

# Some results on the $m$ -Laplace equations with two growth terms

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Abbreviated form of the title:  
Some results on the M-laplace equations..

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**Abstract**

We prove the existence of positive radial solutions of the following equation:

$$\Delta_m u - K^1(r)u|u|^{q-2} + K^2(r)u|u|^{p-2} = 0$$

and give sufficient conditions on the positive functions  $K^1(r)$  and  $K^2(r)$  for the existence and nonexistence of G.S. and S.G.S., when  $q < m^* \leq p$  or  $q = m^* < p$ . We also give sufficient conditions for the existence of radial S.G.S. and G.S. of equation

$$\Delta_m u + K^1(r)u|u|^{q-2} + K^2(r)u|u|^{p-2} = 0$$

when  $q < p \leq m^*$  and  $m^* \leq q < p$  respectively. We are also able to classify all the S.G.S. of this equation.

The proofs use a new Emden-Fowler transform which allow us to use techniques taken from dynamical system theory, in particular the ones developed in [14] for the problems obtained by substituting the ordinary Laplacian  $\Delta$  for the  $m$ -Laplacian  $\Delta_m$  in the preceding equations.

**Key words and phrases:**

M-laplace equations, radial solution, regular/singular ground state, Fowler transformation, invariant manifold, energy function.

**MSC:** 37B55, 35H30, 35J70

**1 Introduction**

In this paper we will discuss positive radial solutions of the following equation:

$$\Delta_m u + k^1(r)u|u|^{q-2} + k^2(r)u|u|^{p-2} = 0 \quad (1.1)$$

where  $\Delta_m u = \operatorname{div}(|Du|^{m-2} Du)$  is the so called  $m$ -Laplacian,  $1 < m < q < p$ ,  $|x| = r$  and  $x \in \mathbb{R}^n$ ,  $n > m$ . We denote by  $m^* = \frac{nm}{n-m}$  the Sobolev critical exponent. The function  $k^2(r)$  is always assumed to be positive; we will consider the case in which  $k^1(r)$  is negative and  $q \leq m^* \leq p$ , and the case in which  $k^1(r)$  is positive and the parameters  $q, p$  are both subcritical or both supercritical. In particular we will focus our attention on the problem of existence of ground states (G.S.), of singular ground states (S.G.S.) and of crossing solutions. By G.S. we mean a positive solution  $u(x)$  defined in the whole of  $\mathbb{R}^n$  such that  $\lim_{|x| \rightarrow \infty} u(x) = 0$ . A Singular Ground State (S.G.S.) of equation (1.1) is a positive solution  $v(x)$  such that

$$\lim_{|x| \rightarrow 0} v(x) = +\infty \quad \text{and} \quad \lim_{|x| \rightarrow +\infty} v(x) = 0.$$

Crossing solutions are solutions  $u(r)$  such that  $u(r) > 0$  for any  $0 \leq r < R$  and  $u(R) = 0$  for some  $R > 0$ , so they can be considered as solutions of the Dirichlet problem in the ball of radius  $R$ . Here and later we write  $u(r)$  for  $u(x)$  when  $|x| = r$  and  $u$  is radially symmetric.

The corresponding autonomous equation is well studied and understood. In particular in [5] it is proved that, when the potentials  $k^i$  have opposite sign, the G.S. of the autonomous problem can only be radial if  $1 < m \leq 2$ . Moreover in [10] the authors state the existence of radial ground states for the autonomous problem when  $m \geq n$  and when  $n > m$  and  $q < p < m^*$ .

In this paper we only deal with radial solutions, so we shall consider the following O.D.E.

$$(u'|u'|^{m-2})' + \frac{n-1}{r}u'|u'|^{m-2} + k^1(r)u|u|^{q-2} + k^2(r)u|u|^{p-2} = 0 \quad (1.2)$$

Here  $'$  denotes the derivative with respect to  $r$ . We will call regular the solutions of (1.2) satisfying the following initial condition

$$u(0) = u_0 > 0 \quad u'(0) = 0 \quad (1.3)$$

and singular the solutions which are not well defined in the origin and such that  $\lim_{r \rightarrow 0} u(r) = +\infty$ . Ni and Serrin, in [17], have also proved that the autonomous equation where the constants  $k^i$  have opposite signs, does not admit any G.S. when  $q < m^* \leq p$  or  $q = m^* < p$ ; the proof is a direct consequence of the Pohozaev identity. In this paper we give a new proof of this fact and generalize it to the non autonomous problem. To be more precise we prove that any solution of the initial value problem (1.2) is positive and has positive lower bound, whenever  $k^1(r) \leq -d$  and  $k^2(r) \leq D$ , where  $d, D$  are positive arbitrary constants and  $k^1(r)$  and  $k^2(r)$  satisfy a monotony condition. This condition is closely related to the one proposed by Kawano, Yanagida and Yotsutani in [16], for the problem with one growth term. In this setting we are also able to prove the non existence of S.G.S. We also give a decay condition on  $K^1$  and a growth condition on  $K^2$ , that are sufficient for the existence of G.S. for Eq. (1.2). Concerning this kind of equation we also wish to quote [9], in which the authors state the existence of G.S. for the autonomous equation of the form (1.2) with  $q < p < m^*$ , and [4] in which it is studied the non-autonomous equation where  $k^1(r) \equiv -1$ , **and**  $k^2(r)$  is a positive function and  $m = q < p < m^*$ .

In the fourth section we use a similar approach to investigate positive solutions of the equation where both  $k^1(r)$  and  $k^2(r)$  are positive. In particular we prove the existence of G.S. for Eq. (1.2), when  $m^* \leq q < p$ , and the functions  $k^1(r)$  and  $k^2(r)$  satisfy certain monotonicity conditions. We also prove the existence of S.G.S., when  $m_* < q < p \leq m^*$  and  $k^1(r)$  and  $k^2(r)$  satisfy opposite monotony conditions. Here  $m_* = \frac{m(n-1)}{n-m}$  is the so called Serrin critical exponent. As a consequence of the techniques applied we also find precise estimates on the asymptotic behavior of positive radial solutions  $u(r)$  of Eq. (1.2), both as  $r \rightarrow 0$  and as  $r \rightarrow \infty$ , both in the case of two positive potentials and in the case of potentials with opposite signs. To be more precise we will prove that, as  $r \rightarrow 0$   $u(r)$  can only be bounded or behave like  $r^{-\alpha}$  where  $\alpha > 0$  is a constant that will be specified later. Moreover decaying solutions can only have two kinds of decay: fast decay, that is  $u(r) \sim r^{-\frac{n-m}{m-1}}$ , or slow decay, that is  $u(r) \sim r^{-\frac{m}{q-m}}$ . With the notation  $u(r) \sim r^{-K}$  as  $r \rightarrow a$  we mean that  $u(r)r^K$  has positive and

finite lower and upper bound as  $r \rightarrow a$ . So, for both the equations, we classify all the admissible S.G.S. of the problem.

The main tool introduced in this paper is a transform of Fowler type which establishes a bijective relationship between the solutions of (1.2) and those of a bidimensional non autonomous dynamical system, thus allowing us to reach a geometrical understanding of the behavior of the solutions. In this way we can apply techniques taken from the theory of dynamical systems to the problem, in particular the ones developed in [13] and in [14] for the problem obtained substituting the ordinary Laplacian  $\Delta$  for the  $m$ -Laplacian  $\Delta_m$  in (1.1). Moreover we have introduced a new energy function closely related to the Pohozaev identity which enables us to deal with the non autonomous setting.

The article is organized as follows: in section 2 we introduce the Fowler transform and give the sufficient condition for the existence of G.S of Eq. (1.2). In section 3 we deal with (1.2) assuming  $k^1(r) \leq -d$  and  $k^2(r) \leq D$ ; in section 4 we discuss the case of two positive potentials, finally in section 5 we prove some technical Lemmas concerning the asymptotic behaviour of the solutions. In section 6 we make some heuristic observations regarding equations with a  $m$ -Laplacian and some growth terms. In order to deal with positive functions we will use the following notation that will be in force throughout all the paper:

$$K^1(r) = |k^1(r)| \quad K^2(r) = |k^2(r)| = k^2(r)$$

We recall now two classical definitions which will be useful in the following sections. Given a system of the form

$$\dot{x} = f(x, t)$$

where  $f$  is Lipschitz continuous and a solution  $x(t)$ , the  $\alpha$ -limit set of  $x(t)$  is the set

$$A = \{P : \exists t_n \rightarrow -\infty \text{ such that } \lim_{n \rightarrow \infty} x(t_n) = P\},$$

while the  $\omega$ -limit set is the set

$$W = \{P : \exists t_n \rightarrow +\infty \text{ such that } \lim_{n \rightarrow \infty} x(t_n) = P\}.$$

One can show that, if  $x(t)$  is bounded on  $\mathbb{R}$ , then those sets are compact. Moreover if the system is autonomous these sets are invariant for the flow generated by the system.

## 2 Fowler transform and existence results

We begin by considering the case in which  $k^1(r)$  is negative; let us introduce the Fowler transform for Eq. (1.2):

$$\begin{aligned} x &= u(r)r^\alpha & y &= u'(r)|u'(r)|^{m-2}r^\beta & r &= e^t \\ h^1(t) &= K^1(e^t)e^{\delta t} = K^1(r)r^\delta & h^2(t) &= K^2(e^t)e^{\eta t} = K^2(r)r^\eta \end{aligned} \quad (2.1)$$

where

$$\alpha = \frac{n-m}{m}, \quad \beta = \frac{n(m-1)}{m}, \quad \delta = \alpha(m^* - q) \quad \eta = \alpha(m^* - p)$$

So Eq. (1.2) can be written as the following dynamical system

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} \alpha & 0 \\ 0 & -\alpha \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} y|y|^{\frac{2-m}{m-1}} \\ h^1(t)\psi_q(x) - h^2(t)\psi_p(x) \end{pmatrix} \quad (2.2)$$

where “.” denotes derivation with respect to  $t$  and  $\psi_l(s) = |s|^{l-2}s$ . To our knowledge this particular transform has not appeared previously in the literature.

**2.1 Remark. [Regularity hypothesis]** It is worthwhile to point out that Eq. (2.2) is  $C^1$  if and only if  $q \geq 2$  and  $1 < m \leq 2$ .

If this hypothesis is not satisfied the right hand side of the differential equation is not even Lipschitz when  $x = 0$  or  $y = 0$ , so that local uniqueness of the solutions on the  $x$  and  $y$  axes is not anymore ensured; thus our use of the term “dynamical system” is not quite rigorous. However we will see that most of the proofs in this paper can be adapted also to the situation in which this regularity hypothesis is not satisfied.

We give now a condition which ensures local existence for Eq. (1.2) with initial values (1.3). We assume that such a condition is in force throughout all the paper. There exists  $\nu < m$  such that the following condition is satisfied

$$\begin{cases} K^1(r) \text{ and } K^2(r) \text{ are continuous with their derivatives for any } r > 0, \\ \text{for every } R > 0, \sup\{r^{-\nu}K^1(r), \quad r^{-\nu}K^2(r) : 0 < r \leq R\} < \infty. \end{cases}$$

What is really needed in our analysis is that the functions  $K^i$  are just Lipschitz continuous. In fact we could substitute the weak derivative to the classical derivative through all the paper. However such regularity investigations are beyond the purposes of this paper so we will use the classical derivative.

Readapting to this setting Proposition (6.1) in [15] we can prove that the existence of solutions of Eq. (1.2) with initial values (1.3) is equivalent to the existence of fixed points of the operator  $T : \mathcal{C}(0, r) \rightarrow \mathcal{C}^1(0, r)$

$$Tu(r) = u_0 - \int_0^r \psi_m^{-1} \left( t^{1-n} \int_0^t f(u, s) s^{n-1} ds \right) dt \quad (2.3)$$

where  $f(u, r) = -K^1(r)u^q + K^2(r)u^p$  or  $f(u, r) = K^1(r)u^q + K^2(r)u^p$ . Using the Schauder's Theorem it is possible to prove that the operator  $T$  has a fixed point for any  $u_0 > 0$ . Furthermore, using the ideas of Proposition (6.1) in [15], we can prove local uniqueness of the solutions of (1.2), (1.3) when  $k^1 > 0$ , and  $q \geq 2$  or when  $k^1 < 0$  and the regularity Hypothesis is satisfied. If this conditions are not satisfied local uniqueness is not ensured.

*2.2 Remark.* When the regularity Hyp. is not satisfied, a priori Eq. (1.2) can have positive solutions  $u(r) \not\equiv 0$  such that  $u(0) = 0$ . For such solutions it can be easily shown that  $u'(r) > 0$  for  $r$  in a right neighborhood of 0. These solutions, if exist, have a behavior similar to regular solutions of Eq. (1.2). However they will not be considered in this paper.

We point out now some elementary correspondences between Eq. (1.2) and system (2.2).

*2.3 Remark.* Positive solutions  $u(r)$  of Eq. (1.2) correspond to trajectories of Eq. (2.2) belonging to the halfplane  $\mathbb{R}_+^2 := \{(x, y) \mid x \geq 0\}$ . Furthermore decreasing solutions  $u(r)$  of Eq. (1.2) correspond to trajectories of Eq. (2.2) belonging to the 4<sup>th</sup> quadrant and viceversa.

**2.4 Proposition.** *Assume that  $\lim_{t \rightarrow -\infty} h^1(t) = A \geq 0$  and  $\lim_{t \rightarrow -\infty} h^2(t) = B > 0$ . Trajectories  $X(t)$  of Eq. (2.2) such that  $\lim_{t \rightarrow -\infty} X(t) = O = (0, 0)$  correspond to regular solutions  $u(r)$  of Eq. (1.2) and viceversa.*

*Trajectories  $X(t)$  of Eq. (2.2), which are well defined and belong to  $\mathbb{R}_+^2$  for  $t$  large, and satisfying  $\lim_{t \rightarrow \infty} X(t) = O$  correspond to solutions  $u(r)$  of Eq. (1.2) which are well defined and positive for  $r$  large and have fast decay that is  $u(r) = o(r^{-\frac{n-m}{m-1}})$ , and viceversa.*

The proof is rather technical so it is postponed to section 5, Lemmas (5.2), (5.3) and (5.6). Note that if the regularity hypothesis is satisfied, using invariant manifold theory for non-autonomous system, see [11] and [12], we can prove the existence of an unstable and a stable manifold:

$$W^u := \{P \mid \lim_{t \rightarrow -\infty} X_t(P; t) = 0\}, \quad W^s := \{P \mid \lim_{t \rightarrow \infty} X_t(P; t) = 0\},$$

where  $X_t(P; t)$  is the solution of Eq. (2.2) or (2.10) passing through  $P$  at  $t = 0$ . Therefore this observation together with Proposition (2.4) gives a proof for the existence and local uniqueness of regular solutions of Eq. (1.2). Furthermore when the regularity hypothesis is not satisfied, the existence of solutions of (1.2) with initial values (1.3), ensures the existence of an unstable set for the dynamical system (2.2), which was not a priori clear.

*2.5 Remark.* We will say that a positive solution  $u(r)$  has fast decay whenever  $u(r)r^{\frac{n-m}{m-1}}$  is bounded as  $r \rightarrow \infty$ , and that it has slow decay whenever  $\lim_{r \rightarrow \infty} u(r)r^{\frac{n-m}{m-1}} = +\infty$ . A priori fast decay solutions may have compact support. It is worthwhile to point out that, if  $p \leq m$ , compact support solutions of (1.2) do exist in the autonomous case, see Proposition 1.3.2 in [9]. However when the regularity hypothesis is satisfied, compact support solutions cannot exist, since trajectory of (2.2) cannot reach the origin for  $t$  finite. **If the regularity Hypothesis is not satisfied and the functions  $K^i(r)$  are unbounded, as in Proposition (2.8), we cannot say whether fast decay solutions have compact support or not.**

The rate of decay of slowly decaying solutions depends on the functions  $K^i$  and the parameters  $q, p$  and will be specified in each case.

Now we define an auxiliary function which will play a crucial role in the following analysis:

$$H(x, y, t) := \alpha xy + \frac{m-1}{m} |y|^{\frac{m}{m-1}} - \frac{h^1(t)}{q} |x|^q + \frac{h^2(t)}{p} |x|^p.$$

First of all we observe that by differentiating we get

$$\frac{d}{dt} H(x(t), y(t), t) = -\frac{dh^1(t)}{dt} \frac{|x|^q}{q} + \frac{dh^2(t)}{dt} \frac{|x|^p}{p} \quad (2.4)$$

We begin by making some strong assumptions (which will be weakened later on), in order to simplify the situation. Let us assume at first

$$h^1(t) \equiv A > 0 \quad h^2(t) \equiv B > 0 \quad (2.5)$$

If hypothesis (2.5) is satisfied then (2.2) becomes the following autonomous system

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} \alpha & 0 \\ 0 & -\alpha \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} y|y|^{\frac{2-m}{m-1}} \\ A\psi_q(x) - B\psi_p(x) \end{pmatrix} \quad (2.6)$$

Let us fix  $A, B > 0$ ; in such a case the function  $H$  does not depend on  $t$ , thus

$$H(x, y) := \alpha xy + \frac{m-1}{m} |y|^{\frac{m}{m-1}} - \frac{A}{q} |x|^q + \frac{B}{p} |x|^p.$$

*2.6 Observation.* Eq. (2.6) admits exactly 3 critical points, which coincides with the critical points of  $H$ : the origin  $O \equiv (0; 0)$ ,  $P \equiv (P_x, P_y)$  and  $-P$  where  $P_x > 0$  and  $P_y < 0$ .

Consider the level sets of the function  $H$   $C_b := \{(x, y) \mid H(x, y) = b\}$ . We claim that  $\lim_{|x|+|y| \rightarrow \infty} H(x, y) = \infty$ . In fact assume for contradiction that there is a sequence  $(x_n, y_n)$  such that  $|x_n| + |y_n| \rightarrow \infty$  such that  $H(x_n, y_n) \rightarrow M < \infty$ . Observing that  $p > q > 1$  we get that  $|x_n| \rightarrow \infty$  implies  $|y_n| \rightarrow \infty$ . Analogously from  $m/(m-1) > 1$  we get that  $|y_n| \rightarrow \infty$  implies  $|x_n| \rightarrow \infty$ ; so we can assume that both  $|x_n|$  and  $|y_n|$  are unbounded. We can assume without losing of generality that  $x_n > 0 > y_n$ . Note that for  $n$  large we have

$$x_n(\alpha y_n + \frac{B}{p} |x_n|^{p-1}) + \frac{m-1}{m} |y_n|^{\frac{m}{m-1}} < 2M$$

(we have neglected the contribution of the term  $|x|^q$  which is small with respect to  $|x|^p$ ). Thus  $|y_n| > \frac{B}{\alpha p} |x_n|^{p-1}$ . Analogously

$$y_n(\alpha x_n - \frac{m-1}{m} |y_n|^{\frac{1}{m-1}} + \frac{B}{p} |x_n|^{p-1}) < 2M$$

Therefore using the previous estimate we get the following

$$\alpha x_n > \frac{m-1}{m} |y_n|^{\frac{1}{m-1}} > \frac{(m-1)B^{1/(m-1)}}{\alpha^{1/(m-1)} p^{1/(m-1)}} |x_n|^{(p-1)/(m-1)}.$$

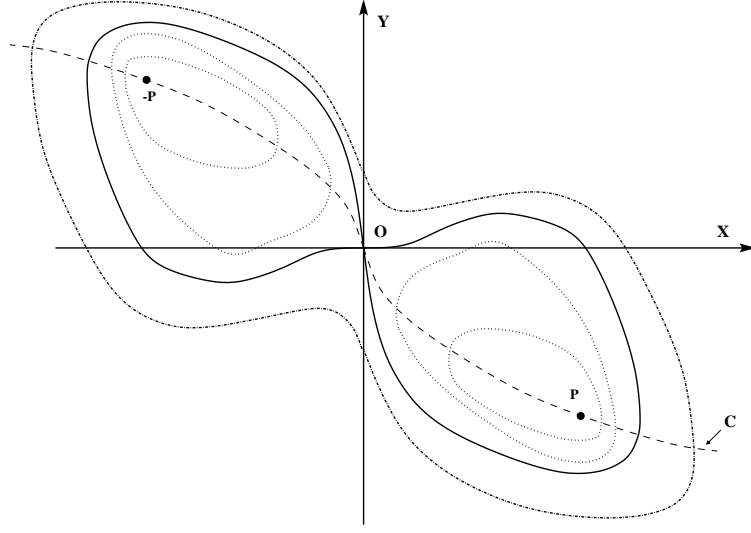


Figure 1: Sketch of the level sets of the function  $H(x_1, x_2, T)$ , for  $T$  fixed. The solid line is the level set  $C_0$ , the dotted and the dashed lines represent some level sets  $C_b$  where respectively  $b < 0$  and  $b > 0$ .

But this is a contradiction, since  $p > m$ , so the claim is proved.

Note that  $H(0) = 0$  and  $H(P) = H(-P) = -b^* < 0$ . Therefore  $-b^*$  is the minimum for  $H$ . We collect some information concerning the shape of the level sets  $C_b$  in the following Lemma, see also Fig. (1).

**2.7 Lemma.** *Each level set  $C_b$  for  $-b^* < b < 0$  is made up by two closed bounded curves symmetric with respect to the origin. One of them is contained in  $\mathbb{R}_+^2$ . The level set  $C_0$  is made up by the union of two closed curves connected by the origin. The level sets  $C_b$  for  $b > 0$  are closed bounded curves which cross the coordinate axes.*

Note that when the assumption (2.5) is satisfied, “ $H$ ” represents a first integral of (2.6), so we are able to draw each trajectory of the system. We are ready to state now a classification result for positive solutions.

**2.8 Proposition.** *Consider Eq. (1.2) where (2.5) is satisfied, and the corresponding autonomous system of the form (2.6). Then we can give the following classification result for positive solutions.*

- A *All the trajectories corresponding to some positive value  $H(x, y) = b > 0$  are periodic and cross the axis. They correspond to singular solutions  $u(r)$  of (1.2) with infinitely many positive maxima and negative minima; moreover there exists  $a > 0$  such that  $-ar^{-\alpha} \leq u(r) \leq ar^{-\alpha} \quad \forall r > 0$ .*
- B *The trajectory corresponding to  $H(x, y) = 0$  is homoclinic to the origin; this means that all the regular solutions  $u(r)$  of (1.2) are G.S., each of*



which has exactly one critical point which is a maximum, and with decay rate  $o(r^{-\frac{n-m}{m-1}})$  at  $\infty$  (fast decay).

*C* All the trajectories corresponding to some negative value  $H(x_1, x_2) = -b > H(P)$  are periodic and belong to  $\mathbb{R}_+^2$ . They represent S.G.S.  $u(r)$  of Eq. (1.2) with rate of decay and growth  $\sim r^{-\alpha}$  respectively at  $\infty$  and at 0. There exists  $-\bar{b} < 0$  such that if  $-b^* < H(x, y) \leq -\bar{b}$ , the corresponding S.G.S. is monotone decreasing, while if  $-\bar{b} < H(x, y) < 0$  it has infinitely many maxima and minima (that corresponds to the trajectory crossing the  $x$  axis).

*D* For the value  $H = H(P)$  we have one fixed point  $P = (P_x, P_y)$ , which corresponds to a monotone decreasing S.G.S of (1.2) of the form  $u(r) = P_x r^{-\alpha}$  where we recall that  $P_x = P_x(A, B)$ .

**2.9 Remark.** No other solutions  $u(r)$  positive in a right neighborhood of  $r = 0$  can exist but the ones described and G.S. with  $u(0) = 0$ , see remark (2.2). We recall that, if the regularity Hyp. are not satisfied, a fast decay solution may have compact support. Correspondingly the homoclinic trajectory may reach the origin in finite  $t$ .

*Proof.* The proof of the Proposition easily follows from Lemma (2.7) and Proposition (2.4).  $\square$

Now we enumerate some Hypotheses that will be used in this and in the following sections. For the definition of  $h_l^1$  and  $h_l^2$  see (2.9).

### Hypotheses

**Mon**  $\dot{h}^1(t) \geq 0$  and  $\dot{h}^2(t) \leq 0$  and one of the inequalities is strict for a certain  $t$ .

**M1** Both  $h^1(t)$  and  $h^2(t)$  are monotone for  $t \rightarrow -\infty$ ;  $\lim_{t \rightarrow -\infty} h^1(t) = A \geq 0$  and  $\lim_{t \rightarrow -\infty} h^2(t) = B > 0$ .

**M2** Both  $h^1(t)$  and  $h^2(t)$  are monotone for  $t \rightarrow \infty$ ;  $\lim_{t \rightarrow \infty} h^1(t) = A \geq 0$  and  $\lim_{t \rightarrow \infty} h^2(t) = B > 0$ .

**M3** There exists  $l > m_*$  but  $l \neq m^*$ , such that  $\lim_{t \rightarrow -\infty} h_l^1(t) = A \geq 0$  and  $\lim_{t \rightarrow -\infty} h_l^2(t) = B > 0$ . Furthermore for a certain  $\xi > 0$

$$\lim_{r \rightarrow 0} (r \frac{dK^1}{dr} + \delta_l K^1) r^{\delta_l + \xi - 1} = 0 \quad \text{and} \quad \lim_{r \rightarrow 0} (r \frac{dK^2}{dr} + \eta_l K^2) r^{\eta_l + \xi - 1} = 0$$

**M4** There exists  $l > m_*$  but  $l \neq m^*$ , such that  $\lim_{t \rightarrow \infty} h_l^1(t) = A \geq 0$  and  $\lim_{t \rightarrow \infty} h_l^2(t) = B > 0$ . Furthermore for a certain  $\xi < 0$

$$\lim_{r \rightarrow \infty} (r \frac{dK^1}{dr} + \delta_l K^1) r^{\delta_l + \xi - 1} = 0 \quad \text{and} \quad \lim_{r \rightarrow \infty} (r \frac{dK^2}{dr} + \eta_l K^2) r^{\eta_l + \xi - 1} = 0$$

**2.10 Theorem.** *Consider Eq. (1.2); assume that Hypotheses Mon and M2 are satisfied.*

- *Then all the regular solutions  $u(r)$  of (1.2) are G.S. with slow decay that is  $u(r) \sim r^{-\alpha}$  as  $r \rightarrow \infty$ .*
- *For any G.S.  $u(r)$ , there exists a S.G.S.  $v(r)$ , solution of (1.2) where  $h^1(t) \equiv A > 0$ , and  $h^2(t) \equiv B > 0$ , such that:*

$$\lim_{r \rightarrow \infty} [u(r) - v(r)]r^\alpha = 0.$$

**2.11 Remark.** The hypothesis of the Theorem are satisfied if we assume  $p = m^*$  and that  $K^2(r)$  is strictly positive and monotone decreasing, and for example  $K^1(r) = \frac{1}{1+r^\delta}$ .

*Proof.* Suppose at first that  $\lim_{t \rightarrow -\infty} h^2(t) < \infty$  and that  $\lim_{t \rightarrow -\infty} h^1(t) > 0$ . Using Proposition (2.4), we know that trajectories  $\hat{X}(t) = (\hat{x}(t), \hat{y}(t))$  of (2.2) corresponding to regular solutions  $\hat{u}(r)$  of (1.2), are such that  $\lim_{t \rightarrow -\infty} H(\hat{x}(t), \hat{y}(t), t) = 0$ . From Lemma (2.7) it follows that the sets  $D(t) := \{(x, y) \mid x > 0, H(x, y, t) < 0\}$  are open bounded subset, for any  $t$ . Since  $\sup_{t \in \mathbb{R}} h^1(t) < \infty$  and  $\inf_{t \in \mathbb{R}} h^2(t) > 0$ , it follows that the sets  $D(t)$  are uniformly bounded with respect to  $t$ .

#### controlla

Hypothesis Mon implies that  $H$  is decreasing along the flow, so we know that, for any  $t$ , the trajectory  $\hat{X}(t)$  is forced to stay in  $D(t)$ , so it cannot cross the  $y$  axis; this means that no crossing solutions  $u(r)$  of (1.2) can exist. Then observing that  $\hat{X}(t)$  is bounded and bounded away from the  $y$  axis for any  $t$ , we deduce that  $u(r) \sim r^{-\alpha}$  as  $r \rightarrow \infty$ , so  $\hat{u}(r)$  is a G.S. with slow decay.

To prove the second claim we need to rewrite (2.2) as an autonomous system by introducing an extra variable  $\tau = r^{-1} = e^{-t}$ .

$$\begin{pmatrix} \dot{x} \\ \dot{y} \\ \dot{\tau} \end{pmatrix} = \begin{pmatrix} \alpha & 0 & 0 \\ 0 & -\alpha & 0 \\ 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \\ \tau \end{pmatrix} + \begin{pmatrix} y|y|^{\frac{2-m}{m-1}} \\ h^1(-\ln \tau)\psi_q(x) - h^2(-\ln \tau)\psi_p(x) \\ 0 \end{pmatrix} \quad (2.7)$$

First of all we observe that the plane  $\tau = 0$  is invariant for the flow and that the  $\omega$ -limit set of each bounded trajectory of (2.2) has to belong to this plane.

Consider again the trajectory  $\hat{X}(t) = (\hat{x}(t), \hat{y}(t), \hat{\tau}(t))$  corresponding to  $u(r)$ , and let  $\Omega$  be its  $\omega$ -limit set. Then  $\Omega$  lies in the plane  $\tau = 0$ . Moreover  $\Omega$  is invariant with respect to the flow on the  $x-y$  plane defined by Eq. (2.6), where  $A = \lim_{t \rightarrow \infty} h^1(t)$  and  $B = \lim_{t \rightarrow \infty} h^2(t)$ . Now recalling Propositions (2.8) we observe that exists a solution of (2.6) for any given admissible negative value of  $H$ . We observe now that the limit  $\lim_{t \rightarrow +\infty} H(\hat{x}(t), \hat{y}(t), t)$  exists since  $H$  is decreasing and bounded below by the value of  $H$  at the critical point  $P$  of Eq. (2.6). Therefore we can conclude that each trajectory of (2.2) converges to one of the periodic trajectory of Eq. (2.6) or to the critical point  $P$ . This proves the claim.

Now assume that  $\lim_{t \rightarrow -\infty} h^1(t) = 0$  or  $\lim_{t \rightarrow -\infty} h^2(t) = \infty$ . Then we observe that the sets  $D(t)$  continues to be bounded for any given  $t$ , but we do not know anything about  $\lim_{t \rightarrow -\infty} H(0, 0, t)$ .

Let us consider a trajectory  $(x(t), y(t))$  of (2.2) corresponding to a regular solution  $u(r)$  of (1.2). Fix  $T$  we define

$$\tilde{H}_T = \begin{cases} H(x(t), y(t), T) & \text{if } t < T \\ H(x(t), y(t), t) & \text{if } t \geq T \end{cases} \quad (2.8)$$

Differentiating with respect to  $t$  we get

$$\frac{d}{dt} \tilde{H}_T = \begin{cases} -[h^1(T) - h^1(t)]\dot{x}|x|^{q-2} + [h^2(T) - h^2(t)]\dot{x}|x|^{p-2} & \text{if } t < T \\ \frac{d}{dt} H(x(t), y(t), t) & \text{if } t \geq T \end{cases}$$

Now observe that for any given  $(x(t), y(t))$  corresponding to a solution  $u(r)$  of (1.2), (1.3) we can choose  $\tilde{T} = \tilde{T}(u_0)$  such that  $\dot{x} \geq 0$  for  $t \leq \tilde{T}$ .

In fact otherwise there would exist a sequence  $t_n = \log(r_n) \rightarrow -\infty$  such that  $\dot{x}(t_k) < 0$ , which implies

$$\exists r_k \rightarrow 0 \quad s.t. \quad \alpha u(r_k) < -u'(r_k)r_k.$$

Therefore for  $n \rightarrow \infty$  we get  $\alpha u(0) \leq 0$ , a contradiction. Note that our definition of  $\tilde{H}$  depends on  $u_0$ ; however, given  $u_0$ , the corresponding sets

$$\tilde{D}(t) := \{(x, y) \mid \tilde{H}_T(x(t), y(t), t) \leq 0\},$$

are uniformly bounded with respect to  $t$ . Since we have assumed that  $-h^1(t)$  and  $h^2(t)$  are decreasing, we have that

$$\frac{d}{dt} \tilde{H}(x(t), y(t), t) \leq 0 \quad \text{and} \quad \lim_{t \rightarrow -\infty} \tilde{H}_T(x(t), y(t), t) = \tilde{H}_T(0, 0, t) = 0.$$

Therefore the solution of (2.2) is forced to stay in the set  $\tilde{D}(t)$  for any  $t$ ; so it represents a G.S. with slow decay  $u(r)$  of (1.2).  $\square$

We prove now two Lemmas concerning the forward and backwards continuability of the trajectories of (2.2). Note that we allow the potentials  $K^i(r)$  to tend to 0 or to  $\infty$  as  $r \rightarrow \infty$ .

**2.12 Lemma.** *Consider Eq. (1.2) and assume that Hypothesis Mon is satisfied. Then any solution  $u(r)$  can be continued forward in  $r$  for any  $r > 0$ .*

*Proof.* Consider a trajectory  $X(t) = (x(t), y(t))$ ; we want to show that it can be continued forward in  $t$ , for any  $t$ . Assume for contradiction that there exists  $T < \infty$  such that that  $X(t)$  is unbounded as  $t \rightarrow T^-$ ; then there exists a sequence  $t_n \rightarrow T^-$  such that  $\lim_{n \rightarrow \infty} H(x(t_n), y(t_n), t_n) = +\infty$ . Observe now that, from Hyp. Mon, it follows that  $H(x(t), y(t), t)$  is decreasing for any  $t$ , so we have found a contradiction. Therefore any solution of (1.2), regular or singular is continuable forward for any  $r > 0$ .  $\square$

We introduce now some notation for system (2.7)

$$c := \{(x_l, y_l, \tau) \mid \dot{x}_l = 0\}$$

$$U^+ := \{(x_l, y_l, \tau) \mid \dot{x}_l > 0\} \quad \text{and} \quad U^- := \{(x_l, y_l, \tau) \mid \dot{x}_l < 0\}.$$

We will use the same notation also for the bi-dimensional systems obtained by removing the variable  $\tau$ , and for system (2.10) to be introduced below.

**2.13 Lemma.** *Consider Eq. (2.2) and assume that Hypothesis Mon is satisfied. If a trajectory  $\hat{X}(t) = (\hat{x}(t), \hat{y}(t))$  is unbounded backwards in  $t$ , then it must cross the coordinate axes indefinitely many times.*

*Proof.* We consider system (2.2) and we want to follow  $\hat{X}(t)$  backwards in  $t$ . From (2.4) we know that  $H(\hat{X}(t), t)$  is monotone decreasing with respect to  $t$ ; therefore there exists  $T$  such that  $0 < M/2 < H(\hat{X}(t), t) < M$ . We denote the level sets  $H(\cdot, T) = 0$  and  $H(\cdot, T) = M$  of the function  $H(\cdot, T)$  by  $C_0(T)$  and  $C_M(T)$ . We call  $E(t)$  the open bounded subset enclosed by  $C_0(t)$ , for  $t \leq T$  and  $C_M(T)$ . We recall that  $D(T) = \{(x, y) \mid H(x, y, T) < 0\}$ . Observe that  $\hat{X}(T)$  is contained in  $E(T)$ ; assume that  $\hat{X}(T) \in U^-$ . When  $t < T$  we have that  $h^1(t) \leq h^1(T)$  and  $-h^2(t) \leq -h^2(T)$  therefore the flow on  $C_M(T) \cap U^-$  is always going towards the exterior of  $E(t)$ . Furthermore the flow on  $C_0(t)$  is always going towards the exterior of  $E(t)$  for any  $t$ . Therefore following  $\hat{X}(t)$  backwards in  $t$  we find that it is forced to stay in  $D(t) \cap U^-$  until it crosses the isocline  $c$ . Since in  $D(t) \cap U^-$   $\dot{x}$  and  $\dot{y}$  are uniformly positive, there exists a value  $t = t_1 < T$  finite such that  $\hat{X}(t_1)$  crosses  $c$ .

Now we want to prove that there exists  $t_3$  such that  $\hat{X}(t_3)$  crosses the positive  $y$  semi axis. Assume for contradiction that  $\hat{X}(t) \in \mathbb{R}_+^2$  for any  $t < t_1$ . We can assume that  $\hat{X}(t) \in U^+$ , for  $t < t_1$  until it crosses the isocline  $c$ . Note that  $H(\hat{X}(t), t) > 0$ , for  $t \leq t_1$ , therefore, analyzing Fig. (1), we see that the trajectory  $\hat{X}(t)$  cannot cross the isocline  $c$  while it is in  $\mathbb{R}_+^2$ , thus  $\hat{x}(t) < \hat{x}(t_1)$  for  $t < t_1$ .

Observe that  $|\dot{H}(\hat{X}(t), t)|$  is finite for  $t$  finite. Therefore  $H(\hat{X}(t), t)$  is bounded for  $t$  finite, which implies that  $\hat{X}(t)$  is bounded. Therefore  $\hat{X}(t)$  can be continued backwards in  $t$  for any  $t$ .

Observing that  $\hat{x}(t_1) < \infty$  and that  $\frac{dx}{dt}$  is strictly positive for  $t < t_1 - \epsilon$  we have that there exists  $t_3$  such that  $\hat{x}(t_3) = 0$ , which is a contradiction. Thus  $\hat{X}(t)$  crosses the positive  $y$  semi axis, for  $t = t_3$ . For continuity reasons there exists  $t_2$  such that  $t_3 < t_2 < t_1$  for which  $\hat{X}(t)$  crosses the  $x$  positive semi axis.

Now we prove that there exists  $t_4 < t_3$  such that  $\hat{X}(t_4) \in c$  and  $\hat{x}(t_4) < 0$ . Consider the function  $\tilde{H}_{t_3}(\hat{X}(t))$  obtained setting  $T = t_3$  in (2.8). Note that  $\frac{\partial}{\partial t} \tilde{H}_{t_3}(\hat{X}(t)) \geq 0$  for  $t < t_3$ , until  $\hat{X}(t) \in U^+$  and  $\hat{x}_1(t) < 0$ . Therefore  $\hat{X}(t)$  is bounded and can be continued backwards until it crosses the isocline  $c$ . Since  $H(\hat{X}(t), t) > 0$  for  $t < t_3$ ,  $\hat{X}(t)$  is bounded away from the critical point, therefore  $\hat{X}(t)$  crosses the isocline for some finite  $t = t_4$ . Iterating the reasoning it is now possible to prove that, if  $\hat{X}(t)$  becomes unbounded going backwards in  $t$ , then it must cross the coordinate axes indefinitely many times.  $\square$

We wish to point out that we have also implicitly proved the following Lemma.

**2.14 Lemma.** *Assume that Hypothesis Mon is satisfied and consider a trajectory  $\hat{X}(t) = (\hat{x}(t), \hat{y}(t))$  of Eq. (2.2). If there exists  $t$  such that  $H(\hat{X}(t), t) > 0$ , then  $\hat{X}(t)$  must cross the coordinate axes backwards in  $t$  indefinitely many times.*

Now we need to introduce a new transform, a bit more general than the one already used.

$$\begin{aligned} \alpha_l &= \frac{m}{l-m}, & \beta_l &= \frac{(m-1)l}{l-m}, & \gamma_l &= \beta_l - (n-1), & l > m \\ x_l &= u(r)r^{\alpha_l} & y_l &= u'(r)|u'(r)|^{m-2}r^{\beta_l} & r &= e^t \\ \phi^1(t) &= K^1(e^t) = K^1(r), & \phi^2(t) &= K^2(e^t) = K^2(r), \\ h_l^1(t) &= \phi^1(t)e^{\delta_l t}, & h_l^2(t) &= \phi^2(t)e^{\eta_l t} \end{aligned} \quad (2.9)$$

where  $\delta_l = \alpha_l(l-q) = m\left(1 - \frac{q-m}{l-m}\right)$ , and  $\eta_l = \alpha_l(l-p) = \left(1 - \frac{p-m}{l-m}\right)$ . This new transform enables us to control the growth of the functions  $h_l^1(t)$  and  $h_l^2(t)$  when  $t \rightarrow \pm\infty$ . Observe that if we set  $l = m^*$  we obtain again the change of variables (2.1). From now on we will denote the quantities obtained through the change of variables (2.9) with  $l = \bar{l}$  with the subscript  $\bar{l}$ ; when we refer to (2.9) with  $l = m^*$  we omit the subscript.

So we will denote by  $\hat{x}(t)$  a trajectory of (2.2), by  $\hat{x}_l(t)$  the corresponding trajectory of (2.10) and by  $\hat{u}(r)$  the corresponding solution of (1.2).

Note that if we set  $l = q$  we obtain  $h_l^1(t) = \phi^1(t)$ , and with  $l = p$  we have  $h_l^2(t) = \phi^2(t)$ . Applying (2.9) to (1.2) we obtain the following dynamical system:

$$\begin{pmatrix} \dot{x}_l \\ \dot{y}_l \end{pmatrix} = \begin{pmatrix} \alpha_l & 0 \\ 0 & \gamma_l \end{pmatrix} \begin{pmatrix} x_l \\ y_l \end{pmatrix} + \begin{pmatrix} y_l |y_l|^{\frac{2-m}{m-1}} \\ h_l^1(t)\psi_q(x_l) - h_l^2(t)\psi_p(x_l) \end{pmatrix} \quad (2.10)$$

We will also make use of the following dynamical system where we have added the extra variable  $\tau = e^{\xi t}$ , which is useful to analyze the asymptotic behaviours:

$$\begin{pmatrix} \dot{x}_l \\ \dot{y}_l \\ \dot{\tau} \end{pmatrix} = \begin{pmatrix} \alpha_l & 0 & 0 \\ 0 & \gamma_l & 0 \\ 0 & 0 & \xi \end{pmatrix} \begin{pmatrix} x_l \\ y_l \\ \tau \end{pmatrix} + \begin{pmatrix} y_l |y_l|^{\frac{2-m}{m-1}} \\ h_l^1(t)\psi_q(x_l) - h_l^2(t)\psi_p(x_l) \\ 0 \end{pmatrix} \quad (2.11)$$

We wish now to stress which are the signs of a constant which will be useful later on.

$$\alpha_l + \gamma_l \begin{cases} > 0 & \text{if and only if } l < m^*, \\ = 0 & \text{if and only if } l = m^*, \\ < 0 & \text{if and only if } l > m^*. \end{cases}$$

We state another Proposition concerning the asymptotic behaviour of positive solutions, which generalizes Proposition (2.4). Once again the proof follows from Lemmas (5.2), (5.3) and (5.6), so it is postponed to section 5.

**2.15 Proposition.** *Assume that Hypothesis Mon and M3 are satisfied. Trajectories  $X(t)$  of Eq. (2.10) such that  $\lim_{t \rightarrow -\infty} X(t) = O$  correspond to regular solutions  $u(r)$  of Eq. (1.2) and viceversa.*

*Assume that Hypothesis Mon and M4 are satisfied. Trajectories  $X_l(t)$  of Eq. (2.10), which are well defined and belong to  $\mathbb{R}_+^2$  for  $t$  large, and satisfying  $\lim_{t \rightarrow \infty} X(t) = O$  correspond to solutions  $u(r)$  of Eq. (1.2) which are well defined and positive for  $r$  large and have fast decay that is  $u(r) = o(r^{-\frac{n-m}{m-1}})$ , and viceversa.*

We introduce now another auxiliary function:

$$\begin{aligned} H_l(x_l(t), y_l(t), t) &:= \frac{n-m}{m} x_l y_l + \frac{m-1}{m} |y_l|^{\frac{m}{m-1}} - \frac{h^1(t)}{q} |x_l|^q + \frac{h^2(t)}{p} |x_l|^p = \\ &= H(x(t), y(t), t) e^{(\alpha_l + \gamma_l)t}. \end{aligned}$$

Observe that differentiating we get

$$\frac{d}{dt} H_l(x_l(t), y_l(t), t) = (\alpha_l + \gamma_l) H_l(x_l(t), y_l(t), t) + e^{(\alpha_l + \gamma_l)t} \frac{d}{dt} H(x(t), y(t), t).$$

Assume that there exists  $l > m_*$ , such that  $h_l^1(t) \equiv A \geq 0$  and  $h_l^2(t) \equiv B > 0$ . Then system (2.10) admits exactly three critical points: the origin,  $P = (P_x, P_y)$  where  $P_y < 0 < P_x$ , and  $-P$ . Moreover, using Poincare-Bendixson criterion we can prove the following.

**2.16 Observation.** System (2.10), admits no periodic trajectories whenever  $h_l^1(t) \equiv A$  and  $h_l^2(t) \equiv B > 0$ .

*Proof.* Note that  $\frac{d\dot{x}}{dx} + \frac{d\dot{y}}{dy} = \alpha + \gamma > 0$ . Assume for contradiction that there exists a periodic trajectory  $X(t)$  of period  $T$ , and call its graph  $\partial B$  and  $B$  the bounded set enclosed by  $\partial B$ . Then

$$0 = \int_0^T (\dot{x}\dot{y} - \dot{y}\dot{x}) dt = \int_{\partial B} \dot{x} dy - \dot{y} dx = \int_B \left( \frac{d\dot{x}}{dx} + \frac{d\dot{y}}{dy} \right) dx dy > 0.$$

So we have found a contradiction and the claim is proved.  $\square$

**2.17 Proposition.** *Assume that either Hyp. M1 or M3 is satisfied. Then there exists at least one singular positive solutions  $v(r)$ , that is a solution  $v(r)$  which is well defined and positive in a right neighborhood of  $r = 0$  and behaves like  $\sim r^{-\frac{m}{l-m}}$  as  $r \rightarrow 0$ .*

*Furthermore assume that the regularity Hyp. and M3 are satisfied and that  $l$  is such that  $m_* < l < m^*$ , then  $v(r)$  is the unique singular solution, so if  $u(r)$  is positive in a right neighborhood of  $r = 0$ , then  $u(r)$  is regular or  $u(r) = v(r)$ .*

*Proof.* Assume that the regularity Hyp. is satisfied and consider system (2.11) where we set  $l$  and  $\xi > 0$  as in Hyp. M3. Assume  $m_* < l < m^*$ . Using Lemma (2.13) we deduce that trajectory of (2.11) which are unbounded backwards in  $t$  cannot correspond to positive solutions  $u(r)$  of (1.2). Note that the  $\alpha$ -limit

set of any bounded trajectory is contained in the plane  $\tau = 0$ . The technical Hypotheses on the functions  $K^{i_l}(r)$  ensures that system (2.11) is Lipschitz continuous when  $\tau = 0$ . From Lemma (2.16) we know that the plane  $\tau = 0$  does not contain any periodic trajectory. Thus bounded solutions of system (2.11) must have  $P$  the origin or  $-P$  as  $\alpha$ -limit set. Therefore the corresponding solutions  $v(r)$  are respectively singular and  $v(r) \sim r^{-\alpha_l}$  as  $r \rightarrow 0$ , regular, or negative as  $r \rightarrow 0$ .

By a straightforward computation we deduce that  $P$  admits a two dimensional stable manifold and a one dimensional unstable manifold. The latter one is transversal to the plane  $\tau = 0$ , so it is made up exactly by one trajectory  $\hat{X}_l(t)$  such that  $\lim_{t \rightarrow -\infty} \hat{X}_l(t) = P$ . This trajectory corresponds to a singular positive solution  $\hat{v}(r)$  such that  $\hat{v}(r) \sim r^{-\alpha_l}$  as  $r \rightarrow 0$ , and no other solutions  $u(r)$  of (1.2) which are positive as  $r \rightarrow 0$  can exist but regular solutions and  $v(r)$ .

If  $l > m^*$  the unstable manifold is 2-dimensional, therefore we lose the uniqueness result. However we can still find at least one trajectory like  $\hat{X}_l(t)$ . Note that for the existence result we do not need the regularity Hyp., since  $P$  is far from the coordinate axes.  $\square$

### 3 Non existence results

In this section we give some non existence result for crossing solutions, G.S. and S.G.S. in a more general setting.

**3.1 Theorem.** *Consider Eq. (1.2) and assume that there are positive constants  $d, D > 0$  such that  $K^1(r) > d$  and  $K^2(r) < D$  for  $r$  large, and that Hypotheses Mon is satisfied. Then all the solutions  $u(r)$  of (1.2) can be continued for any  $r > 0$  and are always strictly positive, thus no crossing solutions can exist. Moreover no G.S nor S.G.S. can exist either.*

*Proof.* Consider any solution  $\bar{v}(r)$  of (1.2) such that  $\lim_{r \rightarrow \infty} \bar{v}(r) = 0$ . Suppose at first that  $\lim_{r \rightarrow \infty} K^1(r)r^{-\frac{n-m}{m-1}} < \infty$ ; we can choose  $l \leq m^*$  in such a way that  $h_l^1(t)$  and  $h_l^2(t)$  are bounded as  $t \rightarrow \infty$ . For example if  $K^1(r)$  is bounded we can choose  $l = q$ . Using Lemma (5.2) we deduce that  $v(r)$  corresponds to a trajectory  $\bar{X}_l(t) = (\bar{x}_l(t), \bar{y}_l(t))$  of (2.10) such that  $\lim_{t \rightarrow \infty} \bar{X}_l(t) \rightarrow (0, 0)$ . Thus  $\lim_{t \rightarrow \infty} H_l(\bar{x}_l(t), \bar{y}_l(t), t) = 0$  and, since  $\alpha_l + \gamma_l > 0$ , we also have  $\lim_{t \rightarrow \infty} H(\bar{x}_l(t), \bar{y}_l(t), t) = 0$ . Therefore for the corresponding trajectory  $(\bar{x}(t), \bar{y}(t))$  obtained setting  $l = m^*$  in (2.9) we have  $H(\bar{x}(t), \bar{y}(t), t) > 0$  for any  $t$ .

Following this trajectory backwards in  $t$ , using Lemma (2.14) we deduce that  $(\hat{x}(t), \hat{y}(t))$  has to cross the coordinate axes indefinitely many times, therefore  $\bar{v}(r)$  is an oscillatory solution.

Now we assume that  $K^1(r)$  grows like  $r^{\frac{n-m}{m-1}}$  or faster. Consider again a decaying solution  $\hat{v}(r)$  and the corresponding trajectory  $(\hat{x}(t), \hat{y}(t))$  obtained through (2.9) with  $l = m^*$ . We want to show that there exists  $T$  such that  $H(\hat{x}(T), \hat{y}(T), T) > 0$  and then conclude with Lemma (2.14). From Lemma

(5.2) we know that  $\lim_{t \rightarrow \infty} \hat{x}(t) = 0$ , and it can be proved easily using (5.2) that there exists  $t_0 \geq -\infty$  such that  $\hat{y}(t_0) = 0$ ,  $\hat{y}(t) < 0$  for  $t > t_0$  and that  $\lim_{t \rightarrow \infty} \hat{y}(t) = 0$ . Moreover, for  $t$  large

$$0 < -\hat{y}(t) = \int_{t_0}^t (-h^1(s)\hat{x}(s)^{q-1} + h^2(s)\hat{x}(s)^{p-1}) ds$$

Recalling that  $h^2(s)\hat{x}(s)^{p-1} \rightarrow 0$  we deduce that  $-h^1(s)\hat{x}(s)^{q-1}$  is bounded with respect to  $s$ . Thus  $\lim_{t \rightarrow \infty} -h^1(s)\frac{\hat{x}(s)^q}{q} = 0$  from which we deduce  $\lim_{t \rightarrow \infty} H(\hat{x}(t), \hat{y}(t), t) = 0$ . Since  $H(\hat{x}(t), \hat{y}(t), t)$  is decreasing we have that it is positive or any  $t$ . Thus we can remove the assumption on the growth of  $K^1(r)$ .

Now we turn to consider the regular solutions  $u(r)$  of (1.2) and the corresponding trajectories  $(x(t), y(t))$  of (2.2). Using the truncation  $\tilde{H}_T$  defined by (2.8), we find that  $\tilde{H}_T(x(t), y(t), t)$  is decreasing and negative, therefore  $H(x(t), y(t), t) < 0$  for  $t > T$ . Then we recall that from lemma (2.7) we know that the curves  $H(x, y, t) = 0$  are bounded for any  $t$  finite, thus we can deduce the continuability and the positiveness of  $u(r)$ . We have seen that decaying solutions  $v(r)$  cannot be always positive. Therefore for any given  $u(r)$  there exists a sequence  $r_n \rightarrow \infty$  such that  $u(r_n) > \delta > 0$  for any  $n$ .  $\square$

Observe that for any fixed value  $R$  there exist a constant  $b(R)$  such that

$$F_u(R) \leq 0 \quad \text{if and only if} \quad 0 \leq u(R) \leq b(R)$$

Let us call  $b^* > 0$  the value such that  $-\frac{A|b^*|^q}{q} + \frac{B|b^*|^p}{p} = 0$ . Now assume that both the functions  $-K^1(r)$  and  $K^2(r)$  are monotone decreasing for  $r$  large. From theorem (3.1) we already know that all the solutions of the problem are positive; we want to show that they oscillate indefinitely between two positive values, and that they are uniformly bounded for  $r$  large. To this purpose we define the following functions taken from [10]:

$$f(u, r) := k^1(r)|u(r)|^{q-1} + k^2(r)|u(r)|^{p-1}, \quad F(u, r) := k^1(r)\frac{|u(r)|^q}{q} + k^2(r)\frac{|u(r)|^p}{p}$$

$$E(u, u', r) := \frac{m-1}{m}|u'(r)|^m + F(u, r).$$

**3.2 Corollary.** *Suppose that Hyp. Mon is satisfied and assume  $q < m^* < p$ . Moreover assume that both the functions  $-K^1(r)$  and  $K^2(r)$  are monotone decreasing for  $r$  large. Then all the regular solutions  $u(r)$  (and the singular, if they exist) of Eq. (1.2) are strictly positive for any  $r > 0$ , and uniformly bounded for  $r$  large. More precisely for any  $u(r)$  we have*

$$0 < \liminf_{r \rightarrow \infty} u(r) < \limsup_{r \rightarrow \infty} u(r) < b^*. \quad (3.1)$$

*Proof.* We already know that the solutions  $u(r)$  are positive. Differentiating  $E(u, u', r)$  we get the following

$$\frac{d}{dr} E(u, u', r) = -\frac{n-1}{r}|u'(r)|^m - \frac{d}{dr} K^1(r)\frac{|u(r)|^q}{q} + \frac{d}{dr} K^2(r)\frac{|u(r)|^p}{p} \leq 0,$$



for  $r$  large. Thus  $E(u, u', r)$  is lower bounded and monotone decreasing for  $r$  large, so it admits a limit. It is enough to prove that  $\lim_{r \rightarrow \infty} E(u, u', r) = M < 0$  then (3.1) follows. Assume that  $\lim_{r \rightarrow \infty} u'(r) = l$  exists, then it easily follows that  $l = 0$ . Therefore we have that  $F(u, r)$  converges as well. Assume for contradiction that  $\lim_{r \rightarrow \infty} F(u, r) > 0$ , then  $\lim_{r \rightarrow \infty} f(u, r) > 0$ , so from (1.2) it follows that  $(u'|u'|^{p-2}(r))' < -\delta$  for  $r$  large and a certain  $\delta > 0$ . But this contradicts  $\lim_{r \rightarrow \infty} u'(r) = 0$ .

Now we assume that  $u'(r)$  does not converge as  $r \rightarrow \infty$ , so it changes sign indefinitely for  $r$  large. Then there exists a sequence of local minima  $r_k \rightarrow \infty$ . So

$$F_u(r_k) = E_u(r_k) > E_u(r_{k+1}) = F_u(r_{k+1}) \rightarrow \lim_{r \rightarrow \infty} E(u, u', r) = M$$

for any  $k$  large enough. From equation (1.2) we easily deduce that, if  $u(r_k)$  is a minimum, then  $f(u(r_k), r_k) < 0$ . Therefore  $F(u(r_k), r_k) < 0$  as well, thus  $M < 0$ . Therefore it is easy to see that  $u(r)$  must oscillate indefinitely between the two positive values  $c_1, c_2$  such that  $-A \frac{|c_1|^q}{q} + \frac{B|c_1|^p}{p} = -A \frac{|c_2|^q}{q} + \frac{B|c_2|^p}{p} = M$   $\square$

**3.3 Remark.** Reasoning as in the proof of the Corollary it can be shown that for each maxima and minima  $u(r_k)$  of a positive solution  $u(r)$  we have  $F_u(r_k) < 0$  and this gives an estimate on the values of  $u(r_k)$ .

## 4 Analysis of the equation with two positive growth terms

Now we look for sufficient conditions for the existence of radial G.S. of the following equation:

$$(\psi_m(u'))' + \frac{n-1}{r} \psi_m(u') + K^1(r) \psi_q(u) + K^2(r) \psi_p(u) = 0 \quad (4.1)$$

Here as usual we assume  $K^1(r) > 0$  and  $K^2(r) > 0$  for  $r > 0$ . We introduce some functions closely related to the one introduced in [16]. Here and later we set  $t = \log(r)$ ; let us define

$$G(r) := r^n \left( \frac{K^1(r)}{q} + \frac{K^2(r)}{p} \right) - \frac{n-m}{m} \int_0^r s^{n-1} (K^1(s) + K^2(s)) ds = \\ = G_q(r) + G_p(r)$$

$$\text{where } G_q(r) = \int_{-\infty}^t \frac{dh^1(s)}{ds} \frac{e^{\alpha qs}}{q} ds \quad \text{and} \quad G_p(r) = \int_{-\infty}^t \frac{dh^2(s)}{ds} \frac{e^{\alpha ps}}{p} ds$$

We recall the definition of the Pohozaev function, taken from [10].

$$P_u(r) := r^{n-1} \frac{n-m}{m} u(r) u'(r) |u'(r)|^{m-2} + r^n E(u, u', r) \quad \text{where} \\ E(u, u', r) := \frac{m-1}{m} |u'(r)|^m + K^1(r) \frac{|u(r)|^q}{q} + K^2(r) \frac{|u(r)|^p}{p}.$$

Note that

$$P_u(r) := H(x(t), y(t), t) = \frac{n-m}{m}xy + \frac{m-1}{m}|y|^{\frac{m}{m-1}} + \frac{h^1(t)}{q}|x|^q + \frac{h^2(t)}{p}|x|^p$$

We can rewrite (2.4) in the following way:

$$\begin{aligned} P_u(r) &= G_q(r)|u(r)|^q + G_p(r)|u(r)|^p - \\ &- \int_0^r (qG_q(s)|u(s)|^{q-2} + pG_p(s)|u(s)|^{p-2}) u(s)u'(s)ds \end{aligned} \quad (4.2)$$

We will also make use of the following auxiliary functions

$$J_q(r) = \int_t^\infty \frac{dh^1(s)}{ds} \frac{e^{\alpha qs}}{q} ds \quad \text{and} \quad J_p(r) = \int_t^\infty \frac{dh^2(s)}{ds} \frac{e^{\alpha ps}}{p} ds.$$

to obtain the following relation

$$\begin{aligned} P_u(r) &= \lim_{r \rightarrow \infty} P_u(r) - J_q(r)|u(r)|^q - J_p(r)|u(r)|^p + \\ &+ \int_r^\infty (qJ_q(s)|u(s)|^{q-2} + pJ_p(s)|u(s)|^{p-2}) u(s)u'(s)ds \end{aligned} \quad (4.3)$$

For convenience of the reader we rewrite system (2.2) in this setting.

$$\begin{pmatrix} \dot{x}_l \\ \dot{y}_l \end{pmatrix} = \begin{pmatrix} \alpha_l & 0 \\ 0 & \gamma_l \end{pmatrix} \begin{pmatrix} x_l \\ y_l \end{pmatrix} + \begin{pmatrix} y_l|y_l|^{\frac{2-m}{m-1}} \\ -h_l^1(t)\psi_q(x_l) - h_l^2(t)\psi_p(x_l) \end{pmatrix} \quad (4.4)$$

First of all we remark that Proposition (2.4) holds also in this setting. The only difference is that we can refine the asymptotic behavior of fast decay solutions giving a lower estimate on the decay. To be more precise a trajectory of (4.4) having the origin as  $\omega$ -limit point correspond to a fast decaying solution  $u(r)$  such that  $u(r) \sim r^{-\frac{n-m}{m-1}}$ . Furthermore we can exclude the existence of nontrivial solutions satisfying the condition  $u(0) = 0$ . The proof of this claim and a more precise statement of the result is postponed to Lemma (5.5) and (5.4) in section 5. We will also use the following autonomous system.

$$\begin{pmatrix} \dot{x}_l \\ \dot{y}_l \\ \dot{\tau} \end{pmatrix} = \begin{pmatrix} \alpha_l & 0 & 0 \\ 0 & \gamma_l & 0 \\ 0 & 0 & \xi \end{pmatrix} \begin{pmatrix} x_l \\ y_l \\ \tau \end{pmatrix} + \begin{pmatrix} y_l|y_l|^{\frac{2-m}{m-1}} \\ -h_l^1(t)\psi_q(x_l) - h_l^2(t)\psi_p(x_l) \\ 0 \end{pmatrix} \quad (4.5)$$

As in section 2 we begin by assuming that system (4.4) with  $l = m^*$  is autonomous, that is  $h^1(t) \equiv C_1 \geq 0$ ,  $h_l^2(t) \equiv C_2 \geq 0$  and  $0 < C_1 + C_2 < \infty$ . We recall that when we refer to the change of variable (2.9) with  $l = m^*$  we leave the subscript unsaid. Also in this case system (4.4) admits exactly 3 critical points: the origin,  $P = (P_x, P_y)$  and  $-P$ , where  $P_y < 0 < P_x$ , and the function  $H$  is a first integral. As done in section 2, fixed  $T$ , we will denote by  $C_b(T)$  the level sets  $\{(x, y) \mid H(x, y, t) = b\}$ . Note that if  $0 < h^1(t) + h^2(t) < \infty$  the level

sets of the function  $H$  are continuous deformations of the one depicted in Fig. (1). The level set  $C_0(T)$  is a 8-shaped curve contained in the  $2^{nd}$  and in the  $4^{th}$  quadrants. The level sets  $C_b(T)$  for  $0 > b > H(P)$  are closed bounded curves contained in the interior of the set enclosed by  $C_0(T)$ . The level sets  $C_b(T)$  for  $b > 0$  are closed bounded curves crossing the coordinate axes.

We are ready to state a result analogous to Proposition (2.8).

**4.1 Proposition.** *Consider Eq. (4.1) and assume that  $h^1(t) \equiv A \geq 0$  and  $h^2(t) \equiv B \geq 0$ , where  $A + B > 0$ . Then we can classify positive solutions as follows.*

- A All the trajectories corresponding to some positive value  $H(x, y) = b > 0$  represent periodic trajectories which cross the axis. They correspond to singular solutions  $u(r)$  of (1.2) with infinitely many positive maxima and negative minima; moreover there exists  $a > 0$  such that  $-ar^{-\alpha} \leq u(r) \leq ar^{-\alpha} \quad \forall r > 0$ .*
- B The trajectory corresponding to  $H(x, y) = 0$  is homoclinic to the origin; this means that all regular solutions  $u(r)$  of (4.1) are monotone decreasing S.G.S. with decay rate  $\sim r^{-\frac{n-m}{m-1}}$  at  $\infty$  (fast decay).*
- C All the trajectories corresponding to some negative value  $H(x_1, x_2) = -b > H(P)$  represent periodic trajectories which belong to  $\mathbb{R}_+^2$ . They represent S.G.S.  $u(r)$  of Eq. (1.2) with rate of growth and decay  $\sim r^{-\alpha}$  respectively at 0 and at  $\infty$ .*
- D For the value  $H = H(P)$  we have one fixed point  $P = (P_x, P_y)$ , which corresponds to a monotone decreasing S.G.S of (1.2) of the form  $u(r) = P_x r^{-\alpha}$  where we recall that  $P_x = P_x(A, B)$ .*

We enumerate now some Hypotheses that will be used in this section.

#### Hypotheses

- Sup**  $G_q(r) \leq 0$  and  $G_p(r) \leq 0$  for any  $r > 0$  and at least one of the inequality is strict for a certain  $r = R > 0$ .
- Sub**  $G_q(r) \geq 0$  and  $G_p(r) \geq 0$  for any  $r > 0$  and at least one of the inequality is strict for a certain  $r = R > 0$ .
- Sub\***  $J_q(r) \geq 0$  and  $J_p(r) \geq 0$  for any  $r > 0$  and at least one of the inequality is strict for a certain  $r = R > 0$ .
- N1**  $\lim_{t \rightarrow -\infty} h^1(t) = A < \infty$  and  $\lim_{t \rightarrow -\infty} h^2(t) = B < \infty$ , where  $A + B > 0$ .
- N2** There exists  $s > m_*$ , such that the limit  $\lim_{t \rightarrow -\infty} h_s^1(t) + h_s^2(t)$  is positive and finite. Furthermore for a certain  $\xi > 0$

$$\lim_{r \rightarrow 0} (r \frac{dK^1}{dr} + \delta_s K^1) r^{\delta_s + \xi - 1} = 0 \quad \text{and} \quad \lim_{r \rightarrow 0} (r \frac{dK^2}{dr} + \eta_s K^2) r^{\eta_s + \xi - 1} = 0.$$

**N3**  $\lim_{t \rightarrow \infty} h^1(t) = A < \infty$  and  $\lim_{t \rightarrow \infty} h^2(t) = B < \infty$ , where  $A + B > 0$ .

**N4** There exists  $L > m_*$ , such that the limit  $\lim_{t \rightarrow \infty} h_L^1(t) + h_L^2(t)$  is positive and finite. Furthermore for a certain  $\xi < 0$

$$\lim_{r \rightarrow \infty} (r \frac{dK^1}{dr} + \delta_L K^1) r^{\delta_L + \xi - 1} = 0 \quad \text{and} \quad \lim_{r \rightarrow \infty} (r \frac{dK^2}{dr} + \eta_L K^2) r^{\eta_L + \xi - 1} = 0.$$

*4.2 Remark.* To satisfy the Hyp. Sup is enough to take  $q \geq m^*$  and  $K^1 > 0$  constant or decreasing and  $K^2$  constant or not increasing too fast (but it can be unbounded or tend to 0 as  $t \rightarrow \infty$ ). Assume that  $K^1(r)$  is strictly positive and bounded, if  $q = m^*$  Hyp. N3 is satisfied, while if  $m_* < q < m^*$  Hyp. N4 with  $L = q$  is satisfied. Analogously to satisfy Hyp. Sub we can take  $m_* < p \leq m^*$  and the functions  $K^i$  monotone increasing.

Consider a trajectory  $\hat{X}(t)$  of Eq. (4.4) corresponding to a positive solution  $u(r)$  of (4.1). Note that whenever  $H(\hat{X}(t), t) < 0$  we have that  $\hat{X}(t)$  is in the 4<sup>th</sup> quadrant, thus  $u'(r) < 0$ . Thus we can deduce the following

*4.3 Remark.* Consider a regular solution  $u(r)$  of Eq. (4.1) defined and positive for any  $r > 0$ . If Hyp. Sup is satisfied then  $P_u(r) \leq 0$  for any  $r$ ; therefore for the corresponding trajectory of Eq. (4.4) we have  $H(x(t), y(t), t) \leq 0$  and  $\limsup_{t \rightarrow \infty} H(x(t), y(t), t) < 0$ . Analogously if Hyp. Sub is satisfied, we have  $P_u(r) \geq 0$  and  $\liminf_{t \rightarrow \infty} H(x(t), y(t), t) > 0$ .

We give now a technical remark that will be useful to analyze asymptotic behaviour of positive solutions.

*4.4 Remark.* Assume that Hyp. Sub is satisfied, then there exists  $\epsilon > 0$  such that  $h^1(t) + h^2(t) > \epsilon$  for  $t$  large. Analogously assume that Hyp. Sup is satisfied, then there exists  $M > 0$  such that  $h^1(t) + h^2(t) < M$  for  $t$  large.

*Proof.* Assume that Hyp. Sub is satisfied. It is enough to prove that  $h^1(t)$  is strictly positive for  $t$  large. Suppose for contradiction that  $\liminf_{t \rightarrow \infty} h^1(t) = 0$ , then for any  $\epsilon > 0$  there exists  $T(\epsilon) > 0$  such that  $h^1(T(\epsilon)) - \epsilon = 0$ . Suppose at first that  $\liminf_{t \rightarrow -\infty} h^1(t) \geq \epsilon$  therefore we can assume that  $\inf_{t < T} h^1(t) = \epsilon$ . We define  $g(t) = h^1(t) - \epsilon$ . Then

$$0 \leq qG_q(r) = \int_{-\infty}^t \frac{dh^1}{ds}(s) e^{\alpha qs} ds = \int_{-\infty}^t \frac{dg}{ds}(s) e^{\alpha qs} ds \quad (4.6)$$

Integrating by parts and setting  $t = T$  we have

$$0 \leq g(T) e^{\alpha q T} - \alpha q \int_{-\infty}^T g(s) e^{\alpha qs} ds = -\alpha q \int_{-\infty}^T g(s) e^{\alpha qs} ds$$

Thus we have found a contradiction since we have assumed  $g(t) > 0$  for  $t < T$ . Now assume that  $\liminf_{t \rightarrow -\infty} h^1(t) = 0$ , then for any  $\epsilon > 0$  there exists  $T_0(\epsilon)$

such that  $g(T_0) = 0$  and  $g(t) > 0$  for any  $T_0 < t < T$ . Thus we can rewrite (4.6) in the following way:

$$\begin{aligned} 0 &\leq \int_{-\infty}^{T_0} \frac{dh^1}{ds}(s) e^{\alpha q s} ds + \int_{T_0}^T \frac{dg}{ds}(s) e^{\alpha q s} ds \leq \\ &\leq h^1(T_0) e^{\alpha q T_0} - \alpha q \left( \int_{-\infty}^{T_0} h^1(s) e^{\alpha q s} ds + \int_{T_0}^T g(s) e^{\alpha q s} ds \right) < \\ &< \epsilon e^{\alpha q T_0} - \int_{T_0}^T g(s) e^{\alpha q s} ds \end{aligned}$$

Note that when  $\epsilon \rightarrow 0$  then  $T_0(\epsilon) \rightarrow -\infty$  and  $T(\epsilon) \rightarrow +\infty$ , thus the right hand side in the last inequality is negative, so we have found a contradiction.  $\square$

Now we reformulate in this setting Lemmas (2.12), (2.13), (2.14); we skip the proofs since they can be obtained working as in section 2 and recalling the previous remark.

**4.5 Lemma.** *Consider Eq. (4.1) and assume that Hyp. Sup (respectively Sub) is satisfied. Then any solution  $u(r)$  can be continued forward (resp. backwards) in  $r$  for any  $r > 0$ . Furthermore consider Eq. (4.4); if a trajectory  $\hat{X}(t) = (\hat{x}(t), \hat{y}(t))$  is unbounded in  $t$ , then it must cross the coordinate axes indefinitely many times. Therefore it cannot correspond to a positive solution  $\hat{u}(r)$  of Eq. (4.1).*

**4.6 Lemma.** *Consider a trajectory  $\hat{X}(t) = (\hat{x}(t), \hat{y}(t))$  of Eq. (4.4).*

- *Assume that  $\liminf_{t \rightarrow -\infty} H(\hat{X}(t), t) > 0$ , then  $\hat{X}(t)$  must cross the coordinate axes backward in  $t$  indefinitely many times;*
- *Assume that  $\liminf_{t \rightarrow \infty} H(\hat{X}(t), t) > 0$ , then  $\hat{X}(t)$  must cross the coordinate axes forward in  $t$  indefinitely many times.*

**4.7 Theorem.** *Assume that Hypotheses Sub and N3 are satisfied, then all the solutions  $u(r)$  of Eq. (4.1) are G.S. with decay of order  $\sim r^{-\alpha}$ .*

*Assume further that the functions  $h^i(t)$  are monotone for  $t$  large. Then for each G.S.  $u(r)$  there exists a S.G.S.  $v(r)$  of the frozen equation Eq. (4.1) where  $K^1(r) = Ar^{-\delta}$  and  $K^2(r) = Br^\eta$  such that  $\lim_{r \rightarrow \infty} (u(r) - v(r))r^\alpha = 0$ .*

**4.8 Remark.** Theorem (4.7) as all the Proposition of this section can be trivially generalized to the situation in which we have a finite sum of terms. To be more specific fix  $v \in \mathbb{N}$  and consider the following equation:

$$(u'|u'|^{m-2})' + \frac{n-1}{r} u'|u'|^{m-2} + K^1(r) \psi_{q_1}(u) + \dots + K^v(r) \psi_{q_v}(u) = 0 \quad (4.7)$$

Assume that  $G_{q_i}(r) \leq 0$  for any  $r > 0$  and any  $i$ ; moreover assume that the functions  $h^i(t)$  are monotone for  $t$  large and  $0 < \lim_{t \rightarrow \infty} \sum_i h^i(t) < \infty$ . Then all the solutions of (4.7) are G.S. with decay of order  $\sim r^{-\alpha}$ . Moreover assume

that  $0 < \sum_{i=1}^v \lim_{t \rightarrow \infty} h^i(t) = \sum_{i=1}^v A_i < \infty$ , then for each G.S.  $u(r)$  there exists a S.G.S.  $v(r)$  of the frozen equation Eq. (4.7) where  $K^i(r) = Ar^{-\delta_i}$  and such that  $\lim_{r \rightarrow \infty} (u(r) - v(r))r^\alpha = 0$ .

The proof is completely analogous to the case of 2 terms, so we will deal with this setting which involves less heavy notation.

*Proof.* We repeat the same reasoning made for Theorem (2.10). We introduce the change of variables (2.9) in order to deal with system (4.4) where  $l = m^*$ . Consider a regular solution  $u(r)$  of (4.1) and the corresponding trajectory  $X(t) = (x(t), y(t))$ . First of all notice that with our assumptions  $H(x(t), y(t), t) \leq 0$ . Thus we know that  $X(t)$  is forced to stay in the set  $H(x, y, t) \leq 0$  for any  $t$ , so we can conclude that it represents a ground state. Furthermore  $\limsup_{t \rightarrow \infty} H(x, y, t) < 0$  so  $X(t)$  is bounded away from the coordinate axes, for  $t$  large. Thus  $u(r) \sim r^{-\alpha}$  as  $r \rightarrow \infty$ .

To prove the statement regarding the asymptotic behavior of the ground states we can repeat the reasoning made at the end of the proof of Theorem (2.10), that is to introduce the system (4.5) where  $\xi < 0$ . Then observe that the  $\omega$ -limit set of the solutions is a subset of the  $\tau = 0$  plane. We recall that the restriction of this system to the  $\tau = 0$  plane corresponds to the autonomous system whose behavior has been described in theorem (4.1). Now recall that, with our assumption  $H(x(t), y(t), t)$ , admits a limit and that the value of this limit characterizes exactly one periodic trajectory of the autonomous problem and conclude.  $\square$

Consider system (4.5) with  $\xi < 0$ . Let us define

$$S_l(\tau) = \{(x_l, y_l, \tau) \mid H_l(x_l, y_l, t) < 0 \text{ where } \tau = e^{\xi t}\}$$

and  $S_l = \cup_{\tau \geq 0} S_l(\tau)$ . Note that if  $h_l^1(t) + h_l^2(t) \rightarrow 0$  as  $t \rightarrow \infty$ , then  $S_l(\tau)$  becomes unbounded as  $\tau \rightarrow 0$ , while if  $h_l^1(t) + h_l^2(t) \rightarrow \infty$  then the closure  $\overline{S}_l(\tau)$  of  $S_l(\tau)$  shrinks to the origin. Therefore the proof of Proposition (4.1) must be modified slightly. From Proposition (5.4) we know that fast decay solutions  $u(r)$  of (4.1), correspond to trajectories  $X(t)$  of (4.4) having the origin as  $\omega$ -limit set. We analyze now positive solutions with slow decay.

**4.9 Proposition.** *Assume that the functions  $h^i(t)$  are monotone for  $t$  large and there exist  $l_2 > l_0 > m^*$  or  $m_* < l_0 < l_2 < m^*$  such that*

$$\limsup_{t \rightarrow \infty} \sum_{i=1}^v h_{l_0}^i(t) = \infty \quad \text{and} \quad \liminf_{t \rightarrow \infty} \sum_{i=1}^v h_{l_2}^i(t) = 0. \quad (4.8)$$

Moreover assume that for a certain  $\xi < 0$

$$\lim_{r \rightarrow \infty} \left( r \frac{dK^1}{dr} + \delta_{l_2} K^1 \right) r^{\delta_{l_2} - \xi} = 0 \quad \lim_{r \rightarrow \infty} \left( r \frac{dK^2}{dr} + \delta_{l_2} K^2 \right) r^{m_2 - \xi} = 0 \quad (4.9)$$

Consider a solution  $u(r)$  which is well defined and positive for  $r$  large. Then  $u(r) \sim r^{-\frac{n-m}{m-1}}$  as  $r \rightarrow \infty$  (fast decay), or  $u(r)$  has slow decay, that is

$$C_1 r^{-\frac{m}{l_0 - m}} \leq u(r) \leq C_2 r^{-\frac{m}{l_2 - m}} \quad \text{as } r \rightarrow \infty. \quad (4.10)$$

*Proof.* Consider system (4.5) with  $\xi < 0$ . Note that, even when the regularity hypothesis is not satisfied, the system is Lipschitz continuous in  $S_l(\tau)$  for any  $\tau > 0$ . Furthermore note that the technical Hyp. (4.9) ensures that the system is Lipschitz continuous also for  $\tau = 0$  when  $l \leq l_2$ .

Consider a positive solution  $\hat{u}(r)$  of (4.1) and the corresponding trajectory  $\hat{X}(t) = (\hat{x}(t), \hat{y}(t), \hat{\tau}(t))$  of (4.5). We recall that the limit  $\lim_{t \rightarrow \infty} H(\hat{X}(t), t) = M$  exists, since  $H(\hat{X}(t), t)$  is monotone for  $t$  large. Note that the only critical point of the system with  $l \leq l_2$  is the origin. Furthermore, applying Poincaré-Bendixon criterion we find that no periodic trajectories can exist in the subset where  $x > 0$  and  $y < 0$ . From Lemma (4.6) we know that if  $M > 0$   $\hat{X}(t)$  must cross the coordinate axes indefinitely many times, therefore it cannot correspond to a positive solution. Thus we can assume  $M \leq 0$ . Note that if  $M = 0$  we have that  $\hat{X}(t)$  converge to the origin so it correspond to a solution  $\hat{u}(r)$  with fast decay. Thus we can assume that  $M < 0$ . Now observe that the origin is the only critical point of the system. Assume for contradiction that  $\hat{X}(t)$  is bounded, then its  $\Omega$ -limit set is contained in the subset where  $x > 0$ ,  $y < 0$  and  $\tau = 0$ , where there are no periodic trajectories or critical points. Thus  $\hat{X}(t)$  is unbounded and it is easily deduced that  $\lim_{t \rightarrow \infty} \hat{x}(t) = \infty$ . Let us fix  $l = l_0$  and consider the corresponding system (4.5) with  $\xi < 0$ . Then once again we have  $\lim_{t \rightarrow \infty} \hat{x}_{l_0}(t) = \infty$ . In fact  $\hat{X}(t)$  cannot be bounded since otherwise we would have the origin as  $\omega$ -limit set. In this case, using Lemma (5.4), we deduce that  $\hat{u}(r) \sim r^{-\frac{n-m}{m-1}}$ , but this is in contradiction with  $\lim_{t \rightarrow \infty} \hat{x}(t) = \infty$ . Let us fix now  $l = l_2$  and consider the corresponding system (4.5) with  $\xi < 0$ . Then we have  $\lim_{t \rightarrow \infty} \hat{x}_{l_2}(t) = 0$ ; in fact  $\hat{X}_{l_2}(t) \in \bar{S}(\tau)$  as  $\tau \rightarrow 0$  and  $\bar{S}(0)$  is the origin.

Thus solutions  $u(r)$  with slow decay must satisfy (4.10)  $\square$

**4.10 Proposition.** *Assume that the functions  $h^i(t)$  are monotone for  $t$  large.*

- *Assume that Hyp. N4 is satisfied then a slow decay solution  $\bar{u}(r)$  is such that  $\bar{u}(r) \sim r^{\frac{m}{L-m+1}}$  as  $r \rightarrow \infty$ .*
- *Assume that Hyp. N2 is satisfied then a singular solution  $\bar{u}(r)$  is such that  $\bar{u}(r) \sim r^{\frac{m}{s-m+1}}$  as  $r \rightarrow 0$ .*

*Proof.* We begin by the first claim. Fix  $l = L$  and consider system (4.5) with  $\xi < 0$ . Repeating the reasoning done in Proposition (4.9) we find that  $\lim_{t \rightarrow \infty} H(\bar{x}(t), \bar{y}(t), t) < 0$ . Note that  $\bar{S}_l(\tau)$  is bounded and have positive measure as  $\tau \rightarrow 0$ . Therefore we have that  $\bar{x}_L(t)$  is bounded. System (4.5) with  $l = L$  has three critical points: the origin,  $P = (P_x, P_y, 0)$  where  $P_y < 0 < P_x$  and  $-P$ . Reasoning as above we see that trajectory converging to the origin correspond to fast decay solutions and trajectories converging to  $P$  correspond to slow decay solutions. The second claim can be proved in the same way considering system (4.5) with  $\xi > 0$  and  $l = s$ .  $\square$

**4.11 Theorem.** *Assume that Hypotheses Sup and N4 are satisfied then all the regular solutions are G.S. with slow decay, (see Proposition (4.10)). Furthermore assume that Hyp. N2 is satisfied then there exists one S.G.S. with slow*

*decay. Moreover if the functions  $h^i(t)$  are monotone as  $t \rightarrow -\infty$  and  $s \neq m^*$  the S.G.S. is unique and no other positive solutions can exist.*

*Proof.* We have already seen that regular solutions  $u(r)$  of (4.1) correspond to trajectories  $X(t)$  of (4.4) such that  $\lim_{t \rightarrow -\infty} H(X(t), t) = 0$ . From Hyp. Sup it follows that  $H(X(t), t) \leq 0$  for any  $t$  and also  $\limsup_{t \rightarrow \infty} H(X(t), t) < 0$ . Thus we deduce that  $X(t) \in S(\tau(t))$  for any  $t$ , which implies that  $u(r) > 0$  and  $u'(r) < 0$  for any  $r > 0$ . Assume that Hyp. N4 is satisfied. From the fact that  $\limsup_{t \rightarrow \infty} H(X(t), t) < 0$  we get that  $X(t)$  cannot have the origin as  $\omega$ -limit set. Therefore, reasoning as in Proposition (4.10) we get that  $u(r)$  must have slow decay.

Now assume that Hyp. N2 is satisfied and consider system (4.5) obtained setting  $l = s$  and  $\xi > 0$ : in the subset  $x \geq 0$  the critical points are the origin and a point  $P = (P_x, -P_y, 0)$  where  $P_x$  and  $P_y$  are positive constants. Assume at first that  $s \neq m^*$  and that the functions  $h^i(t)$  are monotone. Then the point  $P$  admits a one-dimensional unstable manifold  $W_P$ , transversal to the direction of the plane  $\tau = 0$ .  $W_P$  is in fact made up exactly by one trajectory, say  $\hat{X}_s(t) = (\hat{x}_s(t), y_s(t), \hat{\tau}(t))$ . Note that  $\lim_{t \rightarrow -\infty} H_s(\hat{X}_s(t), t) < 0$ , thus repeating the proof just developed for regular solutions, we find that  $H(\hat{X}(t), t) < 0$  for any  $t$  and  $\limsup_{t \rightarrow \infty} H(\hat{X}(t), t) < 0$ . Therefore the corresponding solution  $\hat{u}(r)$  represent a S.G.S. with slow decay. The uniqueness and the non-existence result follow from the asymptotic estimates of Proposition (4.10). If  $s = m^*$  or if the functions  $h^i$  are not monotone we lose the uniqueness result but we can still find an unstable manifold and prove the existence result.  $\square$

**4.12 Remark.** To satisfy the hypotheses of Theorem (4.11) we can take, e. g.,  $p > q > m^*$ ,  $K_1(r)$  and  $K_2(r)$  strictly positive and decreasing. In this case for the singular solution  $u(r)$  we have  $u(r) \sim r^{-\frac{m}{p-m}}$  as  $r \rightarrow 0$  and slow decay solutions  $u(r)$  are such that  $u(r) \sim r^{-\frac{m}{q-m}}$  as  $r \rightarrow \infty$ .

**4.13 Corollary.** *Assume that Hyp. Sup is satisfied. Then*

- *if we are in the Hyp. of Proposition (4.9) we can still get the same classification result for positive solutions but the estimates on the asymptotic behavior of solutions singular in the origin or with slow decay, are the ones described in Proposition (4.9)*
- *Assume that  $\lim_{r \rightarrow \infty} \sum_i K^i(r)r^m = 0$ , then all the solutions  $u(r)$  have positive finite limit and the singular solution  $\hat{u}(r)$  too. No G.S., S.G.S. or crossing solution can exist.*

*Proof.* The first claim can be obtained simply repeating the proof of Theorem (4.11). Suppose that  $\lim_{r \rightarrow \infty} \sum_i K^i(r)r^m = 0$ , then the regular solutions  $u(r)$  of (4.1) are positive and monotone decreasing for any  $r \geq 0$ , and are such that  $\lim_{r \rightarrow \infty} u(r)r^\epsilon = \infty$ , for any  $\epsilon > 0$ . In fact, reasoning as done before we find that for any  $l > 0$  we have  $x_l(t) \rightarrow \infty$ . Repeating the proof at page 738 in [16] we can conclude that, if a solution  $v(r)$  tends to 0 as  $r \rightarrow \infty$ , then  $v(r) \sim r^{-\frac{n-m}{m-1}}$ . Thus regular solutions have positive finite limit. Reasoning in the same way we get the same conclusion for singular solutions as well.  $\square$



We can now give a new proof of a well known result, see [15]

**4.14 Proposition.** *Assume that Hyp. Sub is satisfied. Then all the regular solution  $u(r)$  of Eq. (4.1) are crossing solutions.*

*Proof.* Consider a regular solutions  $u(r)$  of (4.1). The corresponding trajectory  $X(t) = (x(t), y(t))$  of (4.4) is such that  $H(x(t), y(t), t) > 0$  for any  $t > T$  for a certain  $T$ , therefore the Proposition follows from Lemma (4.6).  $\square$

Now we give the following classification result for positive solutions

**4.15 Theorem.** *Consider equation (4.1) and assume that Hyp. Sub\* is satisfied. Assume that Hyp. N2 and N4 are satisfied. Then there exists a S.G.S. with slow decay  $\hat{v}(r)$ , that is  $\hat{v}(r) \sim r^{-\frac{m}{s-m}}$  as  $r \rightarrow 0$  and  $\hat{v}(r) \sim r^{-\frac{m}{L-m}}$  as  $r \rightarrow \infty$ . If we assume that the regularity hypothesis is satisfied there exist also infinitely many S.G.S. with fast decay  $\bar{w}(r)$ :  $\bar{w}(r) \sim r^{-\frac{m}{s-m}}$  as  $r \rightarrow 0$   $\bar{w}(r) \sim r^{-\frac{n-m}{m-1}}$  as  $r \rightarrow \infty$ .*

*Proof.* Consider system (4.5) with  $\xi < 0$  and  $l = L \leq m^*$ . Note that it admits a critical point  $P = (P_x, P_y, 0)$ . We have that  $P$  admits a one-dimensional stable manifold, which is made up of exactly one trajectory, say  $\hat{X}_L(t) = (\hat{x}_L(t), \hat{y}_L(t), \hat{\tau}(t))$ , such that  $\lim_{t \rightarrow \infty} H_L(\hat{x}_L(t), \hat{y}_L(t), (t)) < 0$ . It follows that also  $\lim_{t \rightarrow \infty} H(\hat{x}(t), \hat{y}(t), (t)) \leq 0$ . Consider the corresponding solution  $\hat{v}(r)$  of (4.1). Note that  $\lim_{r \rightarrow \infty} P_{\hat{v}}(r) \leq 0$ . We follow  $\hat{v}(r)$  backwards in  $r$  and from (4.3) we have  $P_{\hat{v}}(r) < 0$  until  $\hat{v}'(r) < 0$ . Since  $\hat{y}(t) < 0$  for  $t$  large, and  $\hat{y}(t) < 0$  whenever  $P_{\hat{v}}(r) < 0$  we have that  $H(\hat{x}(t), \hat{y}(t), (t)) < 0$  for any  $t$  and  $P_{\hat{v}}(r) < 0$  for any  $r$ . Furthermore  $\lim_{t \rightarrow -\infty} H(\hat{x}(t), \hat{y}(t), (t)) < 0$ . Consider now system (4.5) with  $\xi > 0$  and  $l = s$ . Since  $\lim_{t \rightarrow -\infty} H(\hat{x}(t), \hat{y}(t), (t)) < 0$  we have that  $\hat{X}_s(t)$  cannot converge to the origin. Therefore  $\hat{v}(r)$  cannot be a regular solution. Recalling Proposition (4.9) we have that  $\hat{v}(r)$  is a S.G.S. with slow decay.

Now we turn to consider S.G.S. with fast decay. Assume that the regularity Hyp. is satisfied and consider again system (4.5) with  $\xi < 0$  and  $l = L$ . The origin admits a 2-dimensional stable manifold which is transversal to the  $\tau = 0$  plane. Consider any trajectory  $\bar{X}_L(t)$  belonging to this manifold and the corresponding solution  $\bar{w}(r)$  of (4.1). Repeating the reasoning made for slow decay solutions, we find that  $\bar{w}(r)$  is a S.G.S. with fast decay.  $\square$

**4.16 Remark.** If we replace Hyp. N4 by Hyp. N3 we continue to have a S.G.S. with slow decay and infinitely many S.G.S. with fast decay, but we cannot a priori exclude the existence of multiple S.G.S. with slow decay.

Furthermore, if we replace Hyp. N2 with Hyp. N1 or with an estimate of type (4.8) and (4.9) as  $t \rightarrow -\infty$ , Theorem (4.15) continue to hold, but we have respectively  $v(r) \sim r^{-\frac{n-m}{m}}$  and  $C_2 r^{-\frac{m}{l_2-m}} \leq v(r) \leq C_1 r^{-\frac{m}{l_1-m}}$ , as  $r \rightarrow 0$ .

## 5 Asymptotic Behavior

In this section we collect some Lemmas concerning the existence, local uniqueness and asymptotic behavior of positive solutions  $u(r)$  of Eq. (1.2) and Eq.

(4.1) when  $r \rightarrow 0$  and  $r \rightarrow \infty$ . In particular we want to prove Propositions (2.4) and (2.15). In this section we will denote with  $C > 0$  different constants whose value is changing from line to line.

First of all, if we assume that the regularity hypothesis is satisfied, we can prove a partial result exploiting invariant manifold theory. Assume that we can set  $l$  in (2.9) in such a way that both  $h_l^1(t)$  and  $h_l^2(t)$  are bounded for  $t \rightarrow -\infty$ . Let us call  $X_l(Q; t) = (x_l(t), y_l(t))$  the solution of (2.2) passing through  $Q$  at  $t = 0$ . Exploiting a paper by Johnson [12] based on [11], we can prove that there exists an unstable manifold  $W^u$  such that

$$W^u := \{Q \mid \lim_{t \rightarrow -\infty} X_l(Q; t) = 0\}.$$

Consider a solution  $X_l(Q; t)$  such that  $Q \in W^u$ ; for any  $\epsilon > 0$  we have  $\lim_{t \rightarrow -\infty} X_l(Q; t)e^{(-\alpha_l + \epsilon)t} = 0$ . Thus the corresponding solution  $u(r)$  is such that  $\lim_{r \rightarrow 0} u(r)r^\epsilon = 0$ . Furthermore reasoning as in Observation 3.17 in [6] we can prove that  $u(r)$  is strictly positive and bounded for  $r$  small. This way we also give an alternative proof of local existence of solutions of (1.2) with initial values (1.3).

Analogously assume that there exists another value  $L$  in (2.9) such that both  $h_L^1(t)$  and  $h_L^2(t)$  are bounded for  $t \rightarrow \infty$ . Reasoning as above we can prove the existence of a manifold  $W^s := \{Q \mid \lim_{t \rightarrow \infty} X_L(Q; t) = 0\}$ . Furthermore for any  $\epsilon > 0$  we have  $\lim_{t \rightarrow \infty} X_L(Q; t)e^{-\gamma_L + \epsilon} = 0$ . Reasoning as above we can conclude that the corresponding  $u(r)$  is such that  $u(r) \leq Cr^{-\alpha_L + \gamma_L}$  for  $r$  large.

When the regularity Hyp. is not satisfied we cannot anymore apply the previous reasoning. However existence and local uniqueness of regular solutions of (1.2), (1.3) can be proved working directly on the equation, see [9]. We can also improve the estimate on the asymptotic behaviour using some integral manipulations.

First of all we remark that, when  $r$  is positive and finite, we can rewrite Eq. (1.2) and Eq. (4.1) in the following way:

$$(r^{n-1}u'(r)|u'(r)|^{m-2})' = -(k^1(r)u^q + k^2(r)u^p)r^{n-1}.$$

Let us suppose that

$$\lim_{r \rightarrow a} r^{n-1}u'(r)|u'(r)|^{m-2} = 0, \quad (5.1)$$

or equivalently that

$$\lim_{t \rightarrow \log(a)} y_l(t)e^{-\gamma_l t} = 0,$$

then we have the following:

$$-u'(r)|u'(r)|^{m-2} = r^{1-n} \int_a^r f(u, s)s^{n-1}ds. \quad (5.2)$$

*5.1 Remark.* Note that in Section 2 and 3 we are always in the Hypothesis of Lemma (2.13) and in Section 4 we are in the Hypothesis of Lemma (4.5). Therefore for any positive solution we can always find  $a \geq 0$  such that (5.1) is satisfied.

Now suppose that there exists  $a, b$  such that  $u'(r) < 0$  for any  $r \in (a, b)$  and  $\lim_{r \rightarrow a} r^{n-1} u'(r) |u'(r)|^{m-2} = 0$ . Let us consider  $r_1$  and  $r_2$  belonging to  $(a, b)$ , then we have the following formula for  $u(r)$ :

$$u(r_1) - u(r_2) = \int_{r_1}^{r_2} \left( t^{1-n} \int_a^t f(u, s) s^{n-1} ds \right)^{\frac{1}{m-1}} dt \quad (5.3)$$

**5.2 Lemma.** *Consider Eq. (1.2) and assume that there exist two positive constants  $d, D$  such that  $k^1(r) < -d$  and  $0 < k^2(r) < D$  for any  $r$ . Consider a positive solution  $u(r)$ , well defined in a left neighborhood of  $r = \infty$  and converging to 0 as  $r \rightarrow \infty$ ; then it must have fast decay, that is  $\lim_{r \rightarrow \infty} u(r) r^{\frac{n-m}{m-1}}$  is finite.*

*Proof.* We begin by proving that  $u(r)$ , if exists, cannot converge to 0 oscillating indefinitely.

In fact otherwise, applying Eq. (5.2), we could find sequences  $a_k \rightarrow \infty$  and  $b_k \rightarrow \infty$ ,  $a_k < b_k < a_{k+1}$ , such that

$$r^{1-n} \int_{a_k}^{b_k} f(u(s), s) s^{n-1} ds = 0. \quad (5.4)$$

We can find a constant  $l$  large enough so that

$$f(u(l), l) < -du(l)^q + Du(l)^p < 0.$$

So if we choose  $k$  large enough we have  $f(u(s), s) < 0$  for  $a_k < s < b_k$ , but this contradicts Eq. (5.4). Therefore there exists  $a > 0$  such that  $u'(r) < 0$  for any  $r > a$ . Moreover

$$\begin{aligned} 0 \leq \lim_{r \rightarrow \infty} -u'(r) |u'(r)|^{m-2} r^{n-1} &= \int_a^\infty f(u(s), s) s^{n-1} ds < \\ &< \int_a^l f(u(s), s) s^{n-1} ds = M < \infty. \end{aligned} \quad (5.5)$$

Therefore using Eq. (5.5) in Eq. (5.3) we find

$$\begin{aligned} u(r) &= \int_r^\infty \left[ t^{1-n} \int_a^t f(u(s), s) s^{n-1} ds \right]^{\frac{1}{m-1}} dt < \\ &< \int_r^\infty [t^{1-n} M]^{\frac{1}{m-1}} dt < Cr^{-\frac{n-m}{m-1}}. \end{aligned} \quad (5.6)$$

□

We can repeat this kind of reasoning also in slightly different settings.

**5.3 Lemma.** *Consider any positive solution  $u(r)$  of Eq. (1.2) where  $k^1 = -Ar^{-\frac{n-m}{m}(m^*-q)}$  and  $k^2 = Br^{\frac{n-m}{m}(p-m^*)}$ , corresponding to a trajectory having the origin as  $\alpha$ -limit point. Then  $0 \leq u(0) < \infty$  and  $\lim_{r \rightarrow \infty} u(r) r^{\frac{n-m}{m-1}} < \infty$ .*

*Proof.* First of all note that the corresponding dynamical system is (2.6), so it is autonomous. Therefore  $u(r)$  corresponds to a trajectory corresponding to the level set  $C_0$  of the function  $H$ , see section 2. Thus  $u(r)$  is monotone increasing for  $r$  small therefore  $u(0) < \infty$ . Observe that, under the hypothesis of theorem (2.8), we have that for any ground state  $u(r)$  we get  $f(u(r), r)r^{n-1} = (-Ax^q + Bx^p)r^{-1} < 0$  for  $r$  large. Therefore we can repeat the proof just given and conclude.  $\square$

**5.4 Lemma.** *Consider system (4.4) obtained setting  $l > m_*$  in such a way that both  $h_l^1(t)$  and  $h_l^2(t)$  are bounded as  $t \rightarrow \infty$ . Consider any positive solution  $u(r)$  corresponding to a trajectory having the origin as  $\omega$ -limit set. Then  $u(r)$  must have fast decay, that is  $\lim_{r \rightarrow \infty} u(r)r^{\frac{n-m}{m-1}}$  is finite and positive.*

*Proof.* We need to prove that there exist two positive constants  $d$  and  $D$  such that

$$d < |u'|^{m-1}r^{n-1} = \int_a^r f(u(s), s)s^{n-1}ds = I(r) < D \quad (5.7)$$

for any  $r$  large enough. Then using (5.3) with  $r_1 = r$  and  $r_2 = \infty$  we have the thesis. First of all observe that  $f(u(s), s) > 0$  for any  $s > 0$ , therefore the left hand side inequality of (5.7) is trivially satisfied. In this proof we set  $l$  in (2.9) as in the Hypothesis of the Lemma and leave unsaid the subscript. Since we consider solutions such that  $u(r) \rightarrow 0$  as  $r \rightarrow \infty$ . We can assume without losing of generality that  $K_1(r)u^{q-1} < f(u, r) < CK_1(r)u^{q-1}$  for  $r$  large. This way we find:

$$|y(t)|e^{-\gamma t} = I(r) = C \int_{\log(a)}^t x(s)^{q-1}h^1(s)e^{-\gamma s}ds.$$

Observe that there exists a positive constant  $\sigma(0) \leq n - \frac{m(q-1)}{l-m} = -\gamma$  such that  $I(r) < Cr^{\sigma(0)}$ . We begin by proving that  $x(t)$  and  $y(t)$  tends to 0 exponentially so  $\sigma(0) < -\gamma$ . Note that, if the regularity hypothesis is satisfied, this fact is easily observed using invariant manifold theory.

Moreover, if  $h^1(t) \rightarrow 0$  exponentially, as  $t \rightarrow \infty$ , we are done. Otherwise observe that there exists  $T > 0$  such that  $\dot{x}(t) < 0$  for  $t > T$ . In fact assume for contradiction that  $\dot{x}(t)$  changes sign indefinitely many times for  $t$  large, then there exists a sequence  $t_n \rightarrow \infty$  such that  $x(t_n) > P_x(t_n)$ , where  $P(s) = (P_x(s), P_y(s))$  is the critical point of the frozen autonomous system (4.4) where  $h^1 \equiv h^1(s)$  and  $h^2 \equiv h^2(s)$ . But this is a contradiction since  $x(t) \rightarrow 0$  as  $t \rightarrow \infty$ . So  $\alpha x(t) \leq |y(t)|^{\frac{1}{m-1}}$ . Analogously we have that  $\dot{y}(t) > 0$  for  $t$  large.

Suppose for contradiction that there exists a sequence  $s_n \rightarrow \infty$  such that for any  $\epsilon > 0$   $g(s_n) = |y(s_n)e^{\epsilon s_n}| \rightarrow \infty$ . We may assume that  $g(s_n) \geq g(t)$  for

$t < s_n$  Let us define  $J_k = (s_k, s_{k+1})$ .

$$\begin{aligned} |y(s_N)| &= C e^{\gamma s_N} \int_{\log(a)}^{s_N} x(s)^{q-1} h^1(s) e^{-\gamma s} ds < \\ &< C e^{\gamma s_N} \left( C + \sum_{k=0}^{N-1} \int_{J_k} |y(s)|^{\frac{q-1}{m-1}} h^1(s) e^{-\gamma s} ds \right) < \\ &< C e^{\gamma s_N} \sum_{k=0}^{N-1} |y(s_k) e^{\epsilon s_k}|^{\frac{q-1}{m-1}} \int_{J_k} e^{(-\frac{(q-1)\epsilon}{m-1} - \gamma)s} ds < C e^{\gamma s_N} + C |y(s_N)|^{\frac{q-1}{m-1}} \end{aligned}$$

Therefore for  $t$  large enough we have

$$|y(s_N)| < C |y(s_N)|^{\frac{q-1}{m-1}} \quad \text{and} \quad |y(s_N)| \rightarrow 0$$

Since  $q > m$  we have found a contradiction. Therefore there exists  $\epsilon > 0$  such that  $\limsup_{t \rightarrow \infty} |y(t) e^{\epsilon t}| < C$ . Thus  $\sigma(0) + \gamma < 0$ .

Using (5.3) once again, we get

$$u(r) < C \int_r^\infty \left[ s^{1-n+\sigma(0)} \right]^{\frac{1}{m-1}} ds < C r^{\frac{\sigma(0) - (n-m)}{m-1}}$$

Therefore

$$I(r) < C \int_{\log(a)}^{\log(r)} h^1(s) e^{[n-\delta+(q-1)\frac{\sigma(0)-(n-m)}{m-1}]s} ds = C r^{\sigma(1)}$$

Note that

$$\begin{aligned} \sigma(1) - \sigma(0) &= n - \delta - (q-1) \frac{n-m}{m-1} + \frac{q-m}{m-1} \sigma(0) = \frac{q-m}{m-1} (\sigma(0) + \gamma_q - \delta) \\ \sigma(1) - \sigma(0) &= \frac{q-m}{m-1} (\sigma(0) + \gamma) = -j < 0 \end{aligned}$$

Iterating the reasoning we can find a constant  $\sigma(k) < \sigma(0) - kj$  such that

$$n + (q-1) \frac{\sigma(k) - (n-m)}{m-1} < 0,$$

therefore

$$I(r) < C \int_{\log(a)}^{\log(r)} h^1(s) e^{[n+(q-1)\frac{\sigma(k)-(n-m)}{m-1}]s} ds < \infty.$$

□

Arguing in an analogous way we can understand the behavior of the solutions as  $r \rightarrow 0$ .

**5.5 Lemma.** *Consider Eq. (4.1) and system (4.4) where  $l$  is such that both  $h_l^1(t)$  and  $h_l^2(t)$  are bounded as  $t \rightarrow -\infty$ . Consider any solution  $u(r)$  corresponding to a trajectory having the origin as  $\alpha$ -limit set. We have that  $u(0)$  is positive and finite.*

*Proof.* First of all observe that  $u(0) > 0$  since  $u(r) > 0$  and  $u'(r) < 0$  for  $r > 0$  small. We suppose  $l = p$  and leave unsaid the subscript; the proof in the general case is totally analogous. Once again we can assume that  $K^2(r)u^{p-1} \leq f(u, r) \leq K^2(r)u^{p-1}$ . Suppose for contradiction that  $\limsup_{r \rightarrow 0} u(r) = +\infty$ ; we recall that  $\lim_{r \rightarrow 0} u(r)r^{\alpha_p} = 0$ . Let us call

$$S := \sup\{s \mid \limsup_{r \rightarrow 0} u(r)r^s > 0\}.$$

Observe that  $0 \leq S \leq \alpha_p$ ; we will prove that  $S = \alpha_p$ . Let us consider a value  $s_0$  for which there exists a sequence  $r_k \rightarrow 0$  such that  $\lim_{k \rightarrow \infty} u(r_k)r_k^{s_0} > d > 0$ . Then using Eq. (5.3) we get

$$\begin{aligned} u(r)r^{s_0} - u(b)r^{s_0} &= r^{s_0} \int_r^b \left( t^{1-n} \int_0^t f(u, s)s^{n-1} ds \right)^{\frac{1}{m-1}} dt < \\ C r^{s_0} \int_r^b \left( t^{1-n} \int_0^t h^2 s^{n-1-S(p-1)} ds \right)^{\frac{1}{m-1}} &= C r^{s_0 + \frac{m-S(p-1)}{m-1}} \end{aligned} \quad (5.8)$$

where  $C > 0$  is a constant. Now passing to the limit we get the following

$$\begin{aligned} d &< \lim_{k \rightarrow \infty} u(r_k)r_k^{s_0} - u(b)r_k^{s_0} < \lim_{k \rightarrow \infty} C r_k^{s_0 + \frac{m-S(p-1)}{m-1}} \\ d &< \lim_{k \rightarrow \infty} C r_k^{\frac{p-m}{m-1}(\alpha_p - S) + s_0 - S} \end{aligned} \quad (5.9)$$

Thus  $\frac{p-m}{m-1} \left( \frac{m}{p-m} - S \right) + s_0 - S \leq 0$ .

First of all observe that  $S > 0$ ; in fact otherwise we have  $s_0 = 0$  and we find a contradiction in the previous inequality. From the definition of  $S$ , it follows that we can choose  $s_0 = S - \epsilon$  in Eq. (5.9), for any given  $\epsilon > 0$ . This way we obtain  $\frac{p-m}{m-1} \left( \frac{m}{p-m} - S \right) \leq \epsilon$ . Therefore we get  $S = \alpha_p$ .

Now we prove that  $S \neq \alpha_p$  so the thesis is proved. Note that, if the regularity hypothesis is satisfied, using invariant manifold theory we can easily conclude. First of all observe that if  $h^2(t) \rightarrow 0$  exponentially as  $t \rightarrow -\infty$ , then we are done. Otherwise the proof is for contradiction. Therefore we assume that there exists  $0 < \epsilon < \alpha_p$ , such that  $\limsup_{t \rightarrow -\infty} x(t)e^{-\epsilon t} = +\infty$ . Let us call  $g(t) = x(t)e^{-\epsilon t}$ , there exists a monotone decreasing sequence  $t_n \rightarrow -\infty$  such that  $g(t_n) \rightarrow \infty$ . We can assume without losing of generality that  $g(t_n) \geq g(t)$  for any  $t \in I_n = [t_n, t_{n-1}]$ .

Reasoning as in Lemma (5.4) can prove that  $\dot{x} \geq 0$  and  $\dot{y} \leq 0$  in the intervals we are considering, therefore  $y(t) < Cx(t)^{p-1}$ . Moreover using (5.2) and (5.3)

we find

$$\begin{aligned}
x(t_n) &= e^{\alpha t_n} \left( u(\log(t_0)) + \sum_{j=1}^n \int_{I_j} |y(s)|^{\frac{1}{m-1}} e^{-\alpha s} ds \right) < \\
&< e^{\alpha t_n} \left( u(\log(t_0)) + \sum_{j=1}^n \int_{I_j} |x(s)|^{\frac{p-1}{m-1}} e^{-\alpha s} ds \right) < \\
&< e^{\alpha t_n} \left( u(\log(t_0)) + \sum_{j=1}^n (|x(t_j)| e^{\epsilon t_j})^{\frac{p-1}{m-1}} \int_{I_j} e^{(-\alpha - \frac{\epsilon(p-1)}{m-1})s} ds \right)
\end{aligned}$$

Thus we find  $x(t_n) < C|x(t_n)|^{\frac{p-1}{m-1}} \rightarrow 0$ ; but this is a contradiction, so the proof is concluded.  $\square$

**5.6 Lemma.** *Consider Eq. (1.2) and the corresponding system (4.5) with  $\xi > 0$ , obtained setting  $l$  in (2.9) in order to have that both  $h_l^1(t)$  and  $h_l^2(t)$  are bounded. Consider the trajectories of this system having the origin as  $\alpha$ -limit set. Then the corresponding  $u(r)$  is such that  $u(0)$  is finite and nonnegative.*

*Proof.* First of all observe that if  $\lim_{r \rightarrow 0} u(r) = \infty$ , then  $u(r)$  cannot be monotone increasing as  $r \rightarrow 0$ . Therefore we can find a sequence of intervals  $I_k$  in which  $y < 0$ . Moreover we recall that

$$|y_l(t)| e^{-\gamma_l t} \leq \int_{I_k} | -h_l^1(t)x_l(s)^{q-1} + h_l^2(t)x_l(s)^{p-1} | e^{-\gamma_l s} ds$$

Thus in such intervals we have  $0 < |y_l(t)| e^{-\gamma_l t} < \int_{I_k} h_l^2(t)x_l(s)^{p-1} e^{-\gamma_l s} ds$ .

So the inequality (5.8) still holds in such intervals. Suppose that  $|y_l(t)| e^{-\gamma_l t} \sim x_l(t)^{p-1} e^{\epsilon t}$  for a certain  $\epsilon > 0$ , then we are done. Otherwise we can repeat the proof done for Lemma (5.5) and conclude.  $\square$

## 6 Conclusions

In [6] we used methods similar to the ones of this paper to study the equation

$$(u'|u'|^{m-2})' + \frac{n-1}{r} u'|u'|^{m-2} + K(r)\psi_q(u) = 0 \quad (6.1)$$

The results of that paper reduce to the following known results when  $K$  is a constant.

- if  $K < 0$  all the solutions are monotone increasing so they cannot represent G.S.
- when  $K > 0$  we have three different situations:
  - if  $q > m^*$  all the solutions of (6.1) are G.S. with slow decay, that is  $u(r) \sim r^{-\alpha_q}$ , here  $\alpha_q = \frac{m}{q-m}$ ;

- if  $q = m^*$  all the solutions of (6.1) are G.S. with fast decay, that is  $u(r) \sim r^{-\frac{n-m}{m-1}}$ ;
- if  $q < m^*$  all the solutions of (6.1) are crossing solutions.

So the increasing or the decreasing nature of the solutions  $u(r)$  depends strongly on the sign of  $K$ , while the exponent  $q$  controls the rate of decay or of growth at  $\infty$ . When we study the following equation:

$$(u'|u'|^{m-2})' + \frac{n-1}{r}u'|u'|^{m-2} - K^1\psi_q(u) + K^2\psi_p(u) = 0 \quad (6.2)$$

we find a balance between the increasing rate deriving from the negative term  $-K^1\psi_q(u)$  and the positive one  $K^2\psi_p(u)$ . We always find positive solution with positive finite limit at  $\infty$ , but if  $q < p < m^*$  we also find crossing solutions and G.S. Letting  $p$  reach the value  $p = m^*$  the term  $u^q$  has not anymore the strength to force the solution to decay to 0, because the corresponding decay in (6.1) is too slow. So when  $q < m^* \leq p$  or  $q = m^* < p$  all the solutions have positive lower bound.

The statement continues to be true if  $K^1(r)$  and  $K^2(r)$  are positive functions satisfying a rather reasonable monotonicity condition and such that there exist two positive constants  $d, D$  such that  $d < K^1(r)$  and  $K^2(r) < D$ . To have G.S. also in this last case we need to assume some decay hypothesis on the functions  $K^1(r)$  and some increasing hypothesis on  $K^2(r)$ , in order to give more weight to the decay effect related to the positive term.

We also observe that the behaviour of solution of Eq. (4.7) is “ruled” by the behavior of  $K^l(r)$  if  $q_l \leq m^*$  and by  $K^1(r)$  if  $q_1 \geq m^*$  and the functions are bounded.

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