FOWLER TRANSFORMATION AND RADIAL SOLUTIONS FOR QUASILINEAR ELLIPTIC EQUATIONS. PART 1: THE SUBCRITICAL AND THE SUPERCRITICAL CASE.

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ABSTRACT. We illustrate a method, based on a generalized Fowler transformation, to discuss the existence and the asymptotic behavior of positive radial solutions for the following equation:

$$\Delta_p u(\mathbf{x}) + f(u, |\mathbf{x}|) = 0,$$

where $\Delta_p u = div(|Du|^{p-2}Du)$, $\mathbf{x} \in \mathbb{R}^n$, $n > p > 1$. This approach proves to be particularly useful in the spatial dependent case. Moreover it is a good tool to detect singular and fast decay solutions.

We apply it to the case in which $f \geq 0$ is either subcritical or supercritical, obtaining structure results for positive solutions and refining the estimates on the asymptotic behavior.

The equation has been proposed as a reaction diffusion model for a non-Newtonian fluid and can also be regarded as the constitutive law for a problem in elasticity theory.

1. Introduction

The main purpose of this paper is to develop a method, based on a generalized Fowler transformation, to obtain structure results for positive radial solutions of the following quasi-linear elliptic equation

$$\Delta_p u(\mathbf{x}) + f(u, |\mathbf{x}|) = 0, \tag{1.1}$$

where $\Delta_p u = div(|Du|^{p-2}Du)$ is the so called p-Laplacian, $\mathbf{x} \in \mathbb{R}^n$, $n > p > 1$, and $f(u, |\mathbf{x}|)$ is a continuous function which is locally Lipschitz in the u variable and positive for $u > 0$ and it is null for $u = 0$. We also assume f to be super-half linear (see hypotheses $\mathbf{F0}, \mathbf{G0}$ below). We require $f(-u, r) = -f(u, r)$, even if we are just interested in positive solution. We will apply our technique to a rather wide family of nonlinearities f including

$$f(u, |\mathbf{x}|) = k(|\mathbf{x}|)u|u|^{q-2}, \tag{1.2}$$

where $q > p$ and k is positive and continuous, and

$$f(u, |\mathbf{x}|) = k_2(|\mathbf{x}|)\frac{u|u|^{q_2-2}}{1 + k_1(|\mathbf{x}|)u|u|^{q_1-2}}, \tag{1.3}$$

where $q_2 > q_1 > 0$, $q_2 - q_1 > p - 1$ and the functions $k_1(|\mathbf{x}|)$ and $k_2(|\mathbf{x}|)$ are nonnegative and continuous.

We consider just radial solutions so we commit the following abuse of notation: we write $u(r)$ for $u(\mathbf{x})$ where $|\mathbf{x}| = r$. Since we only deal with radial solutions we will in fact consider the following singular O.D.E.

$$(u'|u'|^{p-2})' + \frac{n-1}{r}u'|u'|^{p-2} + f(u, r) = 0. \tag{1.4}$$

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Here $'$ denotes the derivative with respect to r. We call "regular" the positive solution $u(r)$ of (1.4) such that $u(0) = d > 0$ and $u'(0) = 0$, and we denote it by $u(d, r)$. We call "singular" positive solutions $u(r)$ which are singular in the origin, that is $\lim_{r\to 0} u(r) = +\infty$.

In particular we will focus our attention on the problem of existence of ground states (G.S.), of singular ground states (S.G.S.) and of crossing solutions. By G.S. we mean a positive regular solution $u(r)$ defined for any $r \geq 0$ such that $\lim_{r\to\infty} u(r) = 0$. A S.G.S. of equation (1.1) is a singular positive solution $v(r)$ such that $\lim_{|r|\to 0} v(r) = +\infty$ and $\lim_{|r|\to +\infty} v(r) = 0$. Crossing solutions are radial solutions $u(r)$ such that $u(r) > 0$ for any $0 \leq r < R$ and $u(R) = 0$ for a certain $R > 0$, so they can be considered as solutions of the Dirichlet problem in the ball of radius R.

It is possible to prove in a very general context, see Remark 3.8 below, that the limit $\lim_{r\to +\infty} u(r) r^{\frac{n-p}{p-1}}$ exists and it is positive for all the solutions $u(r)$ which are positive for r large. If these limit is finite we say that $u(r)$ has fast decay, while if it is infinity we say that it has slow decay.

In the last 20 years this family of equations has received a lot of interest, both for the intrinsic mathematical interest and for its applications. E. g. when f is of type (1.2) with $p = 2$ and $k(r) = r^{s_1}/(1 + r^{s_2})$, Eq. (1.1) is known as Matukuma equation and the study of G. S. with fast decay is relevant in understanding star clusters in astrophysics. Moreover when $p = 2$ and $q = p^*$ the equation is known as scalar curvature equation and the existence of a G.S. $u(\mathbf{x})$ with fast decay amounts to the existence of a metric g conformal to a standard metric g_0 on $\mathbb{R}^n$ ($g = u^{\frac{4}{n-2}} g_0$), whose scalar curvature is $k(|\mathbf{x}|)$, see [10], [17]. When $p \neq 2$ these equation finds application in theory of non Newtonian fluids and in theory of elasticity. More precisely let us consider the turbulent flow of a polytropic gas, whose concentration is $v(x)$. We assume that it is being produced by a nonlinear reaction, and it diffuses in a porous medium. If we set $v(x) = u(x)^{\alpha}$, where $0 < \alpha < 1/2$ is a constant depending on the pressure, then the steady states are described by (1.1). The function $f(u, x) \geq 0$ describes the nonlinear reaction which generates the gas and the spatial dependence is related to the non-uniform presence of the catalyst. For physical reasons just positive solutions are relevant and we have to assume $p \in [3/2; 2]$, see [3], [7]. Here we consider the diffusion of the gas either in the whole of $\mathbb{R}^n$ or in a ball giving Dirichlet boundary condition. As we said for $p \neq 2$ we have also applications in theory of elasticity. Let us consider a radially symmetric shell subject to the nonlinear stress due to his own weight or to an external force. We denote by $f(u, x)$ the force depending on the axial deflection $u(x)$ from the rest state and on the spatial variable. Then if the material is hyperelastic, so that the stress is in divergence form, the steady states for the function $u(x)$ are given by the solutions of (1.1). We remark that the nonlinear term $\Delta_p(u)$ appears naturally and the value $p \in [1, \infty)$ has to be chosen according to the capacity of the material to resist to flexion. Moreover if the material is not spatially homogenous we have to replace Δ_p by the more general operator of the form $\mathrm{div}(g(|\mathbf{x}|)\nabla u |\nabla u|^{p-2})$. In the appendix we will see how the analysis of radial solution of equations involving this operator and a spatial independent force $f = f(u)$, can be reduced to the investigation of equations involving the usual p-Laplacian and a spatial dependent force $\tilde{f} = \tilde{f}(u, x)$. We remand the interested reader to [1] for a physical explanation of the problem.

Let us denote by $F(u, r) = \int_0^u f(s, r) ds$ and by $\mathfrak{F}(u, r) = f(u, r)/|u|^{p-1}$. We will usually consider functions f satisfying the following:

F0: There are $M > 0$ and $R > 0$ such that $\mathfrak{F}(u, r)$ is increasing in the first variable whenever $(u, r) \in \big([M, +\infty) \times (0, 1/R]\big) \cup \big((0, 1/M] \times [R, +\infty)\big)$.

Hypothesis **F0** is satisfied e. g. when f is of type (1.2) and $q > p$ and when f is of type (1.3) and $q_2 - q_1 > p - 1$. It is well known that in the former case the structure of positive solutions changes drastically when the parameter q is larger or smaller than some critical values $\sigma = p\frac{n-1}{n-p}$ and $p^* := \frac{np}{n-p}$, see Proposition 2.11 below.

When f is of type (1.2) or (1.3), roughly speaking positive solutions exhibit two typical structures separated by a third one that lies in the border between them.

Sub: All the regular solutions $u(d, r)$ are crossing solutions and they have negative slope at their first zero $R(d)$, there is a unique S.G.S. with slow decay and uncountably many S.G.S. with fast decay. No G.S. can exist.

Crit: All the regular solutions are G.S. with fast decay, there are uncountably many S.G.S. with slow decay. No other positive solutions can exist.

Sup: All the regular solutions are G.S. with slow decay, there is a unique S.G.S. and has slow decay. There are uncountably many solutions $u(r)$ of the Dirichlet problem in the exterior of balls, i. e. there is $R > 0$ such that $u(R) = 0$, $u(r) > 0$ for $r > R$ and $u(r)$ has fast decay.

When f is of type (1.2) and it is spatial independent it is well known that we have structure **Sup** for $q > p^*$, **Crit** for $q = p^*$ and **Sub** for $\sigma < q < p^*$. When $p < q \leq \sigma$ regular solutions have structure **Sub** but singular solutions do not exist. Some border phenomena appear when k is uniformly positive and bounded and $q = p^*$, see [11, 18].

In this paper we are mainly interested in developing a flexible method to investigate equations of the family (1.4). The main results are Theorems 4.2 and 4.3, in which we give sufficient conditions for the existence of structure **Sub** and **Sup** respectively. The contribution consists in the possibility to deal together with different spatial dependent non-linearities such as (1.2) and (1.3) inserting them in a more general context, which covers also new non-linearities. This way we link together results such as the ones obtained in [20] and [4], completing them with the analysis of singular solutions, and we generalize the results obtained in [9]. In a forecoming paper we will see how this method can be successfully applied to the more interesting and rich case in which f is subcritical for u large and r small and supercritical for u small and r large.

We follow the way paved by Johnson, Pan and Yi in [17] and later followed by Johnson, Battelli. So we introduce a dynamical system through a change of coordinates that generalizes the well known Fowler transform and we pass to a dynamical system. We spend quite a lot of effort on the analysis of the "rare" autonomous case discussing it in the maximal generality, because it is useful to construct sub and super solutions. Then we use this knowledge to prove the existence of unstable and stable sets $W^u(\tau)$ and $W^s(\tau)$ which are made up of initial condition which correspond respectively to regular and fast decay solutions. Then we establish their mutual position using the transposition of the Pohozaev function, see e.g. [20], [21], for this dynamical context and we conclude with elementary analysis of the phase portrait. We point out that fast decay solutions are not easily detected with different techniques and their existence is usually obtained in the $p = 2$ case through the Kelvin inversion, see e. g. [25], which is not available in the $p \neq 2$ case.

The main disadvantage of our method depends on the fact that it can just deal with radial solutions and it is not suitable to discuss radial domains. However we stress that radially symmetric solutions are particularly important for (1.1). In fact when the domain has radial symmetry and f is spatial independent, G.S. and solutions of the Dirichlet problem in balls have to be radially symmetric under quite weak assumptions, see e.g. [6], [24], [26]. Moreover the ω-limit set of certain parabolic equations associated to (1.1) is made up of the union of radially symmetric ground states of (1.1), see e.g. [23].

The paper is structured as follows: in section 2 we introduce the Fowler transformation and we discuss the autonomous case. In section 3.1 we develop the construction of stable and unstable sets; in section 3.2 we establish some asymptotic estimates on the behavior of positive solutions. In section 4 we give the applications of our method and prove the main results. In the appendix we recall how we can reduce the study of a class of radial equations involving a spatial dependent p-Laplace operator to the study of (1.4), see [14].

2. Preliminary result: autonomous problem

We begin this section by introducing a dynamical system through the following change of coordinates depending on the parameter $l > p$:

$$\begin{aligned} &\alpha_l = \tfrac{p}{l-p}, \qquad \beta_l = \tfrac{(p-1)l}{l-p}, \qquad \gamma_l = \beta_l - (n-1), \\ &x_l = u(r)r^{\alpha_l} \qquad y_l = u'(r)|u'(r)|^{p-2}r^{\beta_l} \qquad r = e^t \\ &g_l(x_l,t) = f(xe^{-\alpha_l t}, e^t)e^{\alpha_l(l-1)t} \end{aligned} \tag{2.1}$$

Using (2.1) we pass from (1.4) to the following system:

$$\begin{pmatrix} \dot{x}_l \\ \dot{y}_l \end{pmatrix} = \begin{pmatrix} \alpha_l & 0 \\ 0 & \gamma_l \end{pmatrix} \begin{pmatrix} x_l \\ y_l \end{pmatrix} + \begin{pmatrix} y_l|y_l|^{\frac{2-p}{p-1}} \\ -g(x_l,t) \end{pmatrix} \tag{2.2}$$

Note that local uniqueness of the solutions of (2.2) on the coordinate axes is guaranteed if and only if f is locally Lipschitz also for $u = 0$, and if $1 < p \le 2$. However most of the difficulties arising in this non-regular setting may be overcome, as we will see below.

We give now some notation that will be in force in the whole paper. We denote by bold letters the trajectory and with normal letters their coordinates as follows: $\bar{\mathbf{x}}_l(t) = (\bar{x}_l(t), \bar{y}_l(t))$. We denote by capital letters the trajectories of the autonomous system to distinguish them from the ones of the non-autonomous system. We denote by $\mathbf{X}_{\bar{l}}(t,\tau;\mathbf{Q},g_{\bar{l}}) = (X_{\bar{l}}(t,\tau;\mathbf{Q},g_{\bar{l}}), Y_{\bar{l}}(t,\tau;\mathbf{Q},g_{\bar{l}}))$ the trajectory of the autonomous system (2.2) where $l = \bar{l}$ and $g_{\bar{l}}(x,t) \equiv g_{\bar{l}}(x)$, departing from $\mathbf{Q}$ at $t = \tau$. We denote by $\mathbf{x}_{\bar{l}}(t,\tau;\mathbf{Q}) = (x_{\bar{l}}(t,\tau;\mathbf{Q}), y_{\bar{l}}(t,\tau;\mathbf{Q}))$ the trajectory of the non-autonomous system (2.2) where $l = \bar{l}$, departing from $\mathbf{Q}$ at $t = \tau$.

Obviously positive solution $u(r)$ of (1.4) correspond to trajectories $\mathbf{x}_l(t)$ such that $x_l(t) > 0$ and $u'(r) < 0$ implies $y_l(t) < 0$ and viceversa. We denote by $G_l(x_l,t) := \int_0^{x_l} g_l(\xi,t)d\xi = F(x_l e^{-\alpha_l t}, e^t)e^{\alpha_l l t}$ and by $\mathfrak{G}_l(x_l,t) := g_l(x_l,t)/|x_l|^{p-1}$ $= \mathfrak{F}(x_l e^{-\alpha_l t}, e^t)e^{pt}$.

When f is of type (1.2) the function f takes the form $g_l(x,t) = h_l(t)x|x|^{q-2}$ and $G_l(x,t) = h(t)\frac{|x|^q}{q}$, where $h_l(t) = k(e^t)e^{\delta_l t}$ and $\delta_l = \alpha_l(l-q)$. Furthermore, if $k(r) = Kr^s$, where $s \in (-p,+\infty)$, then there is $l > p$ such that $g_l(x,t) \equiv Kx|x|^{q-2}$ for any t, so that system (2.2) is autonomous. Moreover if $s \in (-p, \frac{n-p}{p-1}(q-\sigma))$ then $l > \sigma$, hence $\gamma_l < 0$.

We introduce now the Pohozaev function $P(u,u',r)$ and its transposition for this dynamical setting $H_l(x_l,y_l,t)$:

$$\begin{aligned} P(u,u',r) &:= r^n\left[\frac{n-p}{p}\frac{uu'|u'|^{p-2}}{r} + \frac{p-1}{p}|u'|^p + F(u,r)\right] \\ H_l(x_l,y_l,t) &:= \frac{n-p}{p}x_l y_l + \frac{p-1}{p}|y_l|^{\frac{p}{p-1}} + G_l(x_l,t) \end{aligned}$$

If $\mathbf{x}_{p^*}(t) = (x_{p^*}(t), y_{p^*}(t))$ and $\mathbf{x}_l(t) = (x_l(t), y_l(t))$ are the trajectories of (2.2) corresponding to $u(r)$, we have

$$P(u(r),u'(r),r) = H_{p^*}(x_{p^*}(t), y_{p^*}(t), t) = e^{-(\alpha_l+\gamma_l)t}H_l(x_l(t),y_l(t),t)\,. \tag{2.3}$$

When $g_l(x,t) = \bar{g}_l(x)$ the function $H_l(x,y,t)$ does not really depend on t, so we will write simply $H_l(x,y)$. We will exploit these functions in order to evaluate the mutual positions of the stable and unstable sets, both for the t-independent (autonomous) and for the t-dependent (non-autonomous) system (2.2).

The key observation is that when $G_{p^*}(x,t)$ is differentiable with respect to t, for any trajectory $\mathbf{x}_{p^*}(t)$ we have the following:

$$\frac{d}{dt} H_{p^*}(x_{p^*}(t), y_{p^*}(t), t) = \frac{\partial}{\partial t} G_{p^*}(x_{p^*}(t), t) \tag{2.4}$$

This is in fact another way to restate the Pohozaev identity, which is one of the main tool to investigate this equation, see e. g. [20].

From a straightforward computation we find the following.

2.1. *Remark.* For any i, j larger than p we have $\mathfrak{G}_i(x,t) = \mathfrak{G}_j(xe^{(\alpha_j-\alpha_i)t}, t) = \mathfrak{F}(xe^{-\alpha_i t}, e^t)e^{pt}$. So if $\mathbf{x}_i(t)$ and $\mathbf{x}_j(t)$ correspond to the same solution $u(r)$ of (1.4) we have $\mathfrak{G}_i(x_i(t), t) = \mathfrak{G}_j(x_j(t), t)$.

In the whole paper we assume the following condition without mentioning it further:

G0: There is $N > 0$ such that for any $|t| > N$ the function $\mathfrak{G}_l(x,t)$ is such that $\mathfrak{G}_l(0,t) = 0$, $\lim_{x\to\infty} \mathfrak{G}_l(x,t) = \infty$ and it is increasing in x for any $|t| > N$.

Observe that **F0** is a sufficient condition for **G0**, and that if f is of type (1.2) **G0** holds whenever $q > p$. From Remark 2.1 it follows that **G0** is in fact independent on the choice of the parameter l. Moreover we have the following:

2.2. *Remark.* If **G0** is satisfied then for $|t| > N$, the function $\mathcal{G}_l(x,t) := G_l(x,t)/x^p$ is increasing in x and $\lim_{x\to 0} \mathcal{G}_l(x,t) = 0$ and $\lim_{x\to+\infty} \mathcal{G}_l(x,t) = +\infty$.

We start from a remark concerning local uniqueness and continuous dependence on the initial data $d > 0$ for regular solutions $u(d,r)$ of (1.4).

2.3. *Remark.* For any $d > 0$ there is a $\rho(d) > 0$ such that (1.4) admits a unique solution $u(d,r)$ such that $u(d,0) = d$ and $u'(d,r) = 0$, and it is positive and decreasing for $r \in [0, \rho(d))$. Moreover, given $d_1, d_2 > 0$, for any $R < \min\{\rho(d_1), \rho(d_2)\}$ and any $\epsilon > 0$ there is $\delta > 0$ such that $\max_{r\in[0,R]} |u(d_1,r) - u(d_2,r)| + |u'(d_1,r) - u'(d_2,r)| < \epsilon$ whenever $|d_1 - d_2| < \delta$.

This can be proved putting together the ideas of Propositions A3 and A4 in [12], with some trivial modification to adapt them to the non-autonomous problem; see also [14] and the appendix of this paper to understand how we can reduce the case in which $f(u,r)$ is unbounded as $r \to 0$ to the case in which it is bounded. When $g_l(x_l,t)$ does not depend on t, all the solutions of (2.2) are invariant for translations in t. It follows that if $v(r)$ is a solution of (1.4) and $R > 0$ is a constant, then $w(r) = v(rR)R^{\alpha_l}$ is a solution as well. In particular if $u(d,r)$ is a regular solution of (1.4), then $u(dR^\alpha, r) = u(d, rR)R^\alpha$, for any $R > 0$.

When $l = p^*$ the function H_{p^*} is a first integral and we can draw each trajectory of the system, see (2.4). From **G0** it follows that $H_{p^*}(0, y_{p^*})$ is positive for any $y_{p^*} \neq 0$, and that there is $\bar{Q}_x > P_x > 0$ such that if $\mathbf{Q} = (x, (\alpha_{p^*}x)^{p-1}) \in U^0_{p^*}$ then $H_{p^*}(\mathbf{Q})$ is negative and decreasing for $0 < x < P_x$, negative and increasing for $P_x < x < \bar{Q}_x$ and it is positive and increasing for $x > \bar{Q}_x$. Since $H_{p^*}(0,0) = 0$ and H_{p^*} is symmetric with respect to the origin, it is easy to check that the level $H = 0$ is a 8 shaped curve whose junction point is the origin. So it follows that there are two homoclinic trajectories. Moreover the level sets of the function H_{p^*} are bounded, so we can conclude that the phase portrait is similar to the one obtained for non-linearities f of type (1.2) so it is as sketched in Fig. 1.

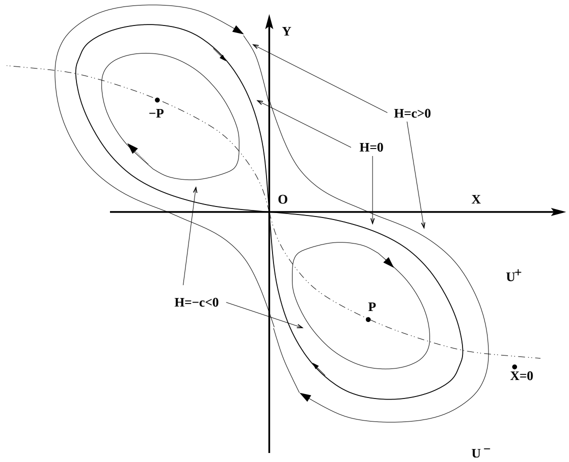

FIGURE 1. The level sets of the function $H(x, y, t)$ for t fixed

When (2.2) is autonomous and locally Lipschitz, using standard invariant manifold theory we find that the origin admits a stable and an unstable manifold. When $p > 2$ or f is not Lipschitz in u we can construct a stable set and an unstable manifold, denoted respectively by $M^s_l(\bar{g}_l)$ and $M^u_l(\bar{g}_l)$, exploiting the argument developed in [11]. In fact the proof in [11] is developed just for $\bar{g}_l$ of the form $\bar{g}_l(x) = kx|x|^{q-2}$, but it is easy to check that it works for any $\bar{g}_l$ satisfying **G0**. In fact $M^u_l(\bar{g}_l)$ exists for any $l > p$ while $M^s_l(\bar{g}_l)$ exists only if $l > \sigma$ (that is $\gamma_l < 0$). As we said $M^u_l(\bar{g}_l)$ is a manifold, while $M^s_l(\bar{g}_l)$ is a closed set, connected by arc and it is made up by the union of locally Lipschitz trajectories. Moreover repeating the argument of Lemmas 5.4 and 5.5 in [9] with some trivial changes we can prove the following.

2.4. **Lemma.** *Assume that $g_l(x_l, t) = \bar{g}_l(x)$ and consider a solution $u(r)$ of (1.4) and the corresponding trajectory $\boldsymbol{X}_l(t)$ of (2.2).*

If $u(r)$ is a regular solutions then $\boldsymbol{X}_l(0) \in M^u_l(\bar{g}_l)$ and viceversa. Moreover assume $l > \sigma$; if $\boldsymbol{x}_l(0) \in M^s_l(\bar{g}_l)$, then $u(r)$ has fast decay and viceversa.

We define here some sets that will be useful in the whole paper.

$$U_l^+ := \{(x_l, y_l) \mid \alpha_l x_l + y_l|y_l|^{\frac{2-p}{p-1}} > 0\} \qquad U_l^- := \{(x_l, y_l) \mid \alpha_l x_l + y_l|y_l|^{\frac{2-p}{p-1}} < 0\}$$
$$U_l^0 := \{(x_l, y_l) \mid \alpha_l x_l + y_l|y_l|^{\frac{2-p}{p-1}} = 0\}$$
$$\mathbb{R}^2_+ := \{(x_l, y_l) \mid x_l \geq 0\} \qquad \mathbb{R}^2_\pm := \{(x_l, y_l) \mid y_l < 0 < x_l\}$$

From now on we will commit the following abuse of notation: we will denote by $M^u_l(\bar{g})$ and $M^s_l(\bar{g})$ the branch of the manifold and of the continuum respectively, which depart from the origin and enters in $\mathbb{R}^2_+$. Follow $M^u_l(\bar{g})$ (respectively $M^s_l(\bar{g})$) from the origin towards $\mathbb{R}^2_+$: it intersects the isocline U^0_l in a point, denoted by $\tilde{\mathbf{Q}}^u_l$ (resp. in a set $\tilde{\xi}^s_l$). Denote by $\tilde{M}^u_l(\bar{g})$ (resp. $\tilde{M}^s_l(\bar{g})$), the branch of $M^u_l(\bar{g})$ (resp. $M^s_l(\bar{g})$) between the origin and $\tilde{\mathbf{Q}}^u_l$ (resp. $\tilde{\xi}^s_l$), deprived of the origin. With some elementary analysis on the phase portrait we get the following useful result.

2.5. **Lemma.** *Assume that there is $l > p$ such that $g_l(x, t) \equiv \bar{g}_l(x)$ and consider a trajectory $\boldsymbol{X}_l(t)$ of (2.2). If there is a sequence T_n, $|T_n| \to +\infty$, such that $\lim_{n\to\infty} |\boldsymbol{X}_l(T_n)| = +\infty$ then the trajectory $\boldsymbol{X}_l(t)$ has to cross the coordinate axes infinitely many times.*

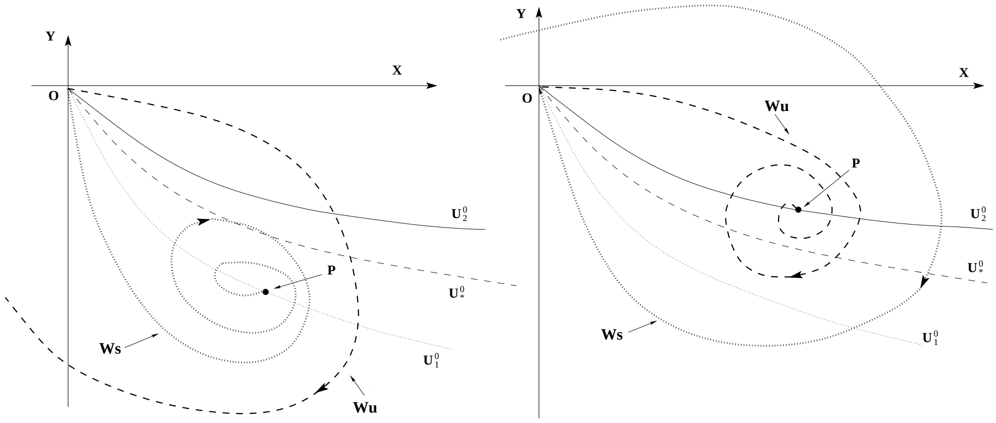

FIGURE 2. A sketch of M^u_l and of M^s_l when $l > p^*$ (on the left) and $\sigma < l < p^*$ (on the right)

Proof. We need to introduce the following modified polar coordinates:

$$x_l|x_l|^{p-2} = \rho_l \cos(\theta_l) \qquad y = \rho_l \sin(\theta_l) \tag{2.5}$$

Note that $\tan(\theta_l(t)) = u'(e^t)e^t/u(e^t)$, so in fact $\theta_l(t) = \theta(t)$ is independent of l. We get

$$\begin{aligned}\dot{\theta} &= \frac{|x_l|^{p-2}}{|x_l|^{2p-2} + |y_l|^2}\Big[[\gamma_l - (p-1)\alpha_l]x_l y_l - (p-1)y_l|y_l|^{\frac{2-p}{p-1}} - x_l g_l(x_l,t)\Big] = \\ &= (p-n)\sin(\theta)\cos(\theta) - (p-1)|\sin(\theta)|^{\frac{p}{p-1}}|\cos(\theta)|^{\frac{p-2}{p-1}} - \\ &- cos^2(\theta)\mathfrak{G}_l(|\rho_l\cos(\theta)|^{\frac{1}{p-1}}\mathrm{sign}[cos(\theta)],t)\end{aligned} \tag{2.6}$$

Observe that $\dot{\theta}_l \to -\infty$ as $\rho_l \to +\infty$, whenever $\cos(\theta_l) \neq 0$ and that it is a negative constant independent of ρ_l when $\cos(\theta_l) = 0$. So if $\mathbf{X}_l(t)$ is unbounded then $\theta_l(t) \to -\infty$ and the trajectory has to cross the coordinate axes indefinitely. □

Now we show that if $\bar{g}$ satisfies hypothesis **G0** (and it is independent of t), then the shape of $M^u_l(\bar{g})$ and $M^s_l(\bar{g})$ is similar to the one depicted in figures 1 and 2.

2.6. **Lemma.** *Assume that $g_l(x,t) \equiv \bar{g}_l(x)$. Then, if $l > p^*$ the unstable manifold M^u_l is contained in $\mathbb{R}^2_+$ and joins the origin and $\mathbf{P}$, while the stable set M^s_l crosses the y_l-positive semi-axis. If $l = p^*$ the unstable manifold M^u_l coincide with the stable set M^s_l and they are the union of the origin and of the graph of a trajectory homoclinic to it. If $\sigma < l < p^*$ the stable set M^s_l is contained in $\mathbb{R}^2_+$ and joins the origin and $\mathbf{P}$ while M^u_l crosses the y-negative semi-axis. If $p < l \leq p^*$, there is neither stable set nor the critical point $\boldsymbol{P}$, while M^u_l exists and crosses the y-negative semi-axis.*

Proof. We recall that $\bar{\mathcal{G}}_l(x) = \bar{G}_l(x)/x^p$ is increasing in the x variable, see Remark 2.2. The key idea is that we can rewrite (2.4) as follows:

$$\frac{d}{dt}H_{p^*}(\mathbf{X}_{p^*}(t),t) = \frac{\partial \bar{G}_{p^*}}{\partial t}(X_{p^*}(t),t) = |X_{p^*}(t)|^p \frac{\partial \bar{\mathcal{G}}_l}{\partial t}(Se^{(\alpha_l - \alpha_{p^*})t})\lfloor_{S=X_{p^*}(t)}.$$

It follows that $H_{p^*}(\mathbf{X}_{p^*}(t),t)$ is increasing in t, whenever $p < l < p^*$, it is constant when $l = p^*$, and it is decreasing when $l > p^*$. When $l = p^*$ the function H_{p^*} is a first integral so the phase portrait is given by the level sets of H_{p^*} and the claim follows, see Fig. 1.

Now we consider the case $l \neq p^*$. First of all observe that, if we denote by $\mathbf{F}_l(x_l,y_l) = \big(F^1_l(x_l,y_l), F^2_l(x_l,y_l)\big)$ the right hand side of (2.2), we have $\frac{\partial}{\partial x_l}F^1_l(x_l,y_l)+$

$\frac{\partial}{\partial y_l}F^2_l(x_l,y_l)=\alpha_l+\gamma_l\neq 0$. Thanks to the Poincarè-Bendixson criterion, we deduce that there are no closed orbits made up of the union of trajectories. So there are nor periodic trajectories neither homoclinic or heteroclinic cycles.

Assume first that $\sigma<l<p^*$, so that $\alpha_l+\gamma_l>0$ and $\mathbf{P}$ exists and it is a repulser. Choose $\mathbf{Q^s}\in M^s_l$ and $\mathbf{Q^u}\in M^u_l$ and consider the trajectories $\mathbf{X}_l(t,\tau,\mathbf{Q^u})$ and $\mathbf{X}_l(t,\tau,\mathbf{Q^s})$; denote by $u(r)$ and $\mathbf{x}^u_{p^*}(t)$ the solutions of (1.4) and of (2.2) with $l=p^*$ corresponding to $\mathbf{X}_l(t,\tau,\mathbf{Q^u})$ and by $s(r)$ and $\mathbf{x}^s_{p^*}(t)$ the ones corresponding to $\mathbf{X}_l(t,\tau,\mathbf{Q^s})$. Observe that $\lim_{t\to-\infty}\mathbf{x}^u_{p^*}(t)=(0,0)=\lim_{t\to+\infty}\mathbf{x}^s_{p^*}(t)$ (see Lemma 2.6), and that $H_{p^*}(\mathbf{x}_{p^*}(t),t)$ is increasing along the trajectories; therefore $H_l(\mathbf{Q^s})<0<H_l(\mathbf{Q^u})$.

Moreover $\mathbf{X}_l(t,\tau,\mathbf{Q^u})$ cannot converge to the origin as $t\to+\infty$ due to the Poincare-Bendixson criterion, nor to $\mathbf{P}$ since $H_l(\mathbf{P})<0$. So from Lemma 2.7 it follows that M^u_l has to cross the y negative semi-axis, and that M^s_l is contained in the set $\{(x_l,y_l)\in\mathbb{R}^2_\pm\mid H_l(x_l,y_l)<0\}$. Then it is easy to check that M^s_l joins the origin and the critical point $\mathbf{P}\in\mathbb{R}^2_\pm\setminus\{(0,0)\}$.

The claim concerning the case $l>p^*$ can be proved in the same way. The claim concerning the case $p<l\leq\sigma$ follows observing that there are no critical points in the interior of $\mathbb{R}^2_+$ and using again Poincarè Bendixson criterion and Lemma 2.7. □

Now we state a Lemma which will be useful to obtain asymptotic estimates also in the non-autonomous case.

2.7. **Lemma.** *Assume that there is $l>p$ such that $g_l(x_l,t)\equiv\bar g_l(x_l)$ and consider a trajectory $\boldsymbol{X}_l(t)$ of (2.2). If there is $t_n\nearrow+\infty$ (respectively $t_n\searrow-\infty$) such that $\boldsymbol{X}_l(t)\in\mathbb{R}^2_+$ for $t>t_1$ (resp. for $t<t_1$) and $X_l(t_n)\to 0$, then $\lim_{t\to+\infty}\boldsymbol{X}_l(t)=(0,0)$ (resp. $\lim_{t\to-\infty}\boldsymbol{X}_l(t)=(0,0)$).*

Proof. The case $l=p^*$ is a consequence of Lemma 2.8; so assume $l>p^*$ and follow the stable set M^s_l from the origin towards $\mathbb{R}^2_+$. From Lemma 2.8 we know that M^s_l crosses the x_l positive semi-axis in a set K. Denote by $A_x=\min\{x\mid(x,0)\in K\}$ and by $\mathbf{A}=(A_x,0)$. Let us denote by B^s the open bounded set enclosed by M^s_l and the segment of the x_l axis between the origin and $\mathbf{A}$. Note that $B^s\subset\mathbb{R}^2_\pm$, so the flow restricted to B^s is continuous. Moreover it is positively invariant.

Choose a point $\mathbf{Q}\in U^0_l\cap B^s$ and set $t_1=\sup\{t>0\mid\dot X_l(s,0,\mathbf{Q})\neq 0$ for $s\in(0,t)\}$ when $\mathbf{Q}\neq\mathbf{P}$ and set $t_1=0$ for $\mathbf{Q}=\mathbf{P}$. When $0<t_1<\infty$ set $t_2=\sup\{t>0\mid\dot X_l(s,0,\mathbf{Q})\neq 0$ for $s\in(t_1,t)\}$; we denote by $\psi^1(\mathbf{Q})=\lim_{t\to t_1}\mathbf{X}_l(t,\mathbf{Q})$ and by $\psi^2(\mathbf{Q})=\lim_{t\to t_2}\mathbf{X}_l(t,\mathbf{Q})$. We set $\psi^2(\mathbf{P})=\psi^1(\mathbf{P})=\mathbf{P}$. From the continuity of the flow of (2.2) we deduce that the functions $\psi^i:U^0_l\cap B^s\to U^0_l\cap B^s$ are continuous for $i=1,2$.

Assume for contradiction that there is a sequence $t_n\nearrow+\infty$ such that $X_l(t_n)\to 0$, but the limit $\lim_{t\to+\infty}X_l(t)$ does not exist. Then we can find a sequence $\tau_n\nearrow+\infty$ such that $\dot X_l(\tau_n)=0$ and $X_l(\tau_n)$ is a positive minimum if n is even and it is a positive maximum if n is odd and $X_l(\tau_{2k})\to 0$ but $X_l(\tau_{2k+1})$ is uniformly positive; moreover we can assume $\psi^1(\mathbf{X}_l(\tau_n))=\mathbf{X}_l(\tau_{n+1})$.

Since $\mathbf{X}_l(\tau_{2k})\in U^0_l$ for any k it follows that $\mathbf{X}_l(\tau_{2k})\to(0,0)$ as $k\to\infty$. Moreover ψ^2 is continuous and monotone in the x variable since the trajectories of (2.2) cannot have self-intersections in B^s. Hence the limit $\psi^2\big((0,0)\big)=\lim_{\mathbf{X}\to(0,0)}\psi^2(\mathbf{X})=\lim_{k\to\infty}\psi^2(\mathbf{X}_l(\tau_{2k}))$ is well defined; but $\psi^2(\mathbf{X}_l(\tau_{2k}))=\mathbf{X}_l(\tau_{2k+2})\to(0,0)$, so $\psi^2\big((0,0)\big)=(0,0)$ and this contradicts the Poincare-Bendixson criterion; thus $\lim_{t\to+\infty}\mathbf{X}_l(t)=(0,0)$.

Assume that there is a sequence $t_n\searrow-\infty$ such that $X_l(t_n)\to 0$, but the limit $\lim_{t\to-\infty}\mathbf{X}_l(t)$ does not exist; in fact again $\mathbf{X}_l(t_n)\to(0,0)$ since the origin is the unique critical point of the y_l axis. We recall that $\tilde{\mathbf{Q}}^{\mathbf{u}}$ and $\tilde{\mathbf{Q}}^{\mathbf{s}}$ are the intersections between U_l^0 and $\tilde{M}_l^u$ and $\tilde{M}_l^s$ respectively, and that $H_l(\tilde{\mathbf{Q}}^{\mathbf{u}}) < 0 < H_l(\tilde{\mathbf{Q}}^{\mathbf{s}})$, so $\tilde{\mathbf{Q}}^{\mathbf{u}}$ is on the left of $\tilde{\mathbf{Q}}^{\mathbf{s}}$. Denote by c the open segment of U_l^0 between $\tilde{\mathbf{Q}}^{\mathbf{u}}$ and $\tilde{\mathbf{Q}}^{\mathbf{s}}$: it is easy to check that, for any $\mathbf{Q} \in c$, there are $T^2(\mathbf{Q}) < T^1(\mathbf{Q}) < 0$ such that $\mathbf{X}_l(t, 0, \mathbf{Q})$ intersects the x and the y positive semi-axis, respectively at $t = T^1(\mathbf{Q})$ and at $t = T^2(\mathbf{Q})$, see figure 2.

For any $\epsilon > 0$ there is $N > 0$ such that $\|\mathbf{X}_l(t_N)\| < \epsilon$. Following $\mathbf{X}_l(t)$ backwards in t we find that there is $T < t_N$ such that $\mathbf{X}_l(T) \in c$. Hence $\mathbf{X}_l(t)$ have to cross the y_l positive semi-axis at a certain $t < T$, a contradiction. So $\lim_{t\to-\infty}\mathbf{X}_l(t) = (0, 0)$. The case $\sigma < l < p^*$ can be obtained reasoning in the same way. In fact if local uniqueness of the solutions on the coordinate axes is not ensured $\mathbf{X}_l(t, 0, \mathbf{Q})$ is not a priori uniquely defined for $t < T^1(\mathbf{Q})$ and $T^2(\mathbf{Q})$ is a multivalued function. However the argument goes through for any of the solutions bifurcating from $\mathbf{X}_l(t, 0, \mathbf{Q})$ for $t < T^1(\mathbf{Q})$. □

From Lemma 2.8 we deduce the following useful results.

2.8. **Proposition.** *Assume that there is $l > p$ such that $g_l(x_l, t) \equiv \bar{g}_l(x_l)$ is independent of t and consider a singular solution $v(r)$ of (1.4). Then $v(r)r^{\alpha_l}$ is uniformly positive and bounded for $0 < r < 1$.*

Analogously assume that $l > \sigma$ and consider a slow decaying solution $w(r)$ of (1.4). Then $w(r)r^{\alpha_l}$ is uniformly positive and bounded for $r > 1$.

Hence putting together Lemma 2.8 and Proposition 2.10 we can prove the following Proposition. The result is known when f is of type (1.2) but in this generality is in fact new, even if the hypothesis of independence in t is rather unusual.

2.9. **Proposition.** *Consider (1.4) and assume that $g_l(x, t) \equiv \bar{g}(x)$.*

- *If $l > p^*$ positive solutions have structure **Sup**.*
- *If $l = p^*$ positive solutions have structure **Crit**.*
- *If $\sigma < l < p^*$ positive solutions have structure **Sub**.*
- *If $p < l < p^*$ all the regular solutions are crossing solutions.*

When f is of type (1.2) and $l = p^*$ in fact we know the exact expression of the G.S. with fast decay, see e.g. [13].
Using the invariance of the system for translations in t, we obtain also the following result that will be useful in the next sections.

2.10. *Remark.* Assume $l > p$; fix $\mathbf{Q} \in M_l^u(\bar{g}_l)$ and consider $\mathbf{X}_l(t, \tau; \mathbf{Q}, \bar{g}_l)$ and the corresponding regular solution $u(d(\tau, \mathbf{Q}), r)$ of (1.4). Then

$$d(\tau, \mathbf{Q}) = C^d(\mathbf{Q})e^{-\alpha_l\tau}\,, \tag{2.7}$$

where $C^d(\mathbf{Q})$ is an injective continuous function such that $C^d((0, 0)) = 0$ (and $C^d(\mathbf{Q}) > 0$ for $\mathbf{Q} \neq (0, 0)$).

Analogously assume $l > \sigma$; fix $\mathbf{P} \in M_l^s(\bar{g}_l)$ and consider the trajectory $\mathbf{X}_l(t, \tau; \mathbf{P}, \bar{g}_l)$ of the autonomous system (2.2) and the corresponding fast decay solution $v(r; \tau, \mathbf{P})$ of (1.4). Let us denote by $L(\tau, \mathbf{P})$ the limit $\lim_{r\to+\infty} v(r; \tau, \mathbf{P})r^{\frac{n-p}{p-1}} = L(\tau, \mathbf{P})$. Then

$$L(\tau, \mathbf{P}) = C^L(\mathbf{P})e^{(\frac{n-p}{p-1}-\alpha_l)\tau} \tag{2.8}$$

where $C^L(\mathbf{P})$ is a function such that $C^L((0, 0)) = 0$ and $C^L(\mathbf{P}) > 0$ for $\mathbf{P} \neq (0, 0)$; if $l = p^*$, then $C^L(\mathbf{P})$ is injective a continuous.

Proof. In this proof the value l of (2.1) is fixed so we omit the subscript. We start from the first claim, so let $\mathbf{Q} \in M^u(\bar{g})$. Then

$$u(d(\tau, \mathbf{Q}), e^t)e^{\alpha t} = \mathbf{X}(t, \tau; \mathbf{Q}, \bar{g}) = \mathbf{X}(t - \tau, 0; \mathbf{Q}, \bar{g}) = u(d(0, \mathbf{Q}), e^{t-\tau})e^{\alpha(t-\tau)}.$$

So if we let t tend to $-\infty$ we get $d(\tau, \mathbf{Q}) = d(0, \mathbf{Q})e^{-\alpha\tau}$ and (2.7) follows simply setting $d(0, \mathbf{Q}) = C^d(\mathbf{Q})$. From Remark 2.3 we also find that $C^d(\mathbf{Q})$ is injective. Since $C^d((0, 0)) = 0$ it follows that $C^d(\mathbf{Q}) > 0$ for $\mathbf{Q} \neq 0$.

Now assume $l > \sigma$ and choose $\mathbf{P} \in M^s(\bar{g})$. Reasoning as above, we get $v(e^t; \tau, \mathbf{P}) = v(e^{t-\tau}; 0, \mathbf{P})e^{-\alpha\tau}$ and $L(\tau, \mathbf{P}) = L(0, \mathbf{P})e^{(\frac{n-p}{p-1}-\alpha)\tau}$. So we can set $C^L(\mathbf{P}) = L(0, \mathbf{P})$. □

3. The t-dependent case

Now we begin to study the case in which $g_l(x, t)$ depends effectively on t, for any l. We state the hypotheses needed to construct stable and unstable sets $\tilde{W}^u_l(\tau)$ and $\tilde{W}^s_l(\tau)$ sharing properties similar to the stable and unstable set $\tilde{M}^u_l$ and $\tilde{M}^s_l$ of the autonomous system. Choose $\tau \in \mathbb{R}$, we introduce the following functions

$$\begin{array}{ll} a^\tau_l(x_l) = \inf_{t\le\tau} 1/2g_l(x_l, t) & b^\tau_l(x_l) = \sup_{t\le\tau} 2g_l(x_l, t) \\ A^\tau_l(x_l) = \inf_{t\ge\tau} 1/2g_l(x_l, t) & B^\tau_l(x_l) = \sup_{t\ge\tau} 2g_l(x_l, t) \end{array} \tag{3.1}$$

Note that these functions are monotone increasing in x for any τ and satisfy **G0**, if they are not identically null or infinity. We need some of these assumptions:

- **G1:** There is $l_1 > p$ such that for any $x > 0$ the function $g_{l_1}(x, t)$ converges to a t-independent locally Lipschitz function $g^{-\infty}_{l_1}(x) \not\equiv 0$ as $t \to -\infty$ uniformly on compact intervals.
- **G1′:** There is $l_1 > p$ such that for any $\tau \in \mathbb{R}$ the functions $a^\tau_{l_1}$ and $b^\tau_{l_1}$ are locally Lipschitz. Moreover for any $x > 0$ and any $\tau < 0$, we have $0 < a^\tau_{l_1}(x) < b^\tau_{l_1}(x) < \infty$.
- **G2:** There is $l_2 > \sigma$ such that for any $x > 0$ the function $g_{l_2}(x, t)$ converges to a t-independent locally Lipschitz function $g^{+\infty}_{l_2}(x) \not\equiv 0$ as $t \to +\infty$ uniformly on compact intervals.
- **G2′:** There is $l_2 > \sigma$ such that for any $\tau \in \mathbb{R}$ the functions $A^\tau_{l_2}$ and $B^\tau_{l_2}$ are locally Lipschitz. Moreover for any $x > 0$ and any $\tau > 0$, we have $0 < A^\tau_{l_2}(x) < B^\tau_{l_2}(x) < \infty$.

Obviously **G1** implies **G1′**, and **G2** implies **G2′**.

3.1. Construction of stable and unstable set assuming G1′, and G2′.

When f has the form (1.2), $k(e^t)$ is uniformly continuous for $t \in \mathbb{R}$ and the system is $\mathbb{C}^1$ we can construct these sets via invariant manifold theory for non-autonomous system, see [17]. In such a case it can be proved that the sets $\tilde{W}^u_l(\tau)$ and $\tilde{W}^s_l(\tau)$ which will be defined below, are actually manifolds. In the general case we need to exploit our knowledge of the autonomous systems in order to construct some barrier sets; then we will apply a topological lemma based on the idea of Wazewski's principle.

We recall that $\tilde{M}^u_l(a^\tau_l)$ is the branch of the unstable manifold of the autonomous systems (2.2) where $g_l(x_l, t) \equiv a^\tau_l(x_l)$, between the origin and the point $\tilde{\mathbf{Q}}^u_l(a^\tau_l)$ of the isocline U^0_l, and that the analogous definition holds for $\tilde{M}^u_l(b^\tau_l)$. Assume that **G1′** and **G2′** hold; observe that for any τ, $\tilde{M}^u_{l_1}(a^\tau_{l_1})$, $\tilde{M}^u_{l_1}(b^\tau_{l_1})$, $\tilde{M}^s_{l_2}(A^\tau_{l_2})$, $\tilde{M}^s_{l_2}(B^\tau_{l_2})$ have positive and finite diameter. If $l_u > l_1$, then $a^\tau_{l_u} \equiv 0$ and if $l_s > l_2$, then $B^\tau_{l_s} \equiv +\infty$; so in such a case $\tilde{M}^u_{l_u}(a^\tau_{l_u})$ is unbounded (it is the x positive semi-axis) and $\tilde{M}^s_{l_s}(B^\tau_{l_s})$ is not well defined while $\tilde{M}^u_{l_u}(b^\tau_{l_u})$ and $\tilde{M}^s_{l_s}(A^\tau_{l_s})$ still have positive finite diameter. Analogously if $l_u < l_1$ and $l_s < l_2$ then $b^\tau_{l_u} \equiv +\infty$ and $A^\tau_{l_s} \equiv 0$.

We denote by $\tilde{c}^u_{l_1}(\tau)$ the branch of $U^0_{l_1}$ between $\tilde{\mathbf{Q}}^u_{l_1}(a^\tau_{l_1})$ and $\tilde{\mathbf{Q}}^u_{l_1}(b^\tau_{l_1})$. $\tilde{M}^u_{l_1}(a^\tau_{l_1})$ is on the right of $\tilde{M}^u_{l_1}(b^\tau_{l_1})$ (here and later we think of the x axis as horizontal and the y axis as vertical), and they do not intersect (this will be proved in Lemma 3.1 below). Finally we denote by $\partial\tilde{E}^u_{l_1}(\tau) = \tilde{M}^u_{l_1}(a^\tau_{l_1}) \cup \tilde{M}^u_{l_1}(b^\tau_{l_1})$ and by $\tilde{E}^u_{l_1}(\tau)$ the bounded set enclosed by $\partial\tilde{E}^u_{l_1}(\tau)$ and $\tilde{c}^u_{l_1}(\tau)$.

Analogously $\tilde{M}^s_{l_2}(A^\tau_{l_2})$ is on the right of $\tilde{M}^s_{l_2}(B^\tau_{l_2})$, and they do not intersect. It follows that $\tilde{\xi}^s_{l_2}(A^\tau_{l_2})$ is on the right of $\tilde{\xi}^s_{l_2}(B^\tau_{l_2})$ as well, see again Lemma 3.1: we denote by $\tilde{\mathbf{Q}}^s_{l_2}(A^\tau_{l_2})$ the left endpoint of $\tilde{\xi}^s_{l_2}(A^\tau_{l_2})$ and by $\tilde{\mathbf{Q}}^s_{l_2}(B^\tau_{l_2})$ the right endpoint of $\tilde{\xi}^s_{l_2}(B^\tau_{l_2})$. We denote by $\tilde{c}^s_{l_2}(\tau)$ the branch of $U^0_{l_2}$ between $\tilde{\mathbf{Q}}^s_{l_2}(A^\tau_{l_2})$ and $\tilde{\mathbf{Q}}^s_{l_2}(B^\tau_{l_2})$.

Finally we denote by $\partial\tilde{E}^s_{l_2}(\tau) = \tilde{M}^s_{l_2}(A^\tau_{l_2}) \cup \tilde{M}^s_{l_2}(B^\tau_{l_2})$ and by $\tilde{E}^s_{l_2}(\tau)$ the bounded set enclosed by $\partial\tilde{E}^s_{l_2}(\tau)$ and $\tilde{c}^s_{l_2}(\tau)$.

3.1. **Lemma.** *Assume that* $\mathbf{G1'}$ *holds. Then the flow of the non-autonomous system (2.2) on* $\partial\tilde{E}^u_{l_1}(\tau)$ *points towards the interior of* $\tilde{E}^u_{l_1}(\tau)$*, for any* $t \le \tau$.

Analogously assume that $\mathbf{G2'}$ *holds. Then the flow of the non-autonomous system (2.2) on* $\partial\tilde{E}^s_{l_2}(\tau)$ *points towards the exterior of* $\tilde{E}^s_{l_2}(\tau)$*, for any* $t \ge \tau$.

Proof. Fix τ and choose $\mathbf{Q}= (Q_x, Q_y) \in \tilde{M}^u_{l_1}(a^\tau_{l_1})$. Note that $\tilde{M}^u_{l_1}(a^\tau_{l_1})$ is contained in the graph of the trajectory $\mathbf{X}_{l_1}(t, T; \mathbf{Q}, a^\tau_{l_1})$ of the autonomous system where $g_{l_1}(x_{l_1}, t) \equiv a^\tau_{l_1}(x_{l_1})$. Recall that $\mathbf{x}_{l_1}(t, T; \mathbf{Q})$ is the trajectory of the non-autonomous system departing from $\mathbf{Q}$ at $t = T$. Observe that $\frac{d}{dt}X_{l_1}(T, T; \mathbf{Q}, a^\tau_{l_1}) = \frac{d}{dt}x_{l_1}(T, T; \mathbf{Q})$ but

$$\left[\dot{Y}_{l_1}(T, T; \mathbf{Q}, a^\tau_{l_1}) - \dot{y}_{l_1}(T, T; \mathbf{Q})\right] = \left[g_{l_1}(Q_x, T) - a^\tau_{l_1}(Q_x)\right] > 0\,,$$

for any $T \le \tau$. Therefore the flow of the non-autonomous system (2.2) on $\tilde{M}^u_{l_1}(a^\tau_{l_1})$ points towards the interior of $\tilde{E}^u_{l_1}(\tau)$. Reasoning in the same way we can conclude that the flow of (2.2) on $\tilde{M}^u_{l_1}(b^\tau_{l_1})$ points towards the interior of $\tilde{E}^u_{l_1}(\tau)$, too. From this argument it also follows that $\tilde{M}^u_{l_1}(a^\tau_{l_1})$ is on the right of $\tilde{M}^u_{l_1}(b^\tau_{l_1})$ and that they do not intersect. The proof concerning $\mathbf{G2'}$ is similar so we will omit it. □

3.2. **Lemma.** *Assume that hypotheses* $\mathbf{G1'}$ *is satisfied. Then there is a connected set* $K^u_{l_1}(\tau) \subset \tilde{c}^u_{l_1}(\tau)$ *such that* $\tilde{\mathbf{Q}}^{\mathbf{u}}_{l_1}(a^\tau_{l_1}) \in K^u_{l_1}(\tau)$ *and the flow of (2.2) on* $\tilde{c}^u_{l_1}(\tau)$ *points towards* $U^-_{l_1}$ *for any* $t \le \tau$. *Analogously assume that hypotheses* $\mathbf{G2'}$ *is satisfied. Then there is a connected set* $K^s_{l_2}(\tau) \subset \tilde{c}^s_{l_2}(\tau)$ *such that* $\tilde{\mathbf{Q}}^{\mathbf{s}}(A^\tau_{l_2}) \in K^s_{l_2}(\tau)$ *and the flow of (2.2) with* $l = l_2$ *on* $\tilde{c}^s_{l_2}(\tau)$ *points towards* $U^-_{l_2}$.

Proof. We just prove the first claim since the second is completely analogous. Since $\tilde{c}^u_{l_1}(\tau)$ is a subset of the isocline $U^0_{l_1}$, the flow of (2.2) on $\tilde{c}^u_{l_1}(\tau)$ is vertical. Set $\tilde{\mathbf{Q}}^u(a^\tau_{l_1}) = (\tilde{Q}_x, \tilde{Q}_y)$ and observe that

$$\dot{y}_{l_1}(\tilde{\mathbf{Q}}^u(a^\tau_{l_1}), t) := -\gamma_{l_1}\tilde{Q}_y - g_{l_1}(\tilde{Q}_x, t) < -\gamma_{l_1}\tilde{Q}_y - a^\tau_{l_1}(\tilde{Q}_x) = -\delta \le 0\,,$$

for any $t \le \tau$. Then the claim follows from a continuity argument. □

We state a topological Lemma which is a slight variant of [22] Lemma 4, which enables us to prove that the sets we are constructing are connected.

3.3. **Lemma.** *Let* $\mathcal{R}$ *be a closed set homeomorphic to a full triangle. We call the vertices* O*,* A *and* B *and* o*,* a*,* b *the edges which are opposite to the respective vertex. Let* $\mathcal{S} \subset \mathcal{R}$ *be a closed set such that* $\sigma \cap \mathcal{S} \ne \emptyset$*, for any path* $\sigma \subset \mathcal{R}$ *joining* a *with* b*. Then* $\mathcal{S}$ *contains a closed connected set which contains* O *and at least one point of* o.

Choose $\mathbf{Q} \in \tilde{E}^u_{l_1}(\tau)$; we denote by

$$\tilde{T}^u(\mathbf{Q}) = \inf\{T \le \tau \,|\, \mathbf{x}_{l_1}(t, \tau; \mathbf{Q}) \in \tilde{E}^u_{l_1}(\tau), \text{ for any } t \in (T, \tau]\}$$

and by $\psi^{u,\tau}_{l_1}(\mathbf{Q}) = \lim_{t\to\tilde{T}^u(\mathbf{Q})} \mathbf{x}_{l_1}(t, \tau; \mathbf{Q})$. For $\mathbf{Q} \in \tilde{E}^s_{l_2}(\tau)$ we set

$$\tilde{T}^s(\mathbf{Q}) = \sup\{T \ge \tau \,|\, \mathbf{x}_{l_2}(t, \tau; \mathbf{Q}) \in \tilde{E}^s_{l_2}(\tau), \text{ for any } t \in [\tau, T)\}$$

and $\psi^{s,\tau}_{l_2}(\mathbf{Q}) = \lim_{t\to\tilde{T}^s(\mathbf{Q})} \mathbf{x}_{l_2}(t,\tau;\mathbf{Q})$. The definitions of $\psi^{u,\tau}_{l_1}(\mathbf{Q})$ and $\psi^{s,\tau}_{l_2}(\mathbf{Q})$ are clearly well given when $\tilde{T}^u(\mathbf{Q}) > -\infty$ and $\tilde{T}^s(\mathbf{Q}) < +\infty$; however with elementary arguments it can be shown that if $\tilde{T}^u(\mathbf{Q}) = -\infty$ then $\lim_{t\to-\infty}\mathbf{x}_{l_1}(t,\tau;\mathbf{Q}) = (0,0)$, and if $\tilde{T}^s(\mathbf{Q}) < +\infty$ then $\lim_{t\to+\infty}\mathbf{x}_{l_2}(t,\tau;\mathbf{Q}) = (0,0)$. The functions $\psi^{u,\tau}_{l_1} : \tilde{E}^u_{l_1}(\tau) \to \partial\tilde{E}^u_{l_1}(\tau)\cup\tilde{c}^u_{l_1}(\tau)$ and $\psi^{s,\tau}_{l_2} : \tilde{E}^s_{l_2}(\tau) \to \partial\tilde{E}^s_{l_2}(\tau)\cup\tilde{c}^s_{l_2}(\tau)$ are well defined, but in general they are not continuous, since the flow of (2.2) on $\tilde{c}^u_{l_1}(\tau)$ and on $\tilde{c}^s_{l_2}(\tau)$ is not transversal. Let us denote by

$$W^u_{l_1}(\tau) := \{\mathbf{Q}\in\tilde{E}^u_{l_1}(\tau)\,|\,\psi^{u,\tau}_{l_1}(\mathbf{Q}) = (0,0)\}$$
$$W^s_{l_2}(\tau) := \{\mathbf{Q}\in\tilde{E}^s_{l_2}(\tau)\,|\,\psi^{s,\tau}_{l_2}(\mathbf{Q}) = (0,0)\}$$

We want to apply Lemma 3.3 to $W^u_{l_1}(\tau)$ and to $W^s_{l_2}(\tau)$. We introduce the following sets

$$\mathcal{A}(\tau) := \{\mathbf{Q}\in\tilde{E}^u_{l_1}(\tau)\,|\,\psi^{u,\tau}_{l_1}(\mathbf{Q}) \in \tilde{M}^u_{l_1}(a^\tau_{l_1})\setminus\{(0,0)\}\}$$
$$\mathcal{B}(\tau) := \tilde{E}^u_{l_1}(\tau)\setminus\big(\mathcal{A}(\tau)\cup W^u_{l_1}(\tau)\big)$$

We claim that $\mathcal{A}(\tau)$ and $\mathcal{B}(\tau)$ are both open in $\tilde{E}^u_{l_1}(\tau)$. Consider the set $\partial\tilde{E}^u_{l_1}(\tau)\cup\tilde{c}^u_{l_1}(\tau)$: we can find a connected set Ω relatively open in $\partial\tilde{E}^u_{l_1}(\tau)\cup\tilde{c}^u_{l_1}(\tau)$ such that $\Omega\supset\tilde{M}^u_{l_1}(a^\tau_{l_1})$, and the flow of (2.2) on $\Omega\setminus\{(0,0)\}$ is transversal (this follows from the existence of $K^u_{l_1}(\tau)$). The flow of (2.2) restricted to the counter image of $\psi^{u,\tau}_{l_1}$ on $\Omega\setminus\{(0,0)\}$ is continuous. Now we prove the continuity of $\psi^{u,\tau}_{l_1}$ on $W^u_{l_1}(\tau)$ and that the flow of (2.2) restricted to the whole

$$D(\tau) := \{\mathbf{Q}\in\tilde{E}^u_{l_1}(\tau)\,|\,\psi^{u,\tau}_{l_1}(\mathbf{Q})\in\Omega\}$$

is continuous; it follows that $W^u_{l_1}(\tau)$ is closed.

Let us choose $\mathbf{Q}\in W^u_{l_1}(\tau)$ and $\mathbf{Q}_n\in\tilde{E}^u_{l_1}(\tau)$ such that $\mathbf{Q}_n\to\mathbf{Q}$ as $n\to\infty$. We claim that for any $\delta>0$ there is N such that $\|\psi^{u,\tau}_{l_1}(\mathbf{Q}_n)\| < \delta$ for any $n>N$. In fact from the continuity of the flow of (2.2) in $\tilde{E}^u_{l_1}(\tau)\setminus\{(0,0)\}$, we know that for any $M>0$ we can find N such that $\tilde{T}^u(\mathbf{Q}_n) < -M$ for any $n>N$. For any $\epsilon>0$ we can choose $M>0$ large enough so that $x_{l_1}(-M,\tau;\mathbf{Q}) < \epsilon/2$. Then, possibly choosing a larger N, we can assume that

$$|x_{l_1}(-M,\tau;\mathbf{Q}_n) - x_{l_1}(-M,\tau;\mathbf{Q})| < \epsilon/2\,,$$

for any $n>N$. Moreover $x_{l_1}(t,\tau;\mathbf{Q}_n) < x_{l_1}(-M,\tau;\mathbf{Q}_n) < \epsilon$ for any $t\in[\tilde{T}^u(\mathbf{Q}_n),-M]$, because $\dot{x}_{l_1}(t,\tau;\mathbf{Q}_n) > 0$ for t in that interval. Since $\mathbf{x}_{l_1}(t,\tau;\mathbf{Q}_n)\in U^+_{l_1}$ we have $\|\psi^{u,\tau}_{l_1}(\mathbf{Q}_n)\| \le \epsilon + [\alpha_{l_1}\epsilon]^{p-1} \le \delta$ if ϵ is small enough, so the continuity of $\psi^{u,\tau}_{l_1}$ in $D(\tau)$ is proved and $W^u_{l_1}(\tau)$ is closed.

Now observe that $\mathcal{A}(\tau)$ is contained in $D(\tau)$ and that it is the counter image of $\Omega\setminus\big(\{(0,0)\}\cup\tilde{M}^u_{l_1}(b^\tau_{l_1})\big)$ through $\psi^{u,\tau}_{l_1}$. Hence $W^u_{l_1}(\tau)$ is closed and $\mathcal{A}(\tau)$ is open in $\tilde{E}^u_{l_1}(\tau)$. In fact $W^u_{l_1}(\tau)\cup\mathcal{A}(\tau)$ is the closure of $\mathcal{A}(\tau)$ in $\tilde{E}^u_{l_1}(\tau)$. Therefore $\mathcal{B}(\tau) = \tilde{E}^u_{l_1}(\tau)\setminus\Big(W^u_{l_1}(\tau)\cup\mathcal{A}(\tau)\Big)$ is open.

Our purpose is to apply Lemma 3.3, with $\mathcal{R} = \tilde{E}^u_{l_1}(\tau)$, $a = \tilde{M}^u_{l_1}(a^\tau_{l_1})$, $b = \tilde{M}^u_{l_1}(b^\tau_{l_1})$, $o = \tilde{c}^u_{l_1}(\tau)$ and $\mathcal{S} = W^u_{l_1}(\tau)$. So consider a continuous path $\Gamma(s) : [0,1] \to \tilde{E}^u_{l_1}(\tau)$ such that $\Gamma(0)\in\tilde{M}^u_{l_1}(a^\tau_{l_1})$ and $\Gamma(1)\in\tilde{M}^u_{l_1}(b^\tau_{l_1})$: we need to show that there is $s\in[0,1]$ such that $\Gamma(s)\in W^u_{l_1}(\tau)$.

Let us define the following two sets:

$$\mathfrak{A}(\tau) := \{s\in[0,1]\,|\,\Gamma(s)\in\mathcal{A}(\tau)\} \qquad \mathfrak{B}(\tau) := \{s\in[0,1]\,|\,\Gamma(s)\in\mathcal{B}(\tau)\}$$

Note that $0\in\mathfrak{A}(\tau)$ while $1\in\mathfrak{B}(\tau)$. Moreover $\mathfrak{A}(\tau) = \Gamma^{-1}(\mathcal{A}(\tau))$ and $\mathfrak{B}(\tau) = \Gamma^{-1}(\mathcal{B}(\tau))$ are both relatively open in $[0,1]$ since $\mathcal{A}(\tau)$ and $\mathcal{B}(\tau)$ are relatively open in $\tilde{E}^u_{l_1}(\tau)$ and Γ is continuous. Since $[0,1]$ is connected there is $s \in (0,1) \setminus \big(\mathcal{A}(\tau) \cup \mathcal{B}(\tau)\big)$, so that $\Gamma(s) \in W^u_{l_1}(\tau)$.

So we can apply Lemma 3.3 and conclude the following.

3.4. **Lemma.** *Assume that* $\mathbf{G1'}$ *is satisfied. Then for any* $\tau \in \mathbb{R}$ $W^u_{l_1}(\tau)$ *contains a compact connected set* $\tilde{W}^u_{l_1}(\tau)$ *to which the origin belongs and which intersects* $\tilde{c}^u_{l_1}(\tau)$ *in a set denoted by* $\tilde{\xi}^u_{l_1}(\tau)$*. Analogously assume that* $\mathbf{G2'}$ *is satisfied, then* $W^s_{l_2}(\tau)$ *contains a compact connected set* $\tilde{W}^s_{l_2}(\tau)$ *to which the origin belongs and which intersects* $\tilde{c}^s_{l_2}(\tau)$ *in a set denoted by* $\tilde{\xi}^s_{l_2}(\tau)$*.*

The proof concerning $\tilde{W}^s_{l_2}(\tau)$ is completely analogous so we omit it.

In order to study the asymptotic behavior of the trajectories departing from $\tilde{W}^u_{l_1}(\tau)$ and $\tilde{W}^s_{l_2}(\tau)$, we need to construct another barrier set, using a similar argument.

We denote by $\bar{\mathbf{B}}^u_{l_1}(\tau) = (\bar{B}^u_x(\tau), \bar{B}^u_y(\tau)) := \tilde{\mathbf{Q}}^u(b^\tau_{l_1})$. We denote by $\bar{\mathbf{A}}^u_{l_1}(\tau)$ the point of intersection between $\tilde{M}^u_{l_1}(a^\tau_{l_1})$ and the line $x = \bar{B}^u_x(\tau)$. Analogously we denote by $\bar{\mathbf{B}}^s_{l_2}(\tau) = (\bar{B}^s_x(\tau), \bar{B}^s_y(\tau)) := \tilde{\mathbf{Q}}^{\mathbf{s}}(B^\tau_{l_2})$. We denote by $\bar{\mathbf{A}}^s_{l_2}(\tau)$ the point of intersection between $\tilde{M}^s_{l_2}(A^\tau_{l_1})$ and the line $x = \bar{B}^s_x(\tau)$. We set

$$\begin{aligned}
\bar{E}^u_{l_1}(\tau) &:= \{\mathbf{x}_{l_1} = (x_{l_1}, y_{l_1}) \in \tilde{E}^u_{l_1}(\tau) \mid x_{l_1} \leq \bar{B}^u_x(\tau)\} \\
\bar{E}^s_{l_2}(\tau) &:= \{\mathbf{x}_{l_2} = (x_{l_2}, y_{l_2}) \in \tilde{E}^s_{l_2}(\tau) \mid x_{l_2} \leq \bar{B}^s_x(\tau)\} \\
\bar{M}^u_{l_1}\big(a^\tau_{l_1}\big) &:= \{\mathbf{x}_{l_1} = (x_{l_1}, y_{l_1}) \in \tilde{M}^u_{l_1}(a^\tau_{l_1}) \mid x_{l_1} \leq \bar{B}^u_x(\tau)\} \\
\bar{M}^s_{l_2}\big(A^\tau_{l_2}\big) &:= \{\mathbf{x}_{l_2} = (x_{l_2}, y_{l_2}) \in \tilde{M}^s_{l_2}\big(A^\tau_{l_2}\big) \mid x_{l_2} \leq \bar{B}^s_x(\tau)\}
\end{aligned}$$

We set $\bar{M}^u_{l_1}\big(b^\tau_{l_1}\big) = \tilde{M}^u_{l_1}\big(b^\tau_{l_1}\big)$ and $\bar{M}^s_{l_2}\big(B^\tau_{l_2}\big) = \tilde{M}^s_{l_2}\big(B^\tau_{l_2}\big)$. We denote by $\bar{c}^u_{l_1}(\tau)$ the line between $\bar{\mathbf{B}}^u_{l_1}(\tau)$ and $\bar{\mathbf{A}}^u_{l_1}(\tau)$ and by $\bar{c}^s_{l_2}(\tau)$ the line between $\bar{\mathbf{B}}^s_{l_2}(\tau)$ and $\bar{\mathbf{A}}^s_{l_2}(\tau)$. We denote by $\partial\bar{E}^u_{l_1}(\tau) = \bar{M}^u_{l_1}\big(b^\tau_{l_1}\big) \cup \bar{M}^u_{l_1}\big(a^\tau_{l_1}\big)$ and by $\bar{E}^u_{l_1}(\tau)$ the bounded set enclosed by $\bar{c}^u_{l_1}(\tau)$ and $\partial\bar{E}^u_{l_1}(\tau)$. Analogously we denote by $\partial\bar{E}^s_{l_2}(\tau) = \bar{M}^s_{l_2}\big(B^\tau_{l_2}\big) \cup \bar{M}^s_{l_2}\big(A^\tau_{l_2}\big)$ and by $\bar{E}^s_{l_2}(\tau)$ the bounded set enclosed by $\bar{c}^s_{l_2}(\tau)$ and $\partial\bar{E}^s_{l_2}(\tau)$.

Note that the flow of (2.2) with $l = l_1$ on $\partial\bar{E}^u_{l_1}(\tau)$ points towards the interior of $\bar{E}^u_{l_1}(\tau)$, while on $\bar{c}^u_{l_1}(\tau)$ it points towards the exterior of $\bar{E}^u_{l_1}(\tau)$, for any $t \leq \tau$ and any $\tau \in \mathbb{R}$. Analogously the flow of (2.2) with $l = l_2$ on $\partial\bar{E}^s_{l_2}(\tau)$ points towards the exterior of $\bar{E}^s_{l_2}(\tau)$, while on $\bar{c}^s_{l_2}(\tau) \setminus \{\bar{\mathbf{A}}^s_{l_2}(\tau), \bar{\mathbf{B}}^s_{l_2}(\tau)\}$ it points towards the interior of $\bar{E}^s_{l_2}(\tau)$ for any $t \leq \tau$ and any $\tau \in \mathbb{R}$.

So repeating the argument of Lemma 3.4 we can prove that there are compact connected sets $\bar{W}^u_{l_1}(\tau)$ and $\bar{W}^s_{l_2}(\tau)$, which contain the origin and intersect respectively $\bar{c}^u_{l_1}(\tau)$ and $\bar{c}^u_{l_2}(\tau)$, endowed with the following property:

$$\begin{aligned}
\bar{W}^u_{l_1}(\tau) &\subset \{\mathbf{Q} \in \mathbb{R}^2_+ \mid \mathbf{x}_{l_1}(t, \tau; \mathbf{Q}) \in \bar{E}^u_{l_1}(\tau) \, \text{for any } t \leq \tau\} \\
\bar{W}^s_{l_2}(\tau) &\subset \{\mathbf{Q} \in \mathbb{R}^2_+ \mid \mathbf{x}_{l_2}(t, \tau; \mathbf{Q}) \in \bar{E}^s_{l_2}(\tau) \, \text{for any } t \geq \tau\}
\end{aligned}$$

Let us denote by $\bar{\xi}^u_l(\tau) = \bar{c}^u_l(\tau) \cap \bar{W}^u_l(\tau)$ and $\bar{\xi}^s_l(\tau) = \bar{c}^s_l(\tau) \cap \bar{W}^s_l(\tau)$, and observe that we can (and we will) choose $\bar{W}^u_l(\tau) \subset \tilde{W}^u_l(\tau)$ and $\bar{W}^s_l(\tau) \subset \tilde{W}^s_l(\tau)$ for any $\tau \in \mathbb{R}$ and for $l = l_1$ and $l = l_2$.

Using the flow of (2.2) we can define global stable and unstable sets as follows:

$$\begin{aligned}
\mathfrak{W}^u_{l_1}(\tau) &:= \cup_{T \in \mathbb{R}} \{\mathbf{P} \mid \exists \mathbf{Q} \in \bar{W}^u_{l_1}(T) \; s.t. \; \mathbf{P} = \mathbf{x}_{l_1}(\tau, T; \mathbf{Q})\}, \\
\mathfrak{W}^s_{l_2}(\tau) &:= \cup_{T \in \mathbb{R}} \{\mathbf{P} \mid \exists \mathbf{Q} \in \bar{W}^s_{l_2}(T) \; s.t. \; \mathbf{P} = \mathbf{x}_{l_2}(\tau, T; \mathbf{Q})\}
\end{aligned}$$

Moreover we have the following estimate on the behaviour of the trajectories of the non-autonomous system departing from these sets. Note that if $\mathbf{G1'}$ holds, a priori $\mathfrak{W}^u_{l_1}(\tau)$ may be contained in $U^+_{l_1}$ and it may not intersect $U^0_{l_1}$.

3.5. **Lemma.** *Assume that* $\mathbf{G1'}$ *and* $\mathbf{G2'}$ *hold; let* $\bar{\mathbf{Q}}^{\mathbf{u}} \in \bar{W}^u_{l_1}(\tau)$ *and* $\bar{\mathbf{Q}}^{\mathbf{s}} \in \bar{W}^s_{l_2}(\tau)$. *Then the solution of (1.4) corresponding to the trajectory* $\boldsymbol{x}_{l_1}(t,\tau;\bar{\mathbf{Q}}^{\mathbf{u}})$ *is a regular solution* $u(d,r)$, *where* $d = d(\tau;\bar{\mathbf{Q}}^{\mathbf{u}})$. *Moreover if* $\bar{\mathbf{Q}}^{\mathbf{u}} \in \bar{\xi}^u_{l_1}(\tau)$ *we have* $d(\tau;\bar{\mathbf{Q}}^{\mathbf{u}}) \to +\infty$ *as* $\tau \to -\infty$ *and viceversa,* $\tau \to +\infty$ *if* $d(\tau;\bar{\mathbf{Q}}^{\mathbf{u}}) \to 0$, *and if* $l_1 \le l_2$ *the viceversa holds as well.*

Analogously the solution of (1.4) corresponding to the trajectory $\boldsymbol{x}_{l_2}(t,\tau;\bar{\mathbf{Q}}^{\mathbf{s}})$ *is a fast decay solution* $v(L,r)$, *where* $L = L(\tau;\bar{\mathbf{Q}}^{\mathbf{s}}) = \lim_{r\to+\infty} v(L,r) r^{(n-p)/(p-1)}$. *Moreover if* $\bar{\mathbf{Q}}^{\mathbf{s}} \in \bar{\xi}^s_{l_2}(\tau)$ *we have* $L(\tau;\bar{\mathbf{Q}}^{\mathbf{s}}) \to +\infty$ *as* $\tau \to +\infty$ *and viceversa,* $\tau \to -\infty$ *as* $L(\tau;\bar{\mathbf{Q}}^{\mathbf{s}}) \to 0$ *as and if* $l_1 \le l_2$ *the viceversa holds as well.*

Proof. The idea is to compare trajectories of the non autonomous system with trajectories of the autonomous system. Choose $\bar{\mathbf{Q}}^u = (\bar{Q}^u_x, \bar{Q}^u_y) \in \bar{\xi}^u_{l_1}(\tau)$. Consider the trajectories $\mathbf{X}_{l_1}(t,\tau;\bar{\mathbf{A}}^u_{l_1}(\tau), a^\tau_{l_1})$ and $\mathbf{X}_{l_1}(t,\tau;\bar{\mathbf{B}}^u_{l_1}(\tau), b^\tau_{l_1})$ of the autonomous system (2.2) and the corresponding regular solutions
$u(d(\tau,\bar{\mathbf{A}}^u_{l_1}(\tau)),r)$ and $u(d(\tau,\bar{\mathbf{B}}^u_{l_1}(\tau)),r)$ of (1.4). Then we have the following

$$X_{l_1}(t,\tau;\bar{\mathbf{A}}^u_{l_1}(\tau), a^\tau_{l_1}) \le x_{l_1}(t,\tau;\bar{\mathbf{Q}}^u) \le X_{l_1}(t,\tau;\bar{\mathbf{B}}^u_{l_1}(\tau), b^\tau_{l_1}) \tag{3.2}$$

for any $t \le \tau$. In fact let us denote by $t_0 = \inf\{T \le \tau \,|\, x_{l_1}(t,\tau;\bar{\mathbf{Q}}^u) \le X_{l_1}(t,\tau;\bar{\mathbf{B}}^u_{l_1}(\tau), b^\tau_{l_1}) \text{ for } t \in [T,\tau]\}$; if $t_0 > -\infty$, then $x_{l_1}(t_0,\tau;\bar{\mathbf{Q}}^u) =$
$X_{l_1}(t_0,\tau;\bar{\mathbf{B}}^u_{l_1}(\tau), b^\tau_{l_1})$ and $y_{l_1}(t_0,\tau;\bar{\mathbf{Q}}^u) > Y_{l_1}(t_0,\tau;\bar{\mathbf{B}}^u_{l_1}(\tau), b^\tau_{l_1})$. Therefore from (2.2) we find $\dot{x}_{l_1}(t_0,\tau;\bar{\mathbf{Q}}^u) > \dot{X}_{l_1}(t_0,\tau;\bar{\mathbf{B}}^u_{l_1}(\tau), b^\tau_{l_1})$ a contradiction, so the second inequality in (3.2) is proved; the other can be proved reasoning in the same way.

Let us denote by $u(r)$ the solution of (1.4) corresponding to $\mathbf{x}_{l_1}(t,\tau;\bar{\mathbf{Q}}^u)$. It follows that for any $r \le \exp(\tau)$ we have $u(d(\tau,\bar{\mathbf{A}}^u_{l_1}(\tau)),r) \le u(r) \le u(d(\tau,\bar{\mathbf{B}}^u_{l_1}(\tau)),r)$, so $u(r)$ is a regular solution. Moreover if $u(0) = d(\tau,\bar{\mathbf{Q}}^{\mathbf{u}})$, we have

$$d(\tau,\bar{\mathbf{A}}^u_{l_1}(\tau)) \le d(\tau,\bar{\mathbf{Q}}^u) \le d(\tau,\bar{\mathbf{B}}^u_{l_1}(\tau)) \tag{3.3}$$

In fact this argument can be repeated for any point $\bar{\mathbf{Q}}^{\mathbf{u}} \in \bar{W}^u_{l_1}(\tau) \cap \{(x,y) \,|\, x = \rho\}$, where $\rho \in (0, \bar{B}^u_x(\tau)]$. Hence if $\bar{\mathbf{Q}}^{\mathbf{u}} \in \bar{W}^u_{l_1}(\tau)$ then $\mathbf{x}_{l_1}(t,\tau,\bar{\mathbf{Q}}^{\mathbf{u}})$ corresponds to a regular solution $u(d,r)$. Now observe that if $\mathbf{G1'}$ is satisfied, $\|\bar{\mathbf{B}}^u_{l_1}(\tau)\|$ and $\|\bar{\mathbf{A}}^u_{l_1}(\tau)\|$ are uniformly positive and bounded for $\tau < 0$. Therefore from Remark 2.12 we find that $d(\tau,\bar{\mathbf{Q}}^{\mathbf{u}}) > d(\tau,\bar{\mathbf{A}}^u_{l_1}(\tau))$ tends to $+\infty$ as $\tau \to -\infty$; viceversa, if $d(\tau,\bar{\mathbf{Q}}^{\mathbf{u}}) \to +\infty$, then $d(\tau,\bar{\mathbf{B}}^u_{l_1}(\tau)) \to +\infty$ and from Remark 2.12 we find that $\tau \to -\infty$.

Now assume that $d(\tau,\bar{\mathbf{Q}}^{\mathbf{u}}) \to 0$, we want to show that $\tau \to +\infty$. Assume for contradiction that there is $M > 0$, a sequence $\tau_n < M$, and $\bar{\mathbf{Q}}^{\mathbf{u}}_{\mathbf{n}} \in \bar{\xi}^u_{l_1}(\tau_n)$, such that the corresponding $d(\tau_n,\bar{\mathbf{Q}}^{\mathbf{u}}_{\mathbf{n}}) \to 0$. It follows that for any $x > 0$, $A^{\tau_n}_{l_1}(x)$ is uniformly positive as $n \to +\infty$. So, from Remark 2.12, we find that there is $\delta > 0$ such that $d(\tau_n,\bar{\mathbf{A}}^u_{l_1}(\tau_n)) > \delta$, for any n. So from (3.3) we find $d(\tau_n,\bar{\mathbf{Q}}^{\mathbf{u}}_{\mathbf{n}}) > \delta$: a contradiction and the claim is proved.

Assume further $\mathbf{G2'}$ with $l_1 \le l_2$; it follows that $\bar{\mathbf{B}}^u_{l_1}(\tau)$ is uniformly positive and bounded for any $\tau \in \mathbb{R}$. Hence from Remark 2.12 we find that $d(\tau,\bar{\mathbf{B}}^u_{l_1}(\tau)) \to 0$ as $\tau \to +\infty$ and viceversa and from (3.3) we find that $d(\tau,\bar{\mathbf{Q}}^{\mathbf{u}}) \to 0$ as $\tau \to +\infty$.

The proof concerning $\bar{W}^s_{l_2}(\tau)$ follows using a similar argument and Remark 2.12. □

With similar arguments we can also prove the following:

3.6. *Remark.* Assume $\mathbf{G1}$ with $l_1 < p^*$. Then there is $D > 0$ such that $u(d,r)$ is a crossing solution for $d > D$ and its first zero $R(d)$ is such that $R(d) \to 0$ as $d \to +\infty$.

From Lemma 3.1 we easily get the following useful Remark.

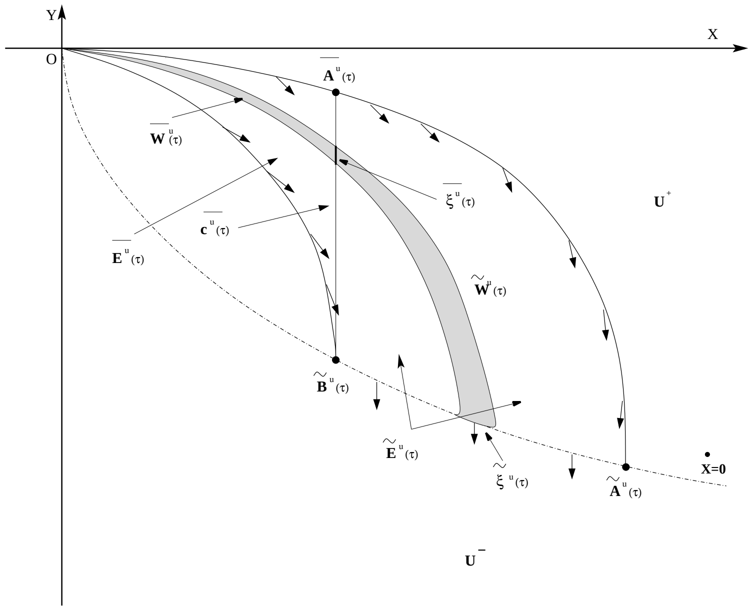

FIGURE 3. Construction of the unstable sets $\tilde{W}^u_{l_1}(\tau)$ and $\bar{W}^u_{l_1}(\tau)$.

3.7. *Remark.* Assume that $\mathbf{G1'}$ is satisfied and consider a trajectory $\mathbf{x}_{l_1}(t)$ such that $\mathbf{x}_{l_1}(t) \searrow 0$ as $t \to -\infty$. Then there is τ such that $\mathbf{x}_{l_1}(t) \in \tilde{E}^u_{l_1}(\tau)$ for any $t \le \tau$. Analogously assume $\mathbf{G2'}$ and consider a trajectory $\mathbf{x}_{l_2}(t)$ such that $x_{l_2}(t) \searrow 0$ as $t \to +\infty$. Then there is τ such that $\mathbf{x}_{l_2}(t) \in \tilde{E}^s_{l_2}(\tau)$ for any $t \ge \tau$.

Now we spend some words on the changes in system (2.2) when we pass from a value $l = i$ to a value $l = j$. We denote by $\aleph^t_{j,i}(\mathbf{x})$ the diffeomorphism such that $\aleph^t_{j,i}\big(\mathbf{x}_i(t)\big) = \mathbf{x}_j(t)$, that is

$$\aleph^t_{j,i}(x,y) = \big(x\exp[(\alpha_j - \alpha_i)t], y\exp[(\beta_j - \beta_i)t]\big)\,. \tag{3.4}$$

If we consider the modified polar coordinates introduced in (2.5), we find that $\aleph^t_{j,i}$ brings (ρ,θ) into $(\rho e^{\delta t},\theta)$ where $\delta = p(p-1)\frac{i-j}{(i-p)(j-p)}$. Since the sets U^0_l can be defined by the relation $\theta = 1/(\alpha_l)\arctan(|\alpha_l|^{p-1})$ so they are invariant for $\aleph^t_{j,i}$, for any $i, j > p$. Also note that $U^0_j \subset U^+_i$ and $U^0_i \subset U^-_j$ if $j > i$.

Assume that $\mathbf{G1'}$ and $\mathbf{G2'}$ are satisfied; using $\aleph$ we can define stable and unstable sets for any $l > p$. We denote by $\tilde{\xi}^u_l(\tau)$ the intersection of $\aleph^\tau_{l,l_1}\big[\tilde{W}^u_{l_1}(\tau)\big]$ with U^0_l for $l \ge l_1$ (note that this intersection does not exist for $l < l_1$), and by $\tilde{\xi}^s_l(\tau)$ the intersection of $\aleph^\tau_{l,l_2}\big[\tilde{W}^s_{l_2}(\tau)\big]$ with U^0_l for $\sigma < l \le l_2$ (again, this intersection does not exist for $l > l_2$). Then we denote by $\tilde{W}^u_l(\tau)$ the connected component of $\aleph^\tau_{l,l_1}\big[\tilde{W}^u_{l_1}(\tau)\big]\setminus U^-_l$ containing the origin and $\tilde{\xi}^u_{l_1}(\tau)$, and we give the analogous definitions for $\tilde{W}^s_l(\tau)$. Then, repeating the argument developed after Lemma 3.4, we can construct the sets $\bar{W}^u_l(\tau)$, $\bar{W}^s_l(\tau)$, $\bar{\xi}^u_l(\tau)$, $\bar{\xi}^s_l(\tau)$. Reasoning in the same way we can define also $\mathfrak{W}^u_l(\tau)$ for any $l > p$ and $\mathfrak{W}^s_l(\tau)$ for any $l > \sigma$.

3.2. **Basic properties of the function H and asymptotic results.** In this subsection we give some further results concerning the asymptotic behavior of positive solutions $u(r)$, both as $r \to 0$ and as $r \to +\infty$.

3.8. *Remark.*
- If a solution $u(r)$ of (1.4) is positive for $0 < r \le R$, then $u'(r) < 0$ for $0 < r < R$.

- Assume that for any $x > 0$ $\limsup_{t\to+\infty} g_\sigma(x,t) < \infty$. If a solution $v(r)$ of (1.4) is positive and decreasing for any $r > R$, then $v(r)r^{\frac{n-p}{p-1}}$ is strictly increasing for any $r > R$. Moreover if $w(r)$ is such that $P(w(r), w'(r), r) < 0$ for $r \in (R_1, R_2)$, then $w(r)r^{\frac{n-p}{p-1}}$ is increasing in that interval.
- Assume that $F(u,r)r^n \to 0$ as $r \to 0$, for any $u > 0$. Then, if $u(d,r)$ is a regular solution of (1.4), for the corresponding trajectory $\mathbf{x}_{p^*}(t)$ of (2.2) we have $\lim_{t\to-\infty} H_{p^*}(\mathbf{x}_{p^*}(t), t) = 0$.
- Assume that for any $x > 0$ $\limsup_{t\to+\infty} g_\sigma(x,t) < \infty$. Then, if a solution $u(r)$ of (1.4) has fast decay, for the corresponding trajectory $\mathbf{x}_{p^*}(t)$ of (2.2) we have $\lim_{t\to+\infty} H_{p^*}(\mathbf{x}_{p^*}(t), t) = 0$.

Note that the assumptions of Remark 3.8 are satisfied if **G1′** and **G2′** hold. Now we extend to the non autonomous case some results already proved in the autonomous case.

3.9. **Lemma.** *Assume* **G1** *and consider a trajectory* $\boldsymbol{x}_{l_1}(t)$ *of (2.2). If there is* $t_n \to -\infty$ *such that* $\lim_{n\to\infty} x_{l_1}(t_n) = 0$*, then* $\lim_{t\to-\infty} \boldsymbol{x}_{l_1}(t) = (0,0)$*. While if* $\lim_{n\to\infty} \|\boldsymbol{x}_{l_1}(t_n)\| = +\infty$ *then* $\boldsymbol{x}_{l_1}(t)$ *has to cross the coordinate axes infinitely many times.*

Analogously assume **G2** *and consider a trajectory* $\boldsymbol{x}_{l_2}(t)$ *of (2.2). If there is* $t_n \to +\infty$ *such that* $\lim_{n\to\infty} x_{l_2}(t_n) = 0$*, then* $\lim_{t\to+\infty} \boldsymbol{x}_{l_2}(t) = (0,0)$*. While if* $\lim_{n\to\infty} \|\boldsymbol{x}_{l_2}(t_n)\| = +\infty$ *then* $\boldsymbol{x}_{l_2}(t)$ *has to cross the coordinate axes infinitely many times.*

The proof can be obtained simply repeating the analysis of Lemma 2.7 and using a continuity argument.

3.10. **Lemma.** *Assume that there is* $l > p$ *such that* $\liminf_{t\to+\infty} g_l(x,t) > 0$ *for any* $x > 0$ *and consider a trajectory* $\mathbf{x}_{p^*}(t)$ *of (2.2) such that* $\liminf_{t\to+\infty} H_{p^*}(\mathbf{x}_{p^*}(t), t) > 0$*. Follow* $\mathbf{x}_{p^*}(t)$ *forward in* t*, then* $\mathbf{x}_{p^*}(t)$ *has to cross the negative* y_{p^*} *semi-axes. Analogously assume that* $\liminf_{t\to-\infty} H_{p^*}(\mathbf{x}_{p^*}(t), t) > 0$*, and that there is* $l > p$ *such that* $\liminf_{t\to-\infty} g_l(x,t) > 0$ *for any* $x > 0$*. Follow* $\mathbf{x}_{p^*}(t)$ *backwards in* t*, then* $\mathbf{x}_{p^*}(t)$ *has to cross the positive* y_{p^*} *semi-axes.*

Proof. Observe that we can rewrite (2.6) as follows

$$\dot{\theta} = -\frac{pH_{p^*}(\mathbf{x}_{p^*}(t),t) - pG_{p^*}(x_{p^*}(t),t) + xg_{p^*}(x_{p^*}(t),t)}{|\cos(\theta)|^{2-p}|\rho_{p^*}|^{\frac{p}{p-1}}} < -p\frac{H_{p^*}(\mathbf{x}_{p^*}(t),t)}{|\cos(\theta)|^{2-p}|\rho_{p^*}|^{\frac{p}{p-1}}},$$

if $|t| > N$, where $N > 0$ is defined in **G0**. If $\liminf_{t\to\pm\infty} H_{p^*}(\mathbf{x}_{p^*}(t), t) > 0$, then $\theta(t)$ is decreasing for $|t|$ large enough and admits a limit $\theta(\pm\infty)$ as $t \to \pm\infty$. If $\theta(\pm\infty) \notin [-\pi/2, \pi/2]$, $\mathbf{x}_{p^*}(t)$ crosses the axes and we are done. If $\theta(\pm\infty) \in [-\pi/2, \pi/2]$, from the previous inequality and (2.6) it follows that $\rho_{p^*}(t)$ is unbounded and $x_{p^*}(t)$ is bounded. Since $x_{p^*}(t)$ is bounded and $\theta(t)$ admits a limit it follows that $\lim_{t\to\pm\infty} \mathbf{x}_{p^*}(t) \in U^0_{p^*}$; so $y_{p^*}(t)$ is bounded as well, a contradiction. So $\mathbf{x}_{p^*}(t)$ has to cross the coordinate axes. □

Now we prove a result concerning the existence and the asymptotic behavior of singular and slow decay solutions.

3.11. **Proposition.** *Assume* **G1** *with* $l_1 > \sigma$*, and that either* $l_1 \neq p^*$ *or* $l_1 = p^*$ *and* $G_{p^*}(x,t)$ *is monotone in* t *for any* x *and any* $t \leq -M$*, for a certain* $M > 0$*. Then there is a trajectory* $\bar{\mathbf{x}}_{l_1}(t)$ *of the non-autonomous system (2.2) such that* $\lim_{t\to-\infty} \bar{\mathbf{x}}_{l_1}(t) = \mathbf{P}^{-\infty}$*, where* $\mathbf{P}^{-\infty}$ *is the unique critical point in* $\mathbb{R}^2_+ \setminus \{(0,0)\}$ *of the autonomous system where* $g_{l_1}(x,t) \equiv g_{l_1}^{-\infty}(x)$*.*

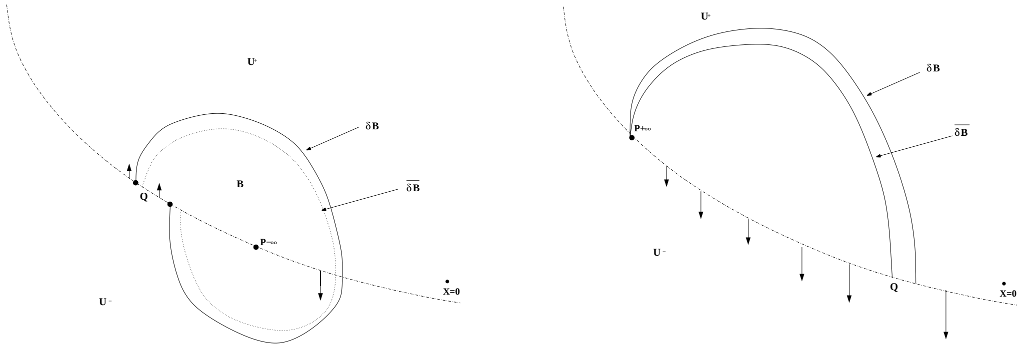

FIGURE 4. Existence of singular solutions when $l_1 > p^*$ and $\mathbf{P}^{-\infty}$ is a focus, and when $\sigma < l_1 < p^*$ and $\mathbf{P}^{-\infty}$ is a node.

Moreover if $l_1 > p^*$ and there is $\zeta > 0$ small so that $\lim_{t\to-\infty} e^{\zeta t} \frac{\partial}{\partial t} g_{l_1}(x,t) = 0$, uniformly for any x in a compact subset of $\mathbb{R}^2_+$, then there are no other singular solutions for (1.4).

Analogously assume that **G2** *is satisfied with $l_2 > \sigma$ and that either $l_2 \neq p^*$ or $l_2 = p^*$ and $G_{p^*}(x,t)$ is monotone in t for any x and any $t \geq M$, for a certain $M > 0$. Then there is a trajectory $\bar{\mathbf{x}}_{l_2}(t)$ of the non-autonomous system (2.2) such that $\lim_{t\to+\infty} \bar{\mathbf{x}}_{l_2}(t) = \mathbf{P}^{+\infty}$, where $\mathbf{P}^{+\infty}$ is the unique critical point in $\mathbb{R}^2_+ \setminus \{(0,0)\}$ of the autonomous system where $g_{l_1}(x,t) \equiv g_{l_1}^{-\infty}(x)$.*

Moreover if $\sigma < l_2 < p^*$ and there is $\zeta > 0$ small so that $\lim_{t\to+\infty} e^{-\zeta t} \frac{\partial}{\partial t} g_{l_2}(x,t) = 0$, uniformly for any x in a compact subset of $\mathbb{R}^2_+$, then there are no other slow decay solutions for (1.4).

Proof. We just prove the claims concerning singular solutions: the ones regarding slow decay solutions can be obtained reasoning in the same way.

In this proof we consider $l = l_1$ in (2.1) fixed, so we leave the subscript unsaid. Assume first $l_1 > p^*$ and consider the autonomous system (2.2) where $g(x,t) \equiv g^{-\infty}(x)$: the critical point $\mathbf{P}^{-\infty}$ is asymptotically stable. Choose a point $\mathbf{Q} \in U^0$ close enough to $\mathbf{P}^{-\infty}$ so that $\lim_{t\to+\infty} \mathbf{X}(t, 0, \mathbf{Q}, g^{-\infty}) = \mathbf{P}^{-\infty}$. Assume that there are $T_2 > T_1 > 0$ such that $\mathbf{X}(t, 0, \mathbf{Q}, g^{-\infty}) \not\in U^0$ for $t \in (0, T_2) \setminus \{T_1\}$ and it crosses U^0 at $t = T_1$ and $t = T_2$. Let us denote by $\partial\bar{B} = \{\mathbf{X}(t, 0, \mathbf{Q}, g^{-\infty}) \mid t \in [0, T_2]\}$, by $C_{\bar{b}}$ the branch of U^0 between $\mathbf{X}(T_2, 0, \mathbf{Q}, g^{-\infty})$ and $\mathbf{Q}$, and by $\bar{B}$ the bounded subset enclosed by $C_{\bar{b}}$ and $\partial\bar{B}$. Note that the flow of the autonomous system on $C_{\bar{b}} \setminus \{\mathbf{Q}\}$ points towards the interior of $\bar{B}$. Now consider the non-autonomous system (2.2). Through a deformation of the paths $\partial\bar{B}$ and $C_{\bar{b}}$ we can construct two closed paths ∂B and C_b with the following properties, see figure 4.

The set C_b is a branch of U^0 which connects the endpoints of ∂B, which are the two only intersections between ∂B and C_b. There is M large enough so that the flow of the non-autonomous system (2.2) on $(\partial B \cup C_b) \setminus \{\mathbf{Q}\}$ points towards the interior of the bounded set B enclosed by ∂B and C_b, for any $t < -M$. Such a construction can be achieved from a continuity argument since $g(x,t) \to g^{-\infty}(x)$ as $t \to -\infty$, uniformly on compact subsets. Using Wazewski's principle, see [15], we find that there is a trajectory $\bar{\mathbf{x}}(t)$ of the non-autonomous system which is forced to stay in B for any $t < -M$. Choosing $M^1 > M$ we can repeat the construction choosing a smaller set $B^1 \subset B$, and eventually we end up with a sequence of numbers $M^n \to +\infty$ and of sets $B^n \subset B^{n-1}$ which shrink to $\{\mathbf{P}^{-\infty}\}$, such that the flow of (2.2) points towards the exterior of B^n for any $t \leq -M^n$. So we obtain a trajectory $\bar{\mathbf{x}}(t)$ which converges to $\mathbf{P}^{-\infty}$ as $t \to -\infty$.

Making some slight modification we obtain the same result also in the case where $\mathbf{P}^{-\infty}$ is a stable node so that we cannot find the values T_2 and T_1.

If $\sigma < l_1 < p^*$ with the same argument we can construct a sequence of sets $B^{n-1} \supset B^n \to \{\mathbf{P}^{-\infty}\}$ and of values $M^n \to +\infty$ such that flow of (2.2) on ∂B and C_b points towards the exterior of B^n for any $t \le -M^n$. So applying once again Wazewski's principle we prove the existence of a trajectory $\bar{\mathbf{x}}(t)$ which converges to $\mathbf{P}^{-\infty}$ as $t \to -\infty$. In fact in such a case we have uncountably many of these trajectories.

When $l_1 = p^*$ and $G_{p^*}(x,t)$ is monotone we proceed as follows. Let us denote by $h^* = \lim_{t\to-\infty} H_{p^*}(\mathbf{P}^{-\infty}, t)$ or equivalently H_{p^*} evaluated in $\mathbf{P}^{-\infty}$ for the autonomous system (2.2) where $g_{p^*}(x,t) \equiv g_{p^*}^{-\infty}(x)$. For any $\epsilon > 0$ we can find $M > 0$ large enough so that for any $t < -M$ we have $\min_{\mathbf{x}\in\mathbb{R}^2_+} H_{p^*}(\mathbf{x},t) < h^* + \epsilon$. Let us denote by $\partial B^\epsilon(t)$ the closed curve in $\mathbb{R}^2_+$ defined by $H_{p^*}(\mathbf{x},t) = h^* + \epsilon$ and by $B^\epsilon(t)$ the bounded set enclosed by $\partial B^\epsilon(t)$. Observe that from (2.4) we know that $H_{p^*}(\mathbf{x}(t),t)$ is monotone in t for any trajectory $\mathbf{x}(t)$ of (2.2). So the flow of (2.2) on $\partial B^\epsilon(t)$ points towards the exterior, respectively the interior, of $B^\epsilon(t)$ for any $t < -M$. Then, using Wazewski's principle, we find a trajectory $\mathbf{x}_{p^*}(t)$ which is in $B^\epsilon(t)$ for any $t \le -M$. Letting $\epsilon \to 0$ (and consequently $M \to -\infty$) we find that there is a trajectory $\bar{\mathbf{x}}_{p^*}(t)$ of the non-autonomous system (2.2) which converges to $\mathbf{P}^{-\infty}$ as $t \to -\infty$.

Now assume $l_1 > p^*$ and that there is $\zeta > 0$ small so that $\lim_{t\to-\infty} \frac{\partial}{\partial t} g(x,t) e^{\zeta t} = 0$. Consider the autonomous system obtained adding to (2.2) the extra-variable $z = e^{\zeta t}$, where $\zeta > 0$ is the small constant defined in the hypotheses. Then $\dot{z} = \zeta z$. It follows that the α-limit set of any trajectory is contained in the $z = 0$ plane, and that this plane admits 3 critical points: the origin , $\mathbf{P}^{-\infty}$, and $-\mathbf{P}^{-\infty}$. Since $l > p^*$, from the Poincarè-Bendixson criterion we find that the system admits no periodic trajectories. Moreover the critical point $\mathbf{P}^{-\infty}$ admits a one-dimensional unstable manifold. In fact this argument also gives a simpler proof of the existence of singular solutions, used e. g. in [18] (which however does not work in the general case). Then using Lemmas 3.5 and 3.9 we get the uniqueness of the singular solution. □

Now we give a further result concerning the asymptotic behavior of singular and slow decay solutions with weaker assumptions.

3.12. **Proposition.** *Consider a trajectory $\mathbf{x}_{p^*}(t)$ of (2.2) such that there are $\delta > 0$ and $T > 0$ such that $x_{p^*}(-T) > 0$ and $H_{p^*}(\mathbf{x}_{p^*}(t),t) < -\delta$ for any $t < -T$. Then the corresponding solution $u(r)$ of (1.4) is a singular solution, hence $\lim_{r\to 0} u(r) = +\infty$. Moreover, if $\mathbf{G1'}$ is satisfied, then*

$$\liminf_{r\to 0} u(r) r^{\frac{n-p}{p}} > 0 \quad \text{and} \quad 0 < \limsup_{r\to 0} u(r) r^{\frac{p}{l_1-p}} < \infty\,.$$

Analogously consider a trajectory $\mathbf{x}_{p^*}(t)$ of (2.2) such that there are $\delta > 0$ and $T > 0$ such that $x_{p^*}(T) > 0$ and $H_{p^*}(\mathbf{x}_{p^*}(t),t) < -\delta$ for any $t > T$. Then the corresponding solution $u(r)$ of (1.4) has slow decay. Moreover, if $\mathbf{G2'}$ is satisfied

$$\liminf_{r\to \infty} u(r) r^{\frac{n-p}{p}} > 0 \quad \text{and} \quad 0 < \limsup_{r\to \infty} u(r) r^{\frac{p}{l_2-p}} < \infty\,.$$

Proof. We just consider the case of slow decay solutions since the case of regular solutions is completely analogous. Let us denote by $K_l(t) = \{\mathbf{x}_l \in \mathbb{R}^2_+ \mid H_l(\mathbf{x}_l,t) < 0\}$. From (2.3) it follows that the trajectory $\mathbf{x}_l(t)$ corresponding to $u(r)$ is such that $\mathbf{x}_l(t) \in K_l(t)$ for any $l > p$ and any $t > T$. Thus $u(r)$ is positive and decreasing for $r > e^T$. So from Remark 3.8 $u(r)r^{(n-p)/(p-1)}$ is increasing for $r > e^T$ and $\frac{n-p}{p-1} x_{p^*}(t) - |y_{p^*}(t)|^{1/(p-1)} \ge 0$. Assume for contradiction that there is a sequence

$t_n \to +\infty$ such that $x_{p^*}(t_n) \to 0$. Then

$$H_{p^*}(\mathbf{x}_{p^*}(t_n), t_n) \geq \frac{n-p}{p} x_{p^*}(t_n) y_{p^*}(t_n) \geq \frac{(n-p)^p}{p(p-1)^{p-1}} |x_{p^*}(t_n)|^p \to 0$$

as $n \to \infty$, a contradiction. So $\liminf_{r\to+\infty} u(r) r^{(n-p)/p} > 0$, and in particular $u(r)$ has slow decay.

From $\mathbf{G2'}$ we find that $K_{l_2}(t)$ is uniformly bounded for $t > T_1$, hence $\limsup_{r\to\infty} u(r) r^{\frac{p}{l_2-p}} < \infty$. Observe also that the diameter of $K_{l_2}(t)$ is uniformly positive. Assume for contradiction that $x_{l_2}(t) \to 0$ as $t \to +\infty$, then either there is $T_2 > T_1$ such that $\dot{x}_{l_2}(t) < 0$ for $t > T_2$, or there is a sequence $t_n \to +\infty$ such that $x_{l_2}(t_n)$ is a local maximum. Assume the forme; from Remark 3.7 we find that $x_{l_2}(t) \searrow 0$ as well as $t \to +\infty$ and we conclude that $u(r)$ has fast decay, a contradiction. Assume the latter; we have that $\mathbf{x}_{l_2}(t_n) \in U^0_{l_2}$ and $\dot{y}_{l_2}(t_n) \leq 0$. From $\mathbf{G2'}$ it follows that there is $\Delta > 0$ such that the flow of (2.2) with $l = l_2$ on $U^0_{l_2} \cap \{(x_{l_2}, y_{l_2}) \,|\, 0 < x_{l_2} < \Delta\}$ points upwards. Therefore $x_{l_2}(t_n) > \Delta$ for any n and $\limsup_{t\to+\infty} x_{l_2}(t) > \Delta > 0$. □

4. Applications

Now we give some new results concerning (1.4) when the non-linearity $f(u, r)$ is either subcritical for any $u \geq 0$ and any $r \geq 0$, or supercritical for any $u \geq 0$ and any $r \geq 0$. In fact this situation has been studied in many papers, see e.g. [9], [20]. The main Hypothesis we will consider in this section is that $G_{p^*}(x, t)$ is monotone in t for any $x > 0$. More precisely we will assume either of these hypotheses:

- $\mathbf{H^+}$: $G_{p^*}(x, t)$ is increasing in t (strictly for some $T \in \mathbb{R}$) for any $x > 0$.
- $\mathbf{H^-}$: $G_{p^*}(x, t)$ is decreasing in t (strictly for some $T \in \mathbb{R}$) for any $x > 0$.

From (2.4) it follows easily that for any trajectory $\mathbf{x}_{p^*}(t)$ the function $H_{p^*}(\mathbf{x}_{p^*}(t), t)$ is increasing in t if $\mathbf{H^+}$ is satisfied and decreasing if $\mathbf{H^-}$ is satisfied. In order to understand better the meaning of Hypotheses $\mathbf{H^+}$ and $\mathbf{H^-}$ we observe the following.

4.1. *Remark.* Assume $\mathbf{G1'}$ and $\mathbf{G2'}$; if $\mathbf{H^+}$ holds then $l_1, l_2 \in (p, p^*]$ while if $\mathbf{H^-}$ holds then $l_1, l_2 \in [p^*, +\infty)$.

Exploiting the results of the previous section we easily get the following.

4.2. **Theorem.** *Assume that* $\mathbf{H^+}$ *holds, then all the regular solution* $u(d, r)$ *of (1.4) are crossing solutions. So* $u(d, r)$ *is a solution of the Dirichlet problem in the ball of radius* $R(d) > 0$. *Assume further* $\mathbf{G1'}$, *then* $\lim_{d\to0} R(d) = +\infty$. *Moreover if* $\mathbf{G1}$ *holds and* $p < l_1 < p^*$, *then* $R(d)$ *is continuous and* $\lim_{d\to+\infty} R(d) = 0$.

Assume that $\mathbf{H^+}$ *and* $\mathbf{G2'}$ *are satisfied, then there are uncountably many S.G.S. with fast decay. Finally assume that* $\mathbf{G2}$ *holds, then there is one S.G.S with slow decay* $w(r)$. *In particular positive solutions have a structure of type* ***Sub***.

Proof. Consider a regular solution $u(d, r)$ of (1.4) and the corresponding trajectory $\mathbf{x}_{p^*}(t)$ of (2.2). From (2.4) we know that $H_{p^*}(\mathbf{x}_{p^*}(t), t) \geq 0$ for any t and that $\liminf_{t\to+\infty} H_{p^*}(\mathbf{x}_{p^*}(t), t) > 0$. From $\mathbf{H^+}$ we know that $\liminf_{t\to+\infty} G_{p^*}(x, t) > 0$ for any $x > 0$. So we can apply Remark 3.8 and Lemma 3.10 to conclude that $u(d, r)$ is a crossing solution, whose first zero is $R(d)$. Assume $\mathbf{G1'}$ and consider a trajectory $\mathbf{x}_{p^*}(t, \tau, \mathbf{Q^u})$ where $\mathbf{Q^u} \in \bar{\xi}^u_{p^*}(\tau)$, and the corresponding solution $u(d, r)$ of (1.4). From Lemma 3.5 we know that $\tau \to +\infty$ as $d \to 0$, hence $R(d) > e^\tau \to +\infty$ as $d \to 0$. If $\mathbf{G1}$ holds with $p < l_1 < p^*$, from Remark 3.6 we find that $\lim_{d\to+\infty} R(d) = 0$. The continuity of $R(d)$ follows from Remark 2.3 and the continuity of the flow of (2.2).

Now assume that $\mathbf{G2'}$ is satisfied, so that the stable set $\bar{W}^s_{l_2}(\tau)$ is well defined for any $\tau \in \mathbb{R}$. Choose $\mathbf{Q} \in \bar{W}^s_{l_2}(\tau)$ and consider the trajectory $\mathbf{x}_{l_2}(t, \tau, \mathbf{Q})$, the corresponding trajectory $\mathbf{x}_{p^*}(t)$ of (2.2) with $l = p^*$ and the corresponding solution

$u(r)$ of (1.4): from Lemma 3.5 we know that $u(r)$ has fast decay. Denote by (T, ∞) the maximal interval of continuation of $\mathbf{x}_{p^*}(t)$. From $\mathbf{H}^+$ and (2.4) we find that $H_{p^*}(\mathbf{x}_{p^*}(t), t) \le 0$ for any $t > T$. Thus $\mathbf{x}_{p^*}(t) \in \mathbb{R}^2_{\pm}$ and it is bounded for any $t \ge T$, hence $T = -\infty$. Moreover $\limsup_{t \to -\infty} H_{p^*}(\mathbf{x}_{p^*}(t), t) < 0$, so $u(r)$ cannot be a regular solution, see Remark 3.8. Hence $u(r)$ is a S.G.S. with fast decay, whose asymptotic behavior as $r \to 0$ can be deduced from Proposition 3.12.

Assume further that hypothesis **G2** is satisfied. Then, from Proposition 3.11 we find that there is a trajectory $\mathbf{x}_{l_2}(t)$ such that $\lim_{t \to +\infty} \mathbf{x}_{l_2}(t) = \mathbf{P}^{+\infty}$, where $\mathbf{P}^{+\infty}$ is the unique critical point in $\mathbb{R}^2_+ \setminus \{(0,0)\}$ of the autonomous system (2.2) where $g_{l_2}(x,t) \equiv g_{l_2}^{+\infty}(x)$; hence $\lim_{t \to +\infty} H_{l_2}(\mathbf{x}_{l_2}(t), t) < 0$. Let $\mathbf{x}_{p^*}(t)$ be the trajectory corresponding to $\mathbf{x}_{l_2}(t)$. Then $\lim_{t \to +\infty} H_{p^*}(\mathbf{x}_{p^*}(t), t) \le 0$ and $H_{p^*}(\mathbf{x}_{p^*}(t), t) \le 0$ for any t and, reasoning as above, we conclude that $\mathbf{x}_{p^*}(t)$ can be continued backwards for any t and $\limsup_{t \to -\infty} H_{p^*}(\mathbf{x}_{p^*}(t), t) < 0$. So the corresponding solution $u(r)$ of (1.4) is a S.G.S. with slow decay. □

Using similar arguments we can prove the following.

4.3. **Theorem.** *Assume that* $\mathbf{H}^-$ *holds, then all the regular solution* $u(d,r)$ *of (1.4) are G. S. with slow decay and no S.G.S. with fast decay can exist. Furthermore if* $\mathbf{G2}'$ *holds there are uncountably many Dirichlet solutions* $w(D,r)$ *in the exterior of a ball, that is* $w(D,r)$ *is null for* $r = D$*, it is positive for* $r > D$ *and it has fast decay. Moreover if* **G2** *holds, then for any* $D > 0$ *there is a solution* $w(D,r)$ *as above.*

Furthermore assume that **G1** *holds, then there is one S.G.S with slow decay* $w(r)$*. In particular positive solution have a structure of type* ***Sup****.*

Proof. Let us consider a regular solution $u(d,r)$ of (1.4) and the corresponding trajectory $\mathbf{x}_{p^*}(t)$ of (2.2). From (2.4) we know that $H_{p^*}(\mathbf{x}_{p^*}(t), t) \le 0$ for any t and that $\liminf_{t \to +\infty} H_{p^*}(\mathbf{x}_{p^*}(t), t) < 0$. So $\mathbf{x}_{p^*}(t)$ is forced to stay in $\mathbb{R}^2_{\pm}$ for any t and it has slow decay, see Proposition 3.12: so it is a G.S. with slow decay.

Now consider a fast decay solution $v(r)$ of (1.4) and the corresponding trajectory $\mathbf{x}_{p^*}(t)$ of (2.2). From Remark 3.8 and (2.4) we have $\lim_{t \to +\infty} H_{p^*}(\mathbf{x}_{p^*}(t), t) = 0$ and $H_{p^*}(\mathbf{x}_{p^*}(t), t) \ge 0$ for any t. So from Lemma 3.10 it follows that there is T such that $x_{p^*}(T) = 0$, so $v(r)$ can be nor a G.S. neither a S.G.S.

Assume $\mathbf{G2}'$ so that we can construct the stable set $\bar{W}^s_{p^*}(\tau)$, and choose $\mathbf{Q} \in \bar{W}^s_{p^*}(\tau)$. Reasoning as above we find there is $T_1 < \tau$ such that $\mathbf{x}_{p^*}(t, \tau, \mathbf{Q}) \in \mathbb{R}^2_{\pm}$ for any $t > T_1$ and it crosses the y positive semi-axis at $t = T_1$. Then a priori we lose uniqueness of the solution, however it is easy to show that for each trajectory bifurcating from $\mathbf{x}_{p^*}(T_1, \tau, \mathbf{Q})$ there is $T_2 < T_1$ such that $x_{p^*}(t, \tau, \mathbf{Q})$ is positive for $t \in (T_2, T_1)$ and it becomes null at $t = T_2$. It follows that the corresponding solution $v(r)$ of (1.4) solves the Dirichlet problem in the exterior of the ball of radius $\ln(T_2)$.

If we assume also **G2** with $l_2 > p^*$, reasoning as in Theorem 4.2, we find that $T_2 \to +\infty$ as $\tau \to +\infty$ and $T_2 \to -\infty$ as $\tau \to -\infty$.

If we assume **G1** with $l_1 \ge p^*$, from Proposition 3.11 we find a trajectory $\mathbf{x}_{l_1}(t)$ which converges to $\mathbf{P}^{-\infty}$ as $t \to -\infty$. So the corresponding solution $w(r)$ of (1.4) is singular and $\lim_{t \to -\infty} H_{p^*}(\mathbf{x}_{p^*}(t), t) \le 0$. So, reasoning as for regular solutions, we find that $w(r)$ is a monotone decreasing S.G.S. with slow decay. □

Note that Theorems 4.2 and 4.3 can be combined with Proposition 3.10 to obtain better estimates on the asymptotic behavior of singular and slow decay solutions, and to prove uniqueness of the S.G.S with slow decay $w(r)$.

4.1. **Some special cases.** In this subsection we show how the hypotheses $\mathbf{H}^+$ and $\mathbf{H}^-$ may be weakened a bit, when we know the exact expression of the function

$f(u,r)$ (and consequently of $g(x,t)$). We discuss the case in which f is a finite sum of terms of type (1.2):

$$f(u,r) = \sum_{j=1}^{m} k_j(r) u|u|^{q_j-2}\,, \tag{4.1}$$

but it is easy to check that a similar argument can be derived for other types of nonlinearities, (1.3) among them. We denote by $h_j(t) = k_j(e^t)\exp[(q_j - p^*)(n-p)/p]$, so that we can write $g_{p^*}(x_{p^*}, t) = \sum_{j=1}^m h_j(t) x_{p^*}|x_{p^*}|^{q_j-2}$, and we introduce the following auxiliary functions which are borrowed from [20].

$$J_j^a(t) := k^j(t)e^{nt} - q_j \frac{n-p}{p} \int_{-\infty}^{t} k^j(s) e^{ns} ds$$

$$J_j^z(t) := k^j(t) e^{(n-q_j\frac{n-p}{p-1})t} + q_j \frac{n-p}{p(p-1)} \int_t^{+\infty} k^j(s) e^{(n-q_j\frac{n-p}{p-1})s} ds$$

If f is of the form (1.2) we may replace hypotheses $\mathbf{H}^+$ and $\mathbf{H}^-$ by the following:

- $\mathbf{H}^+_\mathbf{a}$**:** $J_j^a(t) \ge 0$ for any $t \in \mathbb{R}$ and any j, but $\sum_{j=1}^m J_j^a(t) \not\equiv 0$.
- $\mathbf{H}^+_\mathbf{z}$**:** $J_j^z(t) \ge 0$ for any $t \in \mathbb{R}$ and any j, but $\sum_{j=1}^m J_j^z(t) \not\equiv 0$.
- $\mathbf{H}^-_\mathbf{a}$**:** $J_j^a(t) \le 0$ for any $t \in \mathbb{R}$ and any j, but $\sum_{j=1}^m J_j^a(t) \not\equiv 0$.
- $\mathbf{H}^+_\mathbf{z}$**:** $J_j^z(t) \le 0$ for any $t \in \mathbb{R}$ and any j, but $\sum_{j=1}^m J_j^z(t) \not\equiv 0$.

It is easy to check that $\mathbf{H}^+$ implies $\mathbf{H}^+_\mathbf{a}$ and $\mathbf{H}^+_\mathbf{z}$, while $\mathbf{H}^-$ implies $\mathbf{H}^-_\mathbf{a}$ and $\mathbf{H}^-_\mathbf{z}$. Let $\mathbf{x}^u_{p^*}(t)$ be a trajectory of (2.2) corresponding either to a regular or to a singular solution $u(r)$ of (1.4), and denote by $(-\infty, T^u)$ the maximal interval in which $\mathbf{x}^u_{p^*}(t)$ is positive and decreasing. Analogously let $\mathbf{x}^v_{p^*}(t)$ be a trajectory of (2.2) corresponding either to a fast or to a slow decaying solution $v(r)$ of (1.4), and denote by $(T^v, +\infty)$ the maximal interval in which $\mathbf{x}^v_{p^*}(t)$ is positive and decreasing.

Set $\lim_{t\to-\infty} H_{p^*}(\mathbf{x}^u_{p^*}(t), t) = A \le 0$ and $\lim_{t\to+\infty} H_{p^*}(\mathbf{x}^v_{p^*}(t), t) = B \le 0$; if $\mathbf{H}^+_\mathbf{a}$ is satisfied, using (2.4) we have

$$H_{p^*}(\mathbf{x}^u_{p^*}(t), t) = A + \sum_{j=1}^m \big[J_j^a(t)\frac{|u(e^t)|^{q_j}}{q_j} - \int_\infty^t J_j^a(s)(u(e^s))^{q_j-1}u'(s)ds$$

$$H_{p^*}(\mathbf{x}^u_{p^*}(t), t) = B + \sum_{j=1}^m \big[J_j^z(t)\frac{|x^v_\sigma(t)|^{q_j}}{q_j} + \int_\infty^t J_j^z(s)\dot{x}^v_\sigma(s)|x^v_\sigma(s)|^{q_j-1}ds$$

So if $\mathbf{H}^+_\mathbf{a}$ is satisfied (resp. $\mathbf{H}^-_\mathbf{a}$ is satisfied) from Remark 3.8 we get that $H_{p^*}(\mathbf{x}^u_{p^*}(t), t) \ge 0$ (resp. $H_{p^*}(\mathbf{x}^u_{p^*}(t), t) \le 0$) for $t \le T^u$. Analogously if $\mathbf{H}^+_\mathbf{z}$ is satisfied (resp. $\mathbf{H}^-_\mathbf{z}$ is satisfied) from Remark 3.8 we get that $H_{p^*}(\mathbf{x}^u_{p^*}(t), t) \le 0$ (resp. $H_{p^*}(\mathbf{x}^u_{p^*}(t), t) \le 0$) for $t \ge T^v$. So we can repeat the argument of Theorem 4.2 and 4.3 and prove the following.

4.4. **Corollary.** *Assume f has the form (4.1).*

If $\mathbf{H}^+_\mathbf{a}$ holds all the regular solutions are are crossing solutions. Assume further that **G1** *holds and $p < l_1 < p^*$. Then $R(d)$ is continuous and $R(0) = +\infty$ and $\lim_{d\to+\infty} R(d) = 0$. Assume that $\mathbf{H}^+_\mathbf{z}$ and* **G2′** *are satisfied, then there are uncountably many S.G.S. with fast decay. Finally assume that* **G2** *holds, then there is one S.G.S with slow decay $w(r)$.*

If $\mathbf{H}^-_\mathbf{a}$ holds, then all the regular solution $u(d,r)$ of (1.4) are G. S. with slow decay. Moreover if **G1** *holds, then there is one S.G.S with slow decay $w(r)$. Assume $\mathbf{H}^-_\mathbf{z}$ and* **G2′***, then there are uncountably many Dirichlet solutions $w(D,r)$ in the exterior of a ball, that is $w(D,r)$ is null for $r = D$, it is positive for $r > D$ and it has fast decay. Moreover if* **G2** *holds and $l_2 < p^*$, then for any $D > 0$ there is a solution $w(D,r)$ as above.*

5. Appendix: reduction of $\operatorname{div}(g(|\mathbf{x}|)\nabla u|\nabla u|^{p-2}) + f(u,|\mathbf{x}|) = 0$ and natural dimension.

In this appendix we show how we can reduce the following equation

$$(r^{n-1}g(r)u'|u'|^{p-2})' + r^{n-1}\bar{f}(u,r) = 0. \tag{5.1}$$

to (1.4), see [14] for more details. Set $a(r) = r^{n-1}g(r)$ and assume that one of the Hypotheses below is satisfied

A1: $a^{-1/(p-1)} \in L^1[1,\infty]\setminus L^1[0,1]$
A2: $a^{-1/(p-1)} \in L^1[0,1]\setminus L^1[1,\infty)$

Then we make the following change of variables borrowed from [14]. Let $N > p$ be a constant and assume that Hyp. **A1** is satisfied; we define $s(r) = \left(\int_r^\infty a(\tau)^{-1/(p-1)}d\tau\right)^{\frac{-p+1}{N-p}}$. Obviously $s : \mathbb{R}_0^+ \to \mathbb{R}_0^+$, $s(0) = 0$, $s(\infty) = \infty$ and $s(r)$ is a diffeomorphism of $\mathbb{R}_0^+$ into itself with inverse $r = r(s)$ for $s \geq 0$. If $u(r)$ is a solution of (5.1), $v(s) = u(r(s))$ is a solution of the following transformed equation

$$(s^{N-1}v_s|v_s|^{p-2})_s + s^{N-1}h(s)f(v,s) = 0, \tag{5.2}$$

where $f(v,s) = \bar{f}(v,r(s))$ and

$$h(s) = \left(\frac{N-p}{p-1}\right)^p \left(\frac{g(r(s))^{1/p}r(s)^{n-1}}{s^{N-1}}\right)^{p/(p-1)}.$$

If we replace Hyp. **A1** by Hyp. **A2** we can define $s(r)$ as follows $s(r) = \left(\int_0^r a(\tau)^{-1/(p-1)}d\tau\right)^{\frac{p-1}{N-p}}$ and obtain again (5.2) from (5.1), with the same expression for h. We denote by $f(v,s) = h(s)\bar{f}(v,r(s))$ and obtain (1.4) from (5.2), with r replaced by s.

5.1. *Remark.* Note that, if for any fixed $v > 0$, $\bar{f}(v,r)$ grows like either a positive or a negative power in r for r small, we can play with the parameter N in order to have that, for any fixed $u > 0$, $f(u,0)$ is positive and bounded. E.g., if $g(r) \equiv 1$ and $\bar{f}(u,r) = r^l u|u|^{q-1}$, we can set $N = \frac{p(n+l)-n}{p+l-1}$, so that, switching from r to s as independent variable (5.2) takes the form

$$[s^{N-1}v_s|v_s|^{p-2}]_s + Cs^{N-1}v|v|^{q-1} = 0\,, \tag{5.3}$$

where $C = \left|\frac{N-p}{p-1}\right|^p \left|\frac{p-1}{N-1}\right|^{\frac{n-1}{N-p}p} > 0$. So we can directly study the spatial independent equation (5.3), recalling that the natural dimension is N and this changes the values of the critical exponents σ and p^*.

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