see also [3]).

Lemma 1. *Let $A\in C(\mathbb{R},\mathbf{M}(n))$, where $\mathbf{M}(n)$ be the set of all real $(n\times n)$-matrices. Assume that the linear system $\dot x+A(t)x=0$ has an exponential dichotomy on $\mathbb{R}_+$ and $\mathbb{R}_-$ with constant k and exponent δ, projections P_+ and P_- respectively. Let $P_\pm(T)=X(t)P_\pm X(t)^{-1}$, $X(t)$ being the fundamental solution of $\dot x+A(t)x=0$ such that $X(0)=\mathbb{I}$. Assume that $T_0>0$ exists such that the following hold:*

i) *for any $T>T_0>0$ one has $\mathbb{R}^n=\mathcal{N}P_+(T)\oplus\mathcal{R}P_-(T)$.*

ii) *Let $R_T:\mathbb{R}^n\to\mathbb{R}^n$ be the projection on $\mathbb{R}^n$ such that $\mathcal{R}R_T=\mathcal{R}P_-(T)$ and $\mathcal{N}R_T=\mathcal{N}P_+(T)$. Then $\|R_T\|$ is bounded uniformly with respect to T for $T>T_0$.*

Then, for any $T > T_0$ *and* $(\xi, \eta) \in \mathcal{R}P_+ \times \mathcal{N}P_-$, $h(t) \in L^\infty([-T,T],\mathbb{R}^n)$ *and* $b \in \mathbb{R}^n$, *there exists a unique function* $x(t) = x(t,\xi,\eta,h,b,T) \in W^{1,1}([-T,0]) \cap W^{1,1}([0,T])$ *that satisfies, for almost all* $t \in [-T,T]$, *the equation*

$$\dot{x} + A(t)x = h(t)$$

together with the boundary conditions

$$\begin{aligned} &x(T) - x(-T) = b \\ &x(0^+) = \xi + X(T)^{-1}\varphi_+ - \int_0^T (\mathbb{I} - P_+)X(s)^{-1}h(s)ds \\ &x(0^-) = \eta + X(-T)^{-1}\varphi_- + \int_{-T}^0 P_-X(s)^{-1}h(s)ds \end{aligned} \tag{45}$$

where $(\varphi_+, \varphi_-) \in \mathcal{N}P_+(T) \times \mathcal{R}P_-(-T)$ *is the (unique) solution of the equation*

$$\begin{aligned} \varphi_- - \varphi_+ = \; & X(T)\xi - X(-T)\eta + \int_0^T X(T)P_+X(s)^{-1}h(s)ds \\ & + \int_{-T}^0 X(-T)(\mathbb{I} - P_-)X(s)^{-1}h(s)ds - b. \end{aligned} \tag{46}$$

Finally, $x(t,\xi,\eta,h,b,T)$ *is linear in* (ξ,η,h,b) *for any fixed* T *and the following holds:*

$$\|x(\cdot,\xi,\eta,h,b,T)\|_\infty \le k\{C_1[|\xi| + |\eta|] + C_2(\|h\|_\infty + |b|)\}. \tag{47}$$

We note that Lemma 1 has been proved in [3, **?**] under the assumption that $h(t) \in C_b^0([-T,0]) \cap C_b^0([0,T])$, and in this case $x(\cdot,\xi,\eta,h,b,T) \in C^1([-T,0]) \cap C^1([0,T])$. However the same proof goes through under the assumption about $h(t)$ stated in Lemma 1. Of course then $x(\cdot,\xi,\eta,h,b,T)$ only belongs to $W^{1,1}([-T,0]) \cap W^{1,1}([0,T])$.

As an application of Lemma 1 we now show the following result that does not seem to have been noted previously.

Lemma 2. *Let* $A \in C(\mathbb{R}, \mathbf{M}(n))$. *Assume that*

a) $\lim_{t\to\pm\infty} A(t) \to A_0 \in \mathbf{M}(n)$,

b) A_0 *has no purely imaginary eigenvalues,*

c) *the linear equation* $\dot{x} + A(t)x = 0$ *has a one dimensional space of bounded solutions spanned by, say,* $p(t)$.

Then the adjoint linear equation $\dot{x} - A^*(t)x = 0$ *has a one dimensional space of bounded solutions and there exists* $T_0 > 0$ *such that for any* $T \ge T_0$, $b \in \mathbb{R}^n$ *and* $h(t) \in L^\infty([-T,T],\mathbb{R}^n)$ *there exists a unique solution* $x(t) = x(t,h,b,T) \in W^{1,1}([-T,0]) \cap W^{1,1}([0,T])$ *of equation*

$$\dot{x} + A(t)x = h(t) \tag{48}$$

that satisfies

$$\begin{aligned} &x(T) - x(-T) = b \\ &< p(0), x(0^+) >= 0 \\ &x(0^+) - x(0^-) =< \psi(0), x(0^+) - x(0^-) > \psi(0) \end{aligned} \tag{49}$$

$\psi(t)$ being the unique bounded solution of the adjoint linear equation with $|\psi(0)| = 1$. Moreover there exists a constant $\hat{C}$ independent of T such that

$$\|x(\cdot, h, b, T)\|_\infty \leq \hat{C}(\|h\|_\infty + |b|). \tag{50}$$

Proof. By assumption b) the linear equation $\dot{x} + A_0 x = 0$ has an exponential dichotomy on $\mathbb{R}$ with projection P_0, (P_0 is the projection onto the stable space of A_0 along the unstable space). Then, by roughness, $\dot{x} + A(t)x = 0$ has exponential dichotomies on both $\mathbb{R}_+$ and $\mathbb{R}_-$ and the projections $P_\pm$ can be assumed to satisfy $\lim_{T\to+\infty} P_+(T) = \lim_{T\to+\infty} P_-(-T) = P_0$ (see [12]). In the rest of the proof k and δ will denote the constant and the exponent of the dichotomy of system $\dot{x} = A(t)x$. As a consequence assumptions i) and ii) of Lemma 1 are satisfied. Since the conclusion concerning the adjoint system $\dot{x} - A^*(t)x = 0$ is known (see [12, p. 246]), we only need to prove the last part. Let $x(t, \xi, \eta, h, b, T)$ be the unique solution of (48) that satisfies (45). We show that $(\xi, \eta) \in \mathcal{R}P_+ \times \mathcal{N}P_-$ exist such that (49) holds. From (45) it follows that the second and third conditions in (49) read

$$< p(0), \xi >= -\left\langle p(0), X(T)^{-1}\varphi_+ \right\rangle + \left\langle p(0), \int_0^T (\mathbb{I} - P_+)X(s)^{-1}h(s)ds \right\rangle \tag{51}$$

and

$$\begin{aligned} \xi - \eta = {}& X(-T)^{-1}\varphi_- - X(T)^{-1}\varphi_+ - \psi^*(0)[X(-T)^{-1}\varphi_- - X(T)^{-1}\varphi_+]\psi(0) \\ &+ \int_0^T (\mathbb{I} - P_+)X(s)^{-1}h(s)ds + \int_{-T}^0 P_- X(s)^{-1}h(s)ds - \int_{-T}^T \psi(s)^* h(s)ds\psi(0) \end{aligned} \tag{52}$$

respectively, where we use, according to [12, p. 246], the fact that $\psi(0) \in \mathcal{R}P_+^\perp \cap \mathcal{N}P_-^\perp$, and $\psi(t) = X(t)^{*-1}(\mathbb{I} - P_+^*)\psi(0)$ for $t \geq 0$ and $\psi(t) = X(t)^{*-1}P_-^*\psi(0)$ for $t \leq 0$. Now, from (46) we get

$$|\varphi_\pm| \leq \widetilde{c}\left(ke^{-\delta T}[|\xi| + |\eta|] + 2k\delta^{-1}\|h\|_\infty + |b|\right)$$

and then

$$\begin{aligned} |X(T)^{-1}\varphi_+| &= |X(0)(\mathbb{I} - P_+)X(T)^{-1}\varphi_+| \leq ke^{-\delta T}|\varphi_+| \\ &\leq k^2 e^{-\delta T}\widetilde{c}\left\{[|\xi| + |\eta|]e^{-\delta T} + 2\delta^{-1}\|h\|_\infty + |b|\right\}. \end{aligned} \tag{53}$$

Similarly:

$$|X(-T)^{-1}\varphi_-| \leq k^2 e^{-\delta T}\widetilde{c}\left\{[|\xi| + |\eta|]e^{-\delta T} + 2\delta^{-1}\|h\|_\infty + |b|\right\} \tag{54}$$

for a suitable constant $\widetilde{c} > 0$. Now, the linear map $(\xi, \eta) \to \begin{pmatrix} < p(0), \xi > \\ \xi - \eta \end{pmatrix}$ from $\mathcal{R}P_+ \times \mathcal{N}P_-$ into $\mathbb{R} \times \big(\mathcal{R}P_+ + \mathcal{N}P_-\big)$ is an isomorphism since $\mathcal{R}P_+ \cap \mathcal{N}P_- = \mathrm{span}\{p(0)\}$ and both spaces have the same dimension. Thus, because of (53) and (54) we see that (51) and (52) can be written as

$$\mathcal{L}(\xi, \eta) = r(h, b) \in \mathbb{R} \times \big(\mathcal{R}P_+ + \mathcal{N}P_-\big)$$

where $\mathcal{L}$ is an isomorphism provided T is large enough. Hence there exists a unique $(\xi, \eta) = (\xi(h, b), \eta(h, b)) \in \mathcal{R}P_+ + \mathcal{N}P_-$ satisfying (51) and (52). Moreover, since $\|r(h, b)\| \le C_0(\|h\| + |b|)$, with a constant C_0 independent of T, we can find a constant C independent of T such that

$$|\xi|, |\eta| \le C\big(\|h\|_\infty + |b|\big).$$

Hence (50) easily follows from this and (47). The proof is complete.

4. Existence of periodic solutions

In this section, we prove the existence of $2T/\sqrt{\varepsilon}$-periodic solutions of (1) (Theorem 1). Using the results of the previous sections, we rewrite the periodic problem for (1) as a fixed point problem (see equations (75)–(76)) and solve this last applying a fixed point result (Lemma 3). Finally, we establish the desired $2T/\sqrt{\varepsilon}$-periodic solutions using some symmetry properties of (1).

Let W be the space defined in (34). We set $Z := \{z \in C([-T, T], W) \mid z(-T, x) = z(T, x) \quad a.e.\}$ and $Y = C^1([-T, 0], \mathbb{R}^2) \cap C^1([0, T], \mathbb{R}^2)$ with the norm

$$\|y\| = \sup_{|t| \le T, t \ne 0} \{|y_1(t)|, |\dot{y}_1(t)|\} + k_\varphi \sup_{|t| \le T, t \ne 0} \{|y_2(t)|, |\dot{y}_2(t)|\},$$

Unless otherwise specified, $y(t)$ will denote a function in Y, endowed with this norm. In Z, instead, we consider the norm

$$\|z\| = \|z(x, t)\| = \|\varphi\|_2 \sup_{t \in [-T, T]} \|z(\cdot, t)\|_2.$$

Next, let $\rho > 0$ be a fixed positive number, and take $y(t) \in Y$ and $z(x, t) \in Z$

in such a way that $\|y\| + \|z\| \leq \rho$. For any fixed choice of such functions we set:

$$\begin{aligned} H_1(x,t) =& \ \mu h(x,t) \\ H_2(t) =& \ f([y_1(t) + \Gamma_1(t)] - k_\varphi[y_2(t) + \Gamma_2(t)] + [z(\cdot,t) * \hat{\varphi}](0)) \\ & -f(\Gamma_1(t) - k_\varphi\Gamma_2(t)) - f'(\Gamma_1(t) - k_\varphi\Gamma_2(t)) \cdot \\ & [y_1(t) - k_\varphi y_2(t) + [z(\cdot,t) * \hat{\varphi}](0)] \\ H_3(t) =& \ g([y_1(t) + \Gamma_1(t)] + k_\varphi[y_2(t) + \Gamma_2(t)] + [z(\cdot,t) * \varphi](1)) \\ & -g(\Gamma_1(t) + k_\varphi\Gamma_2(t)) - g'(\Gamma_1(t) + k_\varphi\Gamma_2(t)) \cdot \\ & [y_1(t) + k_\varphi y_2(t) + [z(\cdot,t) * \varphi](1)] \\ \hat{H}_{21}(t) =& \ f(\Gamma_1(t) - k_\varphi\Gamma_2(t)) \\ \hat{H}_{22}(t) =& \ [f'(\Gamma_1(t) - k_\varphi\Gamma_2(t)) - f'(0)][y_1(t) - k_\varphi y_2(t) + [z(\cdot,t) * \hat{\varphi}](0)] \\ \hat{H}_{31}(t) =& \ g(\Gamma_1(t) + k_\varphi\Gamma_2(t)) \\ \hat{H}_{32}(t) =& \ [g'(\Gamma_1(t) + k_\varphi\Gamma_2(t)) - g'(0)][y_1(t) + k_\varphi y_2(t) + [z(\cdot,t) * \varphi](1)] \\ \hat{H}_2(t) =& \ \hat{H}_{21}(t) + \hat{H}_{22}(t), \qquad \hat{H}_3(t) = \hat{H}_{31}(t) + \hat{H}_{32}(t) \\ \widetilde{H}_{21}(t) =& \ f'(0)[y_1(t) - k_\varphi y_2(t)]\,, \quad \widetilde{H}_{22}(t) = f'(0)[z(\cdot,t) * \hat{\varphi}](0) \\ \widetilde{H}_{31}(t) =& \ g'(0)[y_1(t) + k_\varphi y_2(t)]\,, \quad \widetilde{H}_{32}(t) = g'(0)[z(\cdot,t) * \varphi](1)\,. \end{aligned} \tag{55}$$

It is not difficult to see that $H_1(x,t) \in L^1([0,1] \times [-T,T])$, and, for any $z(x,t) \in Z$, $H_j, \hat{H}_j, \widetilde{H}_{ij} \in L^1([-T,T])$. In fact denote by $\Omega^\pm$ a neighborhood of $\Gamma_1(t) \pm k_\varphi\Gamma_2(t)$ respectively and set

$$\begin{aligned} & M_f := \sup\{|f(x)|,\, |f'(x)| \mid x \in \Omega^-\}, \\ & M_g := \sup\{|g(x)|,\, |g'(x)| \mid x \in \Omega^+\}, \\ & \Gamma = \int_{-\infty}^{\infty} \Big(|\dot{\Gamma}_1(t)| + k_\varphi|\dot{\Gamma}_2(t)|\Big)\, dt \\ & \Delta_f(\rho) = \sup_{|y|+|z| \leq \rho} |f'(y_1 + \Gamma_1(t) + y_2 - k_\varphi\Gamma_2(t) + z) - f'(\Gamma_1(t) - k_\varphi\Gamma_1(t))| \\ & \Delta_g(\rho) = \sup_{|y|+|z| \leq \rho} |g'(y_1 + \Gamma_1(t) + y_2 + k_\varphi\Gamma_2(t) + z) - f'(\Gamma_1(t) + k_\varphi\Gamma_1(t))|. \end{aligned}$$

Then, noting that

$$|z(\cdot,t) * \hat{\varphi}(0)| \leq \int_0^1 |z(x,t)\varphi(x)| dx \leq \|\varphi\|_2 \|z(\cdot,t)\|_2 \leq \|z(x,t)\|$$

and similarly:

$$|z(\cdot,t) * \varphi(1)| \leq \|z(x,t)\|,$$

we obtain:

$$\begin{aligned}
&\|H_1\|_1 \le |\mu|\,\|h\|_1 \le 2|\mu|T\,\|h\|_\infty, \quad \|H_1\|_\infty \le |\mu|\,\|h\|_\infty\\
&\|H_{1,t}\|_1 \le |\mu|\,\|h_t\|_1 \le 2|\mu|T\,\|h_t\|_\infty\\
&\|H_2\|_1 \le 2T\Delta_f(\rho)[\|y\| + \|z\|]\\
&\|H_3\|_1 \le 2T\Delta_g(\rho)[\|y\| + \|z\|]\\
&\|\hat{H}_{21}\|_\infty \le M_f, \quad \|\hat{H}_{31}\|_\infty \le M_g\\
&\left\|\frac{d\hat{H}_{21}}{dt}\right\|_1 \le \Gamma M_f, \quad \left\|\frac{d\hat{H}_{31}}{dt}\right\|_1 \le \Gamma M_g \qquad (56)\\
&\|\hat{H}_{22}\|_1 \le 2T(M_f + |f'(0)|)[\|y\| + \|z\|]\\
&\|\hat{H}_{32}\|_1 \le 2T(M_g + |g'(0)|)[\|y\| + \|z\|]\\
&\|\widetilde{H}_{21}\|_\infty \le |f'(0)|\|y\|, \quad \|\widetilde{H}_{31}\|_\infty \le |g'(0)|\|y\|,\\
&\|\widetilde{H}_{21}\|_1, \|\dot{\widetilde{H}}_{21}\|_1 \le 2T|f'(0)|\|y\|, \quad \|\widetilde{H}_{31}\|_1, \|\dot{\widetilde{H}}_{31}\|_1 \le 2T|g'(0)|\|y\|,\\
&\|\widetilde{H}_{22}\|_1 \le 2T|f'(0)|\|z\| \quad \|\widetilde{H}_{32}\|_1 \le 2T|g'(0)|\|z\|\,.
\end{aligned}$$

Then, for fixed $(y_1, y_2) \in Y$, $z(x,t) \in Z$ let $\hat{z}(x,t)$ be the unique solution, given by Proposition 1, of equation (35) where we set $H^1 = H_1$, and $H^i = H_i + \hat{H}_i + \widetilde{H}_{i1} + \widetilde{H}_{i2}$ for $i = 2, 3$. More precisely

$$\hat{z}(x,t) = L_\varepsilon(H_1, H_2, H_3) + L_\varepsilon(0, \hat{H}_2, \hat{H}_3) + L_\varepsilon(0, \widetilde{H}_{21}, \widetilde{H}_{31}) + L_\varepsilon(0, \widetilde{H}_{22}, \widetilde{H}_{32})\,. \quad (57)$$

Let $B_{Z\times Y}(\rho)$ be the ball of radius ρ in $Z \times Y$ with the norm $\|z\| + \|y\|$. We set

$$\begin{aligned}
&\mathcal{F}_1(z, y, \mu, T, \varepsilon) := L_\varepsilon(H_1, H_2, H_3) + L_\varepsilon(0, \hat{H}_2, \hat{H}_3)\,,\\
&L_{1\varepsilon}(y) := L_\varepsilon(0, \widetilde{H}_{21}, \widetilde{H}_{31})\,, \qquad L_{2\varepsilon}(z) := L_\varepsilon(0, \widetilde{H}_{22}, \widetilde{H}_{32}),
\end{aligned}$$

We consider $\mathcal{F}_1(z, y, \mu, T, \varepsilon)$ as a map from $B_{Z\times Y}(\rho) \times \mathbb{R} \times \mathbb{R}_{\theta,\beta,\varepsilon} \to Z$ where $\mathbb{R}_{\theta,\beta,\varepsilon} = \{(T, \varepsilon) \in \mathbb{R}^2 \mid T \ge 1,\ \varepsilon > 0, \text{and } T \text{ satisfies } (32)\}$.

We have the following result.

Proposition 2. Let $(T, \varepsilon) \in \mathbb{R}_{\theta,\beta,\varepsilon}$ and $\Delta(\rho) = \max\{\Delta_f(\rho), \Delta_g(\rho)\}$. Then there exist positive constants k_1 that depends on $\{\|h\|_\infty, \|h_t\|_\infty, M_1, M_f,\ M_g, \beta, \theta\}$ and $k_2, \ldots, k_5$, that depend on $\{|f'(0)|, |g'(0)|, M_1, M_f, M_g, \beta, \theta\}$ such that the following holds:

i)

$$\|\mathcal{F}_1(z, y, \mu, T, \varepsilon)\| \le k_1\varepsilon(1 + |\mu|T) + k_2T\sqrt{\varepsilon}[\Delta(\rho) + k_3](\|y\| + \|z\|), \quad (58)$$

ii)

$$\|L_{1\varepsilon}\| \le k_4T\varepsilon, \quad \|L_{2\varepsilon}\| \le k_5T\sqrt{\varepsilon}, \quad (59)$$

iii)

$$\begin{aligned}
\|\mathcal{F}_1(z', y', \mu, T, \varepsilon) &- \mathcal{F}_1(z'', y'', \mu, T, \varepsilon)\|\\
&\le k_2T\sqrt{\varepsilon}[\Delta(\rho) + k_3](\|y' - y''\| + \|z' - z''\|)
\end{aligned} \quad (60)$$

Proof. From Proposition 1, (56), and $(T,\varepsilon) \in \mathbb{R}_{\theta,\beta,\varepsilon}$ we get

$$\begin{aligned}
&\|L_\varepsilon(H_1,0,0)\| \le 2M_1|\mu|T\varepsilon[c_{1,\beta,\theta}\,\|h\|_\infty + c_{2,\beta,\theta}\|h_t\|_\infty]\\
&\|L_\varepsilon(0,\hat H_{21},\hat H_{31})\| \le M_1(M_f+M_g)\varepsilon[c_{1,\beta,\theta}+c_{2,\beta,\theta}\Gamma]\\
&\|L_\varepsilon(0,\hat H_{22},\hat H_{32})\| \le 2C_{\beta,\theta}M_1T\sqrt{\varepsilon}(M_f+M_g+|f'(0)|+|g'(0)|)(\|y\|+\|z\|)
\end{aligned}$$

and

$$\|L_\varepsilon(0,H_2,H_3)\| \le 4C_{\beta,\theta}M_1T\sqrt{\varepsilon}\Delta(\rho)(\|y\|+\|z\|)$$

Thus (58) follows taking

$$\begin{aligned}
&k_1 = M_1 \max\big\{2[c_{1,\beta,\theta}\,\|h\|_\infty + c_{2,\beta,\theta}\|h_t\|_\infty], (M_f+M_g)[c_{1,\beta,\theta}+c_{2,\beta,\theta}\Gamma]\big\}\\
&k_2 = 4C_{\beta,\theta}M_1\\
&k_3 = \tfrac{1}{2}[M_f+M_g+|f'(0)|+|g'(0)|].
\end{aligned}$$

Similarly, (59) follows from Proposition 1, (56) and $(T,\varepsilon) \in \mathbb{R}_{\theta,\beta,\varepsilon}$, taking

$$k_4 = M_1(|f'(0)|+|g'(0)|)(c_{1,\beta,\theta}+2c_{2,\beta,\theta}), \quad k_5 = 2C_{\beta,\theta}M_1[|f'(0)|+|g'(0)|]\,.$$

Finally we prove iii). Let $H_1'(x,t)$, $H_1''(x,t)$ etc. be the functions we have defined in (55) with (z',y',μ) and (z'',y'',μ) instead of (z,y,μ). Then from

$$\begin{aligned}
&\|H_2'-H_2''\|_1 \le 2T\Delta_f(\rho)[\|y'-y''\|+\|z'-z''\|]\\
&\|H_3'-H_3''\|_1 \le 2T\Delta_g(\rho)[\|y'-y''\|+\|z'-z''\|]\\
&\|\hat H_{22}'-\hat H_{22}''\|_1 \le 2T[M_f+|f'(0)|][\|y'-y''\|+\|z'-z''\|]\\
&\|\hat H_{32}'-\hat H_{32}''\|_1 \le 2T[M_g+|g'(0)|][\|y'-y''\|+\|z'-z''\|]
\end{aligned}$$

and (40) we obtain

$$\begin{aligned}
&\|L_\varepsilon(0,H_2'-H_2'',H_3'-H_3'')\| \le k_2\Delta(\rho)T\sqrt{\varepsilon}[\|y'-y''\|+\|z'-z''\|]\\
&\|L_\varepsilon(0,\hat H_{22}'-\hat H_{22}'',\hat H_{32}'-\hat H_{32}'')\| \le k_2k_3T\sqrt{\varepsilon}[\|y'-y''\|+\|z'-z''\|].
\end{aligned}$$

Inequality (60) now easily follows and the proof is complete.

Now we study the problem of existence of bounded solution to equation (20) with the boundary conditions (44). To this end, it is better to split the problem in two parts. We set

$$\gamma(t) = (\Gamma_1(t),\dot\Gamma_1(t),\Gamma_2(t),\dot\Gamma_2(t))$$

and, for any given $y(t) \in Y$, $z(x,t) \in Z$:

$$h_1(t) = H(t)\begin{pmatrix} [z(\cdot,t)*\hat\varphi](0)\\ \\ [z(\cdot,t)*\varphi](1)\end{pmatrix}$$

$$h_2(t) = F(t,y(t),[z(\cdot,t)*\hat\varphi](0),[z(\cdot,t)*\varphi](1),\mu,\varepsilon) - h_1(t)$$

where $F(t, y, z, \mu, \varepsilon)$ is defined by (22), and $H(t)$ is the $(4 \times 2)-$matrix

$$H(t) = -\begin{pmatrix} 0 & 0 \\ f'(\Gamma_1(t) - k_\varphi \Gamma_2(t)) & g'(\Gamma_1(t) + k_\varphi \Gamma_2(t)) \\ 0 & 0 \\ -\sqrt{3} f'(\Gamma_1(t) - k_\varphi \Gamma_2(t)) & \sqrt{3} g'(\Gamma_1(t) + k_\varphi \Gamma_2(t)) \end{pmatrix}.$$

Then consider the boundary value problems

$$\begin{cases} \dot{y} + A(t)y = h_2(t) \\ y(T) - y(-T) = b(T) \\ < \dot{\gamma}(0), y(0^+) >= 0 \\ y(0^+) - y(0^-) =< \psi(0), y(0^+) - y(0^-) > \psi(0) \end{cases} \tag{61}$$

with $b(T)$ as in (44) and $A(t)$ as in (17), and

$$\begin{cases} \dot{y} + A(t)y = h_1(t) \\ y(T) - y(-T) = 0 \\ < \dot{\gamma}(0), y(0^+) >= 0 \\ y(0^+) - y(0^-) =< \psi(0), y(0^+) - y(0^-) > \psi(0) \end{cases} \tag{62}$$

Note that

$$\lim_{t \to \pm\infty} A(t) = -A$$

where A has been defined in (15). Thus the hypotheses of Lemma 2 are satisfied (see also H2)) and we can solve equations (61) and (62) for

$$\hat{y}_b(t) = \mathcal{F}_2(z(x,t), y(t), \mu, T, \varepsilon) \in Y \tag{63}$$

and

$$\hat{y}_0(t) = Lz(x,t) \in Y \tag{64}$$

respectively. We consider $\mathcal{F}_2$ as a map from $B_{Z \times Y}(\rho) \times \mathbb{R} \times [T_0, \infty) \times (0, \infty)$ into Y, and L as a linear map from Z to Y. In the next Proposition we use the following notation: if $x \in C(I, \mathbb{R}^n)$ where I is an interval, we denote by $\|x\|_\infty = \sup_{x \in I} \sqrt{\sum_{j=1}^n |x_j(t)|^2}$.

We have the following result.

Proposition 3. *Assume the conditions* H1) *and* H2) *are satisfied and let* $\Delta(\rho)$ *be the function defined in Proposition* 2, M_1 *the constant defined in* (39) *and* $c_{1,\beta,\theta}$, $c_{2,\beta,\theta}$ *those defined in* (43). *Then there exist constants* k *and* $\hat{C} > 0$ *independent of* $(z(x,t), y(t), \mu, T, \varepsilon)$, *such that the following properties hold:*

i)

$$\|\mathcal{F}_2(z(x,t), y(t), \mu, T, \varepsilon)\| \le \hat{C}\Big\{ke^{-\hat{\delta}T} + 2[\Delta(\rho)(\|y\| + \|z\|) + |\mu| \|h\|_\infty]\Big\},$$

$\hat{\delta}$ *being the exponent of the dichotomy of* (18);

ii)

$$\|L\| \le 2\hat{C}(k_f + k_g)$$

where

$$k_f = \sup_{t\in\mathbb{R}} |f'(\Gamma_1(t) - k_\varphi \Gamma_2(t))|\,, \quad k_g = \sup_{t\in\mathbb{R}} |g'(\Gamma_1(t) + k_\varphi \Gamma_2(t))|\,.$$

iii)

$$\begin{aligned}&\|L_{1\varepsilon}\mathcal{F}_2(z(x,t), y(t), \mu, T, \varepsilon)\| \le \\ &M_1\varepsilon(|f'(0)| + |g'(0)|)[c_{1,\beta,\theta} + c_{2,\beta,\theta}\sqrt{2T}(\sqrt{T} + \varepsilon^{1/4})] \\ &\hat{C}\Big\{ke^{-\hat{\delta}T} + 2\Big(\Delta(\rho)[\|y\| + \|z\|] + |\mu|\|h\|_\infty\Big)\Big\}\end{aligned}$$

iv) *for any pair* $(z'(t), y'(x,t))$, $(z''(t), y''(x,t)) \in B_{Z\times Y}(\rho)$ *it results*

$$\begin{aligned}&\|\mathcal{F}_2(z'(x,t), y'(t), \mu, T, \varepsilon) - \mathcal{F}_2(z''(x,t), y''(t), \mu, T, \varepsilon)\| \le \\ &2\hat{C}\Delta(\rho)[\|y' - y''\| + \|z' - z''\|]\end{aligned}$$

Proof. Let $(z, y) \in B_{Z\times Y}(\rho)$; from (50) in Lemma 2 we obtain:

$$\begin{aligned}&\|\hat{y}_b(t)\| \le \hat{C}[\|h_2\|_\infty + |b(T)|]\,, \\ &\|\hat{y}_0(t)\| \le \hat{C}\|h_1\|_\infty\,.\end{aligned} \tag{65}$$

Now, using the definition (44) of $b(T)$ we obtain

$$|b(T)| \le 2|\gamma(T)| = 2|Y(T)P_+Y^{-1}(0)\gamma(0)| \le 2\hat{k}e^{-\hat{\delta}T}|\gamma(0)| = 2\hat{k}e^{-\hat{\delta}T}|\Gamma(0)|$$

(we recall that $\hat{k}$, $\hat{\delta}$ and P_+ are the constant, the exponent and the projection of the dichotomy of system $\dot{y} = A(t)y$ where $A(t)$ is the matrix defined in (17)) and it is easy to see that:

$$\begin{aligned}\|h_2(\cdot)\|_\infty &\le 2\,[\Delta(\rho)(\|y\| + \|z\|) + |\mu|\|h\|_\infty] \\ \|h_1\|_\infty &\le 2(k_f + k_g)\|z\|\end{aligned} \tag{66}$$

Hence i) and ii) follow from (65), (66) with $k = 2\hat{k}|\Gamma(0)|$.

Next, if $\mu \in \mathbb{R}$, $(z'(x,t), y'(t))$, $(z''(x,t), y''(t)) \in B_{Z\times Y}(\rho)$, and $\hat{y}'_b(t)$, $\hat{y}''_b(t) \in Y$ denote the corresponding solutions of (61), we see that $\hat{y}_b(t) = \hat{y}'_b(t) - \hat{y}''_b(t)$ is a bounded solution of the boundary value problem

$$\begin{cases} \dot{y} + A(t)y = h'_2(t) - h''_2(t) \\ y(T) - y(-T) = 0 \\ < \dot{\gamma}(0), y(0^+) >= 0 \\ y(0^+) - y(0^-) =< \psi(0), y(0^+) - y(0^-) > \psi(0) \end{cases} \tag{67}$$

the meaning of $h'_2(t)$ and $h''_2(t)$ being obvious. Hence by a similar argument we see that

$$\|\hat{y}'_b(t) - \hat{y}''_b(t)\| \le 2\hat{C}\,\{\Delta(\rho)[\|y' - y''\| + \|z' - z''\|]\} \tag{68}$$

that gives iv). Finally we prove iii). We have

$$L_{1\varepsilon}\mathcal{F}_2(y(t),z(x,t),\mu,T,\varepsilon)=L_\varepsilon(0,\bar H_2,\bar H_3)$$

where

$$\bar H_2(t)=f'(0)[y_1^{(b)}(t)-k_\varphi y_2^{(b)}(t)]$$
$$\bar H_3(t)=g'(0)[y_1^{(b)}(t)+k_\varphi y_2^{(b)}(t)]$$

where we set $\hat y_b(t)=(y_1^{(b)}(t),\dot y_1^{(b)}(t),y_2^{(b)}(t),\dot y_2^{(b)}(t))$. Now:

$$\|\bar H_2(t)\|_\infty\le|f'(0)|\|\hat y_b\|,\quad \|\bar H_3(t)\|_\infty\le|g'(0)|\|\hat y_b\|$$

and

$$\|\dot{\bar H}_2(t)\|_2\le\sqrt{2T}|f'(0)|\|\hat y_b\|,\quad \|\dot{\bar H}_3(t)\|_2\le\sqrt{2T}|g'(0)|\|\hat y_b\|.$$

As a consequence, from Proposition 1-b) we get

$$\begin{aligned}&\|L_{1\varepsilon}\mathcal{F}_2(y(t),z(x,t),\mu,T,\varepsilon)\|\\&\qquad\le M_1\varepsilon(|f'(0)|+|g'(0)|)[c_{1,\beta,\theta}+c_{2,\beta,\theta}\sqrt{2T}(\sqrt{T}+\varepsilon^{1/4})]\|\hat y_b\|\end{aligned}$$

Thus, using Proposition 3-i) and $\hat y_b=\mathcal{F}_2(z(x,t),y(t),\mu,T,\varepsilon)$ we see that:

$$\begin{aligned}\|L_{1\varepsilon}\mathcal{F}_2\|\le\ &M_1\hat C\varepsilon(|f'(0)|+|g'(0)|)[c_{1,\beta,\theta}+c_{2,\beta,\theta}\sqrt{2T}(\sqrt{T}+\varepsilon^{1/4})]\cdot\\&\Big\{ke^{-\hat\delta T}+2\Big[\Delta(\rho)(\|y\|+\|z\|)+|\mu|\|h\|_\infty\Big]\Big\}\end{aligned}$$

which is iii). The proof is complete. $\square$

Now we need a fixed point Lemma, which has been essentially proved in [2, Lemma 6].

Lemma 3. *Let Z, Y be Banach spaces, $B_{Z\times Y}(\rho)$ the closed ball in $Z\times Y$ centered at $(0,0)$ and of radius ρ, $\mathcal{O}\subset\mathbb{R}^m\times(0,\bar\sigma]$ a subset with $(0,0)\in\overline{\mathcal{O}}$, and $F: B_{Z\times Y}(\rho)\times\mathcal{O}\to Z\times Y$ be a map defined as:*

$$F(z,y,\nu,\sigma)=\begin{pmatrix}F_1(z,y,\nu,\sigma)+L_{1\sigma}y+L_{2\sigma}z\\F_2(z,y,\nu,\sigma)+Lz\end{pmatrix},$$

$L_{1\sigma}:Y\to Z$, $L_{2\sigma}:Z\to Z$ and $L:Z\to Y$ being uniformly bounded linear maps for $\sigma>0$ small. Assume that a constant C and a continuous function $\Delta(\rho,\nu,\sigma)$ exist such that $\Delta(0,0,0)=0$, and

$$\begin{aligned}&\|F_1(z,y,\nu,\sigma)\|\le C(\|\nu\|+\sigma)\sigma+\Delta(\rho,\nu,\sigma)(\|z\|+\|y\|)\,,\\&\|F_2(z,y,\nu,\sigma)\|\le C\|\nu\|+\Delta(\rho,\nu,\sigma)(\|z\|+\|y\|)\,,\\&\|L_{1\sigma}F_2(z,y,\nu,\sigma)\|\le C(\|\nu\|+\sigma)\sigma+\Delta(\rho,\nu,\sigma)(\|z\|+\|y\|)\\&\|F_i(z_2,y_2,\nu,\sigma)-F_i(z_1,y_1,\nu,\sigma)\|\le\Delta(\rho,\nu,\sigma)(\|z_2-z_1\|+\|y_2-y_1\|)\end{aligned}\tag{69}$$

when $\|z\|+\|y\|<\rho$, $\|z_1\|+\|y_1\|<\rho$, and $\|z_2\|+\|y_2\|<\rho$.

If there are $0<\lambda<1$ and $\bar\sigma_0>0$ such that

$$\|L_{1\sigma}L+L_{2\sigma}\|<\lambda$$

for any $0 < \sigma \leq \bar{\sigma}_0$, *then there exist* $\nu_0 > 0$, $\sigma_0 > 0$, $\rho_1 > 0$ *and* $\rho_2 > 0$ *such that for* $\|\nu\| \leq \nu_0$, *and* $0 < \sigma \leq \sigma_0$, $(\nu, \sigma) \in \mathcal{O}$, F *has a unique fixed point* $(z(\nu,\sigma), y(\nu,\sigma)) \in B_Z(\rho_1) \times B_Y(\rho_2)$. *Moreover,*

$$\|z(\nu,\sigma)\| + \|y(\nu,\sigma)\| \leq C_1(\|\nu\| + \sigma) \tag{70}$$

for some positive constant C_1 *independent of* (ν,σ), *and*

$$\|z(\nu,\sigma)\|/(\|\nu\| + \sigma) \to 0 \tag{71}$$

as $(\nu,\sigma) \to (0,0)$, $\sigma > 0$.

We emphasize the fact that in [2] the above Lemma has been proved with $m = 1$ (i.e. $\nu = \mu \in \mathbb{R}$). However, it is straightforward to see that the same proof also works when $\nu \in \mathbb{R}^m$, or when ν belongs to a Banach space. Moreover, Lemma 3 can also be extended to the case, where L, $L_{1,\sigma}$ and $L_{2,\sigma}$ also depend on ν but the assumptions of (69) hold uniformly with respect to ν.

Let $\beta \in (1, 3/2)$ and $\theta > 0$ be fixed constants satisfying (89). We set

$$\widetilde{S}_{\beta,\theta,\varepsilon} = \Big\{\chi \in (0, \varepsilon^{-3/4}] \mid |\chi\mu_j^2 - k\pi| \geq \frac{\theta}{j^\beta} \quad \forall (j,k) \in \mathbb{N} \times \mathbb{Z}\Big\}.$$

We are now able to state and prove the main result of this paper.

Theorem 1. *Let* $f(x)$ *and* $g(x)$ *be* C^1-*functions for which* H1) *and* H2) *are satisfied. Then let* $\beta \in (1, 3/2)$ *and* $\theta > 0$ *be fixed constants satisfying* (89), $N > 0$ *be a given constant and* $\hat{\delta}$ *be the exponent of the dichotomy of system* (18) *with* $A(t)$ *defined as in* (17). *Then there exist* $\varepsilon_0 > 0$, $\widetilde{T}_0 > 0$, $\widetilde{T}_0 \leq \varepsilon_0^{-1/4}$, $\mu_0 > 0$ *and a positive constant* $\hat{C}_1 > 0$ *such that for any* $(\varepsilon, \mu) \in (0, \varepsilon_0) \times (-\mu_0, \mu_0)$, *for any* $T \in [\widetilde{T}_0, \varepsilon^{-1/4}]$ *such that* $\frac{T}{\sqrt{\varepsilon}} \in \widetilde{S}_{\beta,\theta,\varepsilon}$ *and any* C^1, $2T-$*periodic function* $h(x,t)$ *such that* $h(x,-t) = h(x,t)$ *with* $\|h\|_\infty, \|h_t\|_\infty \leq N$, *equation* (1) *has a unique weak* $\frac{2T}{\sqrt{\varepsilon}}-$*periodic solution* $u(x,t)$ *such that*

$$u(x, \sqrt{\varepsilon}t) = y_1(t) + \Gamma_1(t) + \sqrt{3}(2x-1)(y_2(t) + \Gamma_2(t)) + z(x,t)$$

with $z(x,t) = \sum_{j=1}^{\infty} z_j(t) w_j(x)$, *and*

$$\|z\| + \|y\| \leq \hat{C}_1(|\mu| + \exp(-\hat{\delta}T) + \sqrt{\varepsilon}) \tag{72}$$

Moreover

$$\sup_{t\in[-T;T]} \|z(\cdot,t)\|_2 = \sqrt{a}\,o(|\mu| + \exp(-\hat{\delta}T) + \sqrt{\varepsilon}) \tag{73}$$

where the norms $\|z\|$ *and* $\|y\|$ *are defined at the beginning of this section,* $\|\cdot\|_2$ *is the usual integral norm on* $L^2([0,1])$, a *is the diameter of the support of* φ *and* $o(|\mu| + \exp(-\hat{\delta}T) + \sqrt{\varepsilon})$ *is independent of* a.

Proof. We want to apply Lemma 3 to our situation, but we need to introduce an extra parameter $\tau \in (0,1]$ to control T. Hence we set $\sigma = \sqrt{\varepsilon}$, $\tau = \exp(-\hat{\delta}T)$,

$\nu = (\mu, \tau)$, $\|\nu\| = \|(\mu, \tau)\| = |\mu| + |\tau|$, and redefine the operators $\mathcal{F}_i$, $L_{i\varepsilon}$ for $i = 1, 2$ and L making the dependence on τ explicit. We set (see (32))

$$\widetilde{S}_{\beta,\theta} = \left\{ \chi > 0 \mid |\chi \mu_j^2 - k\pi| \geq \frac{\theta}{j^\beta} \quad \forall (j,k) \in \mathbb{N} \times \mathbb{Z} \right\}.$$

Then we put

$$\mathcal{O} = \left\{ (\nu, \sigma) \in \mathbb{R}^2 \times (0,1) \,\middle|\, \quad |\mu| < 1,\, \exp(-\hat{\delta}\sigma^{-1/2}) \leq \tau \leq \tau_0,\, -\frac{\ln \tau}{\hat{\delta}\sigma} \in \widetilde{S}_{\beta,\theta} \right\} \tag{74}$$

where $\tau_0 = \exp(-\hat{\delta} T_0)$ and T_0 is from Lemma 2, and we now suppose $T_0 \geq 1$. Note that conditions $T\varepsilon^{1/4} \leq 1$ and $T \geq T_0$ are equivalent to $\exp(-\hat{\delta}\sigma^{-1/2}) \leq \tau \leq \tau_0$, while condition $-\frac{\ln \tau}{\hat{\delta}\sigma} \in \widetilde{S}_{\beta,\theta}$ is equivalent to $T/\sqrt{\varepsilon} \in \widetilde{S}_{\beta,\theta}$. Because of $\widetilde{S}_{\beta,\theta,\varepsilon} = \widetilde{S}_{\beta,\theta} \cap (0, \varepsilon^{-3/4}]$, then under the assumptions of Theorem 1, from (88) and (89) of Appendix 2, we see that $(0,0,0) \in \overline{\mathcal{O}}$ (see also Remark 2 below).

Next, as we have already observed, searching for a weak $\frac{2T}{\sqrt{\varepsilon}}-$periodic solution of (1) is equivalent to the search for a solution of (4) such that $u(x, -T) = u(x, T)$. Writing $u(x,t) = (y_1(t) + \Gamma_1(t)) w_{-1}(x) + (y_2(t) + \Gamma_2(t)) w_0(x) + z(x,t)$, with $z(x,t)$ as in (20)-(21), we see that $u(x,t)$ is a $2T-$periodic solution of (4) if and only if $(z(x,t), y(t)) \in Z \times Y$, where $y(t) = (y_1(t), y_2(t))$, is a fixed point of the system:

$$\begin{cases} z = \mathcal{F}_1(y, z, \nu, T, \varepsilon) + L_{1\varepsilon}(y) + L_{2\varepsilon}(z) \\ y = \mathcal{F}_2(y, z, \nu, T, \varepsilon) + Lz\,, \end{cases} \tag{75}$$

that satisfies also

$$y(0^+) = y(0^-), \quad \dot{y}(0^+) = \dot{y}(0^-) \tag{76}$$

Indeed, the first equation of (75) comes from (21) and (57), while the second one is derived from (20), (63) and (64). First we prove that (75) has a unique solution in $Z \times Y$ whose $y-$component may have a possible jump at $t = 0$. Observe that, from Holder's inequality, we have

$$1 = \|\varphi\|_1 \leq \|\varphi\|_2 \left[\int_0^a ds \right]^{1/2} \leq \sqrt{a} \|\varphi\|_2.$$

Hence

$$\sup_{t \in [-T;T]} \|z(\cdot, t)\|_2 = \|z\| \|\varphi\|_2^{-1} \leq \sqrt{a} \|z\|. \tag{77}$$

Owing to Propositions 2 and 3 we see that the assumptions of Lemma 3 are satisfied with

$$C = \max\{k_1, \hat{C}k, 2\hat{C}N\}$$

and

$$\Delta(\rho, \nu, \sigma) = 2\hat{C}\Delta(\rho) + k_2 \sqrt{\sigma} [\Delta(\rho) + k_3]$$

provided $2\hat{C}k_4\varepsilon^{3/4}(k_f + k_g) + k_5\varepsilon^{1/4} = \lambda < 1$. Thus according to Lemma 3 there are $\nu_0 >$, $\sigma_0 > 0$ such that for any μ, T and ε satisfying

$$\begin{aligned} &|\mu| + \exp(-\hat{\delta}T) \le \nu_0, \quad \sqrt{\varepsilon} \le \sigma_0 \\ &(\mu, \exp(-\hat{\delta}T), \sigma) \in \mathcal{O}, \quad \text{where the set } \mathcal{O} \text{ is defined in (74)}, \end{aligned} \tag{78}$$

we obtain a unique solution of the fixed point equation (75) such that (72) and (73) hold, when (72) is derived from (70), and (73) from (71) and (77). Clearly conditions of (78) are satisfied if

$$\begin{aligned} &|\mu| < \mu_0 := \min\left\{\frac{\nu_0}{2}, 1\right\}, \quad 0 < \varepsilon < \varepsilon_0 := \min\left\{\sigma_0^2, \widetilde{T}_0^{-4}\right\} \\ &\widetilde{T}_0 := \max\left\{T_0, -\frac{\ln(\nu_0/2)}{\hat{\delta}}\right\} \le T \le \varepsilon^{-1/4}, \quad \frac{T}{\sqrt{\varepsilon}} \in \widetilde{S}_{\beta,\theta,\varepsilon}\,. \end{aligned} \tag{79}$$

Consequently, positive constants ε_0, $\widetilde{T}_0$ and μ_0 from the statement of Theorem 1 are established by (79).

We now show that $y(t)$ satisfies (76). We set $\widetilde{z}(x,t) = z(x,-t)$ and $\widetilde{y}(t) = (y_1(-t), -\dot{y}_1(-t), y_2(-t), -\dot{y}_2(-t))$, that is

$$\widetilde{y}(t) = \begin{pmatrix} \widetilde{y}_1(t) \\ \dot{\widetilde{y}}_1(t) \\ \widetilde{y}_2(t) \\ \dot{\widetilde{y}}_2(t) \end{pmatrix} = J \begin{pmatrix} y_1(-t) \\ \dot{y}_1(-t) \\ y_2(-t) \\ \dot{y}_2(-t) \end{pmatrix} = Jy(-t).$$

Then we have, using (24), (25) and $\widetilde{y}_1(t) = y_1(-t)$, $\widetilde{y}_2(t) = y_2(-t)$

$$\begin{aligned} \dot{\widetilde{y}}(t) + A(t)\widetilde{y}(t) &= -J\dot{y}(-t) + A(-t)Jy(-t) = -J[\dot{y}(-t) + A(-t)y(-t)] \\ &= -JF(-t, y_1(-t), y_2(-t), [z(\cdot,-t) * \hat{\varphi}](0), [z(\cdot,-t) * \varphi](1), \mu, \varepsilon) \\ &= F(t, \widetilde{y}_1(t), \widetilde{y}_2(t), [\widetilde{z}(\cdot,t) * \hat{\varphi}](0), [\widetilde{z}(\cdot,t) * \varphi](1), \mu, \varepsilon). \end{aligned}$$

Similarly we see that $(\widetilde{z}(x,t), \widetilde{y}(t))$ satisfies also the second (integral) equation, that is $(\widetilde{z}(x,t), \widetilde{y}(t))$ is another fixed point of equation (75) that satisfies (72), (73). Hence

$$z(x,t) = z(x,-t), \qquad \text{and} \qquad y(t) = Jy(-t).$$

In particular:

$$y_1(t) = y_1(-t), \qquad y_2(t) = y_2(-t) \tag{80}$$

Now, it is not difficult to verify that

$$\psi^*(t) = (-\ddot{\Gamma}_1(t) \quad \dot{\Gamma}_1(t) \quad -\ddot{\Gamma}_2(t) \quad \dot{\Gamma}_2(t))$$

is a bounded solution of the adjoint system $\dot{y} - A^*(t)y = 0$. Hence we can take

$$\psi^*(0) = (-\ddot{\Gamma}_1(0) \quad 0 \quad -\ddot{\Gamma}_2(0) \quad 0).$$

As a consequence:

$$< \psi^*(0), y(0^+) - y(0^-) >= -\ddot{\Gamma}_1(0)[y_1(0^+) - y_1(0^-)] - \ddot{\Gamma}_2(0)[y_2(0^+) - y_2(0^-)]$$

and then, using (80) we get

$$<\psi^*(0), y(0^+) - y(0^-)> = - <\psi^*(0), y(0^+) - y(0^-)>$$

that is $<\psi^*(0), y(0^+) - y(0^-)> = 0$. Thus $y(0^+) = y(0^-)$ (see (61)-(62)). This concludes the proof of Theorem 1. □

Remark 2. The period T satisfies conditions of Theorem 1 if $T \in S_{\theta,\varepsilon}$ where:

$$S_{\theta,\varepsilon} := \left(\sqrt{\varepsilon} S_{5/4,\theta,\varepsilon^{-3/4}}\right) \cap \left[T_0, \varepsilon^{-1/4}\right]$$

where $aB = \{ab \mid b \in B\}$ for any $a \in \mathbb{R}$ and $B \subset \mathbb{R}$.

From (88) in Appendix A2 we get

$$\liminf_{\varepsilon \to 0^+} [m(S_{\theta,\varepsilon})\varepsilon^{1/4}] \geq 1 - \frac{10}{\pi}\theta\,.$$

So for $0 < \theta < \pi/10$ the set of those $T \in [T_0, \varepsilon^{-1/4}]$ satisfying the assumptions of Theorem 1, has a positive measure. Hence, for any $\varepsilon > 0$ sufficiently small, there is a $T \in S_{\theta,\varepsilon} \cap [\frac{1}{2}\left(1 - \frac{10}{\pi}\theta\right)\varepsilon^{-1/4}, \varepsilon^{-1/4}]$.

Remark 3. In particular, when $h(x,t) = 0$, Theorem 1 gives the existence of several layers of free symmetric weak periodic vibrations of (1) for any small $\varepsilon > 0$. Note that in this case the parameter μ does not play any role, so it can be chosen $\mu = 0$. Accumulation of periodic orbits on homoclinic and heteroclinic cycles to hyperbolic equilibria for reversible ordinary differential equations is also studied in [16]. Here we deal with the partial differential equation (1) possessing an infinite dimensional center part and a symmetric homoclinic solution in the first two modes.

5. Appendix 1: Numerical approximations of the eigenvalues

By following [2] and [9], we have

$$\cos\mu_k \cosh\mu_k = 1\,.$$

Then we get $\cos\mu_k = \frac{1}{\cosh\mu_k}$. Numerically we find $\mu_1 \doteq 4.73004075$.

Moreover, $0 < \mu_1 < \mu_2 < \cdots$ and so $\cosh\mu_1 < \cosh\mu_2 < \cdots$. Since $\mu_k \sim \pi(2k+1)/2$ and $\cos(\pi(2k+1)/2) = 0$, we get

$$|\sin\theta_k| \cdot |\mu_k - \pi(2k+1)/2| = |\cos\mu_k - \cos(\pi(2k+1)/2)| = \frac{1}{\cosh\mu_k} \leq 2e^{-\mu_k}$$

for a $\theta_k \in (\mu_k, \pi(2k+1)/2)$. But we have

$$1 \geq |\sin\mu_k| = \sqrt{1 - \cos^2\mu_k} \geq \sqrt{1 - \cos^2\mu_1} \doteq 0.999844212\,,$$

since

$$0 < \cos\mu_k = \frac{1}{\cosh\mu_k} \leq \frac{1}{\cosh\mu_1} = \cos\mu_1\,.$$

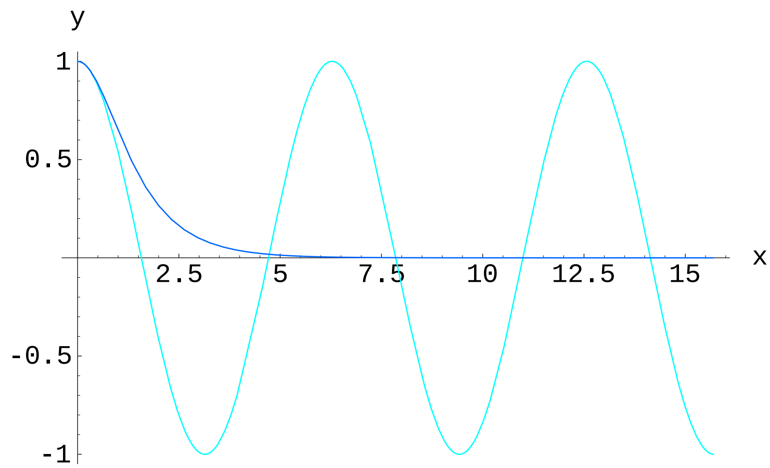

Figure 1. The graphs of functions $y = \cos x$ and $y = \operatorname{sech} x$.

Next, we can easily see (cf. Fig. 1) that in fact $(4k-1)\pi/2 < \mu_{2k-1}$, $\mu_{2k} < (4k+1)\pi/2$ and function $\cos x$ is positive on intervals $(\mu_k, \pi(2k+1)/2)$ for any $k \in \mathbb{N}$. So function $\sin x$ is increasing on these intervals, and it is positive on $\mu_{2k} < (4k+1)\pi/2$ and negative on $(4k-1)\pi/2 < \mu_{2k-1}$. From these arguments we deduce

$$|\sin\theta_k| \geq |\sin\mu_k| \geq |\sin\mu_1| \doteq 0.9998444212\,.$$

This gives

$$|\mu_k - \pi(2k+1)/2| \leq \frac{2}{|\sin\mu_1|}e^{-\mu_1} \doteq 0.017654973\,.$$

So we obtain

$$\mu_k \geq \frac{\pi(2k+1)}{2} - 0.017654973 \geq \pi k\,.$$

Finally, we obtain

$$|\mu_k - \pi(2k+1)/2| \leq \frac{2}{|\sin\mu_1|}e^{-\mu_k} \leq \frac{2}{|\sin\mu_1|}e^{-\pi k} \leq c\frac{\pi}{4}e^{-\pi k} \tag{81}$$

for $c = 2.546875863 < 2.6$.

6. Appendix 2: Diophantine approximations of the eigenvalues

We observe that from (81) we have

$$\mu_j > j\pi > j. \tag{82}$$

Next, for $\chi > 0$ we have

$$\left|\frac{\pi^2}{4}\chi(2j+1)^2 - k\pi\right| + \pi\chi(2j+1)r_j + \chi r_j^2 \geq |\chi\mu_j^2 - k\pi| \geq \left|\frac{\pi^2}{4}\chi(2j+1)^2 - k\pi\right| - \pi\chi(2j+1)r_j - \chi r_j^2 . \tag{83}$$

Lemma A1. *If $0 \leq \beta \leq 1$ then for almost all $\chi > 0$ and each $n \in \mathbb{N}$ there are infinitely many $j, k \in \mathbb{N}$ such that*

$$|\chi\mu_j^2 - k\pi| \leq \frac{1}{nj^\beta} . \tag{84}$$

Proof. We know [14] that for almost all $\chi > 0$ and each $n \in \mathbb{N}$ there are infinitely many $j, k \in \mathbb{N}$ such that

$$\left|\frac{\pi^2}{4}\chi(2j+1)^2 - k\pi\right| \leq \frac{1}{2nj^\beta} . \tag{85}$$

Then from (83) we get

$$|\chi\mu_j^2 - k\pi| \leq \frac{1}{2nj^\beta} + \frac{\pi^2}{4}\chi(2j+1)ce^{-j\pi} + \chi c^2\frac{\pi^2}{16}e^{-2j\pi} \leq \Big(\frac{1}{2n} + \frac{K_\beta\chi}{j}\Big)\frac{1}{j^\beta}$$

for

$$K_\beta = \sup_{j\in\mathbb{N}} \frac{\pi^2}{16}\Big\{4(2j+1)ce^{-j\pi}j^{1+\beta} + c^2e^{-2j\pi}j^{1+\beta}\Big\} .$$

Hence for such j that

$$2\chi nK_\beta \leq j \tag{86}$$

we obtain (84). We note that there are infinitely many $j, k \in \mathbb{N}$ satisfying both (85) and (86). The proof is finished. $\square$

Lemma A2. *If $\beta > 1$, $0 < \theta < \pi$ and $T^* > 0$ then the Lebesgue measure of the set S_{β,θ,T^*} of all $\chi > 0$ satisfying $\chi \leq T^*$ and*

$$|\chi\mu_j^2 - k\pi| \geq \frac{\theta}{j^\beta} \quad \forall (j,k) \in \mathbb{N}\times\mathbb{Z} \tag{87}$$

has an estimate

$$T^* \geq m(S_{\beta,\theta,T^*}) \geq T^*\Big(1 - \frac{2\theta\beta}{\pi(\beta-1)}\Big) - \frac{4\theta^2(\beta+1)}{\pi(2\beta+1)} .$$

Proof. If $k \in \mathbb{Z}$ is negative (87) is trivially satisfied since $0 < \theta < \pi$. Thus we assume $k \in \mathbb{Z}$, $k \geq 0$. Next, if $\chi \notin S_{\beta,\theta,T^*}$ then there are $(j_0, k_0) \in \mathbb{N}\times\mathbb{Z}$, $k_0 \geq 0$, such that

$$|\chi\mu_{j_0}^2 - k_0\pi| < \frac{\theta}{j_0^\beta}$$

which implies

$$\frac{k_0\pi}{\mu_{j_0}^2} - \frac{\theta}{j_0^\beta \mu_{j_0}^2} < \chi < \frac{k_0\pi}{\mu_{j_0}^2} + \frac{\theta}{j_0^\beta \mu_{j_0}^2}$$

and

$$k_0 < \frac{\theta}{\pi j_0^\beta} + \frac{T^* \mu_{j_0}^2}{\pi}\,.$$

Hence, using also $\mu_j > j$:

$$\begin{aligned}
&m([0,T^*] \setminus S_{\beta,\theta,T^*}) \le \sum_{j\in\mathbb{N}} \frac{2\theta}{j^\beta \mu_j^2}\Big(\frac{\theta}{j^\beta \pi} + \frac{T^*\mu_j^2}{\pi}\Big) = \sum_{j\in\mathbb{N}} \frac{2\theta^2}{\pi j^{2\beta}\mu_j^2} + \sum_{j\in\mathbb{N}} \frac{2\theta T^*}{j^\beta \pi} \\
&\le \frac{2\theta^2}{\pi}\sum_{j\in\mathbb{N}} \frac{1}{j^{2\beta+2}} + \frac{2\theta T^*}{\pi}\sum_{j\in\mathbb{N}} \frac{1}{j^\beta} \\
&\le \frac{2\theta^2}{\pi}\Big(1 + \int_1^\infty \frac{1}{x^{2\beta+2}}\,dx\Big) + \frac{2\theta T^*}{\pi}\Big(1 + \int_1^\infty \frac{1}{x^\beta}\,dx\Big) \\
&= \frac{4\theta^2(\beta+1)}{\pi(2\beta+1)} + \frac{2\theta T^*\beta}{\pi(\beta-1)}\,.
\end{aligned}$$

The proof is finished. □

From Lemma A2 we obtain

$$\liminf_{T^*\to+\infty} \frac{m(S_{\beta,\theta,T^*})}{T^*} \ge 1 - \frac{2\theta\beta}{\pi(\beta-1)}\,. \tag{88}$$

So for a given $\beta > 1$, we take $\theta \in (0,\pi)$ such that

$$0 < \theta < \frac{\pi(\beta-1)}{2\beta}. \tag{89}$$

Finally, we note that Lemma A1 implies that, for $0 \le \beta \le 1$ and almost all $\chi > 0$, there is no $\theta > 0$ such that (87) holds. This is the reason why we include Lemma A1 in this paper. Since according to its statement, we necessarily have to suppose $\beta > 1$ in order to get $S_{\beta,\theta,T^*} \ne \emptyset$. Moreover, we do not know in general the structure of the set S_{β,θ,T^*} also if it is nonempty. On the other hand, for $\beta = 0$, we can construct such a χ that (87) holds. We take $\chi = \frac{2q}{\pi p}$ for $p, q \in \mathbb{N}$ with q odd and $(p,q) = 1$. Then from (83) we obtain

$$\begin{aligned}
&\Big|\frac{2q}{\pi p}\mu_j^2 - k\pi\Big| \ge \pi\Big|\frac{q}{2p}(2j+1)^2 - k\Big| - \pi\frac{q}{8p}\Big(4(2j+1)ce^{-j\pi} + c^2e^{-2j\pi}\Big) \\
&\ge \frac{\pi}{8p}(4 - q\Phi(j))
\end{aligned}$$

for

$$\Phi(j) = 4(2j+1)ce^{-j\pi} + c^2e^{-2j\pi}\,.$$

Since $3\Phi(j) \le 3.9985175$ for any $j \in \mathbb{N}$, we can take $\chi = \frac{2q}{\pi p}$ for $p \in \mathbb{N}$ with $q = 1, 3$ and $(p,q) = 1$ for which (87) holds with $\beta = 0$ and $\theta = 0.000582137/p$.

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