Ground states and singular ground states for quasilinear elliptic equation in the subcritical case

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Abstract

We consider the following equation

$$\Delta_p u - k_1(|x|)u|u|^{q_1-1} + k_2(|x|)u|u|^{q_2-1} = 0,$$

where $x \in \mathbb{R}^n$, $n > p > 1$, $q_2 - 1 > p > 1$, $q_1 < q_2 < p^* = \frac{np}{n-p} - 1$ and the functions $k_i(|x|)$ are assumed to be strictly positive and bounded. We prove the existence of radial Ground States under suitable Hypotheses on the functions $k_i(r)$. Furthermore we prove the existence of uncountably many radial Singular Ground States; this last result seems to be new even for the autonomous case and even for $p = 2$.

The proofs combine an energy analysis and a new dynamical systems method.

Key Words: p-laplace equations, radial solution, regular/singular ground state, Fowler inversion, invariant manifold.
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1 Introduction

In this paper we discuss positive radial solutions of the following equation

$$\Delta_p u - k_1(r)u|u|^{q_1-1} + k_2(r)u|u|^{q_2-1} = 0 \tag{1.1}$$

where $\Delta_p u = div(|Du|^{p-2}Du)$ is the so called p-Laplacian, $q_2 > p > 1$, $q_1 < q_2 < p^* = \sigma - 1$, $|x| = r$ and $x \in \mathbb{R}^n$, $n > p$. We denote by $\sigma = \frac{np}{n-p}$ the Sobolev critical exponent. We assume that the functions $k_1(r)$ and $k_2(r)$ are positive for $r > 0$ and locally Lipschitz continuous. We give the following notation

$$f(u,r) = -k_1(r)u|u|^{q_1-1} + k_2(r)u|u|^{q_2-1} \quad \text{and} \quad F(u,r) = \int_0^u f(s,r)ds.$$

We denote by $A(r) > 0$ the positive constant such that $f(A(r), r) = 0$ and by $B(r) > 0$ the constant such that $F(B(r), r) = 0$.

In particular we will focus our attention on the problem of existence of ground states (G.S.), of singular ground states (S.G.S.) and of crossing solutions. By G.S. we mean a positive regular solution $u(x)$ defined in the whole of $\mathbb{R}^n$ such that $\lim_{|x|\to\infty} u(x) = 0$. A Singular Ground State (S.G.S) of equation (1.1) is a singular positive solution $v(x)$ such that

$$\lim_{|x|\to 0} v(x) = +\infty \quad \text{and} \quad \lim_{|x|\to+\infty} v(x) = 0.$$

Crossing solutions are radial solutions $u(r)$ such that $u(r) > 0$ for any $0 \leq r < R$ and $u(R) = 0$ for some $R > 0$, so they can be considered as solutions of the Dirichlet problem in the ball of radius R. Here and later we write $u(r)$ for $u(x)$ when $|x| = r$ and u is radially symmetric. We recall that when k_1 and k_2 are positive constants and $1 < p \leq 2$, ground states, singular ground states and solutions of the Dirichlet problem in the ball, can only be radial, see [2] and [20]. It seems reasonable that this result can be extended also to the case where $-k_1(r)$ and $k_2(r)$ are decreasing. Motivated by this consideration we will consider only radial solutions so we will in fact deal with the following singular O.D.E.:

$$(u'|u'|^{p-2})' + \frac{n-1}{r}u'|u'|^{p-2} - k_1(r)u|u|^{q_1-1} + k_2(r)u|u|^{q_2-1} = 0. \tag{1.2}$$

Here $'$ denotes the derivative with respect to r. We will call "regular" the positive solutions $u(u_0, r)$ of (1.2) satisfying the following initial condition

$$u(0) = u_0 > 0 \qquad\qquad u'(0) = 0. \tag{1.3}$$

We will call "singular" the positive solutions $u(r)$ which are singular in the origin, that is $\lim_{r\to 0} u(r) = \infty$.

The corresponding autonomous equation is well studied and understood. In the pioneering work [10], the authors analyze the case where $p = 2$ and q_1 is sublinear. They prove the existence of a radial G.S. and of both oscillatory and crossing solutions, assuming that the functions $-k_1(r)$ and $k_2(r)$ are constant or decreasing. In [9] the authors state the existence of radial ground states for the autonomous problem when $p \geq n$ and when $n > p$ and $q_1 < q_2 < p^*$. One of the main contribution of this paper is the extension of these existence results for G.S. obtained for the autonomous case in [9], to the non-autonomous case. More precisely we allow k_1 and k_2 to belong to a wide class of bounded positive functions instead of being positive constants.

In [20] it was proved that S.G.S., if they exist, have to be radial. However, to the best of our knowledge, the problem of existence of S.G.S. was still open, even in the autonomous case. The most important result are the ones of Theorem (2.4), where we prove the existence of uncountably many S.G.S. under some weak assumptions on the functions $k_i(r)$. This result holds and it is new even for the autonomous case.

Pucci and Serrin in [19] proved the uniqueness of this G.S. for the spatial independent equation, assuming $q_1 < q_2 < p^*$ and requiring some further conditions on these parameters. We conjecture that the uniqueness of G.S. still holds when the functions $-k_1(r)$ and $k_2(r)$ are decreasing, but we believe that the existence of multiple G.S may be proved if such a condition is violated, perhaps constructing some ad hoc functions $-k_1(r)$ and $k_2(r)$.

We wish to recall, that Ni and Serrin, in [18], have proved that the autonomous equation admits neither G.S. nor crossing solutions, when $q_1 < q_2$ and $q_2 \geq p^*$; the proof is a direct consequence of the Pohozaev identity. Some of these results were generalized to the non-autonomous setting in [4].

We finish this introduction by giving some terminology. Recall that given a system of the form

$$\dot{x} = f(x, t)$$

and a solution $x(t)$, the α-limit set of $x(t)$ is the set

$$A = \left\{ P : \exists t_n \to -\infty \quad \text{such that} \lim_{n\to\infty} x(t_n) = P \right\},$$

while the ω-limit set is the set

$$W = \left\{ P : \exists t_n \to +\infty \quad \text{such that} \lim_{n\to\infty} x(t_n) = P \right\}.$$

One can show that, if $x(t)$ is bounded on $\mathbb{R}$, then these sets are compact. Moreover, if the system is autonomous, these sets are invariant for the flow generated by the system.

2 Statement of the results

The existence of a solution of problem (1.2), (1.3) is equivalent to the existence of a fixed point for the following operator $T : \mathcal{C}(0,r) \to \mathcal{C}^1(0,r)$

$$Tu(r) = u_0 - \int_0^r \left(t^{1-n} \int_0^t f(u,s) s^{n-1} ds \right)^{\frac{1}{p-1}} dt \tag{2.1}$$

Using the Schauder fixed point theorem, local existence for solutions of problem (1.2), (1.3) can be proved, see for example pages 238–242 in [8]. We will give also an alternative proof of this standard result.

We collect here some of the main hypotheses which will be used in the paper

P1 $1 < p \leq 2$ and $q_1 \geq 1$.

P2 $q_1 > p_* = \frac{n(p-1)}{n-p}$

F1 $\lim_{r\to 0} |k_1'(r) r^{1+(q_2-q_1)\frac{p}{q_2-p+1}}| + |k_2'(r) r| < \infty$.

F2 There exist $S_1 > 0$ and $S_2 > 0$ such that $k_1(r) \geq S_1$ and $k_2(r) \leq S_2$ for any $r > 0$. Furthermore $\frac{k_1(0)}{k_2(0)} < \frac{S_1(q_2+1)}{S_2(q_1+1)}$.

F3 $\lim_{r\to\infty} |k_1'(r) r| + |k_2'(r) r^{1-(q_2-q_1)\frac{p}{q_1-p+1}}| < \infty$.

If we assume that the functions $k_i(r)$ are bounded as $r \to 0$, we can prove that regular solutions of (1.2) are such that $u'(r) \leq 0$ for r small, see Lemma (1.1.1) in [8]. From that Lemma we can easily deduce also the following useful result.

2.1 Lemma. *Assume that the functions $k_i(r)$ are bounded for r close to $R > 0$. Then if $u(R)$ is a critical point for $u(r)$ and $f(u(R),R) > 0$, then $u(R)$ is a (local) maximum, while if $f(u(R),R) < 0$, then $u(R)$ is a (local) minimum.*

It can also be proved that any regular solution $u(r)$ of (1.2), (1.3) where $u_0 > A(0)$ can be continued in $J(u_0) = (0, R_{u_0}) = \{r > 0 \mid u'(r) < 0 \text{ and } u(r) \geq 0\}$, where R_{u_0} can also be infinite, see again [8], for example. We will denote by $u(u_0; r)$ the solution of (1.2), (1.3). Note that if $u_0 > A(0)$ there exists the limit $\lim_{r\to R_{u_0}} u(u_0; r) = L(u_0) \geq 0$. Assume that Hypothesis F2 is satisfied, then we construct the following sets:

$$I^- = \{u_0 > A(0) \mid R_{u_0} < \infty \text{ and } L(u_0) = 0\}$$
$$I^+ = \{u_0 > A(0) \mid L(u_0) > 0\}$$

The sets I^+ and I^- are obviously disjoint. Now we are ready to state the following crucial Lemma.

2.2 Lemma. *Assume that Hypotheses F1, F2, P1 are satisfied. Then I^+ and I^- are nonempty and open.*

The proof of Lemma (2.2) is long so we divide it in many steps. In this section we will show that I^+ is open and nonempty. In section 3 we prove that I^- is open and nonempty. The proof of this result involves the introduction of a dynamical system and it is rather long, but it exploits a new method which could be useful also for other families of quasilinear equations. From Lemma (2.2) it follows that there exists a value $A^* \notin (I^+ \cup I^-)$, $A^* > A(0)$ disconnecting I^+ and I^-. Therefore $L(A^*) = 0$ and $R_{A^*} = \infty$. Thus we can easily deduce one of the main Theorems of the paper.

2.3 Theorem. *Assume that Hypotheses F1, F2, P1 are satisfied, then (1.1) admits at least one monotone decreasing radial G.S.*

As a consequence of the dynamical analysis developed in section 3, we also deduce the following Theorem.

2.4 Theorem. *Assume that Hypotheses F1, F2, F3, P1 and P2 are satisfied, then (1.1) admits uncountably many radial S.G.S.*

This is perhaps the most important result of the paper since, to the best of our knowledge, the existence of such solutions had not been observed previously even in the autonomous case and even for the classical Laplacian, that is when $p = 2$.

We introduce now an energy functional closely related to the ones used in many papers involving the autonomous case, see e. g. [9].

$$
\begin{aligned}
E(u,r) &= \frac{p-1}{p}|u'|^p + F(u,r) \\
\frac{d}{dr}E(u,r) &= -\frac{dk_1}{dr}(r)\frac{|u|^{q_1+1}}{q_1+1} + \frac{dk_2}{dr}(r)\frac{|u|^{q_2+1}}{q_2+1} - \frac{n-1}{r}|u'|^p.
\end{aligned}
$$

Note that if $E(u(r),r) < 0$ then $0 < u(r) < B(r)$. Assume at first that $F(u,r)$ is monotone decreasing in r; we want to prove that $B(0) = A \in I^+$. Note that $\frac{d}{dr}E(u,r) \leq 0$ for any $r \geq 0$. Therefore we have

$$
F(u(A,r),r) \leq E(u(A,r),r) < E(u(A,0),0) = F(u(A,0),0) = 0
$$

for r large. In particular $\lim_{r\to R_A} F(u(A,r),r) \leq \lim_{r\to R_A} E(u(A,r),r) < 0$ so the claim is proved. Working a little on this idea we can prove the following more general Lemma. Let us define the following functionals which do not depend explicitly on r.

$$\bar{F}(u) = -S_1 \frac{|u|^{q_1+1}}{q_1+1} + S_2 \frac{|u|^{q_2+1}}{q_2+1} \quad \text{and} \quad \bar{E}(u) = \frac{p-1}{p}|u'|^p + \bar{F}(u).$$

Here and later S_1 and S_2 are the positive constants satisfying the relation in F2

2.5 Remark. Note that if hypothesis F2 is satisfied, we can find $D > A(0)$ such that $f(D, 0) > 0$ and $\bar{F}(D) \leq 0$.

2.6 Lemma. *Assume that Hypothesis F2 is satisfied, then $I^+ \neq \emptyset$.*

Proof. Fix $D > 0$ as in Remark (2.5). We want to show that $D \in I^+$, so we consider the solution $u(D, r)$: note that $u'(D, r) < 0$ for $r > 0$ small. Hence $u'(D, r) < 0$ for $r \in J(D)$ and

$$\frac{d}{dr}\bar{E}(u(r)) = [(S_2 - k_2(r))u|u|^{q_2-1} - (S_1 - k_1(r))u|u|^{q_1-1}]u' - \frac{n-1}{r}|u'|^p < 0.$$

Thus the limit $\lim_{r \to R_D} E(u(D, r), r)$ exists and is negative. It follows that $\lim_{r \to R_D} F(u(D, r), r) < 0$, so $D \in I^+$. $\square$

We point out that the solutions of (1.2), (1.3) depend continuously on initial data and are locally unique in their respective sets $J(u_0)$. This can be proved by putting together the ideas of Propositions A3 and A4 in [8], with some trivial modification to adapt them to the non-autonomous problem; see also Proposition 2.6 in [9].

2.7 Lemma. *Assume $k_1(0) \geq 0$ and $k_2(0) > 0$. Fix $u_0 > A(0)$, then for any $\epsilon > 0$ and $r_0 \in J(u_0)$, there exists $\delta > 0$ such that if $|v_0 - u_0| < \delta$, then $v(r)$ is defined in $[0, r_0]$ and*

$$\sup_{r \in [0, r_0]} (|u(r) - v(r)| + |u'(r) - v'(r)|) < \epsilon.$$

Now reasoning as in [9] we can prove the following Lemma.

2.8 Lemma. *Assume that Hyp. F2 is satisfied, then I^+ is open.*

Proof. Let $c \in I^+$ and consider a sequence $c_k \to c$. We want to show that $c_k \in I^+$ for k large. Choose $R \in J(c) \cap J(c_k)$ such that $\bar{E}(u(c, R)) < 0$. If k is large enough, by Lemma (2.7) we have $\bar{E}(u(c, R)) < \frac{\bar{E}(u(c_k, R))}{2} < 0$. So reasoning as in Lemma (2.6) we have the thesis. $\square$

3 Dynamical Analysis

3.1 Fowler transformation and autonomous system

In this section we want to prove that $I^- \neq \emptyset$; in fact we will show that there exists a positive constant $c > B(0)$ such that $[c; +\infty) \subset I^-$. To achieve our task we need to introduce a dynamical system, through the following change of coordinates which is a generalization of the Fowler transformation for the classical Laplacian, see [3], [4] and [5].

$$\begin{gathered}
\alpha_l = \frac{p}{l-p+1}, \quad \beta_l = \frac{pl}{l-p+1} - 1, \quad \gamma_l = \beta_l - (n-1), \quad l > p-1 \\
x_l = u(r) r^{\alpha_l} \quad y_l = u'(r)|u'(r)|^{p-2} r^{\beta_l} \quad r = e^t \\
\phi_1(t) = k_1(e^t) = k_1(r), \quad \phi_2(t) = k_2(e^t) = k_2(r), \\
h_{1,l}(t) = \phi_1(t) e^{\delta_l t}, \quad h_{2,l}(t) = \phi_2(t) e^{\eta_l t}
\end{gathered} \tag{3.1}$$

where $\delta_l = \alpha_l(l - q_1) = p\left(1 - \frac{q_1-p+1}{l-p+1}\right)$, and $\eta_l = \alpha_l(l - q_2) = p\left(1 - \frac{q_2-p+1}{l-p+1}\right)$. Using (3.1) we change (1.2) into the following dynamical system.

$$\begin{pmatrix} \dot{x}_l \\ \dot{y}_l \end{pmatrix} = \begin{pmatrix} \alpha_l & 0 \\ 0 & \gamma_l \end{pmatrix} \begin{pmatrix} x_l \\ y_l \end{pmatrix} + \begin{pmatrix} y_l |y_l|^{\frac{2-p}{p-1}} \\ h_{1,l}(t)\psi_{q_1}(x_l) - h_{2,l}(t)\psi_{q_2}(x_l) \end{pmatrix} \tag{3.2}$$

where "·" denotes derivation with respect to t and $\psi_l(s) = |s|^{l-1}s$.

3.1 Remark. It is worthwhile to point out that Eq. (3.2) is $\mathcal{C}^1$ if and only if Hypothesis P1 is satisfied and the functions $h_{i,l}(t)$ are $\mathcal{C}^1$. In fact if Hyp. F1 and P1 are satisfied Eq. (3.2) is locally Lipschitz continuous, so local uniqueness of the solutions is ensured. We will use some integral involving the derivative of the functions $h_{i,l}(t)$. When the classical derivatives of $h_{i,l}(t)$ do not exist we could replace it with the weak derivative or rewrite the expression integrating by parts. However for simplicity reasons we will always write $\dot{h}_{i,l}(t)$.

Note that if we set $l = q_1$ we obtain $h_{1,l}(t) = \phi_1(t)$, and with $l = q_2$ we have $h_{2,l}(t) = \phi_2(t)$. We will set $l = q_2$ to investigate the behaviour of trajectories as $t \to -\infty$ and $l = q_1$ to investigate the behaviour of trajectories as $t \to \infty$. We wish to stress that $\gamma_l < 0 < \alpha_l$ whenever $l > p_*$ and $\gamma_l + \alpha_l > 0$ whenever $l < p^*$. We point out now some elementary correspondences between Eq. (1.2) and system (3.2).

3.2 Remark. Positive solutions $u(r)$ of Eq. (1.2) correspond to trajectories of Eq. (3.2) belonging to the halfplane $\mathbb{R}^2_+ := \{(x, y) \mid x \geq 0\}$. Furthermore positive decreasing solutions $u(r)$ of Eq. (1.2) correspond to trajectories of Eq. (3.2) belonging to the 4^{th} quadrant and viceversa.

We give now a Proposition concerning the asymptotic behavior of positive solutions; the proof can be found in [4].

3.3 Proposition. *Assume that* $\lim_{t\to-\infty} h_{1,l}(t) = A \geq 0$ *and* $\lim_{t\to-\infty} h_{2,l}(t) = B > 0$. *Trajectories* $X(t)$ *of Eq. (3.2) such that* $\lim_{t\to-\infty} X(t) = O = (0,0)$ *correspond to regular solutions* $u(r)$ *of Eq. (1.2) and viceversa.*

Assume that $\lim_{t\to\infty} h_{1,l}(t) = A \geq 0$ *and* $\lim_{t\to\infty} h_{2,l}(t) = B > 0$. *Trajectories* $X(t)$ *of Eq. (3.2), which are well defined and belong to* $\mathbb{R}^2_+$ *for* t *large, and satisfying* $\lim_{t\to\infty} X(t) = O$ *correspond to solutions* $u(r)$ *of Eq. (1.2) which are well defined and positive for* r *large and have fast decay that is* $u(r) = o(r^{-\frac{n-p}{p-1}})$, *and viceversa.*

Assume that the functions $k_i(r)$ *are strictly positive and bounded. Then solutions* $u(r)$ *of Eq. (1.2) which are well defined and positive for* r *large and tend to 0 as* $r \to \infty$ *are such that* $u(r) = o(r^{-\frac{n-p}{p-1}})$, *so for the corresponding trajectory* $X(t)$ *of Eq. (3.2) we have* $\lim_{t\to\infty} X(t) = O$

We introduce now some auxiliary functions which are related to the Pohozaev identity.

$$H_l(x_l, y_l, t) := \alpha_l x_l y_l + \frac{p-1}{p}|y_l|^{\frac{p}{p-1}} - \frac{h_{1,l}(t)}{q_1+1}|x_l|^{q_1+1} + \frac{h_{2,l}(t)}{q_2+1}|x_l|^{q_2+1}.$$
$$J_l(x_l, y_l, t) := -\gamma_l x_l y_l + \frac{p-1}{p}|y_l|^{\frac{p}{p-1}} - \frac{h_{1,l}(t)}{q_1+1}|x_l|^{q_1+1} + \frac{h_{2,l}(t)}{q_2+1}|x_l|^{q_2+1}.$$

Note that differentiating with respect to t we find

$$\frac{d}{dt} H_l(x_l(t), y_l(t), t) := (\alpha_l + \gamma_l)\dot{x}_l y_l - \frac{\dot{h}_{1,l}(t)}{q_1+1}|x_l|^{q_1+1} + \frac{\dot{h}_{2,l}(t)}{q_2+1}|x_l|^{q_2+1}.$$
$$\frac{d}{dt} J_l(x_l(t), y_l(t), t) := -(\alpha_l + \gamma_l)x_l \dot{y}_l - \frac{\dot{h}_{1,l}(t)}{q_1+1}|x_l|^{q_1+1} + \frac{\dot{h}_{2,l}(t)}{q_2+1}|x_l|^{q_2+1}.$$

We begin our analysis of Eq. (3.2) assuming that the system is autonomous:

$$h_{1,l}(t) \equiv A > 0 \text{ and } h_{2,l}(t) \equiv B > 0 \text{ and } l < p^*. \tag{3.3}$$

This is a strong assumption: in fact if $l = q_2$ this is equivalent to ask that $k_1(r) = Ar^{-\delta_l}$ and $k_2(r) \equiv B$ for any $r > 0$. We give now some notation which will be in force throughout the whole paper.

$$U^+ := \{(x_l, y_l) \in \mathbb{R}^2 \quad | \quad \dot{x}_l > 0\} \quad \text{and} \quad U^- := \{(x_l, y_l) \in \mathbb{R}^2 \quad | \quad \dot{x}_l < 0\}$$
$$C := \{(x_l, y_l) \in \mathbb{R}^2 \quad | \quad \dot{x}_l = 0\}$$

Note that these definitions depend on l even if it is not explicitly indicated. The autonomous system (3.2), (3.3) admits exactly three critical points: the origin O, $P=(P_x,P_y)$, where $P_y<0<P_x$, and $-P$. The origin is a saddle point and P is an unstable focus. When Hyp. P1 is satisfied, using standard invariant manifold theory, we can prove the existence of a stable and an unstable manifold for (3.2), respectively W^s and W^u. We want to show that they are as depicted in Fig. (1).

3.4 Observation. System (3.2), (3.3) admits no periodic trajectories.

Proof. Note that $\frac{d\dot{x}_l}{dx_l}+\frac{d\dot{y}_l}{dy_l}=\alpha_l+\gamma_l>0$. Assume for contradiction that there exists a periodic trajectory $X_l(t)$ of period T, and call its graph ∂B and B the bounded set enclosed by ∂B. Then

$$0=\int_0^T(\dot{x}_l\dot{y}_l-\dot{y}_l\dot{x}_l)dt=\int_{\partial B}\dot{x}_l dy_l-\dot{y}_l dx_l=\int_B(\frac{d\dot{x}_l}{dx_l}+\frac{d\dot{y}_l}{dy_l})dx_l dy_l>0.$$

So we have found a contradiction and the claim is proved. □

3.5 Observation. We observe now that the level sets of the function H and J are closed bounded curves. The minimum for both is obtained at the critical point P and it is negative. The curves defined by $H=0$ and $J=0$ are 8-shaped: they are made by the union of a closed curve contained in $\mathbb{R}^2_+$, crossing the x axis and to which the origin belong, and its reflection with respect to the origin. Inside the bounded set enclosed by the level set $H=0$, the function H is negative, outside it is positive. The same holds for J.

The proof follows from elementary reasonings so it will be omitted. We will commit the following abuse of notation: when we consider an autonomous system (3.2), satisfying (3.3), we write the functions $H(x,y)$ for $H(x,y,t)$ and $J(x,y)$ for $J(x,y,t)$.

3.6 Lemma. *Consider the autonomous system (3.2) and assume that (3.3) holds. Then any trajectory which becomes unbounded going backwards or forward in t, has to rotate clockwise and cross the coordinate axes indefinitely.*

Proof. Consider a trajectory $X(t)=(x(t),y(t))$ of the autonomous system (3.2) with $p_*<l<p^*$, which is continuable forward for any $t<T$, where $T\le\infty$. Assume that there is a sequence $t_k\to T$ such that $\lim_{k\to\infty}|X(t_k)|=\infty$. Then it follows that $\lim_{k\to\infty}H(X(t_k))=\infty$ and $\lim_{k\to\infty}J(X(t_k))=\infty$. Assume for contradiction that $X(t)\in\mathbb{R}^2_+$ for t in a left neighborhood of T. We can assume without losing of generality that there is T_0 such that $x(T_0)>0$ and $y(T_0)>0$. If $\dot{y}(T_0)\ge 0$ we have that $J(X(t))$ is monotone decreasing for $t\ge T_0$, until $X(t)$ crosses the isocline $\dot{y}=0$. Since we have assumed that $X(t)$ becomes unbounded forward in t there exists $T_1>T_0$ such that $X(t)$ enters in the subset of the 1^{st} quadrant where $\dot{y}(T_1)<0$. We

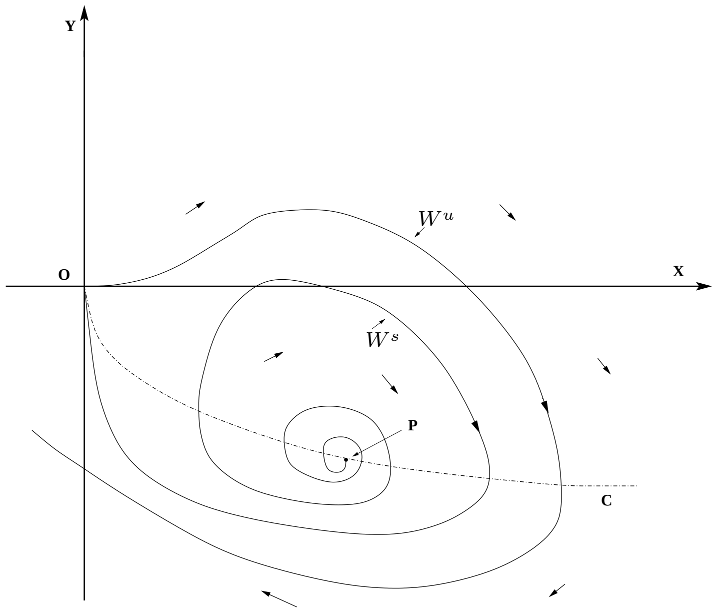

Figure 1: A sketch of the phase portrait for the autonomous system

claim that there exists T_2 at which $X(t)$ crosses the x positive semi-axis. In fact otherwise we would have

$$\frac{\dot{y}(t)}{\dot{x}(t)} \sim \frac{-Bx(t)^{q_2}}{\alpha x(t)} < -\epsilon < 0 \qquad\qquad \text{as } t \to T^-$$

if $q_2 \geq 1$; but this is a contradiction, so the claim is proved. Since $H(\cdot)$ is decreasing in the subset of U^+ where $y < 0$, we have that $X(t)$ has to cross the isocline C for a certain $T_3 > T_2$. From an elementary analysis of the phase portrait it follows that there exists $T_4 > T_3$ for which $X(t)$ has to cross the isocline $\dot{y} = 0$ and enter in the subset of the 4^{th} quadrant where $\dot{y} \geq 0$ and $\dot{x} < 0$. Then $X(t)$ has to cross once again the isocline C and enter in the subset where $\dot{x} > 0$. But, since we have assumed that $X(t)$ becomes unbounded, it must have a self-intersection. But this is in contradiction with the fact that system (3.2) is $\mathcal{C}^1$ and autonomous. Therefore $X(t)$ cannot stay in $\mathbb{R}^2_+$ for every t in a left neighborhood of T. Using the fact that system (3.2) is symmetric with respect to the origin, we prove that any trajectory which is unbounded forward in t must cross the coordinate axes clockwise indefinitely.

An alternative proof can be obtained using polar coordinates (ρ, θ) for system (3.2). In this way we find that

$$\dot{\theta} = (-\alpha+\gamma)\cos\theta\sin\theta - [\rho^{\frac{2-p}{p-1}}|\sin\theta|^{\frac{p}{p-1}} - A\rho^{q_1-1}|\cos\theta|^{q_1+1} + B\rho^{q_2-1}|\cos\theta|^{q_2+1}]$$

Therefore $\dot{\theta} \to -\infty$ for $\rho \to \infty$. Thus if $X(t) \in \mathbb{R}^2_+$ for t close to T and $X(t)$ is unbounded forward in t we can find two sequences $\hat{t}_k$ and $\check{t}_k$, $\hat{t}_k < \check{t}_k < \hat{t}_{k+1}$ both converging to T and such that $X(\hat{t}_k)$ is bounded and $X(\check{t}_k)$ is unbounded. It follows that $X(t)$ must intersect the isocline C infinitely many times. Reasoning as above we find that $X(t)$ must have a self-intersection, so we have found a contradiction.

Reasoning in the same way we can prove that if $X(t)$ is unbounded backward in t, then it must cross the coordinate axes indefinitely. □

We want to prove that the unstable and the stable manifolds W^u and W^s are shaped as in Fig. (1). We follow a trajectory $\hat{X}(t) = (\hat{x}(t), \hat{y}(t))$ of system (3.2), (3.3) belonging to the unstable manifold W^u. We want to prove that $\hat{X}(t)$ is unbounded forward in t. Therefore, from Lemma (3.6) it follows that it has to cross the coordinate axes infinitely many times rotating clockwise. It is easily observed that $\lim_{t \to -\infty} \hat{X}(t) = (0,0) = O$, and that $\hat{X}(t)$ is in the 1^{st} quadrant in the subset where $\dot{x} > 0$ and $\dot{y} > 0$ for $t << 0$. Repeating the reasoning of Lemma (3.6) we can prove that $\hat{X}(t)$ crosses the isocline $\dot{y} = 0$, then the x positive semi-axis and enters the 4^{th} quadrant. Note that the critical point P is an unstable focus, so $\hat{X}(t)$ cannot have P as ω-limit set. Recalling this observation and reasoning as in Lemma (3.6), we have that $\hat{X}(t)$ has to reach the isocline C at some $t = T_1$. From an elementary analysis of the phase portrait it follows that there exists $T_2 > T_1$ for which $\hat{X}(t)$ has to cross the isocline $\dot{y} = 0$.

Since P is an unstable focus, there exists at least one trajectory $\check{X}(t)$ having P as α-limit set. Note that $\check{X}(t)$ cannot cross $\hat{X}(t)$, therefore there are only two possibilities: $\check{X}(t)$ is in $\mathbb{R}^2_+$ for any $t > 0$ and it has the origin as ω-limit set, or there exists $\check{T}$ for which $\check{X}(t)$ crosses the y negative semi-axis. If the former possibility is verified, then W^s is as depicted in figure (1). Assume that the latter holds. Consider a trajectory $\bar{X}(t)$ belonging to the stable manifold W^s, and follow it backwards in t. Observe now that W^u and W^s are both one-dimensional manifolds. Since $W^u \not\equiv W^s$ we have that $W^u \cap W^s = \emptyset$. Thus there exists $T_3 > T_2$ such that $\hat{X}(t)$ crosses the y negative semi-axis. Since $\bar{X}(t)$ cannot intersect $\check{X}(t)$, we have that $\bar{X}(t)$ has P as α-limit set. So once again W^s is as depicted in figure (1). Reasoning in this way we can prove that W^u becomes unbounded rotating clockwise and crossing the coordinate axes indefinitely.

From this dynamical analysis now we can easily deduce the following Proposition.

3.7 Proposition. *Consider system (1.2) and assume that (3.3) and Hyp. P1 are satisfied. Then all the regular solutions $u(r)$ are crossing solutions. Furthermore there exists uncountably many S.G.S. $v(r)$, that is $v(r)$ is well* *defined and positive for any $r > 0$, $v(r)r^{\alpha_l} \to P_x > 0$ as $r \to 0$ and $v(r)r^{\frac{n-p}{p-1}} \to c \geq 0$ as $r \to \infty$.*

Note that system (3.2), (3.3) is invariant for translations in t. Therefore if $X(t)$ is a solution, $X_\tau(t) = X(t+\tau)$ is a solution as well. Equivalently if $u(r)$ is a solution of (1.2), then $v(r) = u(re^\tau)e^{\alpha\tau}$ is a solution as well.

Let us call $R_1(u_0)$ the first value for which the solution $u(u_0, r)$ of (1.2), (1.3) becomes null, that is $u(u_0, R_1(u_0)) = 0$. Then the following corollary holds.

3.8 Corollary. *Consider (1.2) and assume that (3.3) and Hyp. P1 are satisfied. Then, for any $R_1 > 0$ the Dirichlet problem in the ball of radius R_1 admits exactly one solution. Moreover*

$$\lim_{u_0 \to \infty} R_1(u_0) = 0 \qquad \text{and} \qquad \lim_{u_0 \to 0} R_1(u_0) = \infty$$

Furthermore for any $u_0 > 0$, the solution $u(u_0, r)$ admits an infinite sequence of values $R_k \to \infty$ such that $u(u_0, R_k) = 0$.

3.2 Existence of crossing solutions

We turn now to consider the non-autonomous system (3.2). It will be useful to embed this system in the following family of non-autonomous system, where we have added a parameter of translation in t.

$$\begin{aligned} \dot{x}_l =& \alpha_l x_l + y_l |y_l|^{\frac{2-p}{p-1}} \\ \dot{y}_l =& \gamma_l y_l + h_{1,l}(\tau + t)\psi_{q_1}(x_l) - h_{2,l}(\tau + t)\psi_{q_2}(x_l) \end{aligned} \tag{3.4}$$

We want to make a geometrical analysis of the phase portrait comparing solutions of the autonomous and non-autonomous system (3.4). We will carry out the analysis on figure (2) so we introduce some notation in order to explain it. We denote by $W^u(a_1, a_2)$ and $W^s(a_1, a_2)$ respectively the unstable and the stable manifold of the autonomous system (3.4) where $h_{1,l}(t) \equiv a_1$ and $h_{2,l}(t) \equiv a_2$. We call $\xi^+(a_1, a_2) = (x^+(a_1, a_2), y^+(a_1, a_2))$ and $\xi^-(a_1, a_2) = (x^-(a_1, a_2), y^-(a_1, a_2))$ the first intersection of the isocline C respectively with $W^u(a_1, a_2)$ and $W^s(a_1, a_2)$. We call $\tilde{W}^u(a_1, a_2)$ and $\tilde{W}^s(a_1, a_2)$ the branches of $W^u(a_1, a_2)$ and $W^s(a_1, a_2)$ respectively joining the origin and $\xi^+(a_1, a_2)$ and $\xi^-(a_1, a_2)$. Consider the autonomous system such that $h_{i,l}(t) \equiv a_i$; let us call $\bar{X}^u(a_1, a_2, t)$ the trajectory departing from $\xi^+(a_1, a_2)$ at $t = 0$, and $\bar{X}^s(a_1, a_2, t)$ the trajectory departing from $\xi^-(a_1, a_2)$. Set $a_1 = h_{1,l}(\tau)$ and $a_2 = h_{2,l}(\tau)$, we denote by $t = T(\tau)$ the first value for which $\bar{X}^u(a_1, a_2, t)$ intersects the negative y semi-axis. Note that if a_1 is nonnegative and $a_2 \in [c_1, c_2]$, where $0 < c_1 < c_2 < \infty$, then $T(\tau)$ is uniformly bounded. We wish to point out that all these points depend continuously with respect to τ. In fact this dependence is at least as regular as system (3.4). Fix τ, we give the following definitions

$$\begin{aligned} a_1(\tau) &= \inf\{h_{1,l}(t) \mid t \le \tau\}, & a_2(\tau) &= \inf\{h_{2,l}(t) \mid t \le \tau\} \\ b_1(\tau) &= \sup\{h_{1,l}(t) \mid t \le \tau\}, & b_2(\tau) &= \sup\{h_{2,l}(t) \mid t \le \tau\}, \\ \underline{\xi^+}(\tau) &= \xi^+(a_1(\tau), b_2(\tau)) & \overline{\xi^+}(\tau) &= \xi^+(b_1(\tau), a_2(\tau)) \end{aligned}$$

and $c(\tau)$ the segment of C between $\underline{\xi^+}(\tau)$ and $\overline{\xi^+}(\tau)$. We use the following notation $\underline{\xi^+}(\tau) = (\underline{x^+}(\tau), \underline{y^+}(\tau))$ and $\overline{\xi^+}(\tau) = (\overline{x^+}(\tau), \overline{y^+}(\tau))$. Note that $\underline{x^+}(\tau) < \overline{x^+}(\tau)$.

We call $\partial e(\tau)$ the union of $\tilde{W}^u(a_1(\tau), b_2(\tau))$ and $\tilde{W}^u(a_2(\tau), b_1(\tau))$, and $e(\tau)$ the bounded subset enclosed by the origin, $\partial e(\tau)$ and $c(\tau)$. When τ is fixed we use the following notation: we underline the quantities obtained from (3.4) in the autonomous case where $h_{1,l}(t) \equiv a_1(\tau)$ and $h_{2,l}(t) \equiv b_2(\tau)$, e.g. $\underline{X}(t)$, we overline the quantities obtained from (3.4) in the autonomous case where $h_{1,l}(t) \equiv b_1(\tau)$ and $h_{2,l}(t) \equiv a_2(\tau)$, e.g. $\overline{X}(t)$, and we do not underline nor overline quantities of the non-autonomous system, e.g. $X(t)$. We denote by $X(Q, t)$ the trajectory of (3.4) departing at $t = 0$ from $Q \in \mathbb{R}^2$. Now we are ready to state the following Lemma.

3.9 Lemma. *Assume that Hyp. P1 is satisfied and that* $\lim_{t\to-\infty} h_{1,l}(t) \ge 0$ *and* $0 < \lim_{t\to-\infty} h_{2,l}(t) < \infty$. *Then for any τ there exists $Q \in C$ such that for the trajectory $X(Q,t)$ of (3.4) we have* $\lim_{t\to-\infty} X(Q,t) = (0,0)$ *and $X(Q,t) \in U^+$ for any $t < 0$. Let us call $u(u_0, r)$ the corresponding solution of (1.2), then $u_0 \to \infty$ as $\tau \to -\infty$ and $u_0 \to 0$ as $\tau \to \infty$.*

Proof. Observe that $\dot{x}(x,y) = \underline{\dot{x}}(x,y) = \overline{\dot{x}}(x,y)$ on $\partial e(\tau)$, and $\dot{y}(x,y) \ge \underline{\dot{y}}(x,y)$ on $\tilde{W}^u(a_1(\tau), b_2(\tau))$ and $\dot{y}(x,y) \le \overline{\dot{y}}(x,y)$ on $\tilde{W}^u(a_2(\tau), b_1(\tau))$. Now consider the non-autonomous system (3.4); note that the flow on $\partial e(\tau)$ points towards the interior of $e(\tau)$ for any $t \le 0$. Therefore, using Wasewzki's principle, see [7] and [11], it follows that there exists a point $Q = (Q_x, Q_y) \in c(\tau)$ such that the trajectory $X(Q,t)$ departing from Q at $t = 0$ is forced to stay in $e(\tau)$ for any $t \le 0$ and $\lim_{t\to-\infty} X(Q,t) = (0,0)$. This proves the first part of the Lemma.

We recall that from Proposition (3.3) we know that the solution $u(r)$ of (1.2) corresponding to $X(Q,t)$ is regular, therefore $u(0) = b$ for a certain $b > 0$. Fix τ and consider the trajectories $\underline{X^+}(t)$, $X^+(t)$, $\overline{X^+}(t)$ of system (3.4) departing at $t = 0$ respectively from $\underline{\xi^+}(\tau)$, Q and $\overline{\xi^+}(\tau)$. They correspond

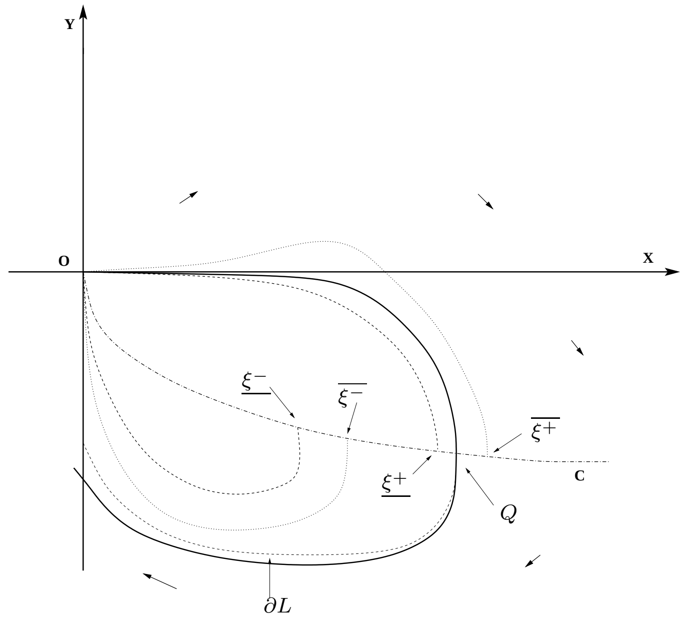

Figure 2: Construction of the crossing solutions. The solid line represents the trajectory $\hat{X}(Q,t)$ of the non-autonomous system. We have used dashed lines for trajectories of the autonomous system where $h_{1,l} \equiv a_1(T_1(\tau))$ and $h_{2,l} \equiv b_2(T_1(\tau))$ and dotted lines for trajectories of the autonomous system where $h_{1,l} \equiv b_1(T_1(\tau))$ and $h_{2,l} \equiv a_2(T_1(\tau))$

to regular solutions, say $\underline{u}(a,r), u(b,r), \overline{u}(c,r)$, of their respective equation of type (1.2). Note that $\underline{x^+}(t) \leq x^+(t) \leq \overline{x^+}(t)$ for any $t < 0$. In fact let us call $\bar{t} = \inf_{t<0}\{\overline{x^+}(t) < x^+(t)\}$, then $\overline{x^+}(\bar{t}) = x^+(\bar{t})$ and $\overline{x^+}(t) < x^+(t)$ for t in a right neighborhood of $t = \bar{t}$. Therefore $\overline{y^+}(\bar{t})|\overline{y^+}(\bar{t})|^{\frac{2-p}{p-1}} < y^+(\bar{t})|y^+(\bar{t})|^{\frac{2-p}{p-1}}$; but this is a contradiction because $X^+(t) \in e(\tau)$.

Therefore for the corresponding solutions of (1.2) we have $u(a,r) \leq u(b,r) \leq u(c,r)$ for r small. Therefore $a \leq b \leq c$. We recall now that a, b, c are functions of τ. Furthermore, from Corollary (3.8) we know that both $a(\tau)$ and $c(\tau)$ go to ∞ as τ goes to $-\infty$ and they go to 0 as τ goes to ∞. Thus the Lemma easily follows. $\square$

Now we are ready to prove the following Proposition

3.10 Proposition. *Assume that Hypothesis P1 is satisfied and that the limits* $\lim_{t\to-\infty} h_{1,l}(t) \geq 0$ *and* $0 < \lim_{t\to-\infty} h_{2,l}(t) < \infty$. *Then there exists* $M > 0$ *such that* $u(u_0, r)$ *is a crossing solution for any* $u_0 > M$. *Furthermore if we denote by* $r = R_1(u_0)$ *the first value for which* $u(u_0, r) = 0$ *we have* $\lim_{u_0\to\infty} R_1(u_0) = 0$.

Proof. In the previous Lemma we have proved that for any τ, there exists a trajectory $X(t)$ such that $X(t) \to 0$ as $t \to -\infty$ and $X(t) \in U^+$ for any $t < 0$. Now we want to show that for $\tau << 0$ these trajectories $X(t)$ have to cross the y negative semi axis for a certain $t = \bar{t}$. We recall that $T(\tau)$ is uniformly bounded as $\tau \to -\infty$. Thus for any $M > 0$ we can find τ_0 such that $T_1(\tau) = \tau + 2T(\tau) < -M$ for any $\tau < \tau_0$. Moreover for any $\epsilon > 0$ small, there exists $M > 0$ such that, for any $\tau < -M$, we have $|b_1(T_1(\tau)) - a_1(T_1(\tau))| + |b_2(T_1(\tau)) - a_2(T_1(\tau))| < \epsilon$. It follows that the points $\overline{\xi^+}(\tau)$ and $\underline{\xi^+}(\tau)$ are arbitrarily close as $\tau \to -\infty$. Thus the set $e(\tau)$ shrinks to a curve as $\tau \to \infty$. So $\overline{x^+}(\tau) - \underline{x^+}(\tau) = \delta(\epsilon)$, where $\delta(0) = 0$ and δ depends continuously on ϵ. Furthermore the distance between stable and unstable manifold of the autonomous system $\tilde{W}^u$ and $\tilde{W}^s$, measured along the isocline C is always uniformly positive. Namely for any constant $N > 0$ there exists a constant $d > 0$ such that $x^+(c_1, c_2) - x^-(c_1, c_2) > d$ for any $c_2 < N$.

Therefore for continuity reasons, we have that $\underline{x^+}(\tau) - x^-(c_1, c_2) > 0$ for any $c_1 \in [a_1(T_1(\tau)), b_1(T_1(\tau))]$ and $c_2 \in [a_2(T_1(\tau)), b_2(T_1(\tau))]$. In particular $\underline{x^+}(\tau) - x^-(b_1(T_1(\tau)), a_2(T_1(\tau))) > 0$.

Fix $\tau < -M$ and consider the non-autonomous system (3.4); from Lemma (3.9) we know that there is $Q = (Q_x, Q_y) \in C$ such that $X(Q, t) \in U^+$ for any $t < 0$ and $\lim_{t \to -\infty} X(Q, t) = (0, 0)$. Consider the trajectory $\hat{X}(Q, t)$ of the autonomous system where $h_{1,l}(t) \equiv b_1(T_1(\tau))$ and $h_{2,l}(t) \equiv a_2(T_1(\tau))$, departing from Q at $t = 0$. Since $Q_x > x^-(b_1(T_1(\tau)), a_2(T_1(\tau)))$ there exists a value $t = \hat{T}(\tau)$ for which $\hat{X}(Q, t) = (\hat{x}(Q, T), \hat{y}(Q, t))$ crosses the negative y semi-axis. Eventually restricting ϵ we can assume that $\hat{T}(\tau) < 2T(\tau)$. We claim that $X(Q, t) = (x(Q, t), y(Q, t))$ has to cross the negative y semi-axis for some $t < \hat{T}(\tau)$. We will prove in fact that $x(Q, t) \le \hat{x}(Q, t)$ for any $0 < t < \hat{T}(\tau)$.

Consider the autonomous system and define $\partial L = \{\hat{x}(Q, t) \mid 0 \le t \le \hat{T}(\tau)\}$. Call L the bounded subset enclosed by ∂L, the segment of the y axis between the origin and ∂L, and the segment of the isocline C connecting the origin with Q. Turn now to consider the non-autonomous system; first of all observe that $\hat{X}(Q, t)$ is not anymore a trajectory. Note that the flow of the non-autonomous system on ∂L points towards the exterior of L, for any $0 < t < \hat{T}(\tau)$. Thus $X(Q, t)$ cannot cross $\hat{X}(Q, t)$ for $0 < t < \hat{T}(\tau)$ so the corresponding $u(r)$ is a crossing solution.

Define $t_0 = \inf\{t > 0 \mid x(Q, t) > \hat{x}(Q, t)\}$, we want to show that $t_0 = 0$. It follows that $\frac{d}{dt} x(Q, t) > \frac{d}{dt} \hat{x}(Q, t)$ for t in a right neighborhood of t_0. But this implies $y(Q, t_0) > \hat{y}(Q, t_0)$, so $X(Q, t)$ has to cross $\hat{X}(Q, t)$, but this is a contradiction so the claim is proved.

Therefore the corresponding solution $u(r)$ of (1.2) is a crossing solution. Then observe that $T(\tau)$ is bounded for any $\tau < -M$, therefore $\tau + \hat{T}(\tau) \to -\infty$ as $\tau \to -\infty$. We point out that $x(Q,t)$ corresponds to a crossing solution $u(r)$ of (1.2), (1.3) and that from Lemma (3.9), $u_0 = b\exp(-\alpha\tau) \to +\infty$ as $\tau \to -\infty$. Moreover if $R_1 > 0$ is the first value for which $u(R_1) = 0$, we have $R_1 = \exp(\tau + \hat{T}(\tau)) \to 0$ as $\tau \to -\infty$. This concludes the proof of the Proposition. $\square$

Now adapting the reasoning in [9] we might prove that I^- is open. However the proof is not completely elementary so we give a new proof which is more natural in this dynamical context. It is worthwhile to point out that this new proof works only if Hyp. P1 is satisfied, while reasoning as in [9] we can prove the result also without this assumption.

3.11 Lemma. *Assume that Hypotheses P1 and F1 are satisfied, then I^- is open.*

Proof. Assume that $d \in I^-$, and consider a sequence $d_k \to d$; we want to prove that $d_k \in I^-$ for k large. Fix $l = q_2$ and consider the trajectories $X(d_k, t)$ of (3.2) corresponding to the solutions $u(d_k, r)$ through (3.1). We know that there exists T_1 and T_2, $T_2 > T_1$ such that $X(d, T_1)$ belongs to the negative y semi-axis, and $X(d, T_2)$ is in the 3^{rd} quadrant. From Lemma (2.7) we know that the solutions $u(c, r)$ of (1.2) depend continuously from the initial data c for any $r \in J(c) = (0, R_c)$. Furthermore note that $R_{d_k} \to R_d$ as $k \to \infty$. Therefore, for any $\epsilon > 0$, we can find $N > 0$ large enough such that there exists $\bar{R} = \exp(\bar{T}) < 1$, $\bar{R} \in J(d_k)$ and $|u(d_k, \bar{R}) - u(d, \bar{R})| < \epsilon$ for any $k > N$. Therefore $|X(d_k, \bar{T}) - X(d, \bar{T})| < \epsilon$ for any $k > N$.

We recall now that, when Hyp. P1 and F1 are satisfied, the solutions of system (3.2) depend continuously on their initial data in each compact set. Therefore, for any $\delta > 0$ we can find $\epsilon > 0$ small enough so that $\sup_{t \in [\bar{T}, T_2]} |X(d_k, t) - X(d, t)| < \delta$. Therefore, possibly choosing a larger N, we can assume that $X(d_k, t)$ is in the 3^{rd} quadrant for $t = T_2$ and $k > N$. So, for continuity reasons, $X(d_k, t)$ has cross the y negative semi-axis for some $t = \hat{T}(k) < \bar{T}$. Thus $u(d_k, r)$ is a crossing solution and $d_k \in I^-$ for k large. $\square$

Putting together Lemma (2.6), Lemma (2.8), Lemma (3.11), Proposition (3.10), we have the proof of Lemma (2.2).

3.3 Existence of Singular Ground States

We want to prove now the existence of radial S.G.S. for (1.1), so we have to investigate the behaviour of positive solutions for r large. Consider system

(3.4) where we set $l = q_1$; if Hypotheses P1 and F3 are satisfied the system is Lipschitz continuous and uniformly continuous with respect to t for any $t > 0$. Therefore using invariant manifold theory for non-autonomous system, we can construct a stable manifold $\bar{W}^s_{q_1}(\tau)$ departing from the origin with the following property:

$$\bar{W}^s_{q_1}(\tau) := \{Q \in \mathbb{R}^2_+ \mid X_{q_1}(Q,t) \in U^- \text{ if } t > 0 \text{ and } \lim_{t\to\infty} X_{q_1}(Q,t) = (0,0)\}.$$

This manifold can be constructed as follows, see [13], [12], [6], [7]. We can take a neighborhood ζ of the origin, in which there exists a set $\bar{W}^s_{q_1,loc}(\tau)$ satisfying the following property, see [13], [12]:

$$\bar{W}^s_{q_1,loc}(\tau) = \{Q \in \zeta \mid X_{q_1}(Q,t) \in U^- \text{ if } t > 0 \text{ and } \lim_{t\to\infty} X_{q_1}(Q,t) = (0,0)\}.$$

Then we can extend this local manifold to a global stable manifold, using the flow as follows:

$$\bar{W}^s_{q_1}(\tau) = \bigcup \{X_{q_1}(Q, \tau - t, t) : Q \in \bar{W}^s_{q_1,loc}(\tau - t)\},$$

where $X_{q_1}(Q, s, \bar{t})$ is the trajectory of system (3.4) with $\tau = s$, departing from Q at $t = 0$, evaluated at $t = \bar{t}$. The manifolds $\bar{W}^s_{q_1}(\tau)$ are Lipschitz continuous and depend continuously on τ.

We want to prove that $\bar{W}^s_{q_1}(\tau)$ departs from the origin, enters U^- and it has its extremum in the isocline C. So we take $Q \in \zeta \cap U^-$, and we have to follow $X_{q_1}(Q,t)$ backwards in t and to prove that it has to cross the isocline C. We can find $B_2 > A_1 > 0$ such that $0 < h_{1,q_1}(\tau + t) < A_1$ and $h_{2,q_1}(\tau + t) > B_2$, for any $t < 0$. Consider the autonomous system (3.4) where $l = q_1$, $h_{1,q_1}(t) \equiv A_1$ and $h_{2,q_1}(t) \equiv B_2$ and call $\underline{X_{q_1}}(Q,t)$ the trajectory of such a system departing from Q. Using Lemma (3.6) it can be shown that $\underline{X_{q_1}}(Q,t)$ has to cross the isocline C at a certain $T < 0$; let us define the following curvilinear segment $\partial L = \{\underline{X_{q_1}}(Q,t); \mid T \le t \le 0\}$. We call L the bounded set enclosed by ∂L, the segment of C between the origin and ∂L, and the rectilinear segment connecting the origin and Q.

Now we go back to the non-autonomous system (3.4); $\underline{X_{q_1}}(Q,t)$ is not anymore a trajectory but, reasoning as in Proposition (3.10), we can prove that the flow on ∂L goes towards the exterior of L, for any $t \le 0$. Therefore going backwards in t, $X_{q_1}(Q,t)$ is forced to stay in L until it crosses the isocline C. Note that for the solution $u(r)$ of (1.2) corresponding to $X_{q_1}(P,t)$ where $P \in \bar{W}^s_{q_1}(\tau)$, we have the following, see Proposition (3.3):

$$\lim_{r\to\infty} u(r) r^{\frac{n-p}{p-1}} = 0 \qquad \text{and} \qquad \lim_{r\to\infty} u'(r) r^{\frac{n+1}{p-1}} = 0. \tag{3.5}$$

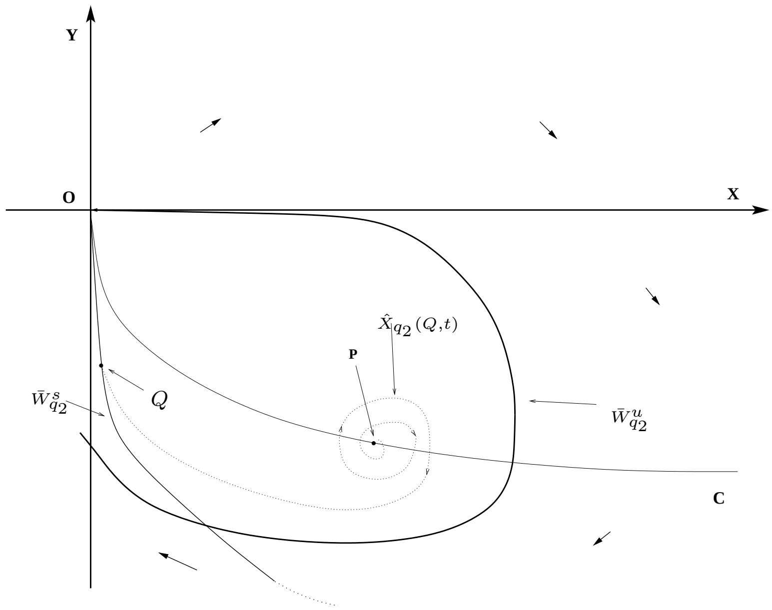

Figure 3: Construction of singular ground states.

Proof of Theorem (2.4). Fix $\tau < -M$, where $M > 0$ is the large constant used in the proof of Proposition (3.10) and consider $\bar{W}^s_{q_1}(\tau)$. Fix $t = 0$; we want to pass from the non-autonomous system (3.4) where $l = q_1$ to system (3.4) where $l = q_2$. Thus we use the following change of coordinates $K_1 : (x_{q_1}, y_{q_1}) \to (x_{q_2}, y_{q_2})$

$$K_1(x_{q_1}, y_{q_1}) = (x_{q_1} \exp((\alpha_{q_2} - \alpha_{q_1})\tau), y_{q_1} \exp((\beta_{q_2} - \beta_{q_1})\tau)).$$

Note that K_1 is a linear transformation which fixes the origin. Thus the manifold $\bar{W}^s_{q_1}(\tau)$ is transformed by K in a connected 1-dimensional manifold, say $\bar{W}^s_{q_2}(\tau)$, contained in the 4^{th} quadrant, and having the origin in one of its extrema. Furthermore observe that if $Q \in \bar{W}^s_{q_2}(\tau)$, the trajectory $X_{q_2}(Q,t)$ is such that $\lim_{t\to\infty} X_{q_2}(Q,t) = (0,0)$. This follows from the fact that for the corresponding solution $u(r)$ of (1.2), (3.5) holds. Consider now the following extended system, where we have added the extra-variable $z = r^s = e^{st}$, where $s > 0$ is a constant.

$$\begin{aligned} \dot{x_l} &= \alpha_l x_l + y_l |y_l|^{\frac{2-p}{p-1}} \\ \dot{y_l} &= \gamma_l y_l + h_{1,l}(\tau + t)\psi_{q_1}(x_l) - h_{2,l}(\tau + t)\psi_{q_2}(x_l) \\ \dot{z} &= sz \end{aligned} \tag{3.6}$$

We set now $l = q_2$. Note that there exists a two dimensional unstable manifold departing from the origin. Such a manifold is made up of all and only the trajectories corresponding to regular solutions of (1.2). From Proposition (3.10) it follows that there exists $D > 0$ such that all the sections $\hat{w}^u(d)$ made up intersecting the stable manifold and the plane $z = d$, where $0 < d < D$, are shaped as in Fig. (3). Namely, departing from the origin, $\hat{w}^u(d)$ enters U^+ and crosses the isocline C, then crosses the y negative semi-axis. Call $B(d)$ the Lipschitz one-dimensional manifold made up by the segment of $\hat{w}^u(d)$ between the origin and the first intersection with the y semi-axis, and by the segment of the y semi-axis between this intersection and the origin. Observe that $B(0, D) = \cup_{0<d<D} B(d) \times \{d\}$ is a Lipschitz manifold, homeomorphic to a cylinder and contained in $\mathbb{R}^2_+ \times (0, D)$. To pass from (3.4) with $l = q_2$ to (3.6), again with $l = q_2$, we adjoin the z-variable therefore $\bar{W}^s_{q_2}(\tau)$ is transformed into $\bar{W}^s_{q_2}(\tau) \times \{e^{s\tau}\}$. Note that there exist uncountably many points $Q \in \bar{W}^s_{q_2}(\tau) \times \{e^{s\tau}\}$ which belong to the bounded set enclosed by $B(e^{s\tau}) \times \{e^{s\tau}\}$. We want to follow backwards in t the trajectories $\hat{X}_{q_2}(Q, t)$ of (3.6) departing, at $t = 0$, from points Q as above. Note that the α-limit set of every bounded trajectory of (3.6) is contained in the plane $z = 0$, since $z(t)$ is always increasing in t. Furthermore, from subsection 3.1 we know that in this plane there are no periodic trajectories and three critical points, which are the origin, $P = (P_x, P_y, 0)$ where $P_y < 0 < P_x$ and $-P$; so bounded trajectories must have one of these points as α-limit set. Note that $\hat{X}_{q_2}(Q, t)$ cannot cross $B(0, D)$ for any $t < 0$, in fact otherwise the corresponding solution $u(r)$ of (1.2) would be a G.S., contradicting Proposition (3.10). Therefore it is bounded going backward in t and continuable for any $t < 0$. Therefore it must have the critical point $P = (P_x, P_y, 0)$ as α-limit set. It follows that $\hat{X}_{q_2}(Q, t) \in \mathbb{R}^2_+$ for any t and that

$$\lim_{t\to-\infty} \hat{X}_{q_2}(Q, t) = P \qquad \text{and} \qquad \lim_{t\to\infty} \hat{X}_{q_2}(Q, t) = O$$

This concludes the proof of Theorem (2.4). □

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