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Positive solutions of semilinear elliptic equations: a dynamical approach.

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POSITIVE SOLUTIONS OF SEMILINEAR ELLIPTIC EQUATIONS: A DYNAMICAL APPROACH.

Abstract. This paper is devoted to the study of the structure of positive radial solutions for the following semi-linear equation:

$$\Delta u + f(u, |x|) = 0\,.$$

We require f to be nonnegative and to exhibit both subcritical and supercritical behavior with respect to the Sobolev critical exponent. More precisely we assume that f is subcritical for u small and $|x|$ large and supercritical for u large and $|x|$ small, and we give existence and non-existence results for ground states regular and singular, with either fast or slow decay. We find a surprisingly rich structure, which is characterized by two different patterns of bifurcations.

We perform a Fowler transformation and we use a dynamical approach, exploiting some ideas borrowed from Bamon, Del Pino, Flores, combining them with the use of the translation of the Pohozaev function for this dynamical context.

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1. Introduction

The purpose of this paper is to describe the structure of positive radial solutions for the following semi-linear equation:

$$\Delta u(x) + f(u, |x|) = 0 \tag{1.1}$$

where $x \in \mathbb{R}^n$, $n > 2$ and f is a continuous function which is assumed to be locally Lipschitz in the u variable, positive and superlinear for $u > 0$, null for $u \leq 0$. We assume that f is subcritical for u small and $|x|$ large and supercritical for u large and $|x|$ small, with respect to the Sobolev critical exponent. We are mainly thinking of two families of functions f; the first is a Matukuma-type equation:

$$f(u, |x|) = k(|x|)|u_+|^{q-1} \tag{1.2}$$

where u_+ stands for $\max\{u, 0\}$, $q > 2$ and e.g. $k(|x|) = k_u|x|^{\delta^u} + k_s|x|^{\delta^s}$, $k_u > 0$, $k_s > 0$ and $-2 < \delta^u < \lambda^* < \delta^s < \lambda_*$, $\lambda_* := (n-2)[q-2\frac{n-1}{n-2}] > \lambda^* := \frac{n-2}{2}[q-2\frac{n}{n-2}]$. The second is

$$f(u, |x|) = k_s(|x|)|u_+|^{q^s-1} + k_u(|x|)|u_+|^{q^u-2} \tag{1.3}$$

where $2_* := \frac{2(n-1)}{n-2} < q^s < 2^* := \frac{2n}{n-2} < q^u$, and k_u, k_s are positive functions.

In fact if the domain is radial (e.g. the whole of $\mathbb{R}^n$), usually positive solutions inherit this symmetry, see [4, 8, 24]. This is the case e.g. for f of type (1.2) and $k(|x|) = k_u|x|^{\delta^u} + k_s|x|^{\delta^s}$, see theorem 2 in [4], and for f of type (1.3) when k_s and k_u are positive constants, see [2]. Therefore we just consider radial solutions and

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we commit the following abuse of notation: we write $u(r)$ for $u(x)$ where $|x| = r$. Then the solutions of (1.1) satisfy the following singular O.D.E.

$$u'' + \frac{n-1}{r}u' + f(u,r) = 0. \tag{1.4}$$

Here and later $'$ denotes the derivative with respect to r. We classify positive solutions in ground states (G.S.), singular ground states (S.G.S.) and crossing solutions. By G.S. we mean a positive solution $u(r)$ defined for any $r \geq 0$ such that $\lim_{r\to\infty} u(r) = 0$. A S.G.S. of equation (1.1) is a positive solution $v(r)$ such that $\lim_{r\to 0} v(r) = +\infty$ and $\lim_{r\to+\infty} v(r) = 0$. Crossing solutions are solutions $u(r)$ such that there is $R > 0$ for which $u(r) > 0$ for any $0 \leq r < R$ and $u(R) = 0$, so they can be considered as solutions of the Dirichlet problem in the ball of radius R. We further distinguish the solutions according to the asymptotic behavior: positive solutions may be regular, i.e. $\lim_{r\to 0} u(r) = d > 0$ and we set $u(r) = u(r;d)$, or singular if $\lim_{r\to 0} u(r) = \infty$ as $r \to 0$; we say that a positive solution $v(r)$ has fast decay (f.d.) if $\lim_{r\to+\infty} v(r)r^{n-2} = L > 0$ and we set $v(r) = v(r;L)$, and that it has slow decay (s.d) if $\lim_{r\to+\infty} u(r)r^{n-2} = \infty$. Usually it is possible to give better estimates on the behavior of both singular solutions and slow decay solutions: in particular it is possible for all the functions f considered in this paper, see subsection 3.1.

Semi-linear equations of this type, and their generalizations to the p-Laplace and ϕ-Laplace case, have received a great interest in the last 30 years. The structure of positive solutions in the purely subcritical and supercritical cases is well known. The situation becomes more interesting and challenging when f exhibits both the behaviors. Such a phenomena is easily obtained for the scalar curvature equation, i.e. f of type (1.2) and $q = 2^* := \frac{2n}{n+2}$ see e.g. [5, 3, 19, 16]. This setting is very sensitive to the behavior at $r = 0$ and at $r = \infty$ of the function k. Another case, well studied in literature, is the one in which f is supercritical for u small and subcritical for u large, see [23, 9, 7, 17]. In this setting the solutions $u(r;d)$ of (1.4) are crossing solutions for d large and G.S. with f.d. for d small, and there is at least a value d^*, usually unique (see [21]), such that $u(r;d^*)$ is a G.S. with f.d. Furthermore there are uncountably many S.G.S. with f.d. and S.G.S. with s.d., see [14]. Comparing [7] and [17], it might be observed that the same structure for positive solutions appears also when f is of type (1.2), $q = \frac{2n}{n+2}$ and $k(r) \sim r^\alpha$ with $\alpha > 0$ as $r \to 0$ and $k(r) \sim r^\beta$ with $\beta < 0$ as $r \to \infty$, see also [14].

In this paper we consider the opposite situations, which seems to be more difficult but more natural: we assume that f is subcritical for u small and supercritical for u large. In fact this case is less studied and understood, and exhibits a strikingly different and richer structure for positive solutions. The seminal papers in this setting are [2] and [11], where the authors consider (1.4) where f is of type (1.3) and $k_u \equiv k_s \equiv 1$. They showed that the structure of positive solutions undergoes different families of bifurcations. More precisely in [2] the following results have been proved, combining the dynamical approach introduced by Johnson Pan and Yi in [20, 19] with new topological ideas.

Theorem 1.1. [2] *Let f be of type (1.3), $k_u \equiv k_s \equiv 1$, $q^s \in (2_*, 2^*)$, then for any $k \in \mathbb{N}$ there is $\varepsilon_k(q^s) > 0$ such that (1.4) admits at least k G.S. with f.d. for any $q^u \in (2^*, 2^* + \varepsilon_k)$. Analogously fix $q^u > 2^*$, then for any $k \in \mathbb{N}$ there is $\varepsilon_k(q^u) > 0$ such that (1.4) admits at least k G.S. with f.d. for any $q^s \in (2^* - \varepsilon_k, 2^*)$.*

Theorem 1.2. [2] *Let f be of type (1.3), $k_u \equiv k_s \equiv 1$. Fix $q^u > 2^*$, then there is $\varepsilon_0(q^u) > 0$ such that (1.4) admits no G.S. with f.d. for any $q^s \in [2_*, 2_* + \varepsilon_0(q^u))$.*

Theorem 1.3. [2] *Let f be of type (1.3), $k_u \equiv k_s \equiv 1$. Fix $q^s \in (2_*, 2^*)$; there is a sequence of values $r^j(q^s) \searrow 2^*$, such that (1.4) with $q^u = r^j(q^s)$ admits either a G.S. with s.d. or a S.G.S. with s.d.*

Analogously fix $q^u > 2^*$; there is a sequence of values $r^j(q^u) \nearrow 2^*$, such that (1.4) with $q^s = r^j(q^u)$ admits either a S.G.S. with f.d. or a S.G.S. with s.d.

We quote [22] where the authors found an explicit formula for a G.S. with s.d for this equation assuming $q^s = 2(q^u - 1)$. These solutions should be "rare", since they may be found as intersection between 2-dimensional and 1 dimensional objects in $\mathbb{R}^3$ exactly as S.G.S. with f.d. But their existence gains more relevance from the following result proved in [11]. Let us denote by $\bar{\sigma}_* := 2\frac{n+2\sqrt{n-1}-2}{n+2\sqrt{n-1}-4}$ and by $\bar{\sigma}^* := 2\frac{n-2\sqrt{n-1}-2}{n-2\sqrt{n-1}-4}$ if $n > 10$ and set $\bar{\sigma}^* = \infty$ if $n \leq 10$. The origin of the values $2_* < \bar{\sigma}_* < 2^* < \bar{\sigma}^*$ will be explained just after Remark 2.1 in relationship with the Fowler transformation.

Theorem 1.4. [11] *Let f be of type (1.3), $k_u \equiv k_s \equiv 1$, and $2_* < \bar{q}^s < 2^* < \bar{q}^u$.*

- **(a):** *Assume (1.4) admits a S.G.S. with either f.d or s.d., and $2^* < \bar{q}^u < \bar{\sigma}^*$. Then (1.4) admits infinitely many G.S. with f.d., too.*
- **(b):** *Assume (1.4) admits either a G.S. with s.d. or a S.G.S. with s.d., and $\bar{\sigma}_* < \bar{q}^s < 2^* < \bar{q}^u$. Then (1.4) admits infinitely many G.S. with f.d., too.*
- **(c):** *If $\bar{q}^s < \bar{q}^u$ satisfy either **(a)** or **(b)** then for any $k \in \mathbb{N}$, $k \geq 1$, there is $\eta_k > 0$ such that (1.4) admits at least k G.S. with f.d. whenever $|q^u - \bar{q}^u| + |q^s - \bar{q}^s| < \eta_k$.*

These results revealed how sensitive the structure of positive solutions is to changes in the exponents. The so called "bubble tower" phenomenon described in theorem 1.1 was reproved by Campos in [6] using a variational approach and a Ljiapunov-Schmidt reduction; in fact in [6] the authors also obtain an asymptotic estimate for G.S. with f.d. in terms of the explicitly known G.S. with f.d. of the critical case.

Similar results where obtained in [1, 10] for f of type (1.2). More precisely in [1], using variational methods and a Ljiapunov-Schmidt reduction, the authors prove the existence of the "bubble tower" phenomenon for (1.2) and $k(r)$ e.g. of type $k(r) = k_u r^{\delta^u} + k_s r^{\delta^s}$, $k_u > 0$, $k_s > 0$ and $-2 < \delta^u < \lambda^* < \delta^s < \lambda_*$. In [10] the authors let the so called "natural dimension" change values and exploit topological methods to prove the coexistence of G.S. with s.d. and of S.G.S. with f.d. for particular values of the parameters and special functions $k(r)$. As a consequence they also find two different sequences of G.S. with f.d. $u(d_k, r)$: one such that $d_k \to d^*$ where $u(d^*, r)$ is a G.S. with s.d. and one for $d_k \to +\infty$.

In [15] we picked up two very special non-linearities f which exhibit the same structure for positive solutions and for which the bifurcation diagrams can be described in all details, i.e. f of type (1.2) with $k(r) = \max\{r^{\delta^u}, r^{\delta^s}\}$ and $f(u) = \max\{u^{q^u-1}, u^{q^s-1}\}$. In fact in these cases we obtain the analogous of theorems 1.1 and 1.4; we also prove the analogous of theorem 1.2 together with its symmetric counterpart (non-existence for q^s large), and we proved 1.3 specifying the type of "rare" solution we have. Moreover the approach is constructive in nature, so it explicitly gives specific values for which the non-existence results hold and it suggests a method to give a computer assisted proof to estimate rigorously the "smallness" of the parameters ε_1 involved in the "bubble tower" phenomenon.
In [15] we conjectured that the very special f analyzed in that paper are the prototype for a more generic class of non-linearities: here we extend most of the results of [15] to a wide family of functions $f(u, r)$ supercritical for u large and r small, and subcritical for u small and r large.

So we extend the results found in [2], to a larger class of spatial dependent functions including (1.2) and (1.3), unifying them with the ones obtained in [1]. In fact we also complete the analysis performed in [2] by extending their non-existence result with its symmetric counterpart, moreover we complete [1] revealing the presence of both the bifurcations phenomena appearing in [2, 15]. However we are not able to generalize to this context the constructive proof developed in [15], so we cannot evaluate numerically the smallness of the parameters involved in the theorems.
Moreover we cannot predict whether the G.S. with fast decay found in theorem 2.4, analogous of 1.3 are regular or singular, while this is possible in [10, 15], but in both the papers just in very special cases.

The paper is divided as follows. In section 2 we introduce the Fowler transformation and we state the main results proved in this paper. In section 3 we develop some tools useful for our analysis: in subsection 3.1 we construct the unstable and the stable manifolds for non-autonomous systems; in subsection 3.2 we combine Kelvin inversion with Fowler transformation to obtain a very clean method to pass from results for regular solutions to results for f.d. solutions; in subsection 3.3 we discuss the critical problems, which will be perturbed in section 4 to prove the existence results. In section 4 we prove the main theorems and we discuss briefly the consequences for the Dirichlet problem in the ball. In the appendix we show how it is possible to weaken slightly the hypotheses if we fix a particular family of functions f, in particular if f is of type (1.2) or (1.3), and we give some examples of functions to which the results apply.

2. Fowler transformation and stating of the results.

In this section we introduce the Fowler transformation for the Laplace operator, which changes equation (1.4) into a two dimensional dynamical system. Setting

$$\alpha_l = \tfrac{2}{l-2}, \qquad \gamma_l = \alpha_l + 2 - n, \qquad l > 2, \qquad r = e^t$$

$$x_l = u(r) r^{\alpha_l}\,, \quad y_l = u'(r) r^{\alpha_l+1}\,, \quad g_l(x,t) = f(xe^{-\alpha_l t}, e^t) e^{(\alpha_l+2)t} \tag{2.1}$$

we pass from (1.4) to the following system

$$\begin{pmatrix} \dot{x}_l \\ \dot{y}_l \end{pmatrix} = \begin{pmatrix} \alpha_l & 1 \\ 0 & \gamma_l \end{pmatrix} \begin{pmatrix} x_l \\ y_l \end{pmatrix} + \begin{pmatrix} 0 \\ -g_l(x_l,t) \end{pmatrix} \tag{2.2}$$

We denote by

$$F(u,r) = \int_0^u f(s,r)ds\,, \qquad G_l(x,t) = \int_0^x g_l(s,t)ds = F(x\mathrm{e}^{-\alpha_l t}, \mathrm{e}^t)\mathrm{e}^{2(\alpha_l+1)t}.$$

We set $\mathbb{R}^2_+ := \{(x_l, y_l) \mid x_l > 0\}$ and $\mathbb{R}^2_\pm := \{(x_l, y_l) \mid y_l < 0 < x_l\}$. We assume first that (2.2) is autonomous and we review quickly some well known facts. To fix the ideas we take $f(u,r) = Kr^\delta |u^+|^{q-1}$, where $K > 0$ and $\delta > -2$, and we set $l = 2\frac{q+\delta}{2+\delta}$ to obtain

$$\begin{pmatrix} \dot{x}_l \\ \dot{y}_l \end{pmatrix} = \begin{pmatrix} \alpha_l & 1 \\ 0 & \gamma_l \end{pmatrix} \begin{pmatrix} x_l \\ y_l \end{pmatrix} + \begin{pmatrix} 0 \\ -K[(x_l)_+]^{q-1} \end{pmatrix} \tag{2.3}$$

We stress that in this case we passed from a singular non-autonomous O.D.E. to an autonomous system from which the singularity has been removed. Moreover note that when $\delta = 0$ we can simply take $l = q$ to obtain (2.3). System (2.3) admits two critical points for $l > 2_* := \frac{2(n-1)}{n-2}$: the origin $O = (0,0)$ and $P = (P_x, P_y)$. The origin is a saddle point and it admits a one-dimensional C^1 stable manifold $\overline{M}^s$ and a one-dimensional C^1 unstable manifold $\overline{M}^u$. Observe that $\overline{M}^s$ (respectively $\overline{M}^u$) is split by the origin into two connected components: a line contained in the $x \le 0$ denoted by M^s_- (resp. denoted by M^u_-), and a smooth manifold which departs from

the origin and enters $\mathbb{R}^2_\pm$, denoted by M^s (resp. denoted by M^u). In the origin $\overline{M}^s$ is tangent to the line $y = -(n-2)x$, while $\overline{M}^u$ is tangent to the x-axis. Since we focus on positive solutions we are just interested on the semi-plane $\mathbb{R}^2_\pm$. From some asymptotic estimate we deduce the following useful result, see e.g. [12, 13] for the proof in the p-Laplace context.

Remark 2.1. The regular solutions $u(r)$ of Eq. (1.4) correspond to the trajectories $\boldsymbol{X_l}(t)$ of system (2.3) departing from points in M^u and viceversa. Positive solutions with fast decay $u(r)$ of (1.4), correspond to trajectories $\boldsymbol{X_l}(t)$ of system (2.3) departing from points in M^s.

The critical point $\boldsymbol{P}$ is asymptotically stable if $l > 2^*$, asymptotically unstable if $2_* < l < 2^*$ and a center if $l = 2^*$.

A key tool in the analysis of equation of type (1.1) is the Pohozaev identity. In this dynamical context it can be rewritten through the following observation: let

$$H_l(x,y,t) = \frac{n-2}{2}xy + \frac{y^2}{2} + G_l(x,t)\,;$$

then, if $\boldsymbol{x_{2^*}}(t) = (x_{2^*}(t), y_{2^*}(t))$ solves (2.2) with $l = 2^*$ we have the following

$$\frac{dH_{2^*}}{dt}(\boldsymbol{x_{2^*}}(t),t) = \frac{\partial G_{2^*}}{\partial t}(x_{2^*}(t),t) \tag{2.4}$$

Moreover if $\boldsymbol{x_{2^*}}(t)$ and $\boldsymbol{x_l}(t)$ are trajectories of (2.2) corresponding to the same solution $u(r)$ of (1.4) we have the following

$$H_{2^*}(\boldsymbol{x_{2^*}}(t),t) = e^{-(\alpha_l+\gamma_l)t}H_l(\boldsymbol{x_l}(t),t)\,. \tag{2.5}$$

We stress that (2.4) and (2.5) hold for the general non-autonomous system (2.2). For any fixed value of t, the 0-level set of the function H_l is made up by a closed curve contained in $\mathbb{R}^2_\pm$, having a corner in the origin and by the lines $y = 0$ and $y = -(n-2)x$ in the $x \le 0$ semiplane. From (2.4) we see that $H_{2^*}(\boldsymbol{x_{2^*}}(t),t)$ is increasing in t (respectively decreasing) along the trajectories $\boldsymbol{x_{2^*}}(t)$ of (2.2) whenever $G_{2^*}(x,t)$ is increasing in t (resp. decreasing in t). Moreover from (2.5) we see that $H_{2^*}(\boldsymbol{x_{2^*}}(t),t)$ and $H_l(\boldsymbol{x_l}(t),t)$ have the same sign. So, if we consider system (2.3), for any $\boldsymbol{Q} \in M^u_l$ and $\boldsymbol{R} \in M^s_l$ we get $H_l(\boldsymbol{Q},t) < 0 < H_l(\boldsymbol{R},t)$ when $l > 2^*$, $H_l(\boldsymbol{R},t) < 0 < H_l(\boldsymbol{Q},t)$ when $2 < l < 2^*$, and $H_l(\boldsymbol{Q},t) = 0 = H_l(\boldsymbol{R},t)$ when $l = 2^*$. Using (2.4) and (2.5), it can be proved that the phase portrait of the autonomous system (2.3) is as depicted in Fig. 1, see e.g. [13]. Then it is easy to classify positive solutions: in the supercritical case ($l > 2^*$) all the regular solutions are G.S. with slow decay, there is a unique S.G.S. with slow decay; in the critical case ($l = 2^*$) all regular solutions are G.S. with fast decay and there are uncountably many S.G.S. with slow decay; in the subcritical case ($2 < l < 2^*$) all the regular solutions are crossing, there are uncountably many S.G.S. with fast decay and a unique S.G.S. with slow decay.

We stress that all the previous discussion concerning the autonomous Eq. (2.3) continues to hold for any autonomous super-linear system (2.2), more precisely whenever $g_l(x,t) \equiv g_l(x)$ and $g_l(x)$ has the following property, denoted by $\boldsymbol{G0}$ (see [13] for a proof in the general p-Laplace context).

$\boldsymbol{G0}$: $g_l(x)/x$ is an increasing function for $x > 0$ and

$$\lim_{x\to 0}\frac{g_l(x)}{x} = 0\,, \qquad \lim_{x\to+\infty}\frac{g_l(x)}{x} = +\infty\,.$$

Note that $\boldsymbol{G0}$ guarantees the uniqueness of the critical point $\boldsymbol{P}$. We introduce two further critical values. Consider first $f = Kr^\delta|u_+|^{q-1}$ and denote by $\sigma_* < \sigma^*$ the real roots of $(\alpha+\gamma)^2 + 4\alpha\gamma(q-2)$ belonging to $(2_*, +\infty)$; we set $\sigma_* = 2_*$ and $\sigma^* = \infty$ if these roots are not real or do not belong to the interval. We have

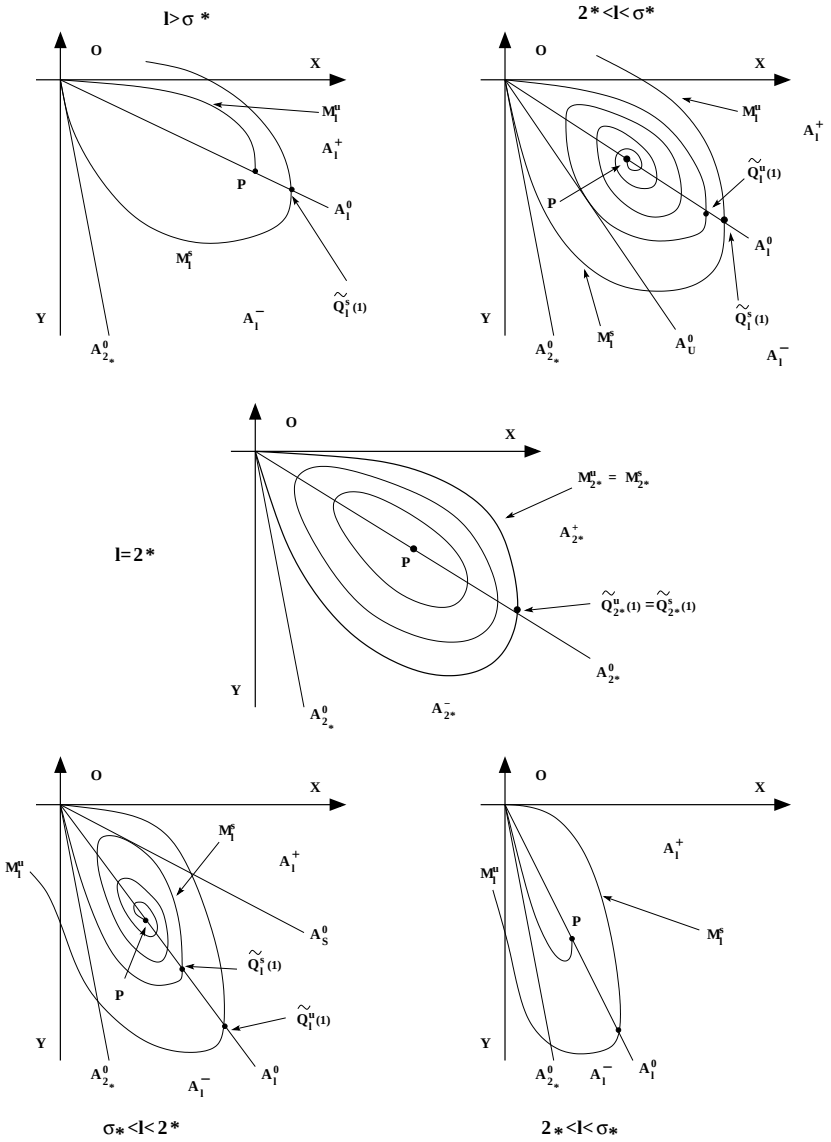

FIGURE 1. Sketches of the phase portrait of (2.3), for $q > 2$ fixed.

$\sigma_* := 4\frac{q-1-\sqrt{(q-1)^2-(q-1)}}{n-2} + 2 < \sigma^* := 4\frac{q-1+\sqrt{(q-1)^2-(q-1)}}{n-2} + 2$; if $\delta = 0$ we get $\bar{\sigma}_* := 2\frac{n+2\sqrt{n-1}-2}{n+2\sqrt{n-1}-4}$, and by $\bar{\sigma}^* := 2\frac{n-2\sqrt{n-1}-2}{n-2\sqrt{n-1}-4}$ if $n > 10$ and $\sigma_* = 2_*$, $\sigma^* = \infty$ if $n \leq 10$. (these are the parameters involved in theorems 1.4 and 2.5). It is easy to show that $\boldsymbol{P}$ is a focus if $\sigma_* < l < \sigma^*$ (a center for $l = 2^*$), an unstable node if $2_* < l < \sigma_*$ and a stable node if $l > \sigma^*$. Now consider an autonomous system (**??**) satisfying $\boldsymbol{G0}$ We denote by $\sigma_* < \sigma^*$ the values such that $\boldsymbol{P}$ is a focus if $\sigma_* < l < \sigma^*$ (a center for $l = 2^*$), an unstable node if $2_* < l < \sigma_*$ and a stable node if $l > \sigma^*$.

As we said in the introduction, we ask for f to be superlinear, i.e. without further mentioning in all the paper we require the following:

$\boldsymbol{F0}$**:** For any $r > 0$ the function $f(u,r)/u$ is strictly increasing in u and

$$\lim_{u\to 0^+} f(u,r) = 0 \qquad \text{and} \qquad \lim_{u\to+\infty} f(u,r) = +\infty\,,$$

We stress that $\boldsymbol{F0}$ implies that, for any fixed $T \in \mathbb{R}$, any autonomous system (2.2) where $g_l(x,t) \equiv g_l(x,T)$ and $l > 2$, satisfies $\boldsymbol{G0}$. Such a property allows us to establish in a standard way that any trajectory $\boldsymbol{x_l}(t)$ of (2.2) is well defined for any $t \in \mathbb{R}$. We list here the hypotheses used in the main theorems.

- $\boldsymbol{G_u}$**:** There is $l_u > 2$ such that for any $x > 0$ the function $g_{l_u}(x,t)$ converges to a t-independent locally Lipschitz function $g_{l_u}^{-\infty}(x) \not\equiv 0$ as $t \to -\infty$, uniformly on compact intervals. The function $g_{l_u}^{-\infty}(x)$ satisfies $\boldsymbol{G0}$. Moreover there is $\varpi > 0$ such that $\lim_{t\to-\infty} \frac{\partial}{\partial t} \mathrm{e}^{-\varpi t} g_{l_u}(x,t) = 0$.
- $\boldsymbol{G_s}$**:** There is $l_s > 2_*$ such that for any $x > 0$ the function $g_{l_s}(x,t)$ converges to a t-independent locally Lipschitz function $g_{l_s}^{+\infty}(x) \not\equiv 0$ as $t \to +\infty$, uniformly on compact intervals. The function $g_{l_s}^{+\infty}(x)$ satisfies $\boldsymbol{G0}$. Moreover there is $\varpi > 0$ such that $\lim_{t\to+\infty} \frac{\partial}{\partial t} \mathrm{e}^{\varpi t} g_{l_s}(x,t) = 0$.
- $\boldsymbol{A_u}$**:** $\frac{\partial}{\partial t} G_{l_u}(x,t) \geq 0$, for any $t \in \mathbb{R}$ and any $x > 0$, strictly for a certain $t \in \mathbb{R}$ and any $x > 0$.
- $\boldsymbol{A_s}$**:** $\frac{\partial}{\partial t} G_{l_s}(x,t) \leq 0$, for any $t \in \mathbb{R}$ and any $x > 0$, strictly for a certain $t \in \mathbb{R}$ and any $x > 0$.

Hypotheses $\boldsymbol{G_u}$ and $\boldsymbol{G_s}$ are needed to ensure the existence of an unstable and a stable manifold in the non-autonomous case, while $\boldsymbol{A_u}$ and $\boldsymbol{A_s}$ are technical conditions (related to (2.4) and to the Pohozaev identity), required to apply the perturbation argument explained in this paper. Now we are ready to state the main results proved in this paper.

Theorem 2.2. *Assume $\boldsymbol{A_u}, \boldsymbol{G_u}$ and $\boldsymbol{G_s}$ with $2_* < l_s < 2^*$. Then for any $k \in \mathbb{N}$ there is $\varepsilon_k(l_s)$ such that (1.4) admits at least k G.S. with fast decay, whenever $2^* < l_u < 2^* + \varepsilon_k(l_s)$. Analogously assume $\boldsymbol{A_s}, \boldsymbol{G_u}$ and $\boldsymbol{G_s}$ with $l_u > 2^*$. Then for any $k \in \mathbb{N}$ there is $\varepsilon_k(l_u)$ such that (1.4) admits at least k G.S. with fast decay, whenever $2^* - \varepsilon_k(l_u) < l_s < 2^*$.*

We stress that theorem 2.2 is a generalization of theorem 1.1 and of the analogous results in [15]. For the proof we have rephrased for this context the topological ideas introduced by Bamon et al. to prove theorem 1.1, connecting them with the Pohozaev function H.

We also have a non-existence counterpart, analogous to theorem 1.2.

Theorem 2.3. *Assume $\boldsymbol{G_u}$ and $\boldsymbol{G_s}$ with $l_u > 2^*$. There is $\varepsilon_0(l_u) > 0$ such that (1.4) admits no G.S. with either fast or slow decay, and no S.G.S. with either fast or slow decay, whenever $2_* < l_s < 2_* + \varepsilon_0(l_u)$.*
Analogously assume $\boldsymbol{G_u}$ and $\boldsymbol{G_s}$ with $2_* < l_s < 2^*$. There is $M_0(l_s) > 2^*$ such that (1.4) admits no G.S. with either fast or slow decay, and no S.G.S. with either fast or slow decay, whenever $l_u > M_0(l_s)$.

We stress that this theorem is completely new for f of type (1.2), apart from the special non-linearity discussed in [15], and the second claim is new for f of type (1.3) even in the spatial independent case. Once again we have no clue on the magnitude of the parameters involved, while for the special non-linearities discussed in [15] we can say that non-existence holds e.g. whenever $2_* < l_s < \sigma_* < \sigma^* < l_u$. We also have a result similar to theorem 1.3.

Theorem 2.4. *Assume $\boldsymbol{A_u}, \boldsymbol{G_u}$ and $\boldsymbol{G_s}$ with $2_* < l_s < 2^*$. There is a sequence $r^k(l_s) \searrow 2^*$ such that whenever $l_u = r^k(l_s)$, (1.4) admits either a G.S. with s.d., or a S.G.S. with f.d, or a S.G.S. with s.d. Analogously assume $\boldsymbol{A_s}, \boldsymbol{G_u}$ and $\boldsymbol{G_s}$ with $l_u > 2^*$.*
There is a sequence $r^k(l_u) \nearrow 2^*$ such that whenever $l_s = r^k(l_u)$, (1.4) admits either a G.S. with s.d., or a S.G.S. with f.d, or a S.G.S. with s.d.

The proof of this result is inspired by the proof of theorem 1.3. However in the original proof in [2], there is a small mistake which is fixed here. Due to this fact we have an alternative between three species of "special" solutions, and not just two as in [2], but we think such a correction is needed even in theorem 1.3. In proposition 4.11 we prove the result of theorem 2.4, but asking for a further assumption, weak but very difficult to be verified (it is verified by the f discussed in [15]): in such a case we can say which type of special solution we have.

As we said in the introduction, these special solutions play a key role since they reveal the presence of a further resonance phenomenon, the one discussed in [11], which drives to the following result.

Theorem 2.5. *Assume that f satisfies $\boldsymbol{G_u}$ and $\boldsymbol{G_s}$ with $2_* < l_u < 2^* < l_s$. Then the conclusion of theorem 1.4 holds, with l_u replaced by q^u and l_s by q^s.*

In fact when f takes the following form (including the motivating cases (1.2) or (1.3))

$$f(u,r) = \sum_{i=1}^{j} k^i(r)|u_+|^{q^i-1} \tag{2.6}$$

we can slightly weaken $\boldsymbol{A_u}$ and $\boldsymbol{A_s}$. As usual we assume $k^i(r) > 0$ for $r > 0$, and we introduce the following functions for $i = 1, \dots, j$:

$$\begin{aligned} J_l^{-,i}(r) &:= \int_0^r s^{\frac{n-2}{2}q^i} \frac{d}{ds}\left[k^i(s)s^{2(l-q^i)/(l-2)}\right] ds\,, \\ J_l^{+,i}(r) &:= \int_r^{+\infty} s^{-\frac{n-2}{2}q^i} \frac{d}{ds}\left[k^i(s)s^{2(l-q^i)/(l-2)}\right] ds\,, \end{aligned} \tag{2.7}$$

We emphasize that integrating by parts we can trivially redefine the functions $J_l^{\pm,i}(r)$ in a way which fits the case where the functions $k^i(r)$ are not differentiable. Assume $\boldsymbol{G_u}$ and $\boldsymbol{G_s}$; when f takes the form (2.6) we can replace $\boldsymbol{A_u}$ and $\boldsymbol{A_s}$ respectively by the hypotheses $\boldsymbol{A'_u}$ and $\boldsymbol{A'_s}$ stated below:

- $\boldsymbol{A'_u}$**:** $J_{l_u}^{-,i}(t) \geq 0$ for any $t \in \mathbb{R}$, and any $i = 1, \dots, j$, and $\sum_{i=1}^{j} J_{l_u}^{-,i}(T) > 0$ for a certain $T \in \mathbb{R}$. There is $M > 0$ such that $\frac{\partial G_{l_u}}{\partial t}(x,t) \geq 0$ for any $x > 0$ and any $t \leq -M$.
- $\boldsymbol{A'_s}$**:** $J_{l_s}^{+,i}(t) \leq 0$ for any $t \in \mathbb{R}$ and any $i = 1, \dots, j$, and $\sum_{i=1}^{j} J_{l_s}^{+,i}(T) < 0$ for a certain $T \in \mathbb{R}$. There is $M > 0$ such that $\frac{\partial G_{l_s}}{\partial t}(x,t) \leq 0$ for any $x > 0$ and any $t \geq M$.

Proposition 2.6. *Assume that f is of type (2.6); then theorems 2.2 and 2.4 hold with $\boldsymbol{A_u}$ replaced by $\boldsymbol{A'_u}$ and $\boldsymbol{A_s}$ replaced by $\boldsymbol{A'_s}$.*

We stress that $\boldsymbol{A_u}$ implies $\boldsymbol{A'_u}$ and $\boldsymbol{A_s}$ implies $\boldsymbol{A'_s}$. In fact we believe that $\boldsymbol{A_u}$ and $\boldsymbol{A_s}$ and their generalization are technical requirements, and might be removed with a different approach (perhaps applying variational techniques and Lijapunov-Schmidt reduction directly on (2.3) as done in [6]).

3. Basic dynamical tools.

In this section we develop some dynamical tools which will be useful for the proofs of the main theorems in section 4.

3.1. **Stable and unstable manifolds for the non-autonomous system.** The following notation will be in force throughout all the paper. We use capital letters for trajectories of autonomous systems and small letters for trajectories of non-autonomous systems; we write $\boldsymbol{x_{\bar{l}}}(t,\tau;\boldsymbol{Q}) = (x_{\bar{l}}(t,\tau;\boldsymbol{Q}), y_{\bar{l}}(t,\tau;\boldsymbol{Q}))$ for a trajectory of (2.2) where $l = \bar{l}$, evaluated at t and departing from $\boldsymbol{Q} \in \mathbb{R}^2$ at $t = \tau$. Assume $\boldsymbol{G_u}$

(respectively $\boldsymbol{G_s}$); we denote by $\boldsymbol{P_{l_u}}(-\infty)$ (resp. by $\boldsymbol{P_{l_s}}(+\infty)$) the unique critical point contained in $\mathbb{R}^2_\pm$ of the autonomous system (2.2) where $g_{l_u}(x,t) \equiv g^{-\infty}_{l_u}(x)$ (resp. $g_{l_s}(x,t) \equiv g^{+\infty}_{l_s}(x)$).

Observe that system (2.3) is invariant for translations in t. Therefore if $\boldsymbol{X_l}(t)$ is a solution, $\boldsymbol{X_l}(t+\tau)$ is a solution as well. Equivalently if $u(r)$ is a solution of (1.4), then $v(r) = u(re^\tau)e^{\alpha\tau}$ is a solution as well. Using this fact we also get the following, see [13].

Remark 3.1. Let $u(r;d)$ be a regular solution of (1.4), and let $\boldsymbol{X_l}(t,\tau;\boldsymbol{Q^u})$ be the corresponding trajectory of the autonomous system (2.2), where $g_l(x,t) \equiv g_l(x)$ satisfy $\boldsymbol{G0}$, so that $\boldsymbol{Q^u} \in M^u$. Then d is a smooth monotone function of τ such that $d(\tau) \to +\infty$ as $\tau \to -\infty$ and $d(\tau) \to 0$ as $\tau \to +\infty$, and viceversa. Furthermore if we fix τ, $d(\boldsymbol{Q^u}) \to 0$ as $\boldsymbol{Q^u} \to (0,0)$ and viceversa, and if $q < 2^*$ then $d(\boldsymbol{Q^u}) \to +\infty$ as $\boldsymbol{Q^u}$ tends to the critical point $\boldsymbol{P}$.

Analogously let $v(r;L)$ be a fast decay solution of (1.4) such that $\lim_{r\to+\infty} v(r;L)r^{n-2} = L > 0$, and let $\boldsymbol{X}(t,\tau;\boldsymbol{Q^s})$ be the corresponding trajectory of (2.2) such that $\boldsymbol{Q^s} \in M^s$. Then L is a smooth monotone function of τ such that $L(\tau) \to +\infty$ as $\tau \to +\infty$ and $L(\tau) \to 0$ as $\tau \to -\infty$, and viceversa. Furthermore if we fix τ, $L(\boldsymbol{Q^s}) \to 0$ as $\boldsymbol{Q^s} \to (0,0)$ and viceversa, and if $q > 2^*$ then $L(\boldsymbol{Q^s}) \to +\infty$ as $\boldsymbol{Q^s}$ tends to the critical point $\boldsymbol{P}$.

Now we turn to consider the non-autonomous systems: before constructing stable and unstable manifolds in this setting we give some simple remarks. Let us introduce polar coordinates in (2.2): set $\rho = \sqrt{x^2+y^2}$ and $\theta = \arctan\left(\frac{y_l}{x_l}\right)$. Then we have

$$\dot\theta = \alpha_l \sin(\theta)\cos(\theta) - \sin^2(\theta) - \frac{\cos(\theta) g_l(\rho[\cos(\theta)]_+, t)}{\rho}, \quad \text{for } \theta \in \left[-\frac{\pi}{2}; \frac{\pi}{2}\right] \tag{3.1}$$

Hence, if $\boldsymbol{G_s}$ holds, large trajectories of (2.2) rotates clockwise and their speed of rotation increases as the radius increases. So if $\boldsymbol{x_{l_s}}(t)$ becomes unbounded as $t \to +\infty$ it must cross the y negative semi-axis transversally for some $T \in \mathbb{R}$. Reasoning similarly in backwards time we get the following.

Remark 3.2. Assume $\boldsymbol{G_u}$ with $l_u > 2$; if $\boldsymbol{x_{l_u}}(t) \in \mathbb{R}^2_+$ for any $t \le 0$, then it is bounded in that interval. Similarly assume $\boldsymbol{G_s}$ with $l_s > 2$; if $\boldsymbol{x_{l_s}}(t) \in \mathbb{R}^2_+$ for any $t \ge 0$, then it is bounded in that interval.

In fact this Remark has been used to draw the phase portraits of the autonomous systems depicted in figure 1, too. Now we construct stable and unstable manifolds, and we show in remark 3.3 below, that regular solutions of (1.4) correspond to trajectories of the unstable manifold while fast decay solutions of (1.4) correspond to trajectories of the stable manifold.

Assume $\boldsymbol{G_u}$; we introduce the following 3-dimensional autonomous system, obtained from (2.2) by adding the extra variable $z = \mathrm{e}^{\varpi t}$:

$$\begin{pmatrix} \dot x_{l_u} \\ \dot y_{l_u} \\ \dot z \end{pmatrix} = \begin{pmatrix} \alpha_{l_u} & 1 & 0 \\ 0 & \gamma_{l_u} & 0 \\ 0 & 0 & \varpi \end{pmatrix} \begin{pmatrix} x_{l_u} \\ y_{l_u} \\ z \end{pmatrix} + \begin{pmatrix} 0 \\ -g_{l_u}(x_{l_u}, \frac{\ln(z)}{\varpi}) \\ 0 \end{pmatrix} \tag{3.2}$$

Observe that (3.2) admits 2 critical points: the origin and $(\boldsymbol{P_{l_u}}(-\infty), 0)$. The restriction of (3.2) to $z = 0$ is the autonomous system (2.2) where $g_{l_u}(x,t) \equiv g^{-\infty}_{l_u}(x)$, and this plane attracts all the trajectories as $t \to -\infty$. The technical hypotheses concerning $\frac{\partial g_{l_u}}{\partial t}$ is needed to ensure (3.2) to be smooth. So this system is useful to get information about the asymptotic behavior of trajectories in the past. Whenever $l_u > 2_*$ the origin admits a 1-dimensional stable manifold and a 3-dimensional unstable manifold; these manifolds are split by the z axis into two

connected components: we denote by $\boldsymbol{W^u_{l_u}}$ the branch of the unstable manifold which enters $x > 0$. We set

$$W^u_{l_u}(\tau) = \{\boldsymbol{Q} \mid (\boldsymbol{Q}, \mathrm{e}^{\varpi\tau}) \in \boldsymbol{W^u_{l_u}}\}\,.$$

It is easy to check that $W^u_{l_u}(\tau)$ is a 1-dimensional manifold for any $\tau \in \mathbb{R}$. Moreover $\boldsymbol{Q} \in W^u_{l_u}(\tau)$ if and only if $\boldsymbol{x_{l_u}}(t, \tau; \boldsymbol{Q})$ converges to the origin as $t \to -\infty$ and the corresponding solution $u(r)$ of (1.4) is a regular solution. We set

$$W^u_{l_u}(-\infty) := \{\boldsymbol{Q} \mid (\boldsymbol{Q}, 0) \in \boldsymbol{W^u_{l_u}}\}\,;$$

it follows that $W^u_{l_u}(-\infty)$ coincide with the unstable manifold $M^u(-\infty)$ of the autonomous system (2.2) where $g_{l_u}(x,t) \equiv g^{-\infty}_{l_u}(x)$.

The critical point $(\boldsymbol{P_{l_u}}(-\infty), 0)$ admits an unstable manifold which is 3-dimensional if $2_* < l_u < 2^*$, and 1-dimensional if $l_u > 2^*$. If $(\boldsymbol{Q}, \mathrm{e}^{\varpi\tau})$ belongs to such a manifold the trajectory $\boldsymbol{x_{l_u}}(t, \tau; \boldsymbol{Q})$ of (2.2) converges to $\boldsymbol{P_{l_u}}(-\infty)$ as $t \to -\infty$ and corresponds to a singular solutions of (1.4). So, we also get the existence of uncountably many singular solutions for $2_* < l_u < 2^*$ and the uniqueness of the singular solution for $l_u > 2^*$.

Similarly if $\boldsymbol{G_s}$ is satisfied we set $l = l_s$ and $\zeta(t) = \mathrm{e}^{-\varpi t}$ and we consider

$$\begin{pmatrix} \dot{x}_{l_s} \\ \dot{y}_{l_s} \\ \dot{z} \end{pmatrix} = \begin{pmatrix} \alpha_{l_s} & 1 & 0 \\ 0 & \gamma_{l_s} & 0 \\ 0 & 0 & -\varpi \end{pmatrix} \begin{pmatrix} x_{l_s} \\ y_{l_s} \\ \zeta \end{pmatrix} + \begin{pmatrix} 0 \\ -g_{l_s}(x_{l_s}, -\frac{\ln(\zeta)}{\varpi}) \\ 0 \end{pmatrix} \tag{3.3}$$

Again (3.3) admits 2 critical points, the origin and $(\boldsymbol{P_{l_s}}(+\infty), 0)$, and its restriction to $z = 0$ gives back the autonomous system (2.2) where $g_{l_s}(x,t) \equiv g^{+\infty}_{l_s}(x)$. Such a plane attracts all the trajectories as $t \to +\infty$. The origin admits a 2-dimensional stable manifold which is split into two connected components by the z axis: we denote by $\boldsymbol{W^s_{l_s}}$ the branch which enters $x > 0$. We set

$$W^s_{l_s}(\tau) := \{\boldsymbol{Q} \mid (\boldsymbol{Q}, \mathrm{e}^{-\varpi\tau}) \in \boldsymbol{W^s_{l_s}}\}\,;$$

so that $W^s_{l_s}(\tau)$ is a 1-dimensional manifold, and $\boldsymbol{Q} \in W^u_{l_u}(\tau)$ if and only if $\boldsymbol{x_{l_s}}(t, \tau; \boldsymbol{Q})$ converges to the origin as $t \to +\infty$ and the corresponding solution $u(r)$ of (1.4) is a fast decay solution, for any $\tau \in \mathbb{R}$. We set

$$W^s_{l_s}(+\infty) := \{\boldsymbol{Q} \mid (\boldsymbol{Q}, 0) \in \boldsymbol{W^s_{l_s}}\} = M^s(+\infty)$$

where $M^s(+\infty)$ is the stable manifold of the autonomous system (2.2) where $g_{l_s}(x,t) \equiv g^{+\infty}_{l_s}(x)$. $(\boldsymbol{P_{l_s}}(+\infty), 0)$ admits a stable manifold which is 1-dimensional if $2_* < l_s < 2^*$, and 2-dimensional if $l_s > 2^*$. Again it follows that we have respectively a unique slow decay solution of (1.4) if $2_* < l_s < 2^*$, and uncountably many slow decay solutions if $l_s > 2^*$. In [13] we proved, with weaker assumptions and in the p-Laplace context, the following result which generalizes Remark 3.1.

Remark 3.3. Let $u(r)$ and $v(r)$ be the solutions of (1.4) corresponding respectively to the trajectories $\boldsymbol{x_{l_u}}(t, \tau; \boldsymbol{Q})$ and $\boldsymbol{x_{l_s}}(t, \tau; \boldsymbol{R})$ of (2.2). Assume $\boldsymbol{G_u}$ with $l_u > 2_*$, then $u(r)$ is a regular solution for (1.4) if and only if $\boldsymbol{Q} \in W^u_{l_u}(\tau)$; analogously assume $\boldsymbol{G_s}$ with $l_s > 2_*$, then $v(r)$ is a fast decay solution for (1.4) if and only if $\boldsymbol{R} \in W^s_{l_s}(\tau)$.

We stress that $W^u_{l_u}(\tau)$ (respectively $W^s_{l_s}(\tau)$) depends smoothly on τ, for any $\tau \in [-\infty, +\infty)$ (resp. for any $\tau \in (-\infty, +\infty]$). More precisely if $W^u_{l_u}(\tau)$ (resp. $W^s_{l_s}(\tau)$) intersects transversally a line L in a point $\boldsymbol{Q}(\tau)$, then $\boldsymbol{Q}(\tau)$ inherits the smoothness of $g_{l_u}(x,t)$, whenever $g_{l_u}(x,t)$ is uniformly continuous in t for $t \le \tau$ (respectively inherits the smoothness of $g_{l_s}(x,t)$ whenever $g_{l_s}(x,t)$ is uniformly continuous in t for $t \ge \tau$), see [19] and [18].

In the next section we look for intersections between stable and unstable manifolds of the origin, corresponding to G.S. with f.d; we also look for S.G.S. with f.d. and for G.S. with s.d corresponding respectively to intersections between the unstable manifold of $(\boldsymbol{P_{l_u}}(-\infty),0)$ and the stable manifold of the origin, and to intersections between the stable manifold of $(\boldsymbol{P_{l_s}}(+\infty),0)$ and the unstable manifold of the origin. So we need to compare (3.2) and (3.3), and to switch between different values of the parameter l in (2.1). Assume $\boldsymbol{Gu}$ and $\boldsymbol{Gs}$; let $u(r)$ be a solution of (1.4) and $\boldsymbol{x_{l_u}}(t,\tau;\boldsymbol{Q})$ and $\boldsymbol{x_{l_s}}(t,\tau;\boldsymbol{R})$ be the corresponding trajectories of (2.2) where l equals respectively l_u and l_s; then $\boldsymbol{R}=\exp[(\alpha_{l_s}-\alpha_{l_u})\tau]$ and

$$\boldsymbol{x_{l_u}}(t,\tau;\boldsymbol{Q})=\exp[(\alpha_{l_u}-\alpha_{l_s})t]\boldsymbol{x_{l_s}}(t,\tau;\boldsymbol{R})$$

So, using also Remark 3.3, we see that if $\boldsymbol{x_{l_u}}(t,\tau;\boldsymbol{Q})$ converges to the origin as $t\to+\infty$ (respectively as $t\to-\infty$), then $\boldsymbol{x_{l_s}}(t,\tau;\boldsymbol{R})$ converges to the origin as $t\to+\infty$ (resp. as $t\to-\infty$), whenever $l_u,l_s>2_*$.

Assume $\mathbf{G_u}$ and $\mathbf{G_s}$ where $2_*<l_s\le 2^*\le l_u$; we introduce the following notation. We denote by $(\boldsymbol{x^u_{l_u}}(t,\downarrow),z(t))$ the unique trajectory of (3.2) contained in the unstable manifold of the critical point $(\boldsymbol{P_{l_u}}(-\infty),0)$, by $u(r,\downarrow)$ the corresponding singular solution of (1.4) and by $(\boldsymbol{x^u_{l_s}}(t,\downarrow),\zeta(t))$ the corresponding trajectory of (3.3). Analogously we denote by $(\boldsymbol{x^s_{l_s}}(t,\uparrow),\zeta(t))$ the unique trajectory of (3.3) contained in the stable manifold of the critical point $(\boldsymbol{P_{l_s}}(+\infty),0)$, by $v(r,\uparrow)$ the corresponding slow decay solution of (1.4) and by $(\boldsymbol{x^s_{l_u}}(t,\uparrow),z(t))$ the corresponding trajectory of (3.2).

Furthermore we introduce the sets:

$$\begin{aligned}W^u_{l_s}(\tau)&:=\{\boldsymbol{Q}\mathrm{e}^{(\alpha_{l_s}-\alpha_{l_u})\tau}\mid \boldsymbol{Q}\in W^u_{l_u}(\tau)\}\\ W^s_{l_u}(\tau)&:=\{\boldsymbol{Q}\mathrm{e}^{-(\alpha_{l_s}-\alpha_{l_u})\tau}\mid \boldsymbol{Q}\in W^s_{l_s}(\tau)\}\end{aligned}$$

Obviously $W^u_{l_s}(\tau)$ and $W^s_{l_u}(\tau)$ are both manifolds for any $\tau\in\mathbb{R}$. Let $u(r)$ be a solution of (1.4), let $\boldsymbol{x_{l_u}}(t,\tau;\boldsymbol{Q})$ and $\boldsymbol{x_{l_s}}(t,\tau;\boldsymbol{R})$ be the corresponding trajectories of (2.2), then $u(r)$ is a regular solution if and only if $\boldsymbol{x_{l_u}}(t,\tau;\boldsymbol{Q})$ and $\boldsymbol{x_{l_s}}(t,\tau;\boldsymbol{R})$ both converge to the origin as $t\to-\infty$, i.e. $\boldsymbol{Q}\in W^u_{l_u}(\tau)$ and $\boldsymbol{R}=\boldsymbol{Q}\mathrm{e}^{(\alpha_{l_s}-\alpha_{l_u})\tau}\in W^u_{l_s}(\tau)$. Similarly $u(r)$ has fast decay if and only if $\boldsymbol{R}\in W^s_{l_s}(\tau)$ and $\boldsymbol{Q}=\boldsymbol{R}\mathrm{e}^{-(\alpha_{l_s}-\alpha_{l_u})\tau}\in W^s_{l_u}(\tau)$.

3.2. **The Kelvin transformation.** Another change of variables which is very useful in the context of equation of type (1.1), is known in literature as "Kelvin transformation". Let us set

$$s=r^{-1}\,,\quad \tilde u(s)=s^{2-n}u(1/s)\,,\quad \tilde f(\tilde u,s)=f(\tilde u s^{n-2},1/s)s^{-2-n}\,. \tag{3.4}$$

From a straightforward computation we see that if $u(r)$ satisfies (1.4) then $\tilde u(s)$ satisfies the following equation and viceversa.

$$\frac{d}{ds}[\tilde u_s(s)s^{n-1}]+\tilde f(\tilde u,s)s^{n-1}=0\,. \tag{3.5}$$

We stress that regular solutions $u(r)$ of (1.4) are driven by (3.4) into fast decay solutions $\tilde v(s)=u(1/s)s^{2-n}$ of (3.5), while fast decay solutions $v(r)$ of (1.4) are driven into regular solutions $\tilde u(s)=v(1/s)s^{2-n}$ of (3.5); moreover $u(0)=\lim_{s\to+\infty}\tilde v(s)s^{n-2}$, and $\lim_{r\to+\infty}v(r)r^{n-2}=\tilde u(0)$. Obviously (3.4) defines an involution, i.e. if we apply it twice we go back to the original equation.

The combination of (3.4) and (2.1) gives rise to a further involution which assumes a more clear form; to the best of our knowledge this observation has not appeared previously in literature. In fact when we apply (2.1) to (3.5) by setting

$$\begin{aligned}&\tau=-t\,,\qquad \tilde x_l(\tau)=\tilde u(\mathrm{e}^\tau)\mathrm{e}^{-\gamma_l\tau}=u(\mathrm{e}^{-\tau})\mathrm{e}^{-\alpha_l\tau}=u(\mathrm{e}^t)\mathrm{e}^{\alpha_l t}\,,\\ &\tilde y_l(\tau)=\tilde u'(\mathrm{e}^\tau)\mathrm{e}^{(-\gamma_l+1)\tau}=-u'(\mathrm{e}^t)\mathrm{e}^{(\alpha_l+1)t}-(n-2)u(\mathrm{e}^t)\mathrm{e}^{\alpha_l t}\end{aligned} \tag{3.6}$$

we simply pass from (2.2) to the following system:

$$\begin{pmatrix} \frac{\partial}{\partial\tau}\tilde{x}_l \\ \frac{\partial}{\partial\tau}\tilde{y}_l \end{pmatrix} = \begin{pmatrix} -\gamma_l & 1 \\ 0 & -\alpha_l \end{pmatrix} \begin{pmatrix} \tilde{x}_l \\ \tilde{y}_l \end{pmatrix} + \begin{pmatrix} 0 \\ -g_l(\tilde{x}_l, -\tau) \end{pmatrix} \tag{3.7}$$

We stress that (3.7) is obtained from (2.2) simply by changing the values of the parameters (α_l, γ_l) into $(-\gamma_l, -\alpha_l)$, and evaluating the function $g_l(x,t)$ in $-\tau$ inspite of t. We give the details of the computation for reader's convenience

$$\begin{aligned}
&\frac{\partial}{\partial\tau}\tilde{x}_l(\tau) = -\gamma_l\tilde{u}(\mathrm{e}^\tau)\mathrm{e}^{-\gamma_l\tau} + \tilde{u}'(\mathrm{e}^\tau)\mathrm{e}^{(-\gamma_l+1)\tau} = -\gamma\tilde{x}_l(\tau) + \tilde{y}_l(\tau) \\
&\frac{\partial}{\partial\tau}\tilde{y}_l(\tau) = \frac{\partial}{\partial\tau}\big[\big(\tilde{y}_l(\tau)\mathrm{e}^{+\alpha_l\tau}\big)\mathrm{e}^{-\alpha_l\tau}\big] = -\alpha_l\tilde{y}_l(\tau) + \mathrm{e}^{-\alpha_l\tau}\frac{\partial}{\partial\tau}\big[\tilde{u}'(\mathrm{e}^\tau)\mathrm{e}^{(n-1)\tau}\big] = \\
&= -\alpha_l\tilde{y}_l(\tau) - \tilde{f}(\tilde{u}(\mathrm{e}^\tau), \mathrm{e}^\tau)\mathrm{e}^{(n-\alpha_l)\tau} = -\alpha_l\tilde{y}_l(\tau) - f(\tilde{u}(\mathrm{e}^\tau)\mathrm{e}^{(n-2)\tau}, \mathrm{e}^{-\tau})\mathrm{e}^{-(\alpha_l+2)\tau} = \\
&= -\alpha_l\tilde{y}_l(\tau) - f(\tilde{x}_l(\tau)\mathrm{e}^{\alpha_l\tau}, \mathrm{e}^{-\tau})\mathrm{e}^{-(\alpha_l+2)\tau} = -\alpha_l\tilde{y}_l(\tau) - g_l(\tilde{x}_l(\tau), -\tau)
\end{aligned}$$

Thus when f satisfies $\boldsymbol{G_u}$ with $l = l_u$ then $\tilde{f}$ satisfies $\boldsymbol{G_s}$ with $l = L_s$, and when f satisfies $\boldsymbol{G_s}$ with $l = l_s$ then $\tilde{f}$ satisfies $\boldsymbol{G_u}$ with $l = L_u$, where L_s and L_u are such that $\alpha_{L_s} = -\gamma_{l_u}$ and $\gamma_{L_s} = -\alpha_{l_u}$, $\alpha_{L_u} = -\gamma_{l_s}$ and $\gamma_{L_u} = -\alpha_{l_s}$, i.e.

$$L_s = 2 - \frac{2}{\gamma_{l_u}} = \frac{2[l_u(n-1) - 2n]}{l_u(n-2) - 2n + 2}\,; \quad L_u = 2 - \frac{2}{\gamma_{l_s}} = \frac{2[l_s(n-1) - 2n]}{l_s(n-2) - 2n + 2} \tag{3.8}$$

Note that (3.8) brings $l = 2^*$ in itself, $l = \sigma_*$ in $l = \sigma^*$, $l_a \in (2_*, \sigma_*)$ and $l_b \in (\sigma_*, 2^*)$ respectively in $L_a > \sigma^*$ and $L_b \in (2^*, \sigma^*)$ and viceversa, and finally $l = 2_*$ in ∞. So a subcritical system is brought in a supercritical one and viceversa.

Moreover the manifolds $W^u_{l_u}(T)$ of (2.2) is changed into the manifold $W^s_{L_s}(-T)$ of (3.7) while $W^s_{l_s}(T)$ of (2.2) is changed into $W^u_{L_u}(-T)$ of (3.7). Finally the trajectories $\boldsymbol{x^u_{l_u}}(t,\downarrow)$ and $\boldsymbol{x^s_{l_s}}(t,\uparrow)$ are changed respectively into $\boldsymbol{x^s_{L_s}}(t,\uparrow)$ and $\boldsymbol{x^u_{L_u}}(t,\downarrow)$. These observations allow us to translate quickly the proofs for regular solutions into proofs for fast decay solutions, and the claims concerning singular solutions into claims concerning slowly decaying solutions, and viceversa.

We recall that Lin and Ni in [22] proved explicitly the existence of a G.S. with s.d $u(r)$ for (1.4) with f of type (1.3) with $k_u \equiv k_s \equiv 1$, and $q^u = 2q^s$. In fact they find $u(r) = A[B+r^2]^{-1/(q^s-2)}$, where A and B are computable constants depending on n and q^s. Using Kelvin inversion we find that

$$v(r) = u(1/r)r^{2-n} = A[B + r^{-2}]^{-1/(q^s-2)}r^{2-n} \tag{3.9}$$

is a S.G.S. with f.d. solving equation (1.4) where $f(u,r)$ is of type (1.3) with $k_u(r) = r^{-C}$ and $k_s(r) = r^D$, where $C = (n-2)(2^* - q^s)$ and $D = (n-2)(q^u - 2^*)$ and $q^u = 2q^s$. We stress that, as pointed out in the introduction, the existence of G.S. with s.d such as $u(r)$, and of S.G.S. with f.d. such as $v(r)$, seems to be a rare phenomenon taking place for precise sequences of values q^u and q^s. However it indicates the presence of the resonance phenomenon discovered by Flores in [11], which is translated in this context by theorem 2.5.

3.3. **Some remarks on the critical case.** The proof of theorem 2.2 is based on a perturbation argument performed on (3.2) and (3.3) respectively in the case $2_* < l_s < l_u = 2^*$ and $l_s = 2^* < l_u$. In this section we deepen our knowledge of these critical cases.

Let us set $H_{2^*}(x,y,\pm\infty) = \lim_{t\to\pm\infty} H_{2^*}(x,y,t)$. When $\boldsymbol{G_u}$ holds and $l_u = 2^*$ (respectively $\boldsymbol{G_s}$ holds and $l_s = 2^*$) there are uncountably many periodic trajectories in the plane $z = 0$ (resp. in the plane $\zeta = 0$), corresponding to the level sets $H_{2^*}(x,y,-\infty) = b$ where $H_{2^*}(\boldsymbol{P_{2^*}}(-\infty), -\infty) < b < 0$ (resp. $H_{2^*}(x,y,+\infty) = b$ where $H_{2^*}(\boldsymbol{P_{2^*}}(+\infty), +\infty) < b < 0$).

From an easy continuity argument we also find the following.

Remark 3.4. Assume $\boldsymbol{G_u}$ with $l_u > 2^*$, then there is a unique singular solution $u(r,\downarrow)$ for (1.4), and it is the one corresponding to the unique trajectory of (3.2), denoted by $(\boldsymbol{x}^{\boldsymbol{u}}_{\boldsymbol{l_u}}(t,\downarrow), \mathrm{e}^{\varpi t})$, converging to $(\boldsymbol{P_{l_u}}(-\infty), 0)$ as $t \to -\infty$. Moreover if a solution $u(r)$ of (1.4) is always positive for $0 < r < R$ (for a certain $R > 0$), then the corresponding trajectory $(\boldsymbol{x_{l_u}}(t), z(t))$ of (3.2) belongs to $\boldsymbol{W}^{\boldsymbol{u}}_{\boldsymbol{l_u}}$ or it coincides with $(\boldsymbol{x}^{\boldsymbol{u}}_{\boldsymbol{l_u}}(t,\downarrow), \mathrm{e}^{\varpi t})$. Assume $\boldsymbol{G_s}$ with $2_* < l_s < 2^*$, then there is a unique slow decay solution $v(r,\uparrow)$ for (1.4), and it is the one corresponding to the unique trajectory of (3.3), denoted by $(\boldsymbol{x}^{\boldsymbol{s}}_{\boldsymbol{l_s}}(t,\uparrow), \mathrm{e}^{-\varpi t})$, converging to $(\boldsymbol{P_{l_s}}(+\infty), 0)$ as $t \to +\infty$. Moreover if a solution $v(r)$ of (1.4) is always positive for $r > R$ (for a certain $R > 0$), then the corresponding trajectory $(\boldsymbol{x_{l_s}}(t), z(t))$ of (3.3) belongs to $\boldsymbol{W}^{\boldsymbol{s}}_{\boldsymbol{l_s}}$ or it coincides with $(\boldsymbol{x}^{\boldsymbol{s}}_{\boldsymbol{l_s}}(t,\uparrow), \mathrm{e}^{-\varpi t})$.

We recall that $u(r;d)$ is the regular solution of (1.4) satisfying $u(0;d) = d$, and $v(r;L)$ is the fast decay solution satisfying $\lim_{r\to+\infty} v(r;L)r^{n-2} = L$. We need a technical result which ensures the existence of crossing solution $u(r)$ of (1.4) and of Dirichlet solutions in exterior domains, i.e. solutions $v(r)$ of (1.4) having fast decay, which are null with positive slope for $r = R$ and are positive for $r > R$.

Lemma 3.5. *Assume $\boldsymbol{G_u}$ and $\boldsymbol{G_s}$ with $2_* < l_s < 2^* < l_u$. Then there are $\bar{D} > 0$ and $\bar{L} > 0$ such that each regular solution $u(r;d)$ is a crossing solution for $0 < d < \bar{D}$, and each fast decay solution $v(r;L)$ is a Dirichlet solution in exterior domain for $0 < L < \bar{L}$. Moreover if $\rho(d)$ is the zero of $u(r;d)$ and $R(L)$ is the zero of $v(r;L)$ then $\rho(d)$ and $R(L)$ are continuous for $0 < d < \bar{D}$ and $0 < L < \bar{L}$, $\rho(d) \to +\infty$ as $d \to 0$ and $R(L) \to 0$ as $L \to 0$.*

Proof. Let us set

$$A^+_l = \{(x,y) \mid -\alpha_l x < y < 0\}, \quad A^0_l = \{(x,y) \mid -\alpha_l x = y < 0\}$$

When we consider (2.2) with $l = \bar{l}$, we see that $\dot{x} > 0$ for any trajectory in $A^+_{\bar{l}}$, and $\dot{x} = 0$ for any trajectory in $A^0_{\bar{l}}$. Moreover A^+_l is invariant for the change of coordinates $\boldsymbol{Q} \to \exp[(\alpha_{l_s} - \alpha_{l_u})\tau]\boldsymbol{Q}$ which allows to pass from trajectories of (3.2) to the corresponding trajectories of (3.3). Assume $\boldsymbol{G_s}$ with $2_* < l_s < 2^*$ and follow $W^s_{l_s}(+\infty)$ from the origin towards $\mathbb{R}^2_\pm$: it intersects transversally $A^0_{l_s}$ a first time in a point denoted by $\tilde{\boldsymbol{Q}}(1)$ and a second time in a point denoted by $\tilde{\boldsymbol{Q}}(2)$. Let $\boldsymbol{R} \in \mathbb{R}^2_+$ and $\tau \in \mathbb{R}$: we denote by $\boldsymbol{X_{l_s}}(t,\tau;\boldsymbol{R};+\infty)$ the trajectory of the autonomous system (2.2) where $g = g^\infty_{l_s}$, departing from $\boldsymbol{R}$ at $t = \tau$. For any $\boldsymbol{R} \in A^+_{l_s}$, $\|\boldsymbol{R}\| < \|\tilde{\boldsymbol{Q}}(2)\|$ and any $\tau \in \mathbb{R}$ there is $T(\boldsymbol{R})$ such that $\boldsymbol{X_{l_s}}(t,\tau;\boldsymbol{R};+\infty)$ intersects transversally the y negative semi-axis and $X_{l_s}(t,\tau;\boldsymbol{R};+\infty) > 0$ for $t \in [\tau, T(\boldsymbol{R}))$. Moreover $T(\boldsymbol{R})$ is continuous and $T(\boldsymbol{R}) \to +\infty$ as $\boldsymbol{R} \to (0,0)$. From a continuity argument we find $\tau^s > 0$ large enough so that, for any $\boldsymbol{R} \in A^+_{l_s}$, $\|\boldsymbol{R}\| < \|\tilde{\boldsymbol{Q}}(2)/2\|$, there is $\tau(\boldsymbol{R})$ such that the trajectory $\boldsymbol{x_{l_s}}(t,\tau^s;\boldsymbol{R})$ of (2.2) intersects transversally the y negative semi-axis and $x_{l_s}(t,\tau;\boldsymbol{R}) > 0$ for $t \in [\tau^s, \tau(\boldsymbol{R}))$; again $\tau(\boldsymbol{R})$ is continuous and tends to $+\infty$ as $\boldsymbol{R} \to (0,0)$.

Assume further $\boldsymbol{G_u}$ and consider (3.3) and the 1-dimensional unstable manifold $W^u_{l_s}(\tau)$ where $\tau > \tau^s$: it is tangent to the x axis in the origin. So for any $\tau > \tau^s$ there is a small branch of $W^u_{l_s}(\tau)$, say $\tilde{W}^u_{l_s}(\tau)$, contained in $\{\boldsymbol{Q} \in A^+_{l_s} \mid \|\boldsymbol{Q}\| < \|\tilde{\boldsymbol{Q}}(2)/2\|\}$. It follows that for any $\boldsymbol{R} \in \tilde{W}^u_{l_s}(\tau)$ and any $\tau > \tau^s$ the trajectory $\boldsymbol{x_{l_s}}(t,\tau^s;\boldsymbol{R})$ intersects the y negative semi-axis at $t = \tau(\boldsymbol{R})$. So the corresponding regular solutions $u(r;d)$ of (1.4) are crossing solutions and their first and unique zero is $R = \exp[\tau(\boldsymbol{R})]$. From the transversality of the crossing and from Remarks 3.3 and 3.1 we find the continuity of $\rho(d)$, as well as the fact that $\rho(d) \to +\infty$ as $d \to 0$.

The proof concerning Dirichlet solutions in exterior domains $v(r)$ can be obtained arguing similarly or using Kelvin inversion, see subsection 3.2. □

Remark 3.6. Assume $\boldsymbol{G_u}$ with $l_u = 2^*$ and $\boldsymbol{A_u}$, then there are uncountably many singular solutions $u(r)$: one of them, $u(r,\downarrow)$, corresponds to $(\boldsymbol{x^u_{2^*}}(t,\downarrow), z(t))$, i.e. the unique trajectory of the unstable manifold of the critical point $(\boldsymbol{P_{l_u}}(-\infty), 0)$. Any singular solution $u(r)$ different from $u(r,\downarrow)$ corresponds to a trajectory $\boldsymbol{x_{2^*}}(t)$ which rotates clockwise indefinitely around $\boldsymbol{P_{l_u}}(-\infty)$ as $t \to -\infty$.
Analogously assume $\boldsymbol{G_s}$ with $l_s = 2^*$ and $\boldsymbol{A_s}$, then there are uncountably many slow decay solutions $v(r)$: one of them, $v(r,\uparrow)$, corresponds to $(\boldsymbol{x^s_{2^*}}(t,\uparrow), \zeta(t))$, i.e. the unique trajectory of the stable manifold of the critical point $(\boldsymbol{P_{l_s}}(+\infty), 0)$. Any slow decay solutions $v(r)$ different from $v(r,\uparrow)$ corresponds to a trajectory $\boldsymbol{x_{2^*}}(t)$ which rotates clockwise indefinitely around $\boldsymbol{P_{l_s}}(+\infty)$ as $t \to +\infty$.

Proof. We just discuss the claims concerning the behavior as $t \to -\infty$, the others being analogous. Assume $\boldsymbol{G_u}$ with $l_u = 2^*$; linearizing close to $(\boldsymbol{P_{l_u}}(-\infty), 0)$ we see that the critical point admits a one-dimensional unstable manifold and a 2-dimensional center manifold. Let $(\boldsymbol{x_{2^*}}(t), z(t))$ be a trajectory of (3.2) corresponding to a solution $u(r)$ which is positive for r small and it is singular. Assume $\boldsymbol{A_u}$: it follows that $H_{2^*}(\boldsymbol{x_{2^*}}(t), t)$ is increasing and it admits a limit b. If $b > 0$ then $(\boldsymbol{x_{2^*}}(t), z(t))$ has to cross the x axis. If $b = 0$, then its α-limit set is either the origin or the union of the homoclinic trajectory and the semi-lines $\{(x, 0, 0) \mid x \le 0\}$ and $\{(x, -(n-2)x, 0) \mid x \le 0\}$: in the former case $u(r)$ is a regular solution, in the latter $u(r)$ becomes negative for r small. So $b < 0$ and the α-limit set of $(\boldsymbol{x_{2^*}}(t), z(t))$ is either the critical point $(\boldsymbol{P_{l_u}}(-\infty), 0)$ or one of the periodic trajectory contained in $\mathbb{R}^2_\pm \times \{0\}$. In the latter case $\boldsymbol{x_{2^*}}(t)$ rotates indefinitely clockwise around $\boldsymbol{P_{l_u}}(-\infty)$ and Remark 3.6 is proved. In the former case either $(\boldsymbol{x_{2^*}}(t), z(t))$ is contained on the unstable manifold of $(\boldsymbol{P_{l_u}}(-\infty), 0)$, so that $u(r)$ is in fact $u(r,\downarrow)$ and we are done, or it is contained in the center manifold of $(\boldsymbol{P_{l_u}}(-\infty), 0)$. Let ρ_P, θ_P denote the polar coordinates on the $x-y$ plane centered in $\boldsymbol{P_{l_u}}(-\infty)$, and let $(\tilde\rho_P(t), \tilde\theta_P(t))$ be the polar coordinates of $\boldsymbol{x_{2^*}}(t)$. Linearizing the system on $(\boldsymbol{P_{l_u}}(-\infty), 0)$, we see that $\tilde\theta_P(t) \sim -\frac{n-2}{2}t$ and $\tilde\rho_P(t) \to 0$ as $t \to -\infty$ slower than exponentially, hence $\tilde\rho_P(t) > 0$ for t finite. So $\boldsymbol{x_{2^*}}(t)$ rotates indefinitely clockwise around $\boldsymbol{P_{l_u}}(-\infty)$.
To conclude the proof of Remark 3.6 we have to show that there are uncountably many singular solutions. So let us choose $\tau \in \mathbb{R}$ and consider the set

$$S = \{(x, y, \mathrm{e}^{\varpi\tau}) \mid H_{2^*}(x, y, \tau) < 0\,, \text{ and } x > 0\}\,.$$

For any $\boldsymbol{Q} \in S$, the trajectories $\boldsymbol{x_{2^*}}(t, \tau; \boldsymbol{Q})$, are such that $H_{2^*}(\boldsymbol{x_{2^*}}(t, \tau; \boldsymbol{Q}), t)$ is negative and increasing for $t < \tau$ and converges to a negative limit; hence the corresponding solutions $u(r)$ of (1.4) is singular. □

From the previous argument we easily get the following useful result.

Remark 3.7. Assume $\boldsymbol{G_u}$ with $l_u > 2$ and consider a trajectory $\boldsymbol{x_{2^*}}(t)$ such that $\liminf_{t\to-\infty} H_{2^*}(\boldsymbol{x_{2^*}}(t), t) > 0$. Then there is T such that $\boldsymbol{x_{2^*}}(t)$ crosses the positive y semi-axis transversally at $t = T$. Analogously assume $\boldsymbol{G_s}$ with $l_s > 2_*$ and consider a trajectory $\boldsymbol{x_{2^*}}(t)$ such that $\liminf_{t\to+\infty} H_{2^*}(\boldsymbol{x_{2^*}}(t), t) > 0$. Then there is T such that $\boldsymbol{x_{2^*}}(t)$ crosses the negative y semi-axis transversally at $t = T$.

To prove theorem 2.2 we look for trajectories $\boldsymbol{x_{l_u}}(t, \tau^u, \boldsymbol{Q^u})$ and $\boldsymbol{x_{l_u}}(t, \tau^s, \boldsymbol{Q^s})$, where $\boldsymbol{Q^u} \in W^u_{l_u}(\tau^u), \boldsymbol{Q^s} \in W^s_{l_u}(\tau^s)$, such that $x_{l_u}(t, \tau^u, \boldsymbol{Q^u}) - x_{l_u}(t, \tau^s, \boldsymbol{Q^s})$ has at least $2k+1$ zeroes. Then using a topological argument borrowed from [2] (proposition 4.1), we infer the existence of k intersections between unstable and stable manifolds, corresponding to k distinct G.S. with fast decay.

We need the following results which generalize Lemmas of [2].

Proposition 3.8. *Assume $\boldsymbol{G_u}$, $\boldsymbol{G_s}$ with $2_* < l_s < l_u = 2^*$, and $\boldsymbol{A_u}$. Then all the regular solutions are crossing, while all the fast decay solutions are S.G.S. with fast*

decay. If $v(r)$ has fast decay and $v(r) \not\equiv u(r,\downarrow)$ then the corresponding trajectory $\boldsymbol{x_{2^*}}(t)$ is bounded for $t \leq 0$ and $x_{2^*}(t) - P_x(-\infty)$ changes sign indefinitely as $t \to -\infty$.

Proof. Let $\tau \in \mathbb{R}$, $\boldsymbol{Q} \in W^u_{2^*}(\tau)$, $\boldsymbol{R} \in W^s_{2^*}(\tau)$: the trajectories $\boldsymbol{x_{2^*}}(t,\tau;\boldsymbol{Q})$ and $\boldsymbol{x_{2^*}}(t,\tau;\boldsymbol{R})$ of (2.2) correspond respectively to a regular solution $u(r)$ and a fast decay solution $v(r)$ of (1.4). Therefore

$$\lim_{t\to-\infty} H_{2^*}(\boldsymbol{x_{2^*}}(t,\tau;\boldsymbol{Q}),t) = 0 = \lim_{t\to+\infty} H_{2^*}(\boldsymbol{x_{2^*}}(t,\tau;\boldsymbol{R}),t)\,,$$

and from (2.4) we find that $H_{2^*}(\boldsymbol{x_{2^*}}(t,\tau;\boldsymbol{Q}),t)$ and $H_{2^*}(\boldsymbol{x_{2^*}}(t,\tau;\boldsymbol{R}),t)$ are both increasing in t. So proposition 3.8 is a straightforward consequence of Remarks 3.6 and 3.7. □

With a specular argument, or using Kelvin inversion, we can prove the following.

Proposition 3.9. *Assume $\boldsymbol{G_u}$, $\boldsymbol{G_s}$ with $2^* = l_u > l_s$, and $\boldsymbol{A_s}$. Then all the regular solutions are G.S. with slow decay while all the fast decay solutions are solutions of the Dirichlet problem in the exterior of a ball. Moreover if $u(r)$ is a regular solution, and $u(r) \neq v(r,\uparrow)$, then the corresponding trajectory $\boldsymbol{x_{2^*}(t)}$ is bounded for $t \geq 0$ and $x_{2^*}(t) - P_x(+\infty)$ changes sign indefinitely as $t \to +\infty$.*

4. Proof of the main theorems.

The proofs of the existence results are based on a topological analysis of the mutual positions of $\boldsymbol{W^u_{l_u}}$, $\boldsymbol{W^s_{l_u}}$, of the singular trajectory $(\boldsymbol{x^u_{l_u}}(t,\downarrow), z(t))$ and of the slow decay trajectory $(\boldsymbol{x^s_{l_u}}(t,\uparrow), z(t))$ of (3.2). We divide this section in 5 parts. In subsection 4.1 we perform the topological analysis needed to prove the existence results, i.e. theorem 2.2 and 2.4, which are actually proved respectively in subsection 4.2 and 4.4. Subsection 4.3 is devoted to the non-existence result, theorem 2.3, and subsection 4.5 to the sketch of the proof of the resonance phenomenon explained in theorem 2.5, and to the consequences of our analysis for solutions of the Dirichlet problem in the ball.

4.1. **The topological construction.** We collect in this page the definitions and the constructions, inspired by [2], which will be relevant in the whole section.

Let $\boldsymbol{\gamma}(t) = (\gamma_x(t), \gamma_y(t)) : [a,b] \to \mathbb{R}^2$ be a curve and $\boldsymbol{Q} = (Q_x, Q_y) \in \mathbb{R}^2$ a point not in $\boldsymbol{\gamma}$. We introduce polar coordinates $(\theta_\gamma(t), \rho_\gamma(t))$ centered in $\boldsymbol{Q}$ for $\boldsymbol{\gamma}(t)$, i.e. we set $\boldsymbol{\gamma}(t) = \boldsymbol{Q} + \rho_\gamma(t)(\cos(\theta_\gamma(t)), \sin(\theta_\gamma(t)))$. We call angular number $\Theta(\boldsymbol{\gamma},\boldsymbol{Q})$ and winding number $w(\boldsymbol{\gamma},\boldsymbol{Q})$ respectively

$$\Theta(\boldsymbol{\gamma},\boldsymbol{Q}) = \frac{\theta_\gamma(b) - \theta_\gamma(a)}{2\pi}\,, \qquad w(\boldsymbol{\gamma},\boldsymbol{Q}) = [\Theta(\boldsymbol{\gamma},\boldsymbol{Q})] = \left[\frac{\theta_\gamma(b) - \theta_\gamma(a)}{2\pi}\right]\,, \tag{4.1}$$

where $[\cdot]$ denotes the integer part. Hence $\Theta(\boldsymbol{\gamma},\boldsymbol{Q})$ is a rotation number and $w(\boldsymbol{\gamma},\boldsymbol{Q})$ is the number of complete rotations of $\boldsymbol{\gamma}$ around $\boldsymbol{Q}$. Let $\boldsymbol{\Gamma_i}(t) = (\boldsymbol{\gamma^i}(t), \phi(t))$ for $i = 1,2$ and $t \in [a,b]$ be curves in $\mathbb{R}^3$ which do not intersect each other; here $\phi(t)$ is a smooth monotone function such as $\phi(t) = z(t) = \mathrm{e}^{\varpi t}$ as in (3.2), or $\phi(t) = \zeta(t) = \mathrm{e}^{-\varpi t}$ as in (3.3) or $\phi(t) = t$ as in [2]. Following again [2], we call linking number of $\boldsymbol{\gamma_1}, \boldsymbol{\gamma_2}$ in $[a,b]$ the number $w(\gamma_1 - \gamma_2, (0,0))$, i.e. the number of complete rotations of a curve around the other. We extend the notion to the case $a = -\infty$ (and to the case $b = +\infty$), assuming that the limit $\lim_{t\to-\infty}\theta_\gamma(t)$ exists (respectively the limit $\lim_{t\to+\infty}\theta_\gamma(t)$ exists). In fact we can go back to the usual notion, introduced in [2], simply by a change of parameters: e.g. passing from t either to $z = \mathrm{e}^{\varpi t}$ or to $\zeta = \mathrm{e}^{-\varpi t}$ as independent variable. We stress that we use winding and linking numbers in the case where such a limit is finite, but the argument goes through even when it is infinite.

By construction the linking number is invariant under homotopies which preserve the endpoints of the curves and their ϕ coordinate, and keep the curves disjoint. Let $u^0(r)$ and $v^\infty(r)$ be solutions of (1.4) such that $u^0(r) > 0$ for $r \in (0, R^b)$, and $v^\infty(r) > 0$ for $r \in (R^a, +\infty)$. We set $T^a = \ln(R^a)$, $z^a = \exp[\varpi T^a]$, $\zeta^a = 1/z^a$, $T^b = \ln(R^b)$, $z^b = \exp[\varpi T^b]$, $\zeta^b = 1/z^b$. We denote by $(\boldsymbol{x}^{\mathbf{0}}_{\boldsymbol{l_u}}(t), z(t))$ and $(\boldsymbol{x}^{\boldsymbol{\infty}}_{\boldsymbol{l_u}}(t), z(t))$ the trajectories of (3.2) corresponding respectively to u^0 and v^∞; analogously we denote by $(\boldsymbol{x}^{\mathbf{0}}_{\boldsymbol{l_s}}(t), \zeta(t))$ and $(\boldsymbol{x}^{\boldsymbol{\infty}}_{\boldsymbol{l_s}}(t), \zeta(t))$ the trajectories of (3.3) corresponding respectively to u^0 and v^∞. Consider (3.2) and choose $z > 0$ and $\tau = \ln(z)/\varpi$.
Assume first that $u^0(r)$ is a regular solution and let $\sigma^u(z, s)$ be a continuous parametrization of the branch of $W^u_{l_u}(\tau)$ between the origin and $\boldsymbol{x}^{\mathbf{0}}_{\boldsymbol{l_u}}(\tau)$, i.e. $\sigma^u(z, s) \in W^u_{l_u}(\tau)$ for any $s \in (0, z)$, $\sigma^u(z, 0) = (0, 0)$, $\sigma^u(z, z) = \boldsymbol{x}^{\mathbf{0}}_{\boldsymbol{l_u}}(\tau)$.
Set $\bigwedge = \{(z, s) \mid 0 \le s \le z\}$ and $\sigma^u((0, 0)) = (0, 0)$. We assume w.l.o.g that the function $\sigma^u(z, s) : \bigwedge \to \mathbb{R}^2$ is continuous in both the variables. We denote by $\Sigma^u(z, s) : \bigwedge \to \boldsymbol{W^u_{l_u}}$ the continuous function defined as $\Sigma^u(z, s) = (\sigma^u(z, s), z)$. Note that $\Sigma^u(z, 0) = (0, 0, z)$ and $\Sigma^u(z, z) = (\boldsymbol{x}^{\mathbf{0}}_{\boldsymbol{l_u}}(\tau), z)$.
Now assume that $u^0(r) = u(r, \downarrow)$ and let $\sigma^u(z, s, *)$ be a continuous parametrization of the whole $W^u_{l_u}(\tau)$, i.e. $\sigma^u(z, s, *) \in W^u_{l_u}(\tau)$ for any $s \in (0, 1)$, $\sigma^u(z, 0, *) = (0, 0)$, $\sigma^u(z, 1, *) = \boldsymbol{x}^{\mathbf{0}}_{\boldsymbol{l_u}}(\tau)$. Again we assume w.l.o.g that the function $\sigma^u(z, s, *) : [0, +\infty) \times [0, 1] \to \mathbb{R}^2$ is continuous, and we denote by $\Sigma^u(z, s, *) : [0, +\infty) \times [0, 1] \to \boldsymbol{W^u_{l_u}}$ the continuous function defined as $\Sigma^u(z, s) = (\sigma^u(z, s), z)$. Again $\Sigma^u(z, 0, *) = (0, 0, z)$ and $\Sigma^u(z, 1, *) = (\boldsymbol{x}^{\mathbf{0}}_{\boldsymbol{l_u}}(\tau), z)$, but $\Sigma^u(0, s, *)$ is a parametrization of the whole $W^u_{l_u}(-\infty) \times \{0\}$.

Similarly, when $v^\infty(r)$ is a fast decay solution, we construct a continuous function $\sigma^s(z, s) : \bigwedge \to \mathbb{R}^2$ such that $\sigma^s(z, s) \in W^s_{l_u}(\tau)$ for any $s \in [0, z]$, $\sigma^s(z, 0) = (0, 0)$, $\sigma^s(z, z) = \boldsymbol{x}^{\boldsymbol{\infty}}_{\boldsymbol{l_u}}(\tau)$. Then we denote by $\Sigma^s(z, s) : \bigwedge \to \boldsymbol{W^s_{l_u}}$ the continuous function defined as $\Sigma^s(z, s) = (\sigma^s(z, s), z)$. While when $v^\infty(r) = v(r, \uparrow)$, we construct a continuous function $\sigma^s(z, s, *) : [0, +\infty) \times [0, 1] \to \mathbb{R}^2$ such that $\sigma^s(z, s, *)$ parameterize $W^s_{l_u}(\tau)$ for any $s \in [0, z]$, $\sigma^s(z, 0, *) = (0, 0)$, $\sigma^s(z, 1, *) = \boldsymbol{x}^{\boldsymbol{\infty}}_{\boldsymbol{l_u}}(\tau)$. Then $\Sigma^s(z, s, *) : [0, +\infty) \times [0, 1] \to \boldsymbol{W^s_{l_u}}$ is the continuous function defined by $\Sigma^s(z, s, *) = (\sigma^s(z, s, *), z)$.

Analogously when we work with (3.3) we fix $\zeta > 0$ and $\tau = -\ln(\zeta)/\varpi$; if $u^0(r)$ and $v^\infty(r)$ are respectively a regular and a fast decay solution, we construct a continuous function $\delta^s(\zeta, s) : \bigwedge \to \mathbb{R}^2$ and $\delta^u(\zeta, s) : \bigwedge \to \mathbb{R}^2$ such that $\delta^s(\zeta, s) \in W^s_{l_s}(\tau)$ for any $s \in [0, \zeta]$, $\delta^s(\zeta, 0) = (0, 0)$, $\delta^s(\zeta, \zeta) = \boldsymbol{x}^{\boldsymbol{\infty}}_{\boldsymbol{l_s}}(\tau)$, and $\delta^u(\zeta, s) \in W^u_{l_s}(\tau)$ for any $s \in [0, \zeta]$, $\delta^u(\zeta, 0) = (0, 0)$, $\delta^u(\zeta, \zeta) = \boldsymbol{x}^{\mathbf{0}}_{\boldsymbol{l_s}}(\tau)$. Then we define $\Delta^s(\zeta, s) = (\delta^s(\zeta, s), \zeta)$ and $\Delta^u(\zeta, s) = (\delta^u(\zeta, s), \zeta)$. When $u^0(r) = u(r, \downarrow)$ and $v^\infty(r) = u(r, \uparrow)$, reasoning as above we construct $\Delta^u(\zeta, s, *) = (\delta^u(\zeta, s, *), \zeta)$ and $\Delta^s(\zeta, s, *) = (\delta^s(\zeta, s, *), \zeta)$, where $\delta^u(\zeta, s, *)$ and $\delta^s(\zeta, s, *)$ for $s \in [0, 1]$ are parameterizations of the whole manifolds $W^u_{l_s}(\tau)$ and $W^s_{l_s}(\tau)$ respectively.

We are ready to state the following key result inspired by proposition 1.4 of [2].

Proposition 4.1. *Assume $\boldsymbol{G_u}$ and $\boldsymbol{G_s}$ with $2_* < l_s < 2^* < l_u < \infty$. Assume that there are $0 < R^a < R^b < \infty$, a solution $u^0(r)$ defined and positive in $(0, R^b)$ and a solution $v^\infty(r)$ of (1.4) defined and positive in (R^a, ∞). Assume that $u^0 \not\equiv v^\infty$ and that $u^0 - v^\infty$ has at least $2k + 1$ zeroes in (R^a, R^b) for some $k \ge 1$.*
If there is $R^1 < R^a$ such that $v^\infty(R^1) = 0$, then the winding number of $s \to \sigma^u(Z, s)$ around $\boldsymbol{x}^{\boldsymbol{\infty}}_{\boldsymbol{l_u}}(T)$ is equal or smaller than $-k$, for any $Z = \exp[\varpi T] \ge z^b$. Similarly, if there is $R^2 > R^b$ such that $u^0(R^2) = 0$ the winding number of $s \to \delta^s(\zeta, s)$ around $\boldsymbol{x}^{\mathbf{0}}_{\boldsymbol{l_s}}(T)$ is equal or larger than k, for any $\zeta = \exp[-\varpi T] \ge \zeta^a$.

Proposition 4.1 is very similar to proposition 1.4 of [2]. However in the proof of proposition 1.4 in [2] the authors require that both u^0 and v^∞ have a non-degenerate zero, while we need just one of them to have this property. In fact such an assumption is not explicitly required in the statement of proposition 1.4 in [2]: such a discordance does not affect the proof of existence of G.S. with f.d., but generates confusion in the proof of theorem 1.3 (of this article but proved in [2]), which is the analogous of theorem 2.4 (proved in this article).

We divide the proof of proposition 4.1 in Lemmas 4.2 and 4.3. The former is obtained repeating word by word Lemma 3.2 in [2], the latter is obtained adapting and simplifying Lemma 3.1 in [2], keeping the main ideas.

Lemma 4.2. *If $u^0 - v^\infty$ has at least $2k+1$ zeroes in (R^a, R^b), then the linking number of the curves $\boldsymbol{x}^{\mathbf{0}}_{\boldsymbol{l_u}}(t)$ and $\boldsymbol{x}^{\boldsymbol{\infty}}_{\boldsymbol{l_u}}(t)$ in (T^a, T^b) is equal or smaller than $-k$.*

Lemma 4.3. *Assume that the linking number of $\boldsymbol{x}^{\mathbf{0}}_{\boldsymbol{l_u}}(t)$ and $\boldsymbol{x}^{\boldsymbol{\infty}}_{\boldsymbol{l_u}}(t)$ in $[T^a, T^b]$ is $-k$.*

If there is $R^1 < R^a$ such that $v^\infty(R^1) = 0$, then the winding number W of $s \to \sigma^u(z,s)$ around $\boldsymbol{x}^{\boldsymbol{\infty}}_{\boldsymbol{l_u}}(T)$ is equal or smaller than $-k$ for any $z \geq z^b$.
Similarly if there is $R^2 > R^b$ such that $u^0(R^2) = 0$ then the winding number of $s \to \delta^s(\zeta, s)$ around $\boldsymbol{x}^{\mathbf{0}}_{\boldsymbol{l_u}}(T)$ is at least k for any $\zeta \geq \zeta^a$.

Proof of Lemma 4.2. The function $h(t) = x^0_{l_u}(t) - x^\infty_{l_u}(t)$ solves a non-autonomous 2nd order linear equation of the form:

$$\ddot{h} - (\alpha_{l_u} + \gamma_{l_u})\dot{h} + a(t)h = 0\,, \tag{4.2}$$

and it has $2k+1$ zeroes. Since the flow of the first order system associated to (4.2) points clockwise on the $\dot{h}$ axis, it follows that $(h, \dot{h})$ cannot make a complete rotation counterclockwise. Therefore we can count the rotations of $(h(t), \dot{h}(t))$ around the origin by the zeroes of $h(t)$, so the Lemma is proved. □

Proof of Lemma 4.3. Assume that there is $R^1 \in (0, R^a)$ such that $v^\infty(R^1) = 0$. From Lemma 4.2 it follows that the linking number of the curves $\boldsymbol{x}^{\mathbf{0}}_{\boldsymbol{l_u}}(t)$ and $\boldsymbol{x}^{\boldsymbol{\infty}}_{\boldsymbol{l_u}}(t)$ in the interval $[T^a, T^b]$ decreases as the interval increases. So the linking number L of $\boldsymbol{x}^{\mathbf{0}}_{\boldsymbol{l_u}}(t)$ and $\boldsymbol{x}^{\boldsymbol{\infty}}_{\boldsymbol{l_u}}(t)$ in $(-\infty, T^b]$ satisfies $L \leq -k$. The proof of Lemma 4.3 is based on the homotopies $HO(Z,S)$ and $HO^*(Z,S)$ depicted in pictures 2 and 3, between $(\boldsymbol{x}^{\mathbf{0}}_{\boldsymbol{l_u}}(t), z(t))$ and the curve obtained following the curve $\Gamma^u(s)$ (and $\Gamma^u(s,*)$) to be defined below. Roughly speaking $\Gamma^u(s)$ (and $\Gamma^u(s,*)$) is obtained following the segment between the origin and the point $(0,0,z_b)$, and then the manifold $W^u(\tau^b)$ between the origin and $(\boldsymbol{x}^{\mathbf{0}}_{\boldsymbol{l_u}}(T^b), z^b)$. We choose $HO(Z,S)$ so that $HO(Z,S) \in \boldsymbol{W}^{\boldsymbol{u}}_{\boldsymbol{l_u}}$, hence it does not intersect the curve $(\boldsymbol{x}^{\boldsymbol{\infty}}_{\boldsymbol{l_u}}(t), z(t))$. Thus $|L|$ equals the number of rotations R of $\Gamma^u(s)$ around $(\boldsymbol{x}^{\boldsymbol{\infty}}_{\boldsymbol{l_u}}(t), z(t))$. Then we show that the winding number W of $s \to \sigma^u(z_b, s)$ around $\boldsymbol{x}^{\boldsymbol{\infty}}_{\boldsymbol{l_u}}(T^b)$ equals either $-R$ or $-R-1$.

The leading idea in the construction of $HO(z,s)$ is the following. Since $v^\infty(r)$ is not a G.S. with f.d. $(\boldsymbol{x}^{\boldsymbol{\infty}}_{\boldsymbol{l_u}}(t), z(t))$ does not intersect the 2-dimensional manifold $\boldsymbol{W}^{\boldsymbol{u}}_{\boldsymbol{l_u}}$: so we can project $(\boldsymbol{x}^{\mathbf{0}}_{\boldsymbol{l_u}}(t), z(t))$ on $W^u_{l_u}(\tau_b) \times \{z_b\}$ following the manifold $\boldsymbol{W}^{\boldsymbol{u}}_{\boldsymbol{l_u}}$, and the homotopy is readily constructed.

We distinguish the case where $\boldsymbol{x}^{\mathbf{0}}_{\boldsymbol{l_u}}(t)$ corresponds to a regular solution of (1.4), from the case where $\boldsymbol{x}^{\mathbf{0}}_{\boldsymbol{l_u}}(t)$ coincides with $\boldsymbol{x}^{\boldsymbol{u}}_{\boldsymbol{l_u}}(t, \downarrow)$ so it corresponds to a singular solution. We begin from the former, so we define the curve

$$\Gamma^u(s) := \begin{cases} (0,0,z^b+s) & \text{for } s \in [-z^b, 0] \\ \Sigma^u(z^b, s) & \text{for } s \in [0, z^b] \end{cases}$$

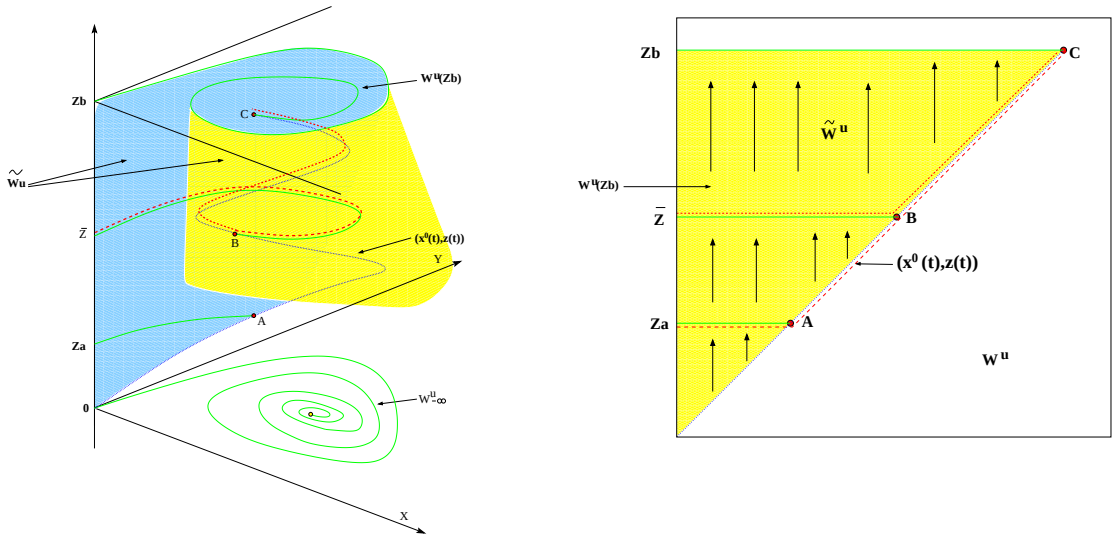

FIGURE 2. This picture gives a further explanation of the construction of the homotopy $ho(z,s)$ and $HO(Z,S)$, in the case where $\boldsymbol{x}^{\mathbf{0}}_{\boldsymbol{l_u}}(t)$ corresponds to a regular solution. Set $z = \mathrm{e}^{\omega\tau}$; the curves $s \to ho(z,s)$ for $s \in [0, z_b]$ are obtained following the manifold $W^u_{l_u}(\tau) \times \{z\}$ from $(0,0,z)$ to $(\boldsymbol{x}^{\mathbf{0}}_{\boldsymbol{l_u}}(\tau), z)$, then following $(\boldsymbol{x}^{\mathbf{0}}_{\boldsymbol{l_u}}(t), z(t))$, for $t \in [\tau, \tau_b]$. On the left we have a 3-dimensional sketch of system (3.2) and of the objects involved in the construction; on the right we have flattened the 2-dimensional manifold $\boldsymbol{W}^{\boldsymbol{u}}_{\boldsymbol{l_u}}$ and represented it on a plane. We have denoted by $\tilde{W}^u$ the 2-dimensional manifold (filled with a yellow pattern) which is the open connected subset of $\boldsymbol{W}^{\boldsymbol{u}}_{\boldsymbol{l_u}}$ between the z-axis and the trajectory $(\boldsymbol{x}^{\mathbf{0}}_{\boldsymbol{l_u}}(t), z(t))$ for $t \le \tau_b$ (denoted by a blue dotted line). In fact $\tilde{W}^u$ is the image of $ho(z,s)$ for $(z,s) \in \bigwedge$. The (green) solid lines indicate the branches of the 1-dimensional manifolds $W^u_{l_u}(\tau) \times \{z(\tau)\}$ between the origin and $(\boldsymbol{x}^{\mathbf{0}}_{\boldsymbol{l_u}}(\tau), z(\tau))$, at different values (i.e. $\tau = \ln(Z_a/\varpi)$, $\tau = \ln(\bar{Z}/\varpi)$, $\tau = \ln(Z_b/\varpi)$). We have denoted with the (red) dashed lines the curves $s \to ho(z,s)$, for $z = \bar{z}$ and on the right for $z = z_a$, too. The homotopy $HO(Z,S)$ between $(\boldsymbol{x}^{\mathbf{0}}_{\boldsymbol{l_u}}(t), z(t))$ for $t \le \tau_b$ and the parametrization $\Sigma^u(Z_b, s)$ of the branch of $W^u_{l_u}(\tau_b)$ is obtained through the projection depicted on the right. Since at each step the homotopic curves lie on the 2-dimensional manifold $\boldsymbol{W^u}$, it follows that $H0(Z,S)$ does not cross the curve $(\boldsymbol{x}^{\boldsymbol{\infty}}_{\boldsymbol{l_u}}(t), z(t))$ for $t \le \tau_b$; in fact such a curve does not intersect $\boldsymbol{W^u}$ for any $t \in \mathbb{R}$.

Let us denote by Θ_1 and Θ_2 the angular numbers $\Theta_1 := \Theta(\boldsymbol{x}^{\boldsymbol{\infty}}_{\boldsymbol{l_u}}(t), (0,0))$ for $t \le T^b$, and by $\Theta_2 := \Theta(\sigma^u(z^b, s), \boldsymbol{x}^{\boldsymbol{\infty}}_{\boldsymbol{l_u}}(T^b))$ for $s \in [0, z^b]$. Then

$$R = [\Theta_2 - \Theta_1]\,, \qquad W = [\Theta_2]\,, \tag{4.3}$$

where $[a]$ denotes the integer part of a. To construct $HO(Z,S)$, we begin by constructing the homotopy $ho(z,s)$, between a curve equivalent to $(\boldsymbol{x}^{\mathbf{0}}_{\boldsymbol{l_u}}(t), z(t))$ for $t < T^b$ and the branch of $W^u(T^b) \times \{z^b\}$ going from $(0,0,z^b)$ to $(\boldsymbol{x}^{\mathbf{0}}_{\boldsymbol{l_u}}(T^b), z^b)$. More

precisely we define the continuous function

$$ho(z,s) = \begin{cases} \Sigma^u(z,s) & \text{if } 0 \le s \le z \\ (\boldsymbol{x}^{\mathbf{0}}_{\boldsymbol{l_u}}(\frac{\ln(s)}{\varpi}), s) & \text{if } z < s \le z^b \end{cases} .$$

Roughly speaking if we set $\mathrm{e}^{\varpi\tau} = z(\tau)$, then $s \to ho(z(\tau), s)$ is obtained following $W^u_{l_u}(\tau) \times \{z(\tau)\}$ from the origin towards $(\boldsymbol{x}^{\mathbf{0}}_{\boldsymbol{l_u}}(\tau), z(\tau))$, and then following $(\boldsymbol{x}^{\mathbf{0}}_{\boldsymbol{l_u}}(t), z(t))$ for $t \in [\tau, T^b]$.

Note that $ho(0, z(t)) = (\boldsymbol{x}^{\mathbf{0}}_{\boldsymbol{l_u}}(t), z(t))$ for $t \le T^b$, while $ho(z^b, s)$ for $0 \le s \le z^b$ is a parametrization of the branch of $W^u(T^b) \times \{z^b\}$ between $(0,0,z^b)$ and $(\boldsymbol{x}^{\mathbf{0}}_{\boldsymbol{l_u}}(T^b), z^b)$; moreover $ho(0,0) = (0,0,0)$. Then we define the homotopy $HO(Z,S) : [0, z^b] \times [-z^b, z^b] \to \boldsymbol{W}^{\boldsymbol{u}}_{\boldsymbol{l_u}}$ by

$$HO(Z,S) = \begin{cases} (0,0,(Z+S)_+) & \text{if } 0 \le Z \le z^b \quad \text{and} \quad -z^b \le S \le 0 \\ ho(Z,S) & \text{if } 0 \le Z \le z^b \quad \text{and} \quad 0 \le S \le z^b \end{cases} ,$$

so that $HO(Z, -z^b) = (0,0,0)$ and $HO(Z, z^b) = ho(Z, z^b) = (\boldsymbol{x}^{\mathbf{0}}_{\boldsymbol{l_u}}(T^b), z^b)$ for any $Z \in [0, z^b]$. Hence the endpoints of the homotopy $s \to HO(Z,s)$ are the endpoints of the curves $(\boldsymbol{x}^{\mathbf{0}}_{\boldsymbol{l_u}}(t), z(t))$ and $\Gamma^u(s)$, for any $Z \in [0, z^b]$. Moreover $HO(0,S) \equiv (\boldsymbol{x}^{\mathbf{0}}_{\boldsymbol{l_u}}(\frac{\ln(S)}{\varpi}), S)$, and $HO(z^b, S) \equiv \Sigma^u(z^b, S)$ whenever $S \in [0, z^b]$. Furthermore $HO(Z,S) \in \boldsymbol{W}^{\boldsymbol{u}}_{\boldsymbol{l_u}}$, so it does not intersect the image of $(\boldsymbol{x}^{\boldsymbol{\infty}}_{\boldsymbol{l_u}}(t), z(t))$.
Hence from the invariance for homotopy we see that the number of rotations R of $\Gamma^u(s)$ around $(\boldsymbol{x}^{\boldsymbol{\infty}}_{\boldsymbol{l_u}}(t), z(t))$ satisfies $L = -R$.

Set $T^1 = \ln(R^1)$ from a straightforward computation it follows that the angular number of $\boldsymbol{x}^{\boldsymbol{\infty}}_{\boldsymbol{l_u}}(t)$ with respect to the origin in $(-\infty, T^1]$ equals $-(\frac{\arctan(n-2)}{2\pi}) \in (-1/4, 0)$. Hence $-3/4 \le \Theta_1 \le -1/4$.
So, from (4.3) we find

$$W = [\Theta_2] \le -R = L \le -k ,$$

and this concludes the first part of the proof of Lemma 4.3.

Now we assume $\boldsymbol{x}^{\mathbf{0}}_{\boldsymbol{l_u}}(t) \equiv \boldsymbol{x}^{\boldsymbol{u}}_{\boldsymbol{l_u}}(\downarrow, t)$. The argument has to be modified slightly since $\boldsymbol{x}^{\mathbf{0}}_{\boldsymbol{l_u}}(t) \not\to (0,0)$ as $t \to -\infty$. We introduce the curve $\Gamma^u(s,*)$ as follows:

$$\Gamma^u(s,*) := \begin{cases} (0,0,z^b+s) & \text{for } s \in [-z^b, 0] \\ \Sigma^u(z^b, s, *) & \text{for } s \in [0,1] \end{cases}$$

We introduce the homotopy $HO^*(z,s) : [0, z_b] \times [-2, 1] \to \boldsymbol{W}^{\boldsymbol{u}}_{\boldsymbol{l_u}}$ defined as follows:

$$HO^*(z,s) = \begin{cases} (0,0,z(s+2)) & \text{if } -2 \le s \le -1 \\ \Sigma^u(z, s+1, *) & \text{if } -1 \le s \le 0 \\ \left(\boldsymbol{x}^{\mathbf{0}}_{\boldsymbol{l_u}}(\frac{\ln((1-s)z+sz_b)}{\varpi}), (1-s)z + sz_b\right) & \text{if } 0 \le s \le 1 \end{cases} ;$$

We stress that the function $s \to HO^*(z,s)$ is obtained following the z axis from the origin to $(0,0,z)$, then following $W^u_{l_u}(\tau) \times \{z\}$ from the origin to $(\boldsymbol{x}^{\boldsymbol{u}}_{\boldsymbol{l_u}}(\tau, \downarrow), z(\tau))$, then following $(\boldsymbol{x}^{\boldsymbol{u}}_{\boldsymbol{l_u}}(t, \downarrow), z(t))$ for $t \in [\tau, \tau_b]$. In particular the function $\Upsilon(s) := HO^*(0,s)$ is obtained following the manifold $W^u_{l_u}(-\infty) \times \{0\}$ from the origin to $\boldsymbol{P}^{\boldsymbol{u}}_{\boldsymbol{l_u}}(-\infty)$, then following $(\boldsymbol{x}^{\boldsymbol{u}}_{\boldsymbol{l_u}}(t, \downarrow), z(t))$ for $t \in (-\infty, \tau_b]$. Thus $\Upsilon(s)$ is homotopic to $HO^*(z^b, s) = \Gamma^u(s,*)$. Moreover the homotopy $HO^*(z,s)$ preserves the endpoints, i.e. $HO^*(z,-2) = (0,0,0)$ and $HO^*(z,1) = (\boldsymbol{x}^{\boldsymbol{u}}_{\boldsymbol{l_u}}(\tau_b, \downarrow), z_b)$ for any $z \in [0, z^b]$. Furthermore $HO^*(z,s) \in \boldsymbol{W}^{\boldsymbol{u}}_{\boldsymbol{l_u}}$ for any (z,s) so it does not cross the curve $(\boldsymbol{x}^{\boldsymbol{\infty}}_{\boldsymbol{l_u}}(t), z(t))$ for any $t \in \mathbb{R}$. Thus the number R of complete rotations of $\Gamma^u(s,*)$ around $(\boldsymbol{x}^{\boldsymbol{\infty}}_{\boldsymbol{l_u}}(t), z(t))$ equals the number of complete rotations of $\Upsilon(s)$ around $(\boldsymbol{x}^{\boldsymbol{\infty}}_{\boldsymbol{l_u}}(t), z(t))$.

Since $x^\infty_{l_u}(t) < 0 < y^\infty_{l_u}(t)$ for $t \le T_1$ and $s \to \sigma^u(0,s,*) \subset \mathbb{R}^2_\pm$ for $s \in (0,1]$, we see that the angular number $\Theta(t)$ of $s \to \sigma^u(0,s,*)$ around $\boldsymbol{x^\infty_{l_u}}(t)$ in the interval $(-\infty; T^1]$, satisfies $\Theta(t) \in [-3/4, -1/4]$ whenever $t \le T^1$, and converges to a finite value $\Theta^\Upsilon \in [-3/4, -1/4]$ as $t \to -\infty$.

Denote by Θ^Γ the angular number of $s \to \sigma^u(z^b, s, *)$ around $\boldsymbol{x^\infty_{l_u}}(T^b)$; note that the linking number of the z axis and $(\boldsymbol{x^\infty_{l_u}}(t), z(t)))$ equals $[-\Theta_1] \in [1/4; 3/4]$; denote by Θ^L the angular number of $\boldsymbol{x^0_{l_u}}(t) - \boldsymbol{x^\infty_{l_u}}(t)$ with respect to the origin for $t \in [T^1; T^b]$, and note that $[\Theta_L + \Theta^\Upsilon] = L$. So, using the invariance for homotopies we see that

$$\Theta^\Upsilon + \Theta^L = -\Theta_1 + \Theta^\Gamma\,;$$

hence $W = [\Theta^\Gamma] = [\Theta^L + \Theta^\Upsilon + \Theta_1]$. Thus $W \le L = -k$.

Once again the converse result can be obtained either from Kelvin inversion, or simply repeating the argument for $W^s_{l_s}(\tau^a) \times \{z^a\}$ and $\boldsymbol{x^0_{l_s}}(t)$. $\square$ From a careful analysis of the previous proof we see that, if we are in the hypotheses of Lemma 4.3 then $L \in \{-k-1, -k\}$ if $\boldsymbol{x^0_{l_s}}(t)$ corresponds to a regular solution, while $L \in \{-k-2, -k-1, -k\}$ if $\boldsymbol{x^0_{l_s}}(t) \equiv (\boldsymbol{x^u_{l_u}}(t, \downarrow), z(t))$.

4.2. **Proof of theorem 2.2.** In this subsection we always assume $\boldsymbol{G_u}$ and $\boldsymbol{G_s}$ with $2_* < l_s < 2^* < l_u$. The proofs we are going to discuss are achieved perturbing the auxiliary critical systems (4.4) and (4.6) we are just going to introduce.

Let us consider $f = f(u,r)$ and the corresponding system (3.2) with $l = l_u$ (and $g_{l_u}(x,t)$ defined by (2.1)). Then we consider the system obtained from the previous one replacing $(\alpha_{l_u}, \gamma_{l_u})$ by (α_l, γ_l) (and maintaining $g = g_{l_u}(x,t)$), i.e.:

$$\begin{pmatrix} \dot{x}^m_l \\ \dot{y}^m_l \\ \dot{z} \end{pmatrix} = \begin{pmatrix} \alpha_l & 1 & 0 \\ 0 & \gamma_l & 0 \\ 0 & 0 & \varpi \end{pmatrix} \begin{pmatrix} x^m_l \\ y^m_l \\ \zeta \end{pmatrix} + \begin{pmatrix} 0 \\ -g_{l_u}(x^m_l, \frac{\ln(z)}{\varpi}) \\ 0 \end{pmatrix} \tag{4.4}$$

We denote with the apex ^{m} quantities referred to systems (4.4) and to the corresponding modified equation (1.4). Set $l = 2^*$ in (4.4); then (4.4) corresponds to (1.4) where we have replaced f by a suitable functions $f^m(u,r)$, i.e.

$$f^m(u,r) := f(ur^{\varepsilon(n-2)\alpha_{l_u}/4}, r) r^{-\varepsilon(n-2)\alpha_{l_u}/4}\,, \tag{4.5}$$

where $\varepsilon = l_u - 2^* > 0$ so that $\alpha_{2^*} - \alpha_{l_u} = \varepsilon(n-2)\alpha_{l_u}/4$. We recall that systems (4.4) admits a critical point $(\boldsymbol{P^m_{2^*}}(-\infty), 0)$ different from the origin and that there is a unique trajectory, denoted by $(\boldsymbol{x^{u,m}_l}(t, \downarrow), z(t))$, whose graph gives the unstable manifold of the critical point $(\boldsymbol{P^m_{2^*}}(-\infty), 0)$ in (4.4). So we can apply proposition 3.8 to (4.4) (i.e. to (1.4) where $f = f^m$) to obtain the following.

Remark 4.4. Assume $\boldsymbol{G_u}, \boldsymbol{G_s}, \boldsymbol{A_u}$ with $2_* < l_s < 2^* < l_u$, and consider system (4.4) where $l = 2^*$. Then all the regular solutions are crossing and all the fast decay solutions are S.G.S. with f.d. All the trajectories $(\boldsymbol{x^m_{2^*}}(t), z(t))$ of (4.4) corresponding to singular solutions are such that $\boldsymbol{x^m_{2^*}}(t)$ rotates indefinitely around the critical point $(\boldsymbol{P^m_{2^*}}(-\infty), 0)$ as $t \to -\infty$, possibly apart from $\boldsymbol{x^{u,m}_{2^*}}(t, \downarrow)$.

Set $\boldsymbol{P^m_{2^*}}(-\infty) = (P^m_x(-\infty), P^m_y(-\infty))$. We stress that a priori $x^{u,m}_{2^*}(t, \downarrow) - P^m_x(-\infty)$ may have no zeroes or just a finite number of zeroes: this fact causes some technical difficulties, which affects the proof of theorem 1.3 borrowed from [2].

Similarly we introduce the following analogous system:

$$\begin{pmatrix} \dot{x}^m_l \\ \dot{y}^m_l \\ \dot{\zeta} \end{pmatrix} = \begin{pmatrix} \alpha_l & 1 & 0 \\ 0 & \gamma_l & 0 \\ 0 & 0 & -\varpi \end{pmatrix} \begin{pmatrix} x^m_l \\ y^m_l \\ \zeta \end{pmatrix} + \begin{pmatrix} 0 \\ -g_{l_s}(x^m_l, -\frac{\ln(\zeta)}{\varpi}) \\ 0 \end{pmatrix} \tag{4.6}$$

Now we are ready to prove theorem 2.2.

Proof of theorem 2.2. Assume $\boldsymbol{A_u}, \boldsymbol{G_u}$ and $\boldsymbol{G_s}$ with $2_* < l_s < 2^*$. We stress that

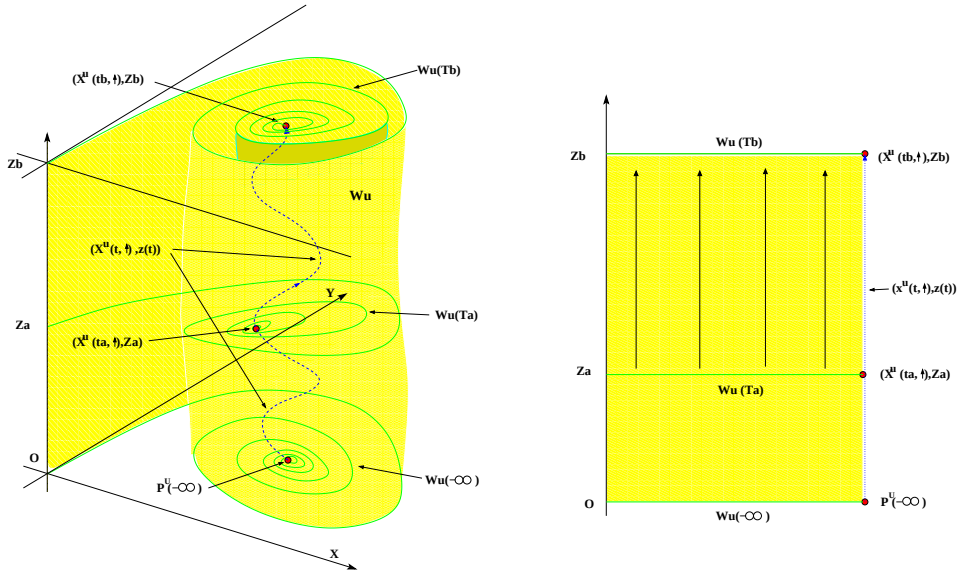

FIGURE 3. In this picture we explain the construction of the homotopy $HO(Z,S)$, in the case where $\boldsymbol{x}^{\mathbf{0}}_{\boldsymbol{l_u}}(t)$ corresponds to the singular solution $u(r,\downarrow)$. On the left we have a 3-dimensional sketch of system (3.2) and of the objects involved in the construction; on the right we have flattened the 2-dimensional manifolds $\boldsymbol{W}^{\boldsymbol{u}}_{\boldsymbol{l_u}}$ and represented it on a plane. Here between the z-axis and the trajectory $(\boldsymbol{x}^{\mathbf{0}}_{\boldsymbol{l_u}}(t), z(t))$ for $t \le \tau_b$ we have the whole 2-dimensional manifold $\boldsymbol{W}^{\boldsymbol{u}}_{\boldsymbol{l_u}}$ (filled with a yellow pattern). The (green) solid lines indicate the 1-dimensional manifolds $W^u_{l_u}(\tau) \times \{z(\tau)\}$ between the origin and $(\boldsymbol{x}^{\mathbf{0}}_{\boldsymbol{l_u}}(\tau), z(\tau))$, at different values (i.e. $\tau = \ln(Z_a/\varpi)$, $\tau = \ln(Z_b/\varpi)$). Again the homotopy $HO(Z,S)$ between $(\boldsymbol{x}^{\mathbf{0}}_{\boldsymbol{l_u}}(t), z(t))$ for $t \le \tau_b$ and the parametrization $\Sigma^u(Z_b, s)$ of the branch of $W^u_{l_u}(\tau_b)$ is obtained through the projection depicted on the right. Since at each step the homotopic curves lie on the 2-dimensional manifold $\boldsymbol{W^u}$ it follows that $H0(Z,S)$ does not cross the curve $(\boldsymbol{x}^{\boldsymbol{\infty}}_{\boldsymbol{l_u}}(t), z(t))$ for $t \le \tau_b$, which does not intersect $\boldsymbol{W^u}$ for any $t \in \mathbb{R}$.

the existence of a G.S. with f.d. corresponds to the existence of an intersection between $W^u_{l_u}(\tau)$ and $W^s_{l_u}(\tau)$ or equivalently of an intersection between $W^u_{l_s}(\tau)$ and $W^s_{l_s}(\tau)$, for some $\tau \in \mathbb{R}$. Let us set $\varepsilon = l_u - 2^* > 0$; then consider system (4.4) where $l = 2^*$: again (4.4) corresponds to (1.4) where we have replaced f by $f^m(u,r)$ given by (4.5) and we denote with the apex ^{m} quantities referred to this equation and the corresponding systems (4.4). By construction $f^m(u,r)$ satisfies $\boldsymbol{G_u}$ with $l^m_u = 2^*$ and $\boldsymbol{G_s}$ with $l^m_s < l_s$. Moreover $0 < l_s - l^m_s = O(\varepsilon)$. Denote by $W^s_{l_s}(\tau)$ the stable manifold and by $W^u_{l_u}(\tau)$ the unstable manifold of system (2.2) obtained from the original f; we denote by $W^{u,m}_{2^*}(\tau)$ the unstable manifold obtained from (4.4) where $l = 2^*$ and by $W^{s,m}_{l^m_s}(\tau)$ the stable manifold obtained from (4.6) where $l = l^m_s$, corresponding to (1.4) with $f = f^m$. Set $D = \alpha_{l_s} - \alpha_{l_u} > 0$ and $D^m = \alpha_{l^m_s} - \alpha_{2^*} > 0$, it is straightforward to check that $D - D^m = O(\varepsilon)$ (in fact if f is of type (1.2) then $D = D^m$).

Let $\tau^b > 0$ to be fixed later and $\zeta^b = \exp[-\tau^b]$; let $\boldsymbol{Q^m}$ be a point in $W^{s,m}_{l^m_s}(\tau^b)$, and correspondingly $\boldsymbol{S^m} = \mathrm{e}^{-D^m\tau^b}\boldsymbol{Q^m} \in W^{s,m}_{2^*}(\tau^b)$. Consider the trajectory

$\boldsymbol{x}^{\boldsymbol{m}}_{\boldsymbol{l}^{\boldsymbol{m}}_{\boldsymbol{s}}}(t,\tau^b;\boldsymbol{Q}^{\boldsymbol{m}})$ of (4.6), the corresponding solution $u^m(r)$ of the auxiliary equation (1.4) where $f = f^m$, and the corresponding trajectory $\boldsymbol{x}^{\boldsymbol{m}}_{\boldsymbol{2^*}}(t,\tau^b;\boldsymbol{S}^{\boldsymbol{m}})$, of (4.4). From Remark 4.4 we see that u^m is a S.G.S. with fast decay and we can assume w.l.o.g. that $\boldsymbol{x}^{\boldsymbol{m}}_{\boldsymbol{2^*}}(t,\tau^b;\boldsymbol{S}^{\boldsymbol{m}})$ rotates indefinitely around the critical point $\boldsymbol{P}^{\boldsymbol{m}}(-\infty) = (P^m_x(-\infty), P^m_y(-\infty))$, where $(\boldsymbol{P}^{\boldsymbol{m}}(-\infty),0)$ is the critical point of (4.4). Therefore $x^m_{2^*}(t,\tau^b;\boldsymbol{S}^{\boldsymbol{m}}) - P^m_x(-\infty)$ has infinitely many zeroes in $(-\infty,\tau^b]$. We denote by $(\boldsymbol{x}^{\boldsymbol{u},\boldsymbol{m}}_{\boldsymbol{l}^{\boldsymbol{m}}_{\boldsymbol{u}}}(t;\downarrow), z(t))$ the unique trajectory of the unstable manifold of the critical point $(\boldsymbol{P}^{\boldsymbol{m}}(-\infty),0)$ of (4.4) with $l = 2^*$, and by $(\boldsymbol{x}^{\boldsymbol{u}}_{\boldsymbol{l_u}}(t;\downarrow), z(t))$ the unique trajectory of the unstable manifold of $(\boldsymbol{P}(-\infty),0)$ of the original system (3.2). It follows that $x^m_{2^*}(t,\tau^b;\boldsymbol{S}^{\boldsymbol{m}}) - x^{u,m}_{2^*}(t;\downarrow)$ has infinitely many zeroes in $(-\infty,\tau^b]$. So we have proved the following:

1) *For any $k \in \mathbb{N}$ we can find $\tau^a < \tau^b$ such that $x^m_{2^*}(t,\tau^b;\boldsymbol{S}^{\boldsymbol{m}}) - x^{u,m}_{2^*}(t;\downarrow)$ has at least $2k+1$ zeroes in $[\tau^a,\tau^b]$.*

The second step is to prove the following claim:

2) *There is $\boldsymbol{Q} \in W^s_{l_s}(\tau^b)\setminus W^u_{l_s}(\tau^b)$ such that $x_{l_s}(t,\tau^b;\boldsymbol{Q}) - x^u_{l_s}(t;\downarrow)$ has at least $2k+1$ zeroes in $[\tau^a,\tau^b]$.*

In fact for any $\sigma > 0$ we can choose τ^b large enough so that there is $\boldsymbol{R}^{\boldsymbol{m}} \in W^{s,m}_{l^m_s}(+\infty)$ such that $\|\boldsymbol{R}^{\boldsymbol{m}} - \boldsymbol{Q}^{\boldsymbol{m}}\| < \sigma/3$. Using continuous dependence on parameters, we see that we can choose $\varepsilon > 0$ small enough so that there is $\boldsymbol{R} \in W^s_{l_s}(+\infty)$ such that $\|\boldsymbol{R} - \boldsymbol{R}^{\boldsymbol{m}}\| < \sigma/3$. Then, possibly choosing a larger τ^b we can find $\boldsymbol{Q} \in W^s_{l_s}(\tau^b)$ such that $\|\boldsymbol{Q} - \boldsymbol{R}\| < \sigma/3$, too; hence we get $\|\boldsymbol{Q} - \boldsymbol{Q}^{\boldsymbol{m}}\| < \sigma$ where $\sigma = \sigma(\varepsilon,\tau^b)$. We can assume w.l.o.g. that $\boldsymbol{Q} \in W^s_{l_s}(\tau^b)\cap\mathbb{R}^2_{\pm}$ and $\boldsymbol{Q} \notin W^u_{l_s}(\tau^b)$, otherwise infinitely many G.S. with f.d. exist and we have concluded. Let $v^\infty(r)$ denote the fast decay solution of (1.4) corresponding to $\boldsymbol{x}_{\boldsymbol{l_s}}(t,\tau^b;\boldsymbol{Q})$ and let $\boldsymbol{x}_{\boldsymbol{l_u}}(t,\tau^b;\boldsymbol{S})$ be the corresponding trajectory of (3.2), so that $\boldsymbol{S} \in W^s_{l_u}(\tau^b)$. Then $\boldsymbol{x}_{\boldsymbol{l_s}}(t,\tau^b;\boldsymbol{Q})$ converges to $\boldsymbol{x}^{\boldsymbol{m}}_{\boldsymbol{l}^{\boldsymbol{m}}_{\boldsymbol{s}}}(t,\tau^b;\boldsymbol{Q}^{\boldsymbol{m}})$ as $\varepsilon \to 0$ uniformly in $[\tau^a,\infty)$. Thus for any fixed τ^b $\boldsymbol{x}_{\boldsymbol{l_u}}(t,\tau^b;\boldsymbol{S})$ converges to $\boldsymbol{x}^{\boldsymbol{m}}_{\boldsymbol{2^*}}(t,\tau^b;\boldsymbol{S}^{\boldsymbol{m}})$ as $\varepsilon \to 0$ uniformly in $[\tau^a,\tau^b]$.

Using continuous dependence on parameters and the fact that $\boldsymbol{P}^{\boldsymbol{m}}(-\infty)$ tends to $\boldsymbol{P}(-\infty)$ as $\varepsilon \to 0$, we see that $\boldsymbol{x}^{\boldsymbol{u},\boldsymbol{m}}_{\boldsymbol{2^*}}(t;\downarrow) - \boldsymbol{x}^{\boldsymbol{u}}_{\boldsymbol{l_u}}(t;\downarrow)$ tends to 0 as $\varepsilon \to 0$ uniformly with respect to $t \in [\tau^a,\tau^b]$. Using these two uniform convergence arguments and point **1)**, we see that $x_{l_u}(t,\tau^b;\boldsymbol{S}) - x^u_{l_u}(t;\downarrow)$ has at least $2k+1$ zeroes in $[\tau^a,\tau^b]$. So $x_{l_s}(t,\tau^b;\boldsymbol{Q}) - x^u_{l_s}(t;\downarrow)$ has at least $2k+1$ zeroes in $[\tau^a,\tau^b]$ too, and claim **2)** is proved.

We have chosen $\boldsymbol{Q} \in W^s_{l_s}(\tau^b)$ (and hence $\boldsymbol{S} \in W^s_{l_u}(\tau^b)$) so that that there is $\tau^1 < \tau^a$ such that $x_{l_s}(\tau^1,\tau^b,\boldsymbol{Q}) = 0$. Thus the corresponding fast decay solution $v^\infty(r)$ of (1.4) has a non-degenerate zero for $r = \mathrm{e}^{\tau^1}$. We denote by $u^0(r)$ the unique singular solution of (1.4) and by $\sigma^u(z^b,s,*)$ a parametrization of the whole $W^u_{l_u}(\tau^b)$. So we can apply proposition 4.1 to conclude the following.

3) *The winding number of $s \to \sigma^u(z^b,s,*)$ around $\boldsymbol{Q}$ is equal or smaller than $-k$.*

Denote by $\tilde{W}^s_{l_s}(\tau^b)$ the branch of $W^s_{l_s}(\tau^b)$ between the origin and $\boldsymbol{Q}$. Possibly choosing a larger τ^b we find that $\tilde{W}^s_{l_s}(\tau^b)$ is close to a segment. In fact $\tilde{W}^s_{l_s}(\tau^b)$ is a graph on the line $A^0_{2_*} = \{(x,-(n-2)x) \mid x > 0\}$ and it is tangent to $A^0_{2_*}$ in the origin; moreover $\boldsymbol{Q} = \boldsymbol{x}_{\boldsymbol{l_s}}(\tau^b,\tau^b;\boldsymbol{Q})$ tends to 0 as $\tau^b \to +\infty$. Let $\tilde{W}^s_{l_u}(\tau^b)$ be the corresponding branch of $W^s_{l_u}(\tau^b)$, i.e. the branch between the origin and $\boldsymbol{S}$: it follows that

4) $\boldsymbol{\tilde{W}^s_{l_u}(\tau^b) = \tilde{W}^s_{l_s}(\tau^b)}$ *$exp[(\alpha_{l_u} - \alpha_{l_s})\tau^b]$ is close to a segment of the line $A^0_{2_*}$.*

Hence (putting together claims 3 and 4) we easily find that $\tilde{W}^s_{l_u}(\tau^b)$ intersects $W^u_{l_u}(\tau^b)$ in at least k points, say $\boldsymbol{Q^j}$ for $j = 1,\dots,k$. In fact $\tilde{W}^s_{l_u}(\tau^b)\cap W^u_{l_u}(\tau^b)$ has at least k connected components. Then $\boldsymbol{x}_{\boldsymbol{l_u}}(t,\tau^b;\boldsymbol{Q^j})$ is a homoclinic trajectory

of (2.2) and the corresponding solution $u^j(r)$ of (1.4) is a G.S. with f.d. for any $j = 1, \dots, k$. $\square$

Remark 4.5. We stress that, if the slow decay solution $v(r, \uparrow)$ has a zero, then we can apply the first part of proposition 4.1 to conclude that $W^u_{l_s}(\tau^b)$ makes k complete rotations around $\boldsymbol{x^s_{l_s}}(\tau^b, \uparrow)$.

From the construction just developed to prove theorem 2.2 we get the following alternatives.
i) There is $\boldsymbol{Q} \in [W^s_{l_s}(\tau^b) \setminus W^u_{l_s}(\tau^b)]$, and a decreasing sequence of values $\varepsilon_k(l_s) \searrow 0$ such that $W^u_{l_s}(\tau^b)$ rotates around $\boldsymbol{Q}$ a finite number of times larger than k, for any $l_u \in (2^* + \varepsilon_{k+1}(l_s); 2^* + \varepsilon_k(l_s))$.
ii) There is $\boldsymbol{Q} \in [W^s_{l_s}(\tau^b) \setminus W^u_{l_s}(\tau^b)]$, and a value $\varepsilon_K(l_s) > 0$ such that $W^u_{l_s}(\tau^b)$ rotates indefinitely around $\boldsymbol{Q}$, for any $l_u \in (2^*; 2^* + \varepsilon_K(l_s))$.
In order to prove theorem 2.4 we need to exclude possibility ii).

Remark 4.6. Assume $\boldsymbol{G_u}$, $\boldsymbol{G_s}$, with $2_* < l_s < 2^* < l_u$ and consider the unique singular trajectory $\boldsymbol{x^u_{l_s}}(t, \downarrow)$ and the unique slow decay trajectory $\boldsymbol{x^s_{l_s}}(t, \uparrow)$ of (2.2). Then, if these trajectories do not coincide, the linking number of $\boldsymbol{x^u_{l_s}}(t, \downarrow)$ and $\boldsymbol{x^s_{l_s}}(t, \uparrow)$ in the whole of $\mathbb{R}$ is finite.

Proof. Consider the angular number $\Theta = \theta(\boldsymbol{x^s_{l_s}}(t, \uparrow) - \boldsymbol{x^u_{l_s}}(t, \downarrow); (0, 0))$ for $t \in [\tau^a, \tau^b]$. Since the angular number is a continuous function, Θ is finite for any given τ^a, τ^b. We can choose τ^b large enough so that $\boldsymbol{x^s_{l_s}}(t, \uparrow)$ is close to the (repulsive) critical point $\boldsymbol{P_{l_s}}(+\infty)$ of the autonomous system (2.2) where $g_{l_s}(x, t) \equiv g^{+\infty}_{l_s}(x)$. Since $\boldsymbol{P_{l_s}}(+\infty)$ is repulsive it follows that $\boldsymbol{x^u_{l_s}}(t, \downarrow)$ cannot rotate indefinitely around it for $t > \tau^b$, so $\theta(\boldsymbol{x^s_{l_s}}(t, \uparrow) - \boldsymbol{x^u_{l_s}}(t, \downarrow); (0, 0))$ is finite in $[\tau^b, +\infty)$ too. Now switch to (2.2) where $l = l_u$ and consider the trajectories $\boldsymbol{x^s_{l_u}}(t, \uparrow)$ and $\boldsymbol{x^u_{l_u}}(t, \downarrow)$ and the critical point $\boldsymbol{P_{l_u}}(-\infty)$ of (2.2) where $g_{l_u}(x, t) \equiv g^{-\infty}_{l_u}(x)$. Reasoning as above and using the fact that $\boldsymbol{P_{l_u}}(-\infty)$ is attractive, we see that the angular number $\theta(\boldsymbol{x^s_{l_u}}(t, \uparrow) - \boldsymbol{x^u_{l_u}}(t, \downarrow); (0, 0))$ is finite for $t \in (-\infty, \tau^a]$. Therefore we find that the linking number of $\boldsymbol{x^u_{l_s}}(t, \downarrow)$ and $\boldsymbol{x^s_{l_s}}(t, \uparrow)$ in the whole of $\mathbb{R}$ is the sum of finite numbers and it is finite. $\square$

From this Remark it follows that possibility ii) can take place just if there is $\delta > 0$ such that (1.4) admits a S.G.S. with s.d. for $l_s \in (2_*, 2^*)$ and any $l_u \in (2^*, 2^* + \delta)$.

4.3. **Proof of theorem 2.3.** The proof of the first part of theorem 2.3 is obtained through a perturbation argument on (1.4) where f satisfies $\boldsymbol{G_u}$ and $\boldsymbol{G_s}$ with $2_* = l_s < 2^* < l_u$. The second part of the theorem is obtained combining the first part with the observations concerning Kelvin inversion (3.4). We begin from the following remark.

Remark 4.7. Consider the autonomous system (2.2) where $l = 2_*$ and $g_{2_*}(x, t) \equiv g_{2_*}(x)$ is t-independent and satisfies $\boldsymbol{G0}$. There are no critical points, no periodic orbits, and for any $\boldsymbol{Q} \in \mathbb{R}^2_+$ there is $T(\boldsymbol{Q}) > 0$ such that $\boldsymbol{X_{2_*}}(t, 0; \boldsymbol{Q})$ crosses transversally the y negative semi-axis at $t = T(\boldsymbol{Q})$.

Proof. The non-existence of periodic orbit is a trivial consequence of Poincare-Bendixson criterion (when $\alpha_l + \gamma_l \neq 0$, i.e. $l \neq 2^*$, no closed orbit can exist). Moreover the flow on the isocline $\dot{x} = 0$ (i.e. $A^0_{2_*} = \{(x, -(n-2)x) \mid x > 0\}$), is vertical and points downwards: here and later we think of the x axis as horizontal and of the y axis as vertical. Using this fact and Remark 3.2, from an easy analysis of the phase portrait we conclude that there is $T(\boldsymbol{Q}) > 0$ such that $X_{2_*}(T(\boldsymbol{Q}), 0; \boldsymbol{Q}) = 0 > Y_{2_*}(T(\boldsymbol{Q}), 0; \boldsymbol{Q})$ for any $\boldsymbol{Q} \in \mathbb{R}^2_+$. $\square$

Now we easily get the following.

Lemma 4.8. *Consider (1.4) and assume* $\boldsymbol{G_u}$ *and* $\boldsymbol{G_s}$ *with* $2_* = l_s < 2^* \le l_u$. *For any* $\boldsymbol{Q} \in \mathbb{R}^2_+$ *there is* $T(\boldsymbol{Q}) > 0$ *such that* $\boldsymbol{x_{l_s}}(t, 0; \boldsymbol{Q})$ *crosses transversally the* y *negative semi-axis at* $t = T(\boldsymbol{Q})$.

Proof. Consider system (3.3); from $\boldsymbol{F0}$ it follows that any trajectory can be continued for any $t \in \mathbb{R}$. Observe that the ω-limit set of $(\boldsymbol{x_{l_s}}(t, 0; \boldsymbol{Q}), \zeta(t))$ is contained in the plane $\zeta = 0$. Then the Lemma is an easy consequence of Remark 4.7. □

If $\boldsymbol{G_s}$ holds with $l_s = 2_*$, from Lemma 4.8 we see that there is $T^* = T^*(l_u)$ such that $y^u_{l_u}(T^*(l_u), \downarrow) < 0 = x^u_{l_u}(T^*, \downarrow)$. We recall that the unstable manifold $W^u_{l_u}(T^*)$ is a smooth path that connects the origin with $\boldsymbol{x^u_{l_u}}(T^*, \downarrow)$. So we can find $\rho > 0$ such that each trajectory $\boldsymbol{x_{l_u}}(t, T^*; \boldsymbol{Q})$ of (2.2), crosses the y negative semi-axis at $t = T(\boldsymbol{Q})$, whenever $\boldsymbol{Q} \in W^u_{l_u}(T^*)$ and $\|\boldsymbol{Q} - \boldsymbol{x^u_{l_u}}(T^*, \downarrow)\| < \rho$. Moreover $T(\boldsymbol{Q})$ is continuous, due to the transversality.

From now on we consider $l_u > 2^*$ fixed and we let l_s vary in $[2_*, 2^*)$, so we stress the dependence on l_s of the objects introduced. Set $z^* = \mathrm{e}^{\varpi T^*}$ and let $\sigma^u(z^*, s, *; l_s)$ be a parametrization of the unstable manifold $W^u_{l_u}(T^*)$ such that $\sigma^u(z^*, 0, *; l_s) = (0, 0)$ and $\sigma^u(z^*, 1, *; l_s) = \boldsymbol{x^u_{l_u}}(T^*, \downarrow)$. Using continuous dependence on parameters we obtain the following Lemma which is the generalization of Lemma 5.2 in [2].

Lemma 4.9. *Assume* $\boldsymbol{G_u}$ *and* $\boldsymbol{G_s}$ *and fix* $l_u > 2^*$. *Then, there are* $B \in (0, 1)$ *(independent of* l_u*) and* $\varepsilon^b_0(l_u) > 0$, *such that for any* $l_s \in (2_*, 2_* + \varepsilon^b_0(l_u))$ *the trajectory* $\boldsymbol{x_{l_u}}(t, T^*; \boldsymbol{Q})$ *is a crossing solution whenever* $\boldsymbol{Q} \in \sigma^u(z^*, s, *; l_s)$ *and* $s \in (B, 1]$. *Correspondingly let* $u(r; \alpha)$ *denote the regular solution such that* $u(0; \alpha) = \alpha$ *and* $\frac{d}{dr}u(0; \alpha) = 0$; *then there is* $D > 0$ *(independent of* l_u*) such that* $u(r; \alpha)$ *is a crossing solution for any* $\alpha > D$.

We wish to stress that Lemma 4.9 might also be proved following the ideas of the proof of Lemma 5.2 in [2], but we have chosen to give a different proof of "dynamical" type. From Lemmas 3.5 and 4.8 we get the following.

Lemma 4.10. *Assume* $\boldsymbol{G_u}$ *and* $\boldsymbol{G_s}$ *and fix* $l_u > 2^*$, *Then, there is* $\varepsilon^a_0(l_u) > 0$ *and* $A \in (0, 1)$ *(independent of* l_u*), such that for any* $l_s \in (2_*, 2_* + \varepsilon^a_0(l_u))$ *the trajectory* $\boldsymbol{x_{l_u}}(t, T^*; \boldsymbol{Q})$ *is a crossing solution whenever* $\boldsymbol{Q} \in \sigma^u(z^*, s, *; l_s)$ *and* $s \in (0, A)$. *Correspondingly there is* $d > 0$ *such that* $u(r; \alpha)$ *is a crossing solution for any* $0 < \alpha < d$.

Proof of theorem 2.3. Set $\tilde{\varepsilon}_0 = \min\{\varepsilon^a_0; \varepsilon^b_0\} > 0$, and assume $\boldsymbol{G_u}$ and $\boldsymbol{G_s}$, where $2_* < l_s < 2_* + \tilde{\varepsilon}_0(l_u) < 2^* < l_u$, so that the hypotheses of Lemmas 4.9 and 4.10 are verified. Consider the unstable manifold $W^u_{l_u}(T^*; l_s)$ and its parametrization $\sigma^u(z^*, s, *; l_s)$: we have shown that the trajectories $\boldsymbol{x}_{l_u}(t, T^*; \boldsymbol{Q}; l_s)$ correspond to crossing solutions of (1.4) whenever $\boldsymbol{Q} \in \sigma^u(z^*, s, *; l_s)$, and $s \in (0, A) \cup (B, 1)$.

Consider now system (4.6) where $l = 2_*$, and the corresponding equation (1.4) obtained replacing f by the function f^m defined as follows

$$f^m(u, r) := f(ur^{n-2-\alpha_{l_s}}, r)r^{-(n-2-\alpha_{l_s})}. \tag{4.7}$$

and observe that f^m satisfies $\boldsymbol{G_u}$ with $l = l^m_u < l_u$ (and obviously $\boldsymbol{G_s}$ with $l = 2_*$). Let $W^{u,m}_{l^m_u}(T^*)$ denote the unstable manifold of the modified problem with $f = f^m$ given by (4.7) and let $\boldsymbol{x^{u,m}_{l^m_u}}(t, \downarrow)$ be the trajectory of (2.2) corresponding to the unique singular solution of such a problem. Let $\sigma^{u,m}(z^*, s, *)$ be a parametrization of $W^{u,m}_{l^m_u}(T^*)$, such that $\sigma^{u,m}(z^*, 0, *) = (0, 0)$ and $\sigma^{u,m}(z^*, 1, *) = \boldsymbol{x^{u,m}_{l^m_u}}(T^*, \downarrow)$. From Lemma 4.8 it follows that any regular solution of the modified equation (1.4) where $f = f^m$ is a crossing solution.

By construction, for any $\boldsymbol{Q} = \sigma^{u,m}(z^*, s, *)$ there is $T^m(\boldsymbol{Q}) \in \mathbb{R}$ such that the trajectory $(\boldsymbol{x}^{\boldsymbol{u},\boldsymbol{m}}_{\boldsymbol{l_u^m}}(t, T^*; \boldsymbol{Q}), z(t))$ of (4.4) crosses transversally the $x = 0$ plane at $t = T^m(\boldsymbol{Q})$, whenever $0 < s \leq 1$. Let $0 < A_0^m < B_0^m < 1$ and denote by K the compact set

$$K := \{\sigma^{u,m}(z^*, s, *) \mid A_0^m \leq s \leq B_0^m\}$$

Denote by $\mathfrak{T} = \sup\{T^m(\boldsymbol{Q}) \mid \boldsymbol{Q} \in K\}$ and observe that $\mathfrak{T}$ is positive and finite.

Let us choose A^0 and B^0 so that $0 < A^0 < A < B < B^0 < 1$; then

$$\bar{W}^u_{l_u}(T^*) = \{\sigma^u(z^*, s, \downarrow; l_s) \mid A^0 \leq s \leq B^0\},$$

is a compact connected set of $W^u_{l_u}(T^*)$. Let $\boldsymbol{Q}$ be a point and C a set; we denote by $B(\boldsymbol{Q}, \rho)$ the open ball centered in $\boldsymbol{Q}$ of radius $\rho > 0$, and by $B(C, \rho) = \cup_{\boldsymbol{Q} \in A} B(\boldsymbol{Q}, \rho)$. For any $\rho > 0$ we can choose $\varepsilon_0(l_u) > 0$ and $A_0^m < B_0^m$ such that $\bar{W}^u_{l_u}(T^*) \subset B(K, \rho)$, whenever $l_s \in (2_*, \varepsilon_0(l_u))$.

Consider the trajectories $\boldsymbol{x_{l_u}}(t, T^*; \boldsymbol{Q})$ of the original problem (2.2) where $\boldsymbol{Q} \in B(K, \rho)$. Using a uniform continuity argument, and possibly choosing a smaller $\varepsilon_0(l_u) > 0$, we can assume that $\rho > 0$ is small enough so that the trajectories $\boldsymbol{x_{l_u}}(t, T^*; \boldsymbol{Q})$ cross the y negative semi-axis whenever $\boldsymbol{Q} \in B(K, \rho)$. So the regular solutions $u^{\boldsymbol{Q}}(r)$ of (1.4) corresponding to $\boldsymbol{x_{l_u}}(t, T^*; \boldsymbol{Q})$ where $\boldsymbol{Q} \in \bar{W}^u_{l_u}(T^*)$ are crossing solutions, whenever $2_* < l_s < 2_* + \varepsilon_0(l_u)$. Thus, if we choose $\varepsilon_0(l_u) < \tilde{\varepsilon}_0(l_u)$, we obtain that all the regular solutions of the original problem are crossing solutions and the first part of the proof of theorem 2.3 is concluded. I.e. for any $l_u > 2^*$ there is $\varepsilon_0(l_u) > 0$ such that (1.4) admits no G.S. with either slow or fast decay, neither S.G.S. with either slow or fast decay, whenever $l_s \in (2_*, 2_* + \varepsilon_0(l_u))$.

Applying Kelvin inversion to a f satisfying $\boldsymbol{G_u}$ and $\boldsymbol{G_s}$ with $l_u > 2^*$ and $l_s \in (2_*, 2_* + \varepsilon_0(l_u))$, we obtain a function $\tilde{f}$ satisfying $\boldsymbol{G_u}$ and $\boldsymbol{G_s}$ with $L_s \in (2_*, 2^*)$ and $L_u > M_0 := 2(n-1) + \frac{4}{(n-2)\varepsilon_0(l_u)}$ where L_u and L_s are given in (3.8) and viceversa (we recall that, according to (3.8), Kelvin inversion brings 2_* into ∞). Using (3.8) it is easy to check that M_0 can be written as a function of L_s. So assume f satisfies $\boldsymbol{G_u}$ and $\boldsymbol{G_s}$ with $l_s \in (2_*, 2^*)$ and $l_u > M_0(l_s)$. Applying Kelvin inversion we pass from such an f to $\tilde{f}$ satisfying $\boldsymbol{G_u}$ and $\boldsymbol{G_s}$ with $L_u > 2^*$ and $L_s \in (2_*, 2_* + \varepsilon_0(L_u))$. So, using the first part of the theorem (already proved), the corresponding equation 3.5 is such that all the fast and slow decay solutions, as well as all the regular and singular solutions, admit a non degenerate zero; hence for the original equation (1.4) (satisfying $\boldsymbol{G_u}$ and $\boldsymbol{G_s}$ with $l_s \in (2_*, 2^*)$ and $l_u > M_0(l_s)$) all the regular and singular solutions, as well as all the fast and slow decay solutions have a non degenerate zero, so the proof of theorem 2.3 is concluded.

4.4. **Proof of theorem 2.4.** We stress that theorem 2.4 is the analogous of theorem 1.3 in [2]. In fact we could prove it simply by adapting to this context the proof of Bamon et al. However there is a point in their proof which is not very clear to us so we prefer to modify it. The problem derives from the following fact. The topological argument used by Bamon et al. in Lemma 3.1 (analogous to Lemma 4.3 of this paper) is developed assuming that the trajectories $x^m_{2^*}(t, \boldsymbol{Q}, \tau)$ rotates indefinitely around the critical point $\boldsymbol{P^m_{2^*}}(-\infty)$, whenever $\boldsymbol{Q} \in W^{s,m}_{2^*}(\tau)$. However there is one solution of (4.4) with $l = 2^*$, $\boldsymbol{x^{u,m}_{2^*}}(t, \downarrow)$, which lies on the unstable manifold of $(\boldsymbol{P^m_{2^*}}(+\infty), 0)$. This trajectory may be such that the difference $x^{u,m}_{2^*}(t, \downarrow) - P^m_x(-\infty)$ has a finite number of zeroes or none. So the topological argument developed in subsection 4.1 works for any solution $v^\infty(r)$ different from the solution $u(r, \downarrow)$ corresponding to the singular solution $\boldsymbol{x^u_{l_u}}(t, \downarrow)$ (which converges uniformly to $\boldsymbol{x^{u,m}_{2^*}}(t, \downarrow)$ for $t \leq 0$).

So we have no problems when we want to prove theorem 1.1, or its analogous theorem 2.2, since we can choose any fast decay solution $v^\infty(r)$. However in the proof

of theorem 1.3 u^0 and v^∞ are assumed to be specific solutions, respectively the singular and the slow decay solution of (1.4). So we prefer to refine the argument needed to prove the existence of k zeroes of $u^0(r)-v^\infty(r)$, to prevent the trajectory corresponding to v^∞ to be close to $x^{u,m}_{2^*}(t,\downarrow)$.
We begin the discussion on the existence of S.G.S. with f.d. and of G.S. with s.d. considering the following hypotheses.

- $\boldsymbol{B_u}$**:** Assume $\boldsymbol{G_u}, \boldsymbol{G_s}, \boldsymbol{A_u}$ with $2_* < l_s < 2^* < l_u$. The solution $(\boldsymbol{x^{u,m}_{2^*}}(t,\downarrow), z(t))$ of (4.4) with $l = 2^*$ crosses transversally the $x = 0$ semi-plane.
- $\boldsymbol{B_s}$**:** Assume $\boldsymbol{G_u}, \boldsymbol{G_s}, \boldsymbol{A_s}$ with $2_* < l_s < 2^* < l_u$. The solution $(\boldsymbol{x^{s,m}_{2^*}}(t,\uparrow), \zeta(t))$ of (4.6) with $l = 2^*$ crosses transversally the $x = 0$ semi-plane.

We give a stronger result, proposition 4.11 below, which requires $\boldsymbol{B_u}$ or $\boldsymbol{B_s}$, and a weaker result which does not, theorem 2.4. These hypotheses seem to be generic: e.g. $\boldsymbol{B_u}$ is verified when a two dimensional object, $W^{s,m}_{l^m_u}(\tau)$, does not intersect a one dimensional object in $\mathbb{R}^3$, i.e. the stable manifold of the critical point $\boldsymbol{P}_{l^m_u}(-\infty)$, and similarly for $\boldsymbol{B_s}$. However it is difficult to prove that $\boldsymbol{B_u}$ and $\boldsymbol{B_s}$ are actually verified. In fact it is possible, and straightforward, for the nonlinearities f discussed in [15], i.e. $f(u,r)$ of type (1.2) and $k(r) = \max\{r^{\delta^u}, r^{\delta^s}\}$ or $f(u) = \max\{u^{q^u-1}, u^{q^s-1}\}$. In such a case it is in fact enough to observe respectively that $\boldsymbol{x^{u,m}_{2^*}}(t,\downarrow) \equiv \boldsymbol{P^{u,m}_{2^*}}(-\infty)$ for $-t$ large enough, and $\boldsymbol{x^{s,m}_{2^*}}(t,\uparrow) \equiv \boldsymbol{P^{s,m}_{2^*}}(+\infty)$ for t large enough, and to make some trivial geometrical observations. So for these non-linearities proposition 4.11 gives an alternative (but more difficult) proof of the existence of G.S. with s.d. and of S.G.S. with f.d.

Proposition 4.11. *Assume $\boldsymbol{A_s}, \boldsymbol{G_s}$ and $\boldsymbol{G_u}$ with $l_u > 2^*$ and $\boldsymbol{B_s}$. Then there is an increasing sequence $r^j(l_u) \nearrow 2^*$ as $j \to \infty$ such that (1.4) where $l_s = r^j(l_u)$ admits a (unique) S.G.S. with f.d.*

Analogously assume $\boldsymbol{A_u}, \boldsymbol{G_u}$ and $\boldsymbol{G_s}$ with $2_* < l_s < 2^*$. Assume further $\boldsymbol{B_u}$. Then there is a decreasing sequence $r^j(l_s) \searrow 2^*$ as $j \to \infty$ such that (1.4) where $l_u = r^j(l_s)$ admits a (unique) G.S. with s.d.

This proposition is similar to theorem 2.4, but it requires the hard to be proved hypotheses $\boldsymbol{B_u}$ and $\boldsymbol{B_s}$. However it allows to specify the type of special solution obtained. To prove such a proposition we need the following result, which follows from continuous dependence on parameters.

Lemma 4.12. *Assume $\boldsymbol{A_u}, \boldsymbol{G_u}$ and $\boldsymbol{G_s}$ with $2_* < l_s < 2^*$. Assume further $\boldsymbol{B_u}$. Then there is $\varepsilon_*(l_s) > 0$, such that whenever $2^* < l_u < 2^* + \varepsilon_*(l_s)$, the unique singular solution $u(r,\downarrow)$ of (1.4) is a crossing solution, i.e. there is a unique value $R > 0$ such that $u'(R,\downarrow) < 0 = u(R,\downarrow)$. Moreover R depends continuously on l_u and l_s.*

Analogously assume $\boldsymbol{A_s}, \boldsymbol{G_s}$ and $\boldsymbol{G_u}$ with $l_u > 2^*$ and $\boldsymbol{B_s}$. Then there is $\varepsilon_*(l_u) > 0$, such that whenever $2^* - \varepsilon_*(l_u) < l_s < 2^*$, the unique slow decay solution $v(r,\uparrow)$ of (1.4) is a crossing solution, i.e. there is a unique value $R > 0$ such that $v'(R,\uparrow) < 0 = v(R,\uparrow)$; again R depends continuously on l_u and l_s.

To prove both proposition 4.11 and theorem 2.4 we have to repeat the topological argument developed in subsection 2.1 and 2.2.
Proof of proposition 4.11. We just prove the existence of S.G.S. with f.d., since the existence of G.S. with s.d. is analogous and can be deduced by Kelvin inversion. So we assume $\boldsymbol{A_s}, \boldsymbol{G_s}$ and $\boldsymbol{G_u}$ with $l_u > 2^*$ and $\boldsymbol{B_s}$. From Lemma 4.12 we know that the unique slow decay solution $v(r,\uparrow)$ of (1.4) has a non-degenerate zero, whenever $l_s \in (2^* - \varepsilon_*(l_u), 2^*)$, so no S.G.S. with s.d neither G.S. with s.d. can exist. It follows that the unique singular solution $(\boldsymbol{x^{u,m}_{2^*}}(t,\downarrow), \zeta(t))$ of the modified problem (4.6) has a periodic orbit as α-limit set, so it rotates indefinitely clockwise around

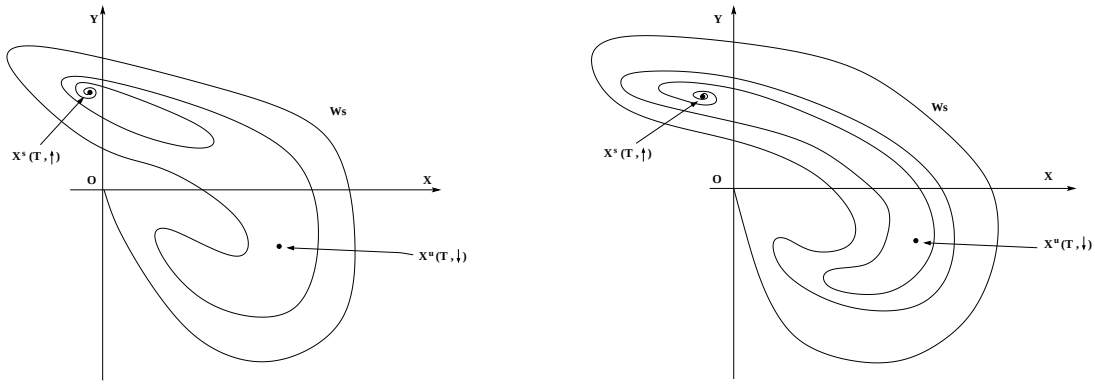

FIGURE 4. Proof of the existence of a S.G.S. with slow decay. The angular number $\Theta(Z, l_s)$ indicates the rotation of the manifold $W^s_{l_s}(\tau)$ around the point $\boldsymbol{x^0_{l_s}}(\tau) = \boldsymbol{x^u_{l_s}}(\tau, \downarrow)$, corresponding to the unique singular solution $u(r, \downarrow)$ of (1.4). $\Theta(Z, l_s)$ jumps from the interval $(2 - 1/4; 2)$ to the interval $(3 - 1/4; 3)$, violating the continuity of $\Theta(Z, l_s)$, thus driving to a contradiction.

the critical point $(\boldsymbol{P_{2^*}}(+\infty), 0)$ (we stress that to obtain such a conclusion $\boldsymbol{B_s}$ is required).

Consider the unique singular solution $u(r, \downarrow)$ of the original problem (1.4) and the corresponding trajectories $(\boldsymbol{x^u_{l_u}}(t, \downarrow), z(t))$ of (3.2) and $(\boldsymbol{x^u_{l_s}}(t, \downarrow), \zeta(t))$ of (3.3). Fix $\tau^a \in \mathbb{R}$, since $\boldsymbol{P_{l_s}}(+\infty) \to \boldsymbol{P_{2^*}}(+\infty)$ as $l_s \to 2^*$, for any $k \in \mathbb{N}$ we can find $\varepsilon_k > 0$ small and $\tau^b > 0$ large such that $\boldsymbol{x^u_{l_s}}(t, \downarrow)$ rotates clockwise around $\boldsymbol{P_{l_s}}(+\infty)$ at least k times, for $t \in [\tau^a, \tau^b]$, whenever $l_s \in (2^* - \varepsilon_k(l_u), 2^*)$. So, possibly choosing a smaller ε_k, we see that $x^u_{l_s}(t, \downarrow) - x^s_{l_s}(t, \uparrow)$ has at least $2k + 1$ zeroes for $t \in [\tau^a, \tau^b]$. Let us consider the corresponding trajectories of (3.2), i.e. $(\boldsymbol{x^u_{l_u}}(t, \downarrow), z(t))$ and $(\boldsymbol{x^s_{l_u}}(t, \uparrow), z(t))$: obviously $x^u_{l_u}(t, \downarrow) - x^s_{l_u}(t, \uparrow)$ has at least $2k+1$ zeroes for $t \in [\tau^a, \tau^b]$ too.

We fix the parameter l_u, while we allow the parameter l_s to vary in the interval $(2_*, 2^*)$. So we stress the dependence on l_s of the objects we introduce: i.e. we denote the stable manifold $W^s_{l_u}(z)$ by $W^s_{l_u}(z, l_s)$ and the singular solution $\boldsymbol{x^u_{l_u}}(t, \downarrow)$ by $\boldsymbol{x^u_{l_u}}(t, \downarrow; l_s)$. From Lemma 4.12 we already know that no S.G.S. with s.d. may exist for $l_s \in (2^* - \varepsilon_*(l_u), 2^*)$, hence the singular solution $u(r, \downarrow)$ cannot have slow decay (note that we can and will assume $\varepsilon_k(l_u) < \varepsilon_*(l_u)$, and this is certainly possible if k is large enough, thanks to Remark 4.6).

Let $l_s \in (2^* - \varepsilon_k(l_u), 2^*)$; assume first that $u(r, \downarrow)$ is a crossing solution. Let $\sigma^s(z, \cdot, *; l_s) : [0, 1] \to \mathbb{R}^2$ be a parametrization of $W^s_{l_u}(\ln(z)/\varpi; l_s)$ such that $\sigma^s(z, 0, *; l_s) = (0, 0)$ and $\sigma^s(z_1, 1, *; l_s) = \boldsymbol{x^s_{l_u}}([\ln(z)/\varpi], \uparrow; l_s)$, see subsection 4.1. We can apply proposition 4.1 to conclude that the winding number $w(\bar{Z}, l_s)$ of $\sigma^s(\bar{Z}, \cdot, *; l_s)$ around $\boldsymbol{x^u_{l_u}}(\bar{T}, \downarrow)$ is at least k, for any $l_s \in (2^* - \varepsilon_k(l_u), 2^*)$ and any $\bar{T} = \ln(\bar{Z})/\varpi < \tau_a$.

Moreover from theorem 2.3 we know that there is $\varepsilon_0(l_u)$ such that for $l_s \in (2_*, 2_* + \varepsilon_0(l_u))$ no G.S. with f.d. exist, hence the winding number $w(\bar{Z}; l_s)$ is 0 or 1 whenever $l_s \in (2^* - \varepsilon_0(l_u), 2^*)$. Therefore **the winding number $\boldsymbol{w(\bar{Z}, l_s)}$ is less than 2 for any $\boldsymbol{l_s \in (2_*, 2_* + \varepsilon_0(l_u))}$ and it is at least $\boldsymbol{k}$ for $\boldsymbol{l_s \in (2^* - \varepsilon_k(l_u), 2^*)}$ and $\boldsymbol{k}$ large enough.** Hence we have the following two possibilities:

i) there is a sequence of values $r^k \nearrow 2^*$ such that for $l_s = r^k$ the singular solution $u(r, \downarrow)$ is not a crossing solution,

ii) there is $\delta(l_u) > 0$ such that $u(r, \downarrow)$ is a crossing solution for $l_s \in (2^* - \delta(l_u), 2^*)$.

In the former case no S.G.S. with s.d. are allowed for $0 < 2^* - l_s < \varepsilon_*$, so for $l_s = r^k$ the unique singular solution has to be a S.G.S. with f.d and we have done. So assume for contradiction the latter; possibly changing slightly the values of the parameters ε_k, we can assume that $w(\bar{Z}, l_s)$ equals exactly k for $l_s \in (2^* - \varepsilon_k(l_u), 2^* - \varepsilon_{k+1}(l_u))$ whenever $0 < \bar{Z} < z^a$. We focus on $l_s \in [2^* - \varepsilon_{k-1}(l_u), 2^* - \varepsilon_{k+1}(l_u)]$. We recall that the unique slow decay solution $v(r, \uparrow)$ has a unique zero, say $R(l_s) = \mathrm{e}^{T(l_s)}$. Let us choose $Z = \min\{\mathrm{e}^{\varpi T(l_s)} \mid l_s \in [2^* - \varepsilon_{k-1}(l_u), 2^* - \varepsilon_{k+1}(l_u)]\}$ and note that $Z > 0$. By construction the point $\sigma^s(Z, 1; l_s) = \boldsymbol{x}^{\boldsymbol{s}}_{\boldsymbol{l_u}}(T, \uparrow)$ lies in the quadrant $x \le 0 < y$ (we recall that the semi-line $\{(x, -(n-2)x) \mid x \le 0\}$ is part of the unstable manifold, so it is invariant for the flow and cannot be crossed). It follows that the angular number $\Theta(Z; l_s)$ of $\sigma^s(Z, \cdot; l_s)$ around $\boldsymbol{x}_{\boldsymbol{l_u}}(T, \downarrow)$ satisfies

$$\begin{array}{ll} k - 1/4 \le \Theta(Z, l_s) < k & \text{when } l_s \in (2^* - \varepsilon_{k-1}(l_u), 2^* - \varepsilon_k(l_u)) \\ k + 3/4 \le \Theta(Z, l_s) < k+1 & \text{when } l_s \in (2^* - \varepsilon_k(l_u), 2^* - \varepsilon_{k+1}(l_u)), \end{array} \tag{4.8}$$

see figure 4. But this contradicts the continuity in l_s of the angular number $\Theta(Z, l_s)$. So we have a value $r^k(l_u) \in (2^* - \varepsilon_{k-1}(l_u), 2^* - \varepsilon_{k+1}(l_u))$ such that $u(r, \downarrow)$ is a S.G.S. with f.d. for $l_s = r^k(l_u)$. □

Remark 4.13. From the proof it is in fact clear that we can assume $r^k(l_u) = 2^* - \varepsilon_k(l_u)$, i.e. we have a S.G.S. with f.d. at the value for which the winding number $w(l_s, Z)$ increases. So $r^k(l_u)$ separates the values of l_s for which we have at least k G.S. with f.d. from the ones for which we have at least $k+1$ G.S. with f.d., see picture 4.

As we have already stressed it is difficult to verify hypothesis $\boldsymbol{B_u}$ and $\boldsymbol{B_s}$, even if they seem to be generic. However the relevance of proposition 4.11 lies on the fact that it is the first step to prove theorem 2.4.

Lemma 4.14. *Assume $\boldsymbol{A_s}, \boldsymbol{G_u}$ and $\boldsymbol{G_s}$ with $l_u > 2^*$. Consider the unique solution $(\boldsymbol{x}^{\boldsymbol{s,m}}_{\boldsymbol{2^*}}(t, \uparrow), \zeta(t))$ of (4.6) asymptotic to $\boldsymbol{P}^{\boldsymbol{m}}_{\boldsymbol{2^*}}(+\infty)$ as $t \to +\infty$, the corresponding trajectory $(\boldsymbol{x}^{\boldsymbol{s,m}}_{\boldsymbol{l^m_u}}(t, \uparrow), z(t))$ of (4.4), and the corresponding solution $v^m(r, \uparrow)$ of the modified equation (1.4). Then, if either*
a) $v^m(r, \uparrow)$ is a G.S. with s.d. *or*
b) $v^m(r, \uparrow)$ is a S.G.S. with s.d. (i.e. $\boldsymbol{x}^{\boldsymbol{s,m}}_{\boldsymbol{l^m_u}}(t, \uparrow) \to \boldsymbol{P}^{\boldsymbol{m}}_{\boldsymbol{l^m_u}}(+\infty)$ as $t \to -\infty$)

then there is a sequence $r^j(l_u) \nearrow 2^*$ such that the original problem with $l_s = r^j(l_u)$ admits either a G.S. with s.d., or a S.G.S. with f.d. or a S.G.S. with s.d.

Analogously assume $\boldsymbol{A_u}, \boldsymbol{G_u}$ and $\boldsymbol{G_s}$ with $2_* < l_s < 2^*$. Consider the unique solution $(\boldsymbol{x}^{\boldsymbol{u,m}}_{\boldsymbol{2^*}}(t, \downarrow), z(t))$ of (4.4) asymptotic to $\boldsymbol{P}^{\boldsymbol{m}}_{\boldsymbol{2^*}}(-\infty)$ as $t \to -\infty$ and the corresponding trajectory $(\boldsymbol{x}^{\boldsymbol{u,m}}_{\boldsymbol{l^m_s}}(t, \downarrow), \zeta(t))$ of (4.6), and the corresponding solution $u^m(r, \downarrow)$ of the modified equation (1.4). Then, if either
c) $u^m(r, \downarrow)$ is a S.G.S. with f.d. *or*
d) $\boldsymbol{x}^{\boldsymbol{u,m}}_{\boldsymbol{l^m_s}}(t, \downarrow) \to \boldsymbol{P}^{\boldsymbol{m}}_{\boldsymbol{l^m_s}}(+\infty)$ as $t \to +\infty$ (i.e. $u^m(r, \downarrow)$ is a S.G.S. with s.d.)

then there is a sequence $r^j(l_s) \searrow 2^*$ such that the original problem with $l_u = r^j(l_s)$ admits either a G.S. with s.d., or a S.G.S. with f.d. or a S.G.S. with s.d.

Proof. Assume $\boldsymbol{G_u}$, $\boldsymbol{G_s}$ with $l_u > 2^*$ and $\boldsymbol{A_s}$; assume further that a) holds. Consider the unique singular solution $\boldsymbol{x}^{\boldsymbol{u,m}}_{\boldsymbol{l^m_u}}(t, \downarrow)$ of system (4.4) where $l = l^m_u$ and the corresponding solutions $u^m(r, \downarrow)$ of (1.4) and $\boldsymbol{x}^{\boldsymbol{u,m}}_{\boldsymbol{2^*}}(t, \downarrow)$ of (4.6), and set $\boldsymbol{P}^{\boldsymbol{m}}_{\boldsymbol{2^*}}(+\infty) = (P^m_x(+\infty), P^m_y(+\infty))$. Since $u^m(r, \downarrow)$ does not coincide with $v^m(r, \uparrow)$, it follows that $x^{u,m}_{2^*}(t, \downarrow) - P^m_x(+\infty)$ has infinitely many zeroes in any interval of the form $[\tau_a, +\infty)$, see Lemma 3.9.

Once again we fix l_u and we let l_s vary. So let $(\boldsymbol{x}^{\boldsymbol{s}}_{\boldsymbol{l_s}}(t, \uparrow; l_s), \zeta(t))$ be the unique trajectory of (3.3) asymptotic to $(\boldsymbol{P}_{\boldsymbol{l_s}}(+\infty), 0)$, and let $v(r, \uparrow; l_s)$ and $(\boldsymbol{x}^{\boldsymbol{s}}_{\boldsymbol{l_u}}(t, \uparrow$

$;l_s), z(t))$ be the corresponding singular solution of (1.4), and the corresponding trajectory of (3.2). If $\boldsymbol{x}^{\boldsymbol{s}}_{\boldsymbol{l_s}}(t,\uparrow;l_s) \equiv \boldsymbol{x}^{\boldsymbol{u}}_{\boldsymbol{l_s}}(t,\downarrow;l_s)$, there is a S.G.S. with s.d. and we have done; so we assume that these trajectories do not coincide. Let $\boldsymbol{P_{l_s}}(+\infty) = (P_x(+\infty;l_s), P_y(+\infty;l_s))$. We can find $\varepsilon_k > 0$ and τ_b large enough so that $x^u_{l_s}(t,\downarrow;l_s) - P_x(+\infty;l_s)$ has at least $2k+1$ zeroes in $[\tau^a,\tau^b]$, whenever $l_s \in (2^* - \varepsilon_k(l_u), 2^*)$. Then, possibly choosing a smaller $\varepsilon_k > 0$ we see that $x^u_{l_s}(t,\uparrow;l_s) - x^s_{l_s}(t,\downarrow;l_s)$ has at least $2k+1$ zeroes in $[\tau^a,\tau^b]$, when $l_s \in (2^* - \varepsilon_k(l_u), 2^*)$ too. As in proposition 4.11 we can assume w.l.o.g. that the linking number of $x^s_{l_s}(t,\downarrow;l_s)$ and $x^u_{l_s}(t,\uparrow;l_s)$ is exactly k when $l_s \in (2^* - \varepsilon_k(l_u), 2^* - \varepsilon_{k+1}(l_u))$ for $t \in [\tau^a,\tau^b]$.

Assume for contradiction that both $v(r,\uparrow;l_s)$ and $u(r,\downarrow;l_s)$ have a non-degenerate zero for any $l_s \in (2^* - \varepsilon_{k-1}(l_u), 2^* - \varepsilon_{k+1}(l_u))$. Repeating the argument of proposition 4.11 we find a contradiction; hence for $l_s = r^k(l_u) = 2^* - \varepsilon_k(l_u)$ either $v(r,\uparrow;l_s)$ is a G.S. with s.d. or a S.G.S. with s.d., or $u(r,\downarrow;l_s)$ is a S.G.S. with f.d. and the proof of the Lemma in case a) is concluded.

Now we assume b), so that the trajectory $(\boldsymbol{x}^{\boldsymbol{u,m}}_{\boldsymbol{2^*}}(t,\downarrow), \zeta(t))$ is asymptotic to $(\boldsymbol{P}^{\boldsymbol{m}}_{\boldsymbol{2^*}}(+\infty), 0)$.
Consider the autonomous system (2.2) where $l = l_s$ and $g_{l_s}(x,t) \equiv g^{+\infty}_{l_s}(x)$. Let $\boldsymbol{Q} \in B(\boldsymbol{P_{l_s}}(+\infty),\bar\rho)\setminus \{\boldsymbol{P_{l_s}}(+\infty)\}$ and consider the trajectory $\boldsymbol{X_{l_s}}(t,\bar\tau;\boldsymbol{Q},+\infty)$. Since $\boldsymbol{P_{l_s}}(+\infty)$ is repulsive, for any $k \in \mathbb{N}$ we can choose $\bar\rho > 0$ and $\delta_k(\bar\rho) > 0$ such that $\boldsymbol{X_{l_s}}(t,\bar\tau;\boldsymbol{Q},+\infty)$ rotates at least k times around $\boldsymbol{P_{l_s}}(+\infty)$ for $t > \bar\tau$, before getting out from $B(\boldsymbol{P_{l_s}}(+\infty),\sqrt{\bar\rho})$, whenever $l_s \in (2^* - \delta_k(\bar\rho), 2^*)$.

Now we fix l_u and we let l_s vary. Since $\boldsymbol{x}^{\boldsymbol{u,m}}_{\boldsymbol{2^*}}(t,\downarrow) \equiv \boldsymbol{x}^{\boldsymbol{s,m}}_{\boldsymbol{2^*}}(t,\uparrow)$, for any $\bar\rho > 0$, $\bar\tau > 0$ we can find $\varepsilon_k > 0$ such that $\boldsymbol{x}^{\boldsymbol{u}}_{\boldsymbol{l_s}}(\bar\tau,\downarrow;l_s) \in B(\boldsymbol{x}^{\boldsymbol{s}}_{\boldsymbol{l_s}}(\bar\tau,\uparrow;l_s),\bar\rho)$ whenever $l_s \in (2^* - \varepsilon_k(l_u), 2^*)$. If $\boldsymbol{x}^{\boldsymbol{u}}_{\boldsymbol{l_s}}(\bar\tau,\downarrow;l_s) = \boldsymbol{x}^{\boldsymbol{s}}_{\boldsymbol{l_s}}(\bar\tau,\uparrow;l_s)$ then the singular solution $u(r,\downarrow;l_s)$ is a S.G.S. with s.d. and we have done. Otherwise, using continuous dependence on parameters of (3.3), we see that we can choose $\bar\tau > 0$ large enough $\bar\rho > 0$, $0 < \varepsilon_k(\bar\rho,\bar\tau;l_u) < \delta_k(\bar\rho)$, such that $\boldsymbol{x}^{\boldsymbol{u}}_{\boldsymbol{l_s}}(t,\downarrow)$ is in $B(\boldsymbol{P_{l_s}}(+\infty),\bar\rho)$ for $t = \bar\tau$. So we can also assume that $\boldsymbol{x}^{\boldsymbol{u}}_{\boldsymbol{l_s}}(t,\downarrow)$ rotates around $(\boldsymbol{x}^{\boldsymbol{s}}_{\boldsymbol{l_s}}(t,\uparrow)$ exactly k times clockwise, for $t \geq \bar\tau$. Hence the linking number of $(\boldsymbol{x}^{\boldsymbol{u}}_{\boldsymbol{l_s}}(t,\downarrow),\zeta(t))$ and $(\boldsymbol{x}^{\boldsymbol{s}}_{\boldsymbol{l_s}}(t,\uparrow),\zeta(t))$ for $t \in (\bar\tau,+\infty)$ is $-k$ whenever $l_s \in (2^* - \varepsilon_k(l_u), 2^*)$.

So we can find $\tau^a < \tau^b$, τ^b large enough so that $\boldsymbol{x}^{\boldsymbol{u}}_{\boldsymbol{l_s}}(t,\downarrow) - \boldsymbol{x}^{\boldsymbol{s}}_{\boldsymbol{l_s}}(t,\uparrow)$ has at least $2k+1$ zeroes, whenever $l_s \in (2^* - \varepsilon_k(l_u), 2^*)$. Now assume for contradiction that $u(r,\downarrow)$ and $v(r,\uparrow)$ are crossing solutions. Applying again proposition 4.1 and repeating the reasoning developed for the proof of point a) we reach a contradiction and we conclude the proof.

The proof of Lemma 4.14 when $\boldsymbol{A_u}$, $\boldsymbol{G_u}$, $\boldsymbol{G_s}$ and either c) or d) are assumed, can be developed reasoning in the same way but reversing time, or directly using Kelvin inversion, see subsection 3.2. □

Now the proof of theorem 2.4 is a straightforward consequence of proposition 4.11 and Lemma 4.14.

4.5. **Discussion of theorem 2.5, and consequences for the Dirichlet problem in the ball.** Flores in [11] discovered the resonance phenomenon described in theorem 1.4 which fits this context perfectly too. In [15] we have modified its proof very slightly keeping all the main ideas. Here we just sketch it remanding the interested reader to [11] for details. The proof developed by Flores is of topological flavour and is rather general, so it may work also in different contexts.

Consider (2.2) and assume $\boldsymbol{G_u}$ and $\boldsymbol{G_s}$ so that we can construct the manifolds $W^u(\tau)$ and $W^s(\tau)$. We start from claim a) and we assume first that a G.S. with s.d. exist. Such a solution corresponds to the trajectory $\boldsymbol{x}^{\boldsymbol{s}}_{\boldsymbol{l_s}}(t,\uparrow)$ of (2.2) since it has slow decay. Moreover $\boldsymbol{Q}(\tau) := \boldsymbol{x}^{\boldsymbol{s}}_{\boldsymbol{l_s}}(\tau,\uparrow)$ belongs to the unstable manifolds $W^u_{l_s}(\tau)$

(changing with τ), for any $\tau \in \mathbb{R}$; we stress that $\boldsymbol{Q}(\tau)$ is in fact in the interior of $W^u_{l_s}(\tau)$ since it does not coincide either with $\boldsymbol{x}^{\boldsymbol{u}}_{\boldsymbol{l_s}}(\tau,\downarrow)$, nor with the origin. We want to prove the existence of infinitely many points $\boldsymbol{R_k} \in [W^u_{l_s}(\tau) \cap W^s_{l_s}(\tau)]$. Then the trajectories $\boldsymbol{x_{l_s}}(t,\tau;\boldsymbol{R_k})$ are homoclinic to the origin, so they correspond to solution $u_k(r)$ of (1.4) which are G.S. with f.d.
The argument is based on the following facts.

(1) The critical point $\boldsymbol{P_{l_s}}(+\infty)$ of the autonomous system (2.2) where $g_{l_s}(x,t) \equiv g^{+\infty}_{l_s}(x)$ is an unstable focus (this holds whenever $l_s \in (\sigma_*, 2^*)$).
(2) For any $\tau \in \mathbb{R}$ the manifold $W^u_{l_s}(\tau)$ is a smooth path connecting the origin with $\boldsymbol{x}^{\boldsymbol{u}}_{\boldsymbol{l_s}}(\tau,\downarrow)$, and the manifold $W^s_{l_s}(\tau)$ is a smooth path connecting the origin with $\boldsymbol{Q}(\tau) = \boldsymbol{x}^{\boldsymbol{s}}_{\boldsymbol{l_s}}(\tau,\uparrow)$.
(3) There is a G.S. with s.d. if and only if the trajectory $\boldsymbol{x}^{\boldsymbol{s}}_{\boldsymbol{l_s}}(\tau,\uparrow) \in W^u_{l_s}(\tau)$ for any $\tau \in \mathbb{R}$. There is a S.G.S. with s.d. if and only if $\boldsymbol{x}^{\boldsymbol{s}}_{\boldsymbol{l_s}}(t,\uparrow) \equiv \boldsymbol{x}^{\boldsymbol{u}}_{\boldsymbol{l_s}}(t,\downarrow)$ for any $t \in \mathbb{R}$.

From point 1 we see that the stable manifold $W^s_{l_s}(+\infty)$ is a spiral rotating indefinitely around $\boldsymbol{P_{l_s}}(+\infty)$ and which connects such a point with the origin. Then from point 2 it follows that we can choose τ large enough so that $W^s_{l_s}(\tau)$ is a spiral rotating indefinitely around $\boldsymbol{Q}(\tau) := \boldsymbol{x}^{\boldsymbol{s}}_{\boldsymbol{l_s}}(\tau,\uparrow)$. From point 3 we see that $\boldsymbol{Q}(\tau) \in W^u_{l_s}(\tau)$. Let U be a neighborhood of $\boldsymbol{Q}(\tau)$ and denote by $\bar{W}^u_{l_s}(\tau)$ the connected component of $W^u_{l_s}(\tau) \cap U$ containing $\boldsymbol{Q}(\tau)$. Since $\bar{W}^u_{l_s}(\tau)$ is a C^1 manifold we can choose U small enough so that $\bar{W}^u_{l_s}(\tau)$ is C^1 close to a segment. So it is easy to realize (use e.g. polar coordinates centered in $\boldsymbol{Q}(\tau)$, see [11] for details) that $\bar{W}^u_{l_s}(\tau)$ intersects the spiral $W^s_{l_s}(\tau)$ infinitely many times. Thus we get the existence of infinitely many G.S. with f.d.
Assume now that (1.4) admits a S.G.S. with s.d., i.e. $u(r,\downarrow) \equiv v(r,\uparrow)$ for any $r > 0$: this is a very degenerate case corresponding to the intersection of two one-dimensional objects in $\mathbb{R}^3$, however we cannot exclude this possibility as an alternative to the other "rare" solutions in theorem 2.4. In such a case, as observed in [11], the unstable manifold $W^u_{l_s}(\tau)$ is a spiral that winds around $\boldsymbol{x}^{\boldsymbol{u}}_{\boldsymbol{l_s}}(\tau,\downarrow)$ clockwise, while $W^s_{l_s}(\tau)$ is a spiral that winds around $\boldsymbol{x}^{\boldsymbol{s}}_{\boldsymbol{l_s}}(\tau,\uparrow)$ counterclockwise. So repeating the previous argument we find again infinitely many intersections between $\bar{W}^u_{l_s}(\tau)$ and $W^s_{l_s}(\tau)$, and we get infinitely many G.S. with f.d.; so assertion **(a)** is proved. Assertion **(b)** is completely analogous and might be obtained using again Kelvin inversion.

Assertion **(c)** follows observing that this topological argument is someway robust. So if we perturb the system (changing the values of α_l and γ_l) the manifold $W^s_{l_s}(\tau)$ is a spiral, but its center is not anymore a point $\boldsymbol{Q}(\tau) \in W^u_{l_s}(\tau)$ but it is close to it. So a large number of (transversal) intersections between $\bar{W}^u_{l_s}(\tau)$ and $W^s_{l_s}(\tau)$ persist: so we still have a large number of G.S. with f.d.

From Remark 3.3 and the discussion of theorem 2.5 we get the following, see also [10, 15].

Proposition 4.15. *Assume $\boldsymbol{G_u}$ and $\boldsymbol{G_s}$ with $2_* < l_s < 2^* < l_u$. Assume further that $l_s \in (\sigma_*, 2^*)$ and that there is a G.S. with s.d. $u(\bar{d}, r)$, then there is a sequence $d_j \to \bar{d}$ such that $u(r; d_j)$ is a G.S. with f.d. Analogously assume that $l_u \in (2^*, \sigma^*)$, and that there is a S.G.S. with f.d. Then there is a sequence $d_j \to +\infty$ such that $u(r; d_j)$ is a G.S. with f.d.*

We see now briefly which are the consequences of our analysis for the Dirichlet problem in the ball. Assume $\boldsymbol{G_u}$ and $\boldsymbol{G_s}$ with $2_* < l_s < 2^* < l_u$, and let $u(r; d)$ be the regular solution of (1.4) with $u(0; d) = d$. It is easy to check that the set

$$C := \{d > 0 \,|\, u(r; d) \text{ is a crossing solution } \} \tag{4.9}$$

is open. From Lemma 3.5 we also know that there is $D > 0$ (depending on l_u, l_s and the type of non-linearity f chosen) such that $(0, D) \subset C$. Denote by $R(d)$ the first zero of $u(r; d)$: using continuous dependence on initial data we find that $R(d)$ is continuous on C. Furthermore from Remarks 3.1 and 3.3 we get $R(d) \to +\infty$ as $d \to 0$. Let $v(r)$ be the unique singular solution and R^* its first zero (we set $R^* = +\infty$ if v is a S.G.S.), then $R(d) \to R^*$ as $d \to +\infty$: this follows using Remark 3.3 and continuous dependence from initial data. Finally note that, if $u(d^*, r)$ is a G.S. then $R(d) \to +\infty$ as $d \to d^*$. Using these observations we find the following.

Proposition 4.16. *Assume $\boldsymbol{G_u}$ and $\boldsymbol{G_s}$ with $2_* < l_s < 2^* < l_u$. Then there are $\rho_2 \geq \rho_1$ such that the Dirichlet problem in the ball of radius R for (1.4) admits no solutions whenever $0 < R < \rho_1$, at least two solutions for $R \in (\rho_1, \rho_2)$ and at least one for $R \geq \rho_2$.*
Moreover assume that there are exactly k G.S. with f.d. (or infinitely many of them). Then there are $\rho_0 \leq \rho_1 \leq \rho_2 \leq \ldots \leq \rho_k < +\infty$ (respectively an increasing sequence $\rho_k \to \infty$), such that the Dirichlet problem in the ball of radius R admits no solutions for $0 < R < \rho_0$, at least $2j+1$ solutions for any $R \geq \rho_j$ for $j = 0, \ldots, k$ (respectively no solutions for $0 < R < \rho_0$, at least $2j + 1$ solutions for any $R \geq \rho_j$ for $j \in \mathbb{N}$).

5. Appendix

5.1. **The technical hypotheses $\boldsymbol{A'_u}$ and $\boldsymbol{A'_s}$.** In this subsection we always assume that f is of type (2.6) and we prove proposition 2.6. I.e. we show that, in order to prove theorems 2.2 and 2.4, we can replace the technical requirement $\boldsymbol{A_u}$ by the weaker assumption $\boldsymbol{A'_u}$, and $\boldsymbol{A_s}$ by $\boldsymbol{A'_s}$. In fact such assumptions are needed to prove propositions 3.8 and 3.9. So we just need to reprove those propositions with these modified assumptions, then it is straightforward to check that the proofs of theorems 2.2 and 2.4 go through without further changes.

When f is of type (2.6) we can rewrite (2.4) for the auxiliary system (2.2) as follows:

$$\frac{d}{dt} H_{2^*}(\boldsymbol{x_{2^*}}(t, \tau; \boldsymbol{Q}), t) = \frac{\partial}{\partial t} G_{2^*}(\boldsymbol{x_{2^*}}(t, \tau; \boldsymbol{Q}), t) = \frac{\partial}{\partial t} G_{2^*}(\boldsymbol{x_{2^*}}(t, \tau; \boldsymbol{Q}), t) =$$
$$= \sum_{i=1}^{j} \left\{ \frac{\left[\boldsymbol{x_{2^*}}(t, \tau; \boldsymbol{Q})\right]^{q^i}}{q^i} \frac{d}{dt} \left[k^i (e^t) \mathrm{e}^{2(2^* - q^i)/(2^* - 2)t} \right] \right\}$$

for any $\boldsymbol{Q} \in \mathbb{R}^2_+$ and any $t, \tau \in \mathbb{R}$. So, assume $\boldsymbol{G_u}$ and consider system (4.4) and the modified equation (1.4) with $f = f^m$ as in (4.5). Let $\bar{u}^m(r)$ be a solution of the modified equation (1.4) with $f = f^m$ and let $\bar{\boldsymbol{x}}^{\boldsymbol{m}}_{\boldsymbol{2^*}}(t, \tau; \boldsymbol{Q^m})$ be the corresponding trajectory of (4.4); integrating by parts we get

$$H_{2^*}(\bar{\boldsymbol{x}}^{\boldsymbol{m}}_{\boldsymbol{2^*}}(t, \tau; \boldsymbol{Q^m}), t) = \int_{-\infty}^{t} \frac{\partial}{\partial t} G_{l_u}(\bar{\boldsymbol{x}}^{\boldsymbol{m}}_{\boldsymbol{2^*}}(s, \tau, \boldsymbol{Q^m}), s) ds =$$
$$= \sum_{i=1}^{j} \left\{ J^{-,i}(r) [\bar{u}^m(r)]^{q^i} - q^i \int_0^r J^{-,i}(s) [\bar{u}^m(s)]^{q^i - 1} \frac{d}{ds} [\bar{u}^m(s)] ds \right\}$$

whenever $\boldsymbol{Q^m} \in W^{u,m}_{2^*}(\tau)$. So if $\boldsymbol{A'_u}$ holds, as long as $\frac{d}{dr} \bar{u}^m(r) < 0 < \bar{u}^m(r)$, we find $H_{2^*}(\bar{\boldsymbol{x}}^{\boldsymbol{m}}_{\boldsymbol{2^*}}(t, \tau; \boldsymbol{Q^m}), t) > 0$, where $r = \mathrm{e}^t$. Then the proof of proposition 3.8, and consequently of theorems 2.2 and 2.4 goes through without further changes.

To reprove proposition 3.9 we need the following well known observation which holds whenever $f(u, r) \geq 0$.

Remark 5.1. Assume that the solution $v(r)$ of (1.4) is positive and decreasing for $r > R$. Then $v(r) r^{n-2}$ is increasing for $r > R$.

Assume $\boldsymbol{G_s}$ and consider system (4.6) and the modified equation (1.4) with $f = f^m$ as in (4.7). Let $\bar{v}^m(r)$ be a fast decay solution of the modified equation (1.4) with $f = f^m$ and let $\bar{\boldsymbol{x}}^{\boldsymbol{m}}_{\boldsymbol{2^*}}(t,\tau;\boldsymbol{R^m})$ be the corresponding trajectory of (4.6) where $\boldsymbol{R^m} \in W^{s,m}_{2^*}(\tau)$. Assume that $\bar{v}^m(r)$ is positive and decreasing for $r > R$. From remark 5.1, integrating by parts we find

$$H_{2^*}(\bar{\boldsymbol{x}}^{\boldsymbol{m}}_{\boldsymbol{2^*}}(t,\tau;\boldsymbol{R^m}),t) = -\int_t^{+\infty} \frac{\partial}{\partial t} G_{l_s}(\bar{\boldsymbol{x}}^{\boldsymbol{m}}_{\boldsymbol{2^*}}(s,\tau,\boldsymbol{R^m}),s)ds =$$

$$= \sum_{i=1}^{j} \left\{ J^{+,i}(r)[\bar{v}^m(r)r^{n-2}]^{q^i} + q^i \int_r^{+\infty} J^{+,i}(s)[\bar{v}^m(s)s^{n-2})]^{q^i-1} \frac{d}{ds}[\bar{v}^m(s)s^{n-2})]ds \right\}$$

So $H_{2^*}(\bar{\boldsymbol{x}}^{\boldsymbol{m}}_{\boldsymbol{2^*}}(t,\tau;\boldsymbol{R^m}),t)$ is positive if $\boldsymbol{A'_s}$ hold and $r = \mathrm{e}^t > R$. Then again the proof of proposition 3.9, and consequently of theorems 2.2 and 2.4 goes through without further changes. We stress that in fact this second part of the proof could be obtained also directly from Kelvin inversion.

5.2. **Applications.** Here we give some examples of non-linearities f to which our results apply. Assume that f is of type (1.2), $q > -2$ and set $\lambda_* := (n-2)[q - 2\frac{n-1}{n-2}] > \lambda^* := \frac{n-2}{2}[q - 2\frac{n}{n-2}]$. Assume further that $k(r)$ is a Lipschitz function and that there exist $A > 0$, $B > 0$, $-2 < \delta^u < \lambda^* < \delta^s < \lambda_*$ and $\varpi > 0$ small enough such that

$$\begin{array}{ll} \lim_{r\to 0} k(r)r^{-\delta_u} = A\,, & \lim_{r\to+\infty} k(r)r^{-\delta_s} = B\,, \\ \lim_{r\to 0} k'(r)r^{1-\varpi} = 0\,, & \lim_{r\to+\infty} k'(r)r^{1+\varpi} = 0\,, \end{array} \tag{5.1}$$

then from a straightforward computation we see that $\boldsymbol{G_u}$ and $\boldsymbol{G_s}$ are satisfied, with $l_u = 2\frac{\delta^u+q}{\delta^u+2}$ and $l_s = 2\frac{\delta^s+q}{\delta^s+2}$. Note that if $\delta^u = \Sigma^* := \frac{2(q-\sigma^*)}{2-\sigma^*}$, or $\delta^s = \Sigma_* := \frac{2(q-\sigma_*)}{2-\sigma_*}$ then $\boldsymbol{G_u}$ and $\boldsymbol{G_s}$ hold respectively with $l_u = \sigma^*$ and $l_s = \sigma_*$. So if (5.1) holds we can apply theorem 2.3 and conclude that, given $\lambda^* < \delta^s < \lambda_*$ there is $n_0(\delta^s) > 0$ such that (1.4) admits no G.S. with either fast or slow decay and no S.G.S. with either fast or slow decay whenever $\delta^u \in (-2; -2 + n_0(\delta^s))$. Similarly given $-2 < \delta^u < \lambda^*$ there is $\varepsilon_0(\delta^u) > 0$ such that (1.4) admits no G.S. with either fast or slow decay and no S.G.S. with either fast or slow decay whenever $\delta^s \in (\lambda_* - \varepsilon_0(\delta^u), \lambda_*)$.

Moreover if $k(r)r^{-\delta_u}$ is strictly increasing for r small and $J^-_{l_u}(r) > 0$ for any $r > 0$ then $\boldsymbol{A'_u}$ holds; thus, when (5.1) is satisfied we can apply theorem 2.2. I.e. if we fix $\delta^s \in (\lambda^*; \lambda_*)$, then for any integer $k > 0$ there exists $\varepsilon_k(\delta^s) > 0$ such that (1.4) admits at least k G.S. with f.d. whenever $\delta^u \in (\lambda^* - \varepsilon_k(\delta^s), \lambda^*)$. Moreover, via theorem 2.4, we also get the existence of a sequence of values $r^k(\delta^s) \nearrow \lambda^*$ such that (1.4) with $\delta^u = r^k(\delta^s)$ admits either a G.S. with s.d., or a S.G.S. with f.d. or a S.G.S. with s.d. In fact for k large enough we can also assume that $r^k(\delta^s) = \lambda^* - \varepsilon_k(\delta^s)$. Moreover if we also assume that $\delta^s \in (\lambda^*; \Sigma_*)$, when $\delta^u = r^k(\delta^s)$, we can apply theorem 2.5 to conclude the existence of infinitely many G.S. with f.d. and the persistence of a large number of them for small variations in the parameters.

Similarly if $k(r)r^{-\delta_s}$ is strictly decreasing for r large and $J^+_{l_s}(r) < 0$ for any $r > 0$ then $\boldsymbol{A'_s}$ holds; hence, when (5.1) holds we can apply theorem 2.2. Thus for any integer $k > 0$ there exists $\epsilon_k(\delta^u) > 0$ such that (1.4) admits at least k G.S. with f.d. whenever $\delta^s \in (\lambda^*, \lambda^* + \epsilon_k(\delta^u))$. Then, via theorem 2.4, we get the existence of a sequence of values $R^k(\delta^u) \searrow \lambda^*$ such that (1.4) with $\delta^s = R^k(\delta^u)$ admits either a G.S. with s.d., or a S.G.S. with f.d. or a S.G.S. with s.d., and for k large enough we can also assume that $R^k(\delta^u) = \lambda^* + \epsilon_k(\delta^u)$. Moreover for $\delta^u \in (\Sigma^*; \lambda^*)$, and $\delta^s = R^k(\delta^u)$, via theorem 2.5 we find infinitely many G.S. with f.d. a large number of which persists for small variations in the parameters.

We emphasize that if $k(r)r^{-\delta_u}$ is increasing for any $r > 0$ (strictly in some interval) then $J^-_{l_u}(r)$ is positive for any $r > 0$, and if $k(r)r^{-\delta_s}$ is decreasing for any

$r > 0$ (strictly in some interval) then $J^+_{l_s}(r)$ is positive for any $r > 0$. So we can apply our construction e.g. to a function f of type (1.2) where $k(r)$ is of type

$$k(r) = Ar^{\delta^u} + \sum_{i=1}^{j} C_i r^{\delta^j} + Br^{\delta^s} \tag{5.2}$$

and A, B are positive constants $C_i \geq 0$, and $-2 < \delta^u < \lambda^* < \delta^s < \lambda_*$ and $\delta^i \in (\delta^u, \delta^s)$ for any $i = 1, \dots, j$. Here again the leading parameters l_u and l_s are determined just by δ^u and δ^s respectively. In such a case theorems 2.3, 2.4, and 2.5 give results which have not appeared previously in literature at all (to the best of our knowledge), while theorem 2.2 has already been proved via variational techniques in [1].

Our results can be applied also to functions f of the form (2.6), i.e.

$$f(u, r) = \sum_{i=1}^{j} k^i(r)|u_+|^{q^i - 1} \tag{5.3}$$

which in fact are not discussed in literature in the spatial dependent case. Set $\zeta^i = \frac{2(q^j - q^i)}{q^j - 2} \geq 0$, $\eta^i = \frac{2(q^1 - q^i)}{q^1 - 2} \leq 0$. Assume that the functions $k^i(r)$ are positive and, for simplicity, that the limits $\lim_{r\to 0} k^i(r)$, $\lim_{r\to+\infty} k^i(r)$ are positive and finite for any $i = 1, \dots, j$. Then define

$$h^i(r) = k^i(r) r^{\zeta^i}, \qquad \text{and} \qquad \tilde{h}^i(r) = k^i(r) r^{\eta^i}$$

for $i = 1, \dots, j$ and assume that there is $\varpi > 0$ small enough so that

$$\lim_{r\to 0} \frac{dk^i}{dr}(r) r^{\zeta^i + 1 - \varpi} = 0 = \lim_{r\to+\infty} \frac{dk^i}{dr}(r) r^{\eta^i + 1 + \varpi}$$

for $i = 1, \dots, j$; then $\boldsymbol{G_u}$ and $\boldsymbol{G_s}$ hold with $l_u = q^j$ and $l_s = q^1$. Assume further $2_* < q^1 < 2^* < q^j$, and $q_i < q_{i+1}$, for $i = 1, \dots, j-1$; then we can apply theorem 2.3; so if we fix q^i for any $i \geq 2$, we can find $\varepsilon_0(q^j) > 0$ such that (1.4) admits no G.S. neither S.G.S. (with either fast or slow decay), for $q^1 \in (2_*, 2_* + \varepsilon_0(q^j))$. Similarly if we fix q^i for $i \leq j-1$, we can find $N_0(q^1) > 0$ such that (1.4) admits no G.S. neither S.G.S. (with either fast or slow decay), for $q^j > N_0(q^1)$.

Now assume $2_* < q^i \leq 2^* < q^j$ for any $i = 1, \dots, j-1$. If the functions $k^j(r)$, and $h^i(r)$ are increasing in r for any $r > 0$, for $i = 1, \dots j-1$, then $\boldsymbol{A'_u}$ holds. So we can apply theorem 2.2 and 2.4; i.e. if we fix $q^i \in (2_*; 2^*)$ for $i \leq j-1$, we see that for any $k > 0$ there is $\varepsilon_k(q^1) > 0$ such that (1.4) admits at least k G.S. with f.d. whenever $q^j \in (2^*; 2^* + \varepsilon_k(q^1))$. Moreover there is a sequence $r^k(q^1) \searrow 2^*$ such that (1.4) with $q^j = r^k(q^1)$ admits either a G.S. with s.d. or a S.G.S. with either f.d. or s.d. (again we can also assume that $r^k(q^1) = 2^* + \varepsilon_k(q^1)$). Moreover if $q^1 \in (\sigma_*, 2^*)$ then we can also apply theorem 2.5 and we see that for $q^j = r^k(q^1)$ we also have infinitely many G.S. with f.d. a large number of which persist for small variations in the exponents q^i and ζ^i.

Analogously assume that $2_* < q^1 < 2^* \leq q^i < q^j$ for any $i = 2, \dots, j-1$. If the functions $k^1(r)$, and $\tilde{h}^i(r)$ are decreasing in r for any $r > 0$, for $i = 2, \dots j$, then $\boldsymbol{A'_s}$ holds, and we can apply theorem 2.2 and 2.4. So let $q^i > 2^*$ be fixed, for $i \geq 2$; we see that for any $k > 0$ there is $\varepsilon_k(q^j) > 0$ such that (1.4) admits at least k G.S. with f.d. whenever $q^1 \in (2^* - \varepsilon_k(q^j); 2^*)$. Moreover there is a sequence $R^k(q^j) \nearrow 2^*$ such that (1.4) with $q^1 = R^k(q^j)$ admits either a G.S. with s.d. or a S.G.S. with either f.d. or s.d. Moreover if $q^j \in (2^*; \sigma^*)$ then we can also apply theorem 2.5 and we see that for $q^1 = R^k(q^j)$ we also have infinitely many G.S. with f.d. a large number of which persist for small variations in the exponents q^i and η^i. These results extend to the spatial dependent case [2] and [11]. In fact the

nonexistence result for q^j large (i.e. the second part of theorem 2.3) is new even in the spatial independent case with $j = 2$.

We emphasize that the whole argument applies to f of type (5.3) even when the functions $k^i(r)$ are not uniformly positive and bounded but there are constants δ_u^i and δ_s^i such that $k^i(r)r^{\delta_u^i}$ and and $k^i(r)r^{\delta_s^i}$ tend to positive constants respectively as $r \to 0$ and as $r \to \infty$, for any $i = 1, \dots, j$. Moreover the condition on the monotonicity of the functions $h^i(r)$ and $\tilde{h}^i(r)$ are sufficient to satisfy $\boldsymbol{A'_u}$ and $\boldsymbol{A'_s}$ respectively, but they are not necessary (i.e. $\boldsymbol{A'_u}$ and $\boldsymbol{A'_s}$ are more general).

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